Chapter 5 Fluid Motion

5-1 Introduction

In this Chapter we consider the dynamics of a fluid. By this concept we mean an interacting many-particle system containing a large number of particles per unit volume so that even a macroscopically small volume element in space contains a large number of particles. We may then examine the motion of such a system by considering the motion of a representative "fluid particle", i.e. a volume element in space containing many particles but macroscopically "small". We characterize the motion of such a fluid particle by the velocity of the fluid $\mathbf{u}(\mathbf{r}, t)$ at the position $\mathbf{r}$ of the fluid particle and the density of the fluid at this point. Strictly speaking the "position" referred to by $\mathbf{r}$ is fuzzy up to the size of the volume element delineating our fluid particle. Likewise, the velocity $\mathbf{u}$ at the point $\mathbf{r}$ is to be thought of as an average velocity throughout this volume element. One should carefully bear in mind these physical idealizations of the system in order to appreciate what is to follow.

A complete treatment of fluid motion within the confines of a single chapter is not practical. This extensive field deserves a text book for itself. Our purpose here is rather a much more modest one: to introduce the reader to the basic ideas and equations, with some applications of the laws of motion governing *perfect* fluids. A perfect fluid is one devoid of viscosity and hence also of most physical interest. Nevertheless, a study of such systems is essential as a first step to understanding the incredibly complicated field of real fluids.

Some kinematical concepts are in order first. We consider a finite volume ΔV in a fluid. It contains an amount of mass

$$\int_{\Delta V} \rho \, dV \tag{5.1}$$

if $\rho = \rho(\mathbf{r}, t)$ designates the fluid density at $\mathbf{r}$ at time t and $dV = dx\,dy\,dz$ the volume element at position $\mathbf{r}$. If the fluid is in motion a net amount of fluid

$$\oint \rho u_n \, da \tag{5.2}$$

will pass out of ΔV in unit time. In this expression u_n is the fluid velocity normal to the surface element da pointing out of the volume ΔV and the integral goes over the entire bounding surface of ΔV. This result follows from the observation that ρu_n is the mass of fluid transported per unit time across unit area perpendicular to u_n. A net outflow means mass is being lost from ΔV. This can only occur at the expense of a decrease in mass in ΔV since mass is conserved in a fluid. The decrease in mass per unit time is

$$-\frac{\partial}{\partial t}\int_{\Delta V}\rho\, dv = -\int_{\Delta V}\frac{\partial \rho}{\partial t}\, dv, \tag{5.3}$$

since ΔV is fixed. Equating this to the mass outflow, we obtain the conservation law

$$\int \frac{\partial \rho}{\partial t}\, dv + \oint \rho u_n\, da = 0. \tag{5.4}$$

The second member of this equation can be transformed into a volume integral using Gauss' theorem in reverse,

$$\oint \rho u_n\, da = \int_{\Delta V} \mathrm{div}(\rho \mathbf{u})\, dv \tag{5.5}$$

so that

$$\int_{\Delta V}\{\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u})\}\, dv = 0. \tag{5.6}$$

But the volume ΔV is arbitrary. Therefore, if we assume it is small enough so that ρ and $\mathbf{u}$ can be replaced by their values "at the center" of ΔV (but still large enough to contain many particles), this equation reads

$$\{\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u})\}\int_{\Delta V} dv = 0, \tag{5.7}$$

or, equivalently,

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = 0. \tag{5.8}$$

This is the *equation of continuity* expressing the conservation of mass in fluid flow in differential form.

5-2 *Euler's Equations for Fluid Motion*

So much for kinematical preliminaries. We now go over to the task of examining the dynamics of fluid flow. The action principle of Chapter 1 will again be our guide. We take up this principle in the form of (1.31),

$$\int_{t_1}^{t_2}(\delta T + \delta W)\, dt = 0, \tag{5.9}$$

and tailor it to suit our problem. We straightaway state that we are only going to study *incompressible* fluids (in practice ordinary liquids like water or alcohol) that are characterized by exhibiting no noticeable change in their density in space or time when subjected to external stresses. This means

$$\rho = \rho_0 \quad \text{a constant}, \quad \frac{\partial \rho}{\partial t} = 0, \quad \operatorname{grad} \rho = 0, \tag{5.10}$$

and so

$$\operatorname{div} \mathbf{u} = 0, \tag{5.11}$$

from the equation of continuity. Equation (5.9) is a *condition* the flow velocity must obey at each point in a fluid if the fluid is incompressible.

