Chapter 3 Rigid Body Dynamics

3-1 Introduction

We start our discussion of specialized many-body systems by considering the motion of a *rigid body*. By this term one understands a system of N interacting particles in which the relative separations of all particles remain constant even when the system is acted upon by external forces. The use of N discrete particles is no restriction on the validity of the results that we derive in the following sections. In the limit where the mass distribution can be regarded as continuous rather than discrete ($N \to \infty$ in such a way that the mass density ρ at every point in the body remains finite) we simply replace summations over all particles by integrals over the mass density, i.e. $\sum_i \dots m_i \to \int \dots d^3r$, where d^3r is the volume element at position $\mathbf{r}$ in the body.

Perfectly rigid bodies do not exist in nature. However, the idealization we have just described is completely adequate for discussing the motion of bodies that are rigid enough so that small distortions induced by external forces do not matter. In the questions that relate to the motion of a body in space, the idealization of a perfectly rigid body will be adhered to. It is well to realise, however, that situations can arise in the laboratory in which the internal structure of the body becomes important. A simple illustration is provided by a rapidly rotating flywheel, which behaves like a perfectly rigid body until angular velocities are reached where the accelerations that are required to force each element of the flywheel to move in a circle cannot be provided by the internal interactions holding the various parts of the flywheel together and it simply flies apart[39].

[39] This phenomenon actually happens in high speed centrifuges.

The subject of rigid body dynamics is an extensive one and it is not the purpose of this chapter to provide a complete treatment of the field. Rather the aim is to illustrate the methods and the difficulties that arise where one uses Newton's equations of motion to study the motion of a rigid body. The reader who is interested in a much more detailed treatment of the subject would find the classic, if somewhat old-fashioned, treatment in Routh's Rigid Body Dynamics[40] delightful, and well worth the time spent in sorting out a notation that was in vogue at the turn of the century.

[40] E.J. Routh, *Elementary Rigid Dynamics*, Seventh Edition, Macmillan and Co., Ltd., London, 1905; *Advanced Rigid Dynamics*, Sixth Edition, MacMillan and Co., Ltd., London, 1905.

3-2 *Frames of Reference*

In order to describe the motion of a rigid body it is necessary to introduce a suitable set of axes to serve as a frame of reference from which to view the motion. Two systems of axes occur quite naturally in this subject. The first system is the *space-fixed* or laboratory system of axes. Call this set O_{xyz}; it forms an inertial frame of reference. The directions of x, y, z are fixed in space, as is the origin O. Newton's laws apply in this frame. The second system of axes is the *body-fixed* system $O'_{x'y'z'}$ whose origin is attached to a fixed point in the body, and whose axes $x'y'z'$ are *frozen* into the rigid body and rotate when the body rotates[41]. The body-fixed system is *not* an inertial system since when the body accelerates it does also, and so Newton's laws as written in an inertial frame do not apply. Much of the complexity of describing the motion of a rigid body in fact arises just because, while quantities like the total angular momentum or kinetic energy of rotation of a rigid body assume very simple forms in the body-fixed system of axes, the equations of motion become much more complicated.

[41] An olive with three toothpicks stuck into it to form an orthogonal set of axes, and frozen into an ice cube, brings this situation to mind very graphically.

The origin O' of the body-fixed system can be attached to any point that moves with the rigid body. Most commonly however, O' is either a fixed point about which the rigid body is moving (hence O' is also fixed in space), or is taken at the center-of-mass of the rigid body. The two systems of axes are displayed in Fig. 3.1.

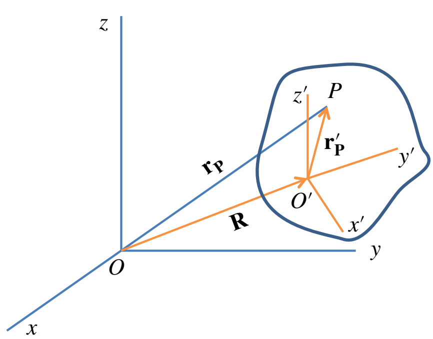

Figure 3.1: Coordinates of body-fixed system relative to the space-fixed system.

Consider the particle P of mass m_0 located at $\mathbf{r}_p$ with respect to O and $\mathbf{r}'_p$ with respect to O'. The separation of O and O' is $\mathbf{R}$. Naturally, $\mathbf{r}_p$, $\mathbf{r}'_p$ and $\mathbf{R}$ in general vary with time. An infinitesimal displacement of the particle P at $\mathbf{r}_p$ can be accomplished by displacing O' by an amount $\delta\mathbf{R}$ with respect to O and then displacing P relative to O' by an amount $\delta\mathbf{r}'_p$, so that the displacement $\delta\mathbf{r}_p$ of the particle at P appears as

$$\delta\mathbf{r}_p = \delta\mathbf{R} + \delta\mathbf{r}'_p. \tag{3.1}$$

The result (3.1) is obvious, but not particularly useful until we use the fact that the particle at P forms part of a rigid body. For then it is clear that the displacement $\delta\mathbf{r}'_p$ of the particle at P located with respect to O' can only come about from a change in orientation of $\mathbf{r}'_p = O'P$, since $|\mathbf{r}'_p|$ =constant. Moreover, the distances between all particles are fixed. If we rotate the body-fixed axes $O'_{x'y'z'}$ through an angle $\delta\phi$ about a direction $\mathbf{n}$ passing through O' the displacement of any particle is $\delta\mathbf{r}'_p = \delta\phi\mathbf{n} \times \mathbf{r}_p$ since all particles undergo a common angular displacement $\delta\phi\mathbf{n}$. Then we have $\delta\mathbf{r}_p = \delta\mathbf{R} + \delta\phi\mathbf{n} \times \mathbf{r}'_p$ and if this displacement occurs in time δt the velocity of the particle at P is given as

$$\mathbf{v}_p = \lim_{\delta t \to 0} \frac{\delta \mathbf{r}_p}{\delta t} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}'_p, \tag{3.2}$$

where $\mathbf{V} = \lim_{\delta t \to 0} \delta\mathbf{R}/\delta t$ is the velocity of O' as seen from O. The vector $\boldsymbol{\omega}$ is called the angular velocity of the rigid body. Obviously,

$$\boldsymbol{\omega} = \lim_{\delta t \to 0} \frac{\delta\phi}{\delta t}\mathbf{n}. \tag{3.3}$$

If we consider the point O' to be instantaneously at rest we see from (3.2) that the velocity of all particles is perpendicular to $\boldsymbol{\omega}$ at this instant. Furthermore, there exists a set of positions $\mathbf{r}_p$ lying on a straight line passing through O' along which the particle velocities are zero at the instant considered (see Prob. 3-1). This line is called the instantaneous axis of rotation. We see then that the most general motion of a rigid body moving about a fixed point can be described as a pure rotation of the body about the instantaneous axis passing through that point (Euler's theorem). It is clear therefore that the *most general* displacement of a rigid body can be reduced to a translation followed by a rotation about some point.

We remark in passing that $\boldsymbol{\omega}$ in (3.3) is a vector by construction. Furthermore, two simultaneous infinitesimal rotations, $\delta\boldsymbol{\phi}_1 = \delta\phi_1\mathbf{n}_1$ and $\delta\boldsymbol{\phi}_2 = \delta\phi_2\mathbf{n}_2$ say, behave like vectors in the sense that the quantity $\delta\boldsymbol{\phi} = \delta\phi_1\mathbf{n}_1 + \delta\phi_2\mathbf{n}_2$, combined vectorially, represents the result of carrying out the infinitesimal rotations $\delta\boldsymbol{\phi}_1$ and $\delta\boldsymbol{\phi}_2$. For if $\mathbf{r}'_p$ positions the particle P with respect to O', then the first rotation moves P to $\mathbf{r}'_p + \delta\phi_1\mathbf{n}_1 \times \mathbf{r}'_p$; the second rotation moves P to