We can now fill in (5.9). The kinetic energy density of fluid contained in a volume element dv is obviously

$$\frac{1}{2}\rho_0 u^2, \tag{5.12}$$

so that

$$T = \int \frac{1}{2}\rho_0 u^2 \, dv, \quad \delta T = \int \rho_0 (\mathbf{u} \cdot \delta \mathbf{u}) \, dv, \tag{5.13}$$

if $\delta \mathbf{u}$ is the variation induced in $\mathbf{u}$ by the virtual displacement $\delta \mathbf{r}$ of the fluid particle at $\mathbf{r}$. The virtual work is simply calculated as

$$\delta W = \int (\mathbf{F} \cdot \delta \mathbf{r}) \, dv \tag{5.14}$$

at any internal point in the fluid if $\mathbf{F}$ is the applied force (per unit volume) at this point. Thus (5.9) reads

$$\int_{t_1}^{t_2} \int \{\rho_0 \mathbf{u} \cdot \delta \mathbf{u} + \mathbf{F} \cdot \delta \mathbf{r}\} \, dv \, dt = 0 \tag{5.15}$$

for an incompressible fluid subjected to a force distribution $\mathbf{F}$ per unit volume throughout its volume. Equation (5.15) does not account for the *internal* stresses in a fluid that render it incompressible. Microscopically these are due to the strong interparticle forces that act at liquid densities. Macroscopically we replace these forces of constraint (for that is what they are) by a condition of constraint of the fluid motion. This condition is just (5.11). This equation must be obeyed by all virtual motions of our fluid. Thus, $\operatorname{div}\mathbf{u} = 0$ implies also

$$\operatorname{div}(\delta \mathbf{u}) = 0, \quad \text{or} \quad \frac{d}{dt}[\operatorname{div}(\delta \mathbf{r})] = 0, \tag{5.16}$$

since

$$\delta \mathbf{u} = \frac{d}{dt}(\delta \mathbf{r}), \tag{5.17}$$

according to a by now familiar argument. We would like (5.16) to be obeyed for all times. This is certainly the case if we restrict ourselves to virtual displacements that are subject to the constraint

$$\operatorname{div}(\delta \mathbf{r}) = 0. \tag{5.18}$$

A Lagrange multiplier λ is necessary to include this constraint in (5.15), which then becomes

$$\int_{t_1}^{t_2}\int\{\rho_0\mathbf{u}\cdot\delta\mathbf{u}+\mathbf{F}\cdot\delta\mathbf{r}+\lambda\mathrm{div}(\delta\mathbf{r})\}\,dv\,dt=0. \tag{5.19}$$

Two transformations are now necessary. First we make use of the relation (5.17) to rewrite the first term as

$$\int_{t_1}^{t_2}\int\rho_0\mathbf{u}\cdot\delta\mathbf{u}\,dv\,dt=-\int_{t_1}^{t_2}\int\rho_0\frac{d\mathbf{u}}{dt}\cdot\delta\mathbf{r}\,dv\,dt, \tag{5.20}$$

after imposing the usual boundary conditions $\delta\mathbf{r}(t_1)=0$, $\delta\mathbf{r}(t_2)=0$, at the endpoints of the time integration. The second transformation involves the last integral whose integrand may be written as

$$\lambda\mathrm{div}(\delta\mathbf{r})=\mathrm{div}(\lambda\delta\mathbf{r})-\mathrm{grad}\,\lambda\cdot\delta\mathbf{r}. \tag{5.21}$$

Carrying out a spatial integration and using Gauss' theorem once more, one obtains (δr_n is the component of $\delta\mathbf{r}$ normal to the surface element da)

$$\int\lambda\mathrm{div}(\delta\mathbf{r})\,dv=\oint\lambda\delta r_n\,da-\int\mathrm{grad}\,\lambda\cdot\delta\mathbf{r}\,dv. \tag{5.22}$$

Reserving remarks on the effect of the surface integral in this expression for later, we temporarily ignore it and obtain

$$\int_{t_1}^{t_2}\int\{-\rho_0\frac{d\mathbf{u}}{dt}+\mathbf{F}-\mathrm{grad}\,\lambda\}\cdot\delta\mathbf{r}\,dvdt=0 \tag{5.23}$$

for the final form of the action principle. As before, we consider the time interval $t_2-t_1=\Delta t$ to be vanishingly small and conclude that

$$[\int(-\rho_0\frac{d\mathbf{u}}{dt}+\mathbf{F}-\mathrm{grad}\,\lambda)\cdot\delta\mathbf{r}\,dv]\cdot\int_{\Delta t}dt=0 \tag{5.24}$$

where the variables in the round brackets () now refer to a particular instant of time t. But $\delta\mathbf{r}$ can now be chosen arbitrarily. We assume it vanishes everywhere but in a small volume ΔV surrounding the point $\mathbf{r}$. Then,

$$[(-\rho_0\frac{d\mathbf{u}}{dt}+\mathbf{F}-\mathrm{grad}\,\lambda)\cdot\delta\mathbf{r}]\cdot\int_{\Delta v}dv\int_{\Delta t}dt=0, \tag{5.25}$$

showing that the relation

$$-\rho_0\frac{d\mathbf{u}}{dt}+\mathbf{F}-\mathrm{grad}\,\lambda=0 \tag{5.26}$$

holds at each point $\mathbf{r}$ in the fluid at time t. The components of this equation are easily identified physically. The term $-\rho_0 d\mathbf{u}/dt$ is the negative of the rate of change of momentum (per unit volume) of a fluid moving with velocity $\mathbf{u}$, and $\mathbf{F}$ is the force on this unit volume. But what is $-\text{grad}\lambda$? This clearly must be a *force of constraint* that arises in order to accomodate the condition (5.11) of incompressible flow. In this context λ can be identified with the *pressure* p at a point in the fluid since

$$-\text{grad}\, p \tag{5.27}$$

represents a pressure gradient or force per unit volume a fluid particle at $\mathbf{r}$ will experience. This interpretation of pressure gives it the physically understandable meaning of an internal reaction in the fluid in order to obey the constraint of incompressibility. We note that p is an unknown that has to be determined along with $\mathbf{u}$ from the equations of motion.