$$\begin{aligned}\mathbf{r}'_p + \delta\phi_1\mathbf{n}_1 \times \mathbf{r}'_p + \delta\phi_2\mathbf{n}_2 \times (\mathbf{r}'_p + \delta\phi_1\mathbf{n}_1 \times \mathbf{r}'_p) \\ \simeq \mathbf{r}'_p + (\delta\phi_1\mathbf{n}_1 + \delta\phi_2\mathbf{n}_2) \times \mathbf{r}'_p,\end{aligned}$$

since $\delta\phi_1$ and $\delta\phi_2$ are infinitesimal, which proves the assertion. However, the appearance of $\boldsymbol{\omega}$ in the combination $\boldsymbol{\omega} \times \mathbf{r}'_p$ in Eq. (3.2) demonstrates that it is not an ordinary vector, but rather a pseudo- or axial-vector that does not change sign under coordinate reflections as do "ordinary" vectors. This property of $\boldsymbol{\omega}$ guarantees that the vector $\mathbf{v}_p$ in (3.2) will change sign properly under coordinate reflections.

3-3 The Inertia Tensor

Our next task is to construct the kinetic energy and the angular momentum of a moving rigid body and it is in so doing that the advantages of the body-fixed frame of reference will become apparent. Let us therefore examine the form of the kinetic energy T when looked at from the space- and body-fixed systems. In the space-fixed system we simply have $T = \sum_{p=1}^{N} \frac{1}{2} m_p v_p^2$; if we substitute for $\mathbf{v}_p$ its equivalent in terms of $\mathbf{V}$ and ω we have

$$T = \sum_p \frac{1}{2} m_p v_p^2 = \frac{1}{2}(\sum_p m_p)V^2 + \sum_p \frac{1}{2} m_p(\omega \times \mathbf{r}'_p)^2 [\omega \times (\sum_p m_p \mathbf{r}'_p)] \cdot \mathbf{V}. \tag{3.4}$$

We have mentioned previously that O' is most commonly taken to be either a fixed point in space about which the body is rotating, or the center of mass of the body. In the former case, both the first and last terms in (3.4) are zero and we are left with the kinetic energy of rotational motion about O':

$$T_{rot} = \sum_p \frac{1}{2} m_p (\omega \times \mathbf{r}'_p)^2. \tag{3.5}$$

In the second case where O' is at the center of mass of the moving body, we have $\sum_p m_p \mathbf{r}'_p = 0$. Then, the last term in (3.4) still drops out while the first term $\frac{1}{2}(\sum_p m_p)V^2 = \frac{1}{2}MV^2$, where $M = \sum_p m_p$ is the total mass, simply adds the kinetic energy of translational motion to T_{rot}. In either case the interesting quantity is T_{rot}. Writing out the square of the vector product in (3.5) we have

$$\begin{aligned} T_{rot} &= \sum_p \frac{1}{2} m_p [\omega^2 (\mathbf{r}'_p \cdot \mathbf{r}'_p) - (\omega \cdot \mathbf{r}'_p)^2] \\ &= \sum_{i,j=1}^{3} \sum_{p=1}^{N} \frac{1}{2} m_p \omega_i (r_p^2 \delta_{i,j} - x_{p,i} x_{p,j}) \omega_j, \end{aligned}$$

if we introduce the components ω_i and $x_{p,i}$, $i = (1, 2, 3)$ of the vectors ω and $\mathbf{r}_p$ along the body-fixed system O'_{xyz}. Since the ω_i are the same for all particles they can be taken outside of the p summation in (3.63). Then, the set of quantities

$$I_{i,j} = \sum_p m_p (r_p^2 \delta_{i,j} - x_{p,i} x_{p,j}) \tag{3.6}$$

can be calculated once and for all for a given rigid body, and the kinetic energy of rotation becomes