Equation (5.26), with the interpretation (5.27) of λ constitute the equations of motion we seek. They were first obtained by Leonhard Euler in 1755. To give Euler's form of these equations, some further transformations are necessary. We first carefully note the meaning of the differential quotient

$$\frac{d\mathbf{u}}{dt} \tag{5.28}$$

in (5.26). Since $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$, changes in $\mathbf{u}$ will occur through its explicit dependence on time at a fixed point in space and due to the fact that the position $\mathbf{r}$ of the fluid element we are examining changes with time. For example,

$$\frac{du_x}{dt} = \frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}, \tag{5.29}$$

where $\mathbf{u} = \{u_x, u_y, u_z\}$. Similar equations are found for u_y and u_z. One may write the summarizing vector expression as

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}, \tag{5.30}$$

where $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, but this is only true if the cartesian components of $\mathbf{u}$ are used. Now, the vector operation can be applied in the sense of the following formula

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \text{grad}(\frac{1}{2}u^2) - \mathbf{u}\times\text{curl}\,\mathbf{u}. \tag{5.31}$$

Therefore, we can write Euler's equation (5.26) in the symbolic form

$$\rho_0(\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u}\times\text{curl}\,\mathbf{u}) + \text{grad}(\frac{1}{2}\rho_0 u^2 + p) = \mathbf{F} \tag{5.32}$$

that is valid in any system of coordinates for $\mathbf{u}$. This has to be supplemented by the incompressible flow condition $\text{div}\mathbf{u} = 0$ in all our applications of course, giving us two equations for finding the unknowns u and p.

We yet have to account for the role placed by the ignored surface integral

$$\oint \lambda\delta r_n\, da = \oint p\delta r_n\, da \tag{5.33}$$

in (5.22). Its role can be elucidated by the following considerations. First, suppose the fluid surface is in contact with a rigid bounding wall. Then $\delta r_n = 0$ at every point on such a rigid surface and the integral must vanish. This cannot be the case, however, if the fluid has a *free surface*. For then, there is certainly no reason to maintain $\delta r_n = 0$. However, a second contribution to the virtual work comes into play at a free surface: One has to invoke a microscopic picture of the fluid to realize that an additional force $-F'_n$ per unit area pointing along the inward normal arises at a free surface due to the unbalanced internal forces on particles "at" the surface coming from particles below them. Consequently, δW has to be supplemented by

$$-\int F'_n \delta r_n \, da \tag{5.34}$$

at a free surface. Inclusion of this term in the action principle, together with (5.37), leads to a contribution

$$\int_{t_1}^{t_2} \int (p - F'_n) \delta r_n \, da \, dt, \tag{5.35}$$

that vanishes provided

$$p = F'_n. \tag{5.36}$$

Thus, the action principle also provides us with a *surface boundary condition* on the pressure at a free surface.

Before discussing specific applications of (5.32), some general remarks are in order. The first thing that strikes one about Euler's equation is its *non-linear* character. This fact alone accounts for the immense complexity of fluid flow problems, for it negates entirely the principle of superposing simple solutions of a problem to build up a general solution. One's natural inclination is to try to restore this principle by *linearizing* (5.32), that is, by dropping terms quadratic in the velocity **u**. This underscores the problem of slowly-moving fluids and is the subject of subsequent sections. However, there is a particular case of (5.32) for which a first integral may readily be found. This is the case of irrotational flow:

$$\operatorname{curl} \mathbf{u} = 0, \tag{5.37}$$

which is also steady ($\frac{\partial \mathbf{u}}{\partial t} = 0$). Then that equation reduces to

$$\operatorname{grad}(\frac{1}{2}\rho_0 u^2 + p) = \mathbf{F} = -\operatorname{grad} U, \tag{5.38}$$

if we assume in addition that $\mathbf{F}$ is derivable from a potential field U per unit volume. Then space integration results in

$$\frac{1}{2}\rho_0 u^2 + p + U = \text{constant}, \tag{5.39}$$

a relation first obtained by Bernoulli in 1728. (Note the date! It was obtained *before* the work of Euler). Bernoulli's equation is perhaps the most fundamental relation in elementary fluid mechanics. In this equation $\rho_0 u^2/2$ is the kinetic energy per unit volume of fluid, U the external potential, and the pressure p a kind of potential energy per unit volume arising from internal forces. One surprising consequence of (5.39) is the drop in pressure it dictates with increasing velocity of fluid flow, a well-known (but often not appreciated) fact to most students of elementary physics.

5-3 Potential Flow

We showed in the previous section that Bernoulli's equation was the result of integrating Euler's equation under the condition (5.37) corresponding to irrotational flow. But we know from vector analysis that it is always possible to construct an irrotational vector by taking the gradient of any scalar function φ. Thus, curl $\mathbf{u} = 0$ identically if

$$\mathbf{u} = -\operatorname{grad}\varphi \tag{5.40}$$

(the minus sign is by convention only). Since the flow we consider is also incompressible we can superpose this condition on (5.39) to obtain

$$\nabla^2\varphi = 0 \tag{5.41}$$

that is Laplace's equation for φ. The function φ is called the *velocity potential* by analogy with potential theory of conservative force fields and flow governed by (5.40) and (5.41) is called *potential flow*. It is important to notice that (5.41) does not yet contain any dynamics. To obtain this we need to substitute (5.40) into Bernoulli's equation,

$$\frac{1}{2}\rho_0(\operatorname{grad}\varphi)^2 + p + U = \text{constant}, \tag{5.42}$$

an equation which now determines the pressure field p of a given velocity potential.

A more general version of (5.42) is obtained if we keep the irrotationality condition (5.37) but allow the flow to be non-steady $(\partial U/\partial t \neq 0)$. Then we find that

$$\operatorname{grad}(-\rho_0\frac{\partial\varphi}{\partial t} + \frac{1}{2}\rho_0(\operatorname{grad}\varphi)^2 + p + U) = 0, \tag{5.43}$$

after using (5.40) for the first term in (5.32) and interchanging the order of differentiation $\partial/\partial t$ with grad. It therefore follows that

$$-\rho_0\frac{\partial\varphi}{\partial t} + \frac{1}{2}\rho_0(\operatorname{grad}\varphi)^2 + p + U = \text{constant} \tag{5.44}$$

once more, but where the "constant" can now be a function of time.

5-4 *Water Waves*

Let us apply (5.44) to the phenomenon of surface waves on an open sea. The coordinate system O_{xyz} we use is situated on the surface of the sea as illustrated in Fig. 5.1. The first step is to find a suitable velocity potential φ. We assume that all quantities associated with surface waves are *periodic* in time, with period $2\pi/\omega$.

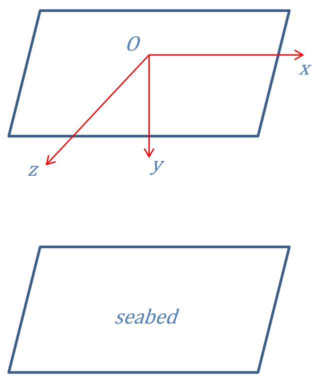

Figure 5.1: Coordinate system for discussing water waves on an open sea.

This means φ has the form

$$\varphi(x,y)e^{-i\omega t} \tag{5.45}$$

if we restrict the discussion to waves propagating in the x direction. Thus, we must find the function $\varphi(x,y)$ that satisfies

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})\varphi = 0. \tag{5.46}$$

Assuming separability, $\varphi = f(x)g(y)$, one finds that

$$\frac{1}{f}\frac{d^2 f}{dx^2}+\frac{1}{g}\frac{d^2 g}{dy^2} = 0, \tag{5.47}$$

or that

$$\frac{d^2 f}{dx^2}+k^2 f = 0, \quad \frac{d^2 g}{dy^2}-k^2 g = 0, \tag{5.48}$$

where k^2 is a separation constant. These two equations admit a variety of solutions. The ones we require are determined by *boundary conditions* at the sea bed and water surface. Suppose for simplicity that the sea is very deep. The φ should certainly vanish at the sea bed, $y \to \infty$, since the velocity u_y must vanish there. This dictates the choice

$$g(y) = e^{-ky} \tag{5.49}$$

for the function $g(y)$. The solutions for f are oscillatory in space as a function of x. We want here to study *travelling* waves on the surface of the sea, and so take

$$f(x) = e^{ikx}. \tag{5.50}$$

Our final solution for the velocity potential thus becomes

$$\varphi(x,y)e^{-i\omega t} = Ae^{-ky}e^{i(kx-\omega t)}, \tag{5.51}$$

where A is an arbitrary amplitude. We now have to ensure that the velocity of flow given by this potential is consistent with Bernoulli's equation at the surface of the water. By doing so, we introduce a dynamic element into the calculation. This compatibility problem is only tractable in the event that we consider small deviations of the surface of the water from horizontal. Call this displacement η and count the downward direction as positive, see Fig. 5.2.

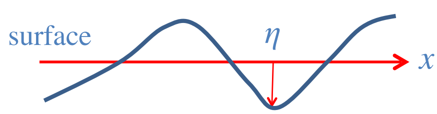

Figure 5.2: Deviations from the surface are designated by η.