$$T_{rot} = \sum_{i,j} \frac{1}{2} \omega_i I_{ij} \omega_j. \tag{3.7}$$

The quantity I_{ij} is called the inertia tensor. It is a quantity which characterizes the distribution of mass of a rigid body about the point O' rather than the mass itself. We see from (3.7) that rigid bodies of the same total mass can possess quite different kinetic energies of rotational motion depending on how the mass in the body is distributed. Thus, the I_{ij} play the role of a set of inertial parameters which determine the energy content of the rotational motion in the same way that the total mass M determines the energy of translational motion of the body as a whole. From the definition (3.6) we see that I_{ij} is real and symmetric, $I_{ij} = I_{ji}$. We may therefore display the inertia tensor as a real 3×3 symmetric matrix (writing now x, y, z for x'_1, x'_2, x'_3),

$$I = (I_{ij}) = \begin{bmatrix} \sum_p m_p(y_p^2 + z_p^2) & -\sum m_p x_p y_p & -\sum m_p x_p z_p \\ -\sum m_p x_p y_p & \sum m_p(z_p^2 + x_p^2) & -\sum m_p y_p z_p \\ -\sum m_p x_p z_p & -\sum m_p y_p z_p & \sum m_p(x_p^2 + y_p^2) \end{bmatrix}.$$

It is customary to refer to the diagonal elements of this matrix as the moments of inertia, and the off-diagonal elements as the products of inertia of the system, all taken with respect to the prescribed set of axes O'_{xyz}.

Like all real, symmetric matrices, I can be transformed into a diagonal form

$$M = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}, \tag{3.8}$$

where the I_1, I_2, I_3 are called the principal moments of inertia. The inertia tensor only assumes the simple form (3.8) with respect to a special set of body-fixed axes, called the *principal axes* of the rigid body. Clearly, it is to our advantage to exploit this simplicity by expressing the kinetic energy T_{rot} in terms of the components of the angular velocity ω taken along the principal axes, rather than along an arbitrary set of body-fixed axes. Then, (3.7) reduces to a sum of squares

$$T_{rot} = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2, \tag{3.9}$$

where ω_1, ω_2, ω_3 are now the components of ω taken along the principal axes. The reduction of the expression for T_{rot} in (3.7) to a sum of squares is a familiar procedure from our discussion of the theory of small oscillations. We need therefore only remark that if we were unlucky enough to choose our original set of body-fixed axes not to be simultaneously also principal axes, we would have to diagonalize the matrix I. This procedure obviously also provides us with the necessary information of how the principal axes are oriented with respect to the original body-fixed set we started out with. For simple rigid bodies, one can usually determine the principal axes by inspection, using the symmetries that the rigid body exhibits. The general case, however, requires that we construct the inertia tensor I_{ij} in a particular frame of reference and then carry out the diagonalization to obtain the principal moments of inertia and principal axes.

The angular momentum of the rigid body about the point O' can also be expressed in terms of the inertia tensor. Let us again regard O' as either a fixed point in space about which the rigid body rotates, or its center-of-mass. The total angular momentum of the rigid body about O' is

$$\begin{aligned} L &= \sum_p (\mathbf{r}'_p \times m_p \mathbf{v}'_p) \\ &= \sum_p m_p \mathbf{r}'_p \times (\omega \times \mathbf{r}'_p) \\ &= \sum_p m_p [(\mathbf{r}'_p \cdot \mathbf{r}'_p)\omega - (\omega \cdot \mathbf{r}'_p)\mathbf{r}'_p]. \end{aligned} \tag{3.10}$$

Therefore, the component of $\mathbf{L}$ along the body-fixed axis i is

$$\begin{aligned} L_i &= \sum_p m_p [r'^2_p \omega_i - (\sum_j \omega_j x_{p,j}) x_{p,i}] \\ &= \sum_j \sum_p m_p (r'^2_p \delta_{i,j} - x_{p,i} x_{p,j}) \omega_j \\ &= \sum_j I_{ij} \omega_j, \end{aligned} \tag{3.11}$$

or

$$L_1 = I_1 \omega_1, \quad L_2 = I_2 \omega_2, \quad L_3 = I_3 \omega_3 \tag{3.12}$$

if we project $\boldsymbol{\omega}$ along the principal axes of the rigid body.

Equations (3.9) and (3.12) provide us with simple expressions for the kinetic energy and angular momentum of a rotating rigid body in terms of its angular velocity components along the principal axes.