Specifically we then assume that (i) this displacement is so small that all terms quadratic in η can be ignored, (ii) that the only restoring force is due to gravity so that $U = -\rho_0 g\eta$ and (iii) the atmospheric pressure at the surface can be ignored so that $p_s \simeq 0$. These assumptions, together with the assumed form (5.51) of the velocity potential yield

$$i\omega\rho_o Ae^{-k\eta}e^{i(kx-\omega t)} - \rho_0 g\eta = 0 \tag{5.52}$$

for (5.44), after setting the constant on the right hand side equal to zero. This is the only choice that does not interfere with the propagation of a travelling wave on the surface. But

$$k\eta \ll 1, \tag{5.53}$$

so $\exp(-k\eta)$ can be replaced by unity in (5.52) and we get

$$\eta(x,t) = \frac{i\omega}{g}Ae^{i(kx-\omega t)}. \tag{5.54}$$

This relation provides an alternative way of calculating the velocity of the water at the surface, in particular the "sinking speed"

$$\frac{\partial\eta}{\partial t} = \frac{\omega^2}{g}Ae^{i(kx-\omega t)} \tag{5.55}$$

in the y direction. But this must also be given by

$$u_y = -\text{grad}_y\,\varphi e^{-i\omega t} = kAe^{i(kx-\omega t)} \quad \text{at} \quad y = 0. \tag{5.56}$$

Consequently, these two expressions are compatible only if ω depends on the wave number k like

$$\omega = \sqrt{gk}. \tag{5.57}$$

In turn this means that the *phase velocity* (the velocity with which planes of equal phase propagate)

$$u_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g}{2\pi}} \cdot \sqrt{\lambda} \tag{5.58}$$

is dependent on the wave length $\lambda = 2\pi/k$ of the oscillation. *Dispersion* is present; long wave lengths travel faster than short ones.

Since the restoring force causing this wave motion is gravity, the phenomenon we have just discussed is usually called a *gravity wave.* Waves encountered on the open sea are of this type.

Actually, the dispersion relation $\omega = \sqrt{gk}$ is only valid if the wave number k is not too high (the wave length not too short). At short wave lengths a new type of wave, called a *capillary wave*, can exist. This circumstance comes about because the hitherto neglected effect of the surface tension of the water enters the picture. The pressure at the surface due to surface tension is proportional to the *curvature* of the water surface (just like the restoring force on a string under tension):

$$p = T_0 \frac{\partial^2 \eta}{\partial x^2}. \tag{5.59}$$

Here, T_0 is a constant of proportionality called the surface tension (units: dynes/cm or Newtons/meter). For short wave lengths $\partial^2\eta/\partial x^2$ is large and this term can no longer be neglected. Supplementing (5.52) on the left by this additional pressure term one has

$$i\omega\rho_0 A e^{i(kx-\omega t)} + T_0 \frac{\partial^2 \eta}{\partial x^2} - \rho_0 g \eta = 0 \tag{5.60}$$

for η. Assuming a travelling plane wave profile

$$\eta = \eta_0 e^{i(kx-\omega t)}, \tag{5.61}$$

for η once more, one finds that

$$i\omega A - k^2 \frac{T_0}{\rho_0} \eta_0 - g\eta_0 = 0. \tag{5.62}$$

The condition of matched sinking speeds becomes

$$kA = -i\omega\eta_0 \tag{5.63}$$

that provides a second relation between the unknown amplitudes A and η_0. The two equations for A and η_0 are compatible only if

$$\begin{vmatrix} i\omega & -k^2 \frac{T_0}{\rho_0} - g \\ k & i\omega \end{vmatrix} = 0, \tag{5.64}$$

or

$$\omega = \sqrt{gk + k^3 \frac{T_0}{\rho_0}}. \tag{5.65}$$

If the wave length is short enough, the last term under the radical dominates, and we obtain the dispersion relation for pure capillary waves

$$\omega = \sqrt{T_0/\rho_0}\, k^{3/2}, \tag{5.66}$$

moving with a phase velocity

$$u'_p = \sqrt{\frac{T_0}{\rho_0}}\sqrt{k} = \sqrt{\frac{2\pi T_0}{\rho_0}}\,\frac{1}{\sqrt{\lambda}} \tag{5.67}$$

that increases with decreasing wave length. The phase velocity for a combined gravity-capillary wave motion can consequently be given by the curious form

$$u = \sqrt{u_p^2 + u_p'^2}. \tag{5.68}$$

The wave length dependence of (5.68) as well as its gravity and

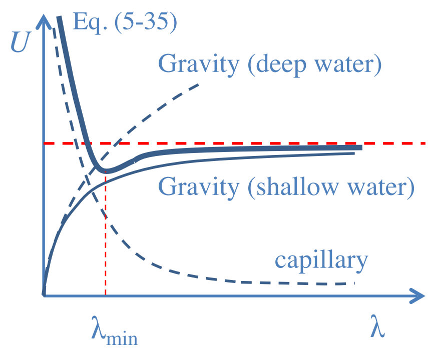

Figure 5.3: Phase velocity - wave-length relations for water waves.

capillary contributions are shown in Fig. 5.3. Notice that a minimum phase velocity occurs at the wave length

$$\lambda_{\min} = 2\pi\sqrt{\frac{T_0}{\rho_0 g}} \tag{5.69}$$

that serves as a rough dividing line between the wave lengths of capillary dominated ($\lambda \ll \lambda_{\min}$) and gravity dominated ($\lambda \gg \lambda_{\min}$) waves. For water ($T_0 = 73.05$ dynes/cm, $\rho_0 = 1$ gm/cc, $g = 980$ cm/sec^2) this wave length is

$$\lambda_{\min} = 1,72\,\text{cm}, \tag{5.70}$$

giving a minimum phase velocity

$$u_{\min} = 23,13\,\text{cm/sec}. \tag{5.71}$$

Waves of a smaller velocity than this cannot propagate in water!