Our next task is to express $\boldsymbol{\omega}$ and $\mathbf{L}$ in terms of a suitable set of coordinates (and their time derivatives) that describe the motion of the rigid body. We restrict the discussion to a rigid body moving about a fixed point in space. The first question that comes up is how many degrees of freedom are there in this case? This number is simple to determine. If we fix our attention on any point P of the rigid body moving about O', then P is constrained to move on the surface of a sphere centered at O' and thus has two degrees of freedom. A second point Q of the body can then be uniquely located in terms of a rotation about the line $O'P$. The three points O', P, Q uniquely determine the position of the rigid body as a whole which therefore possesses three degrees of freedom. Therefore, three independent coordinates $q_k(t)$ are necessary to describe the motion. How shall we choose the $q_k(t)$? For a body moving about a fixed point O it is obviously convenient to choose O as a common origin for the space-fixed and body-fixed systems of axes as in Fig. 3.2.

Then, we see that by knowing the orientation of the axes $O_{x'y'z'}$, which move with the body, with respect to the space-fixed axes O_{xyz} at each instant of time, we are provided with a complete picture of the motion of the rigid body itself.

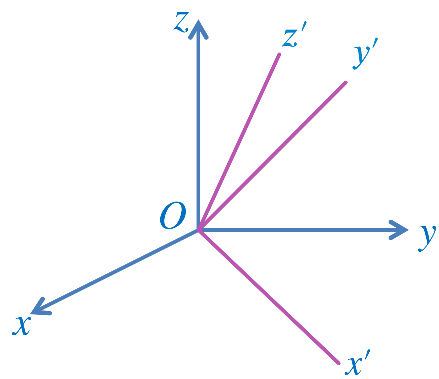

Figure 3.2: The origins of the body-fixed and space-fixed systems coincide, $O = O'$.

We have also remarked that the rotational motion about O can be described at each instant of time as the rotation of the body about some axis passing through O. We may therefore regard the final position of the axes $O_{x'y'z'}$ at time t as having been reached via a succession of infinitesimal rotations of $O_{x'y'z'}$ about O' that were dictated by Newton's equations of motion and the forces acting on the system. The entire problem of determining the motion of a rigid body moving about a fixed point therefore reduces to finding the finite rotation of the axes $O_{x'y'z'}$ that brings them from some initial position at $t = 0$ to occupy their final position in the moving body at time t. Two important questions immediately come up. How shall we parametrize the position of the rotated set $O_{x'y'z'}$ with respect to the space-fixed set O_{xyz}, and how do the components of vector quantities like the angular velocity and angular momentum $\boldsymbol{\omega}$ and $\mathbf{L}$ relate to each other when viewed from the two sets of axes? To answer such questions, we have to study the transformation properties of the components of a vector when the coordinate axes, to which these components refer, are rotated.

3-4 Coordinate Transformations

We begin our discussion of coordinate transformations with a simple example. Suppose the body-fixed axes coincide with O_{xyz} at $t = 0$. Now, rotate them through an angle ϕ about the common z axis to take up the position $O_{x'y'z'}$ shown in Fig. 3.3. If $\mathbf{Q}$ is any vector lying in the $x - y$ plane we see geometrically that the components of $\mathbf{Q}$ along $O_{x'y'z'}$ can be expressed as linear combinations of the components of $\mathbf{Q}$ along O_{xyz}

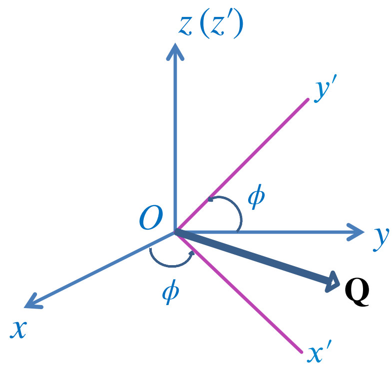

Figure 3.3: The body-fixed and space-fixed origins coincide ($O = O'$). In addition, the z axes are common $z = z'$. The x and y axes are rotated by ϕ to x' and y'.