In the other limit where the wave number in (5.51) becomes very small (long wave lengths) we again get into trouble, but from a different source; the sea bed. For if $\lambda \sim d$, the depth of the water, the boundary condition at the sea bed on the velocity potential has to be considered. We leave it to the reader to find the appropriate velocity potential and to show that there is no dispersion of water waves in the *shallow water* limit $\lambda \gg d$ (see Fig. 5.1 and Prob. 5-3).

5-5 Circular Waves on a Pond

Up to this point we have inquired into the manner of propagation of waves on the surface of a liquid without asking how the wave was generated in the first place. However, anyone who has ever thrown stones into a quiet pond or watched an insect swimming on its surface must have wondered about the beautiful wave patterns that develop on the surface of the water. As an interesting conclusion to our discussion of wave motion, we give a short mathematical analysis of the former problem.

Consider then, a small stone dropped into a pond, or a small insect settling on it. We assume that either event causes a small depression in the surface without imparting appreciable velocity to the water. Set up polar coordinates $x = r\cos\phi$, $z = r\sin\phi$ for a point P on the surface of the pond, see Fig. 5.4.

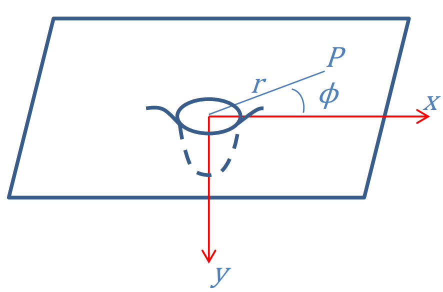

Figure 5.4: Initial disturbance of pond.

Now consider an initial disturbance

$$\eta(r, t = 0) = \frac{v_0}{\pi a^2} e^{-r^2/a}, \quad r = \sqrt{x^2 + a^2}, \tag{5.72}$$

v_0 being the volume of water displaced and **a** the "size" of the stone (or insect). The problem is then to determine how this disturbance propagates on the surface of the pond. Obviously such a disturbance is going to cause a wave pattern having a *cylindrical* symmetry about the vertical y axis, i.e. we are now dealing with a cylindrical rather than a plane wave propagation pattern.

We confine the discussion to a *deep* pond and consider a periodic disturbance first. If the wave pattern is to have a cylindrical symmetry instead of a plane one, we can expect the velocity potential to be given by the expression (5.51) but with $\exp(ikx)$ replaced by a radial function $f(r)$:

$$\phi e^{-i\omega t} = A e^{-ky} f(r) e^{-i\omega t}. \tag{5.73}$$

Substituting this expression into Laplace's equation one readily finds that $f(r)$ must satisfy

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + kf = 0, \tag{5.74}$$

that is Bessel's equation in the variable kr. Thus $f(r)$ is proportional to the Bessel function

$$f(r) = J_0(kr) \tag{5.75}$$

of order zero since ϕ must be regular at the origin.

The rest of the calculation for deep water plane waves transfers over to the present case with $J_0(kr)$ replacing $\exp(ikx)$ everywhere. Thus, the depression of the surface is now

$$\eta(r,t) = \frac{i\omega}{g} A J_0(kr) e^{-i\sqrt{gk}t}, \tag{5.76}$$

instead of (5.54), while the disperson law (5.57) for deep water waves remains unchanged: $\omega = \sqrt{gk}$.

The problem is now to find a solution like $\eta(r,t)$ in (5.76) that satisfies Bernoulli's equation and *in addition* coincides with the prescribed shape (5.72) of η at $t = 0$. The solution to this problem can be solved by using the superposition principle. We only have to realize that the linear superposition

$$\int_0^\infty A(k) J_0(kr) e^{-i\sqrt{gk}t}\, dk \tag{5.77}$$

of the solutions (5.76) for arbitrary amplitudes $A(k)$ is also a solution of Laplace's equation. Replacing $A\exp(i(kx - \omega t))$ in (5.52) by this form of solution leads to the expression

$$\eta(r,t) = \int_0^\infty \frac{i\sqrt{gk}}{g} A(k) J_0(kr) e^{-i\sqrt{gk}t} k\, dk \tag{5.78}$$

for η. Equation (5.78) solves our problem. We only have to determine $A(k)$ such that $\eta(r, t = 0)$ assumes the form (5.72) at $t = 0$:

$$\frac{v_0}{\pi a^2} e^{-r^2/a^2} = \int_0^\infty i\frac{\sqrt{gk}}{g} A(k) J_0(kr) k\, dk. \tag{5.79}$$

The value of the coefficient $i\sqrt{gk}/gA(k)$ is given by the standard Fourier-Bessel inversion formula[64]

$$i\frac{\sqrt{gk}}{g} A(k) = \frac{v_0}{\pi a^2}\int_0^\infty J_0(kr) e^{-r^2/a^2} r\, dr = \frac{v_0}{2\pi} e^{-(ka/2)^2}. \tag{5.80}$$

[64] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill Book Company Inc., New York, 1953, Vol. I, p 766.

Hence,

$$\eta(r,t) = \frac{v_0}{2\pi} \int_0^\infty e^{-(ka/2)^2} J_0(kr) e^{-i\sqrt{gk}t} k\, dk. \tag{5.81}$$

This equation gives η for all r and t but does not yet tell us how the surface moves. We can obtain a qualitative idea of this by performing the integral of k approximately. Consider the case of a disturbance with a very small spatial extent ($a \to 0$) but keep the volume v_0 of water displaced constant. This is called a point disturbance. Also recede to a distance r from the point disturbance such that

$$kr \gg 1 \tag{5.82}$$

for all wave numbers k that count (this procedure will be justified *a posteriori*). Then $J_0(kr)$ in (5.81) can be replaced by its asymptotic form[65] for large kr,

$$J_0(kr) \to \sqrt{\frac{2}{\pi}} \frac{\cos(kr - \pi/4)}{\sqrt{kr}}. \tag{5.83}$$

[65] See for example W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics*, Chelsea Publishing Company, New York, 1949.

This turns the expression for η into

$$\begin{aligned} \eta(r,t) &\simeq \sqrt{\frac{2}{\pi}} \frac{v_0}{2\pi} \int_0^\infty \frac{\cos(kr - \pi/4)}{\sqrt{kr}} e^{-i\sqrt{gk}t} k\, dk \\ &= \sqrt{\frac{2}{\pi}} \frac{v_0}{2\pi r^2} \int_0^\infty \cos(u - \pi/4) e^{-2i\sqrt{\xi}\sqrt{u}} \sqrt{u}\, du, \end{aligned} \tag{5.84}$$

where the last form follows on a change of variable from k to $kr = u$, and upon collecting the r and t dependence of the integrand into the single parameter

$$\xi = \frac{gt^2}{4r}. \tag{5.85}$$

If this parameter is large, the integral in (5.84) can be evaluated approximately by the method of *saddle point* integration. To appreciate this procedure we first write the cosine as a sum of exponentials, so that

$$\int_0^\infty \cos(u - \frac{\pi}{4}) e^{-2i\sqrt{\xi}\sqrt{u}} \sqrt{u}\, du = I_1 + I_2, \tag{5.86}$$

where

$$I_1 = \frac{1}{2} \int_0^\infty e^{i(u - 2\sqrt{\xi}\sqrt{u} - \frac{\pi}{4})} \sqrt{u}\, du \tag{5.87}$$

and

$$I_2 = \frac{1}{2} \int_0^\infty e^{-i(u + 2\sqrt{\xi}\sqrt{u} - \frac{\pi}{4})} \sqrt{u}\, du. \tag{5.88}$$

Take I_1 first. If ξ is large then the phase of the exponential

$$e^{if(u)}, \quad f(u) = u - 2\sqrt{\xi}\sqrt{u} - \frac{\pi}{4} \tag{5.89}$$

changes rapidly with u giving rise to strong cancellations in the integrand. Thus the main contribution to the integral I_1 must come from a region around a value for $u = u_0$ that makes the phase $f(u)$ *stationary* (if there is such a value). This value of u is determined by the condition $f'(u) = 0$, or

$$1 - \sqrt{\xi/u} = 0. \tag{5.90}$$

Thus $u_0 = \xi$ in the case of I_1. Expanding $f(u)$ about this point we find

$$f(u) = -(\xi + \frac{\pi}{4}) + \frac{(u-\xi)^2}{4\xi} + \ldots \tag{5.91}$$

Consequently

$$I_1 \simeq \frac{1}{2} e^{-i(\xi+\frac{\pi}{4})} \sqrt{\xi} \int_0^\infty e^{\frac{i(u-\xi)^2}{4\xi}}\, du. \tag{5.92}$$

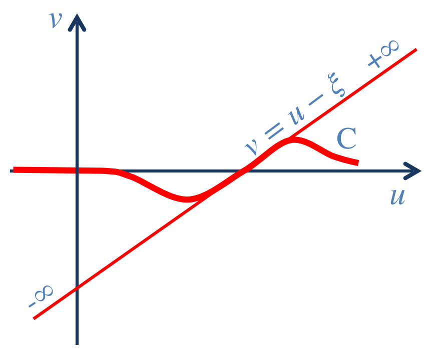

Figure 5.5: Contour C for the integral.