according to

$$\begin{aligned} Q_{x'} &= \cos\phi\, Q_x + \sin\phi\, Q_y \\ Q_{y'} &= -\sin\phi\, Q_x + \cos\phi\, Q_y, \end{aligned}$$

where $Q_{x'}$, $Q_{y'}$ are the components of the *same* vector **Q** in the rotated coordinate system. A more elegant way of writing (**??**) is to use a matrix notation and write

$$\begin{bmatrix} Q_{x'} \\ Q_{y'} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \end{bmatrix},$$

or, more briefly,

$$(\mathbf{Q}') = A(\mathbf{Q}), \tag{3.13}$$

where $(\mathbf{Q}')$ is a column matrix of the components of **Q** along the new axes $O_{x'y'z'}$, $(\mathbf{Q})$ is a column matrix of the components of **Q** along the old axes O_{xyz}. A is a square matrix that depends upon the parameters that described the rotation (in this case the angle ϕ). We will speak of A as a rotation matrix inducing a rotation of the axes O_{xyz} to some final position $O_{x'y'z'}$. The length of **Q** is unchanged by the rotation of axes "under it". We verify immediately from (**??**) that this means that

$$AA^T = A^T A = I, \quad \text{and Det } A = +1. \tag{3.14}$$

We do not consider the inversion of coordinates and so discard Det $A = -1$ where A^T again denotes the transpose of A, Det A is the determinant of the matrix A and I is the unit matrix. We now turn this statement around and characterize *all linear transformation* of the type (3.13) where A is a real matrix satisfying the conditions (3.14) as a rotation of axes in the space of n dimensions, where n is the number of linearly independent components of **Q**. For our purpose, $n = 2$ or 3. Such rotations are often also referred to as proper orthogonal transformations, and the rotation matrix A that satisfies $AA^T = A^TA = I$ is called an orthogonal matrix. We note that the inverse rotation, or transformation, also exists because Det $A \neq 0$ so that A has an inverse $A^{-1} = A^T$. The inverse rotation is therefore

$$(\mathbf{Q}) = A^{-1}(\mathbf{Q}') = A^T(\mathbf{Q}').$$

It will prove convenient to indicate the directions of the rotated and unrotated sets of axes by introducing two sets of orthogonal unit vectors at the origin O. These sets of unit vectors lie along the O_{xyz} and $O_{x'y'z'}$ axes respectively. Let

$$\mathbf{e}_x, \quad \mathbf{e}_y, \quad \mathbf{e}_z \tag{3.15}$$

be unit vectors lying along the coordinate axes of O_{xyz} (the unprimed or unrotated axes) and let

$$\mathbf{f}_{x'}, \quad \mathbf{f}_{y'}, \quad \mathbf{f}_{z'} \tag{3.16}$$

be unit vectors lying along the coordinate axes of $O_{x'y'z'}$ (the primed or rotated axes). Then

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad \text{and} \quad \mathbf{f}_\alpha \cdot \mathbf{f}_\beta = \delta_{\alpha\beta}, \tag{3.17}$$

where $i, j = (x, y, z)$ and $\alpha, \beta = (x', y', z')$ express the orthogonaltiy and normalization properties of the $\mathbf{e}$'s and $\mathbf{f}$'s. The $\mathbf{e}_i$ and $\mathbf{f}_\alpha$ constitute a complete set of unit vectors in three dimensions in the sense that an arbitrary vector $\mathbf{Q}$ may be expressed in terms of either set, according to

$$\mathbf{Q} = \sum_i Q_i \mathbf{e}_i = \sum_\alpha Q'_\alpha \mathbf{f}_\alpha, \tag{3.18}$$

where the Q_i or the Q'_α are the components of $\mathbf{Q}$ along the corresponding axes. Equation (3.18) immediately allows us to relate the components of the vector $\mathbf{Q}$ along the primed axes with its components along the unprimed axes. Taking the scalar product of (3.18) with $\mathbf{f}_\alpha$ and using the orthogonality relations (3.17) we get