To evaluate this integral, we deform the path of integration along the real axis into the contour C shown in Fig. 5.5. This is legitimate since we have crossed no singularities of the integrand in the process. Thus, setting $z = u + iv$, we calculate

$$\int_C e^{\frac{i(z-\xi)^2}{4\xi}}\, dz. \tag{5.93}$$

But the point $z = \xi$ is a saddle point for the integrand in (5.93) (a minimum or a maximum depending on how you approach it), which therefore grows or receeds as we move away from ξ, depending on the path followed. Now, since

$$(z-\xi)^2 = (u - \xi + iv)^2 = [(u-\xi)^2 - v^2] + 2iv(u-\xi), \tag{5.94}$$

it is clear that along the path

$$v = u - \xi, \quad dv = du, \tag{5.95}$$

the oscillatory part of (5.93) is removed and the value of the integrand receeds. Consequently, we may extend the limits of integration to $\pm\infty$ along *this* path with impunity to obtain the approximate evaluation

$$\int_C e^{\frac{i(z-\xi)^2}{4\xi}}\, dz \simeq (1+i) \int_{-\infty}^{+\infty} e^{-\frac{v^2}{2\xi}}\, dv = 2\sqrt{\xi\pi} e^{\frac{i\pi}{4}}. \tag{5.96}$$

Putting all this together, we find that

$$I_1 \simeq \sqrt{\pi}\xi e^{-i\xi}. \tag{5.97}$$

Turning now to I_2, we find its phase is stationary at the root of

$$1 + \sqrt{\frac{\xi}{u}} = 0, \tag{5.98}$$

which does not lie in the region of integration $u > 0$. Consequently,

$$I_2 \simeq 0 \tag{5.99}$$

to the same order of approximation as our expression for I_1. Thus

$$\int_0^\infty \cos(u - \frac{\pi}{4})e^{-2i\sqrt{\xi}\sqrt{u}}\sqrt{u}\,du \simeq \sqrt{\pi}\xi e^{-i\xi}, \tag{5.100}$$

and the value of the depression η is

$$\eta(r,t) \simeq \sqrt{\frac{2}{\pi}} \cdot \frac{v_0}{2\pi r^2}\sqrt{\pi}\xi e^{-i\xi} = \frac{v_0}{\sqrt{2}\pi r^2}\xi e^{-i\xi}, \tag{5.101}$$

or, taking the real part,

$$\eta(r,t) \simeq \frac{v_0}{\sqrt{2}\pi r^2}\xi\cos\xi, \quad \xi = \frac{gt^2}{4r} \gg 1. \tag{5.102}$$

Thus $\eta(r,t)$ represents a circular wave pattern moving away from the center of disturbance with a wave length that increases with increasing r at fixed time, but with an amplitude that decreases like r^{-3}. The steady build-up without limit of the amplitude like t^2 at a fixed point on the surface of the pond is of course unphysical; it arises from the use of a point disturbance with infinite amplitude [(5.72) becomes infinite at $r = 0$ as $a \to 0$] and the neglect of all dissipative processes.

Because the wave length of η decreases as we approach the source of the disturbance, one may well wonder what role the surface tension of the water plays; for we saw in Sec. 5-3 that the propagation of wave lengths below 1,72 cm (wave numbers above 3,7 cm^{-1}) in water are controlled more by surface tension effects than by gravity. We leave it to the reader to show that for capillary waves on the surface of a pond, the controlling parameter is

$$\xi' = \frac{9}{4}\frac{T_0 t^2}{\rho_0 r^3} \gg 1 \tag{5.103}$$

instead of ξ, where T_0 is the surface tension and ρ_0 the density of the water. Surface tension effects dominate gravity effects if $\xi' \gg \xi$ and *vice versa* if $\xi' \ll \xi$ (both factors of course remaining $\gg 1$). The dividing line separating the regions of capillary and gravity dominated wave motion occurs at roughly $\xi = \xi'$, or a distance

$$r_{\text{crit}} = \sqrt{\frac{9T_0}{\rho_0 g}} \simeq 1\,\text{cm for water} \tag{5.104}$$

from the source of the disturbance. This value of r is independent of t. At distances further away from the source of disturbance than r_{crit} gravity effects rapidly dominate the motion, in agreement with what is seen when a quiet pond is disturbed by a small stone or an insect alighting on it.

Problems

5-1. Obtain Euler's equations for fluid motion by applying Newton's laws to the motion of a small volume of fluid.

5-2. Construct the *streamlines* for the velocity potential (5.51) given in the text. (A streamline is a line that has the fluid velocity tangent to it at every point in space). Hint: the streamlines are determined by the set of equations

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}, \tag{5.105}$$

where $\mathbf{u} = (u_x, u_y, u_z)$ is the fluid velocity at (x, y, z).

5-3. Show that $\omega = \sqrt{gd}k$ for waves on a shallow $(kd \ll 1)$ pond of depth d.

5-4. Determine the wave form on the surface of a pond initiated by the disturbance (5.72) if the pond is very shallow. (See previous problem).

5-5. Worry about the convergence of the integral appearing in (5.84).

5-6. Verify the statements made in the text in connection with (5.103). Investigate the role of surface tension by calculating $\eta(r, t)$ on the assumption that surface tension alone is responsible for the wave motion. Indicate any modifications of Fig. 5.5 that your results suggest to be important.

5-7. Develop equations describing a sound wave in a gas. Obtain an expression for the velocity of sound in the gas paying particular attention to the condition under which you derive your result. Consider further the propagation of a periodic sound wave past a sharp edge, like a vertical wall. Indicate what one might expect as to the distribution of sound energy on the other side of the wall. (Hint: look up some characteristics of sound waves in, for example P.M. Morse, *Vibration and Sound*, McGraw-Hill Book Company, Inc., New York, 1948, and develop methods to treat this particular problem).