$$Q'_\alpha = \sum_i (\mathbf{f}_\alpha \cdot \mathbf{e_i}) Q_i = \sum_i A_{\alpha i} Q_i, \tag{3.19}$$

if we call

$$(\mathbf{f}_\alpha \cdot \mathbf{e_i}) = A_{\alpha i}. \tag{3.20}$$

Equation (3.19) is identical with the matrix equation (3.13) written out in detail. We also see from (3.20) that the matrix elements of A in (3.13) are given by the direction cosines of the rotated unit vectors $\mathbf{f}_\alpha$ with respect to the unrotated unit vectors $\mathbf{e}_i$. The unit vectors $\mathbf{f}_\alpha$ can therefore also be expressed in terms of the unrotated basis according to the equation

$$\mathbf{f}_\alpha = \sum_i A_{\alpha i} \mathbf{e}_i, \tag{3.21}$$

which follows directly from (3.20). We urge the reader to notice the suble difference between this relation and (3.13). Equation (3.13) is a statement about the components of the *same* vector on both sides of the equation looked at from two different coordinate systems, whereas (3.21) merely relates two *different* vectors to each other, i.e. $\mathbf{f}_\alpha$ with the set $\mathbf{e}_i$. This difference is most easily appreciated if $\mathbf{Q}$ happens to be the vector $\mathbf{f}_\alpha$. Then, by contrast with (3.21), equation (3.13) reads

$$(\mathbf{f}'_\alpha) = A(\mathbf{f}_\alpha) \tag{3.22}$$

where $(\mathbf{f}'_\alpha)$ contains the components of $\mathbf{f}_\alpha$ along the rotated axes and $(\mathbf{f}_\alpha)$ contains the components of $\mathbf{f}_\alpha$ along the unrotated axes. Let us ask for the latter components. We invert (3.22) to find

$$(\mathbf{f}_\alpha) = A^T(\mathbf{f}'_\alpha)$$

or (writing $f_{\alpha,i}$ for the component of $\mathbf{f}_\alpha$ along the $\mathbf{e}_i$, etc.)

$$f_{\alpha,i} = \sum (A^T)_{i\beta} f'_{\alpha,\beta} = (A^T)_{i\alpha} = A_{\alpha i},$$

since $f'_{\alpha,\beta} = \delta_{\alpha\beta}$ because the $\mathbf{f}_\alpha$ happen to be unit vectors along the rotated axes. Thus, we have $\mathbf{f}_\alpha = \sum_i f_{\alpha,i}\mathbf{e}_i = \sum_i A_{\alpha i}\mathbf{e}_i$, which reconstructs (3.21).

Equation (3.13) or, equivalently (3.19), gives the law of change for the components of a vector under rotations of the coordinate system, or basis set of unit vectors $\mathbf{e}_i$ to which the vector was referred. The components of a vector in a particular basis are said to form a *representation* of the vector in that basis. A change of basis according to (3.21) therefore produces a new representation of the same vector. In (3.18) the components Q_i and Q'_α provide an example of two representations of the vector $\mathbf{Q}$. Equation (3.21) is the connection between these two representations for a given transformation matrix A.

So far, we have talked only about vectors and their representations, a concept which distinguishes between the components of a vector and the vector "itself" so to speak. However, the same concepts extend to a wider class of entities which are called (linear) operators. Consider for example (3.11) relating the angular momentum $\mathbf{L}$ and the angular velocity $\boldsymbol{\omega}$. Since $\mathbf{L}$ and $\boldsymbol{\omega}$ are vectors, it is a useful point of view to regard the inertia tensor I as an *operator* which operates on $\boldsymbol{\omega}$ to produce a *new* vector $\mathbf{L}$,

$$I\boldsymbol{\omega} = \mathbf{L}. \tag{3.23}$$

In a representation with respect to a particular basis $\mathbf{e}_i$, say, $\mathbf{L} = \sum_i \mathbf{e}_i L_i$, $\boldsymbol{\omega} = \sum_j \mathbf{e}_j \omega_j$, (3.23) becomes

$$I \sum_j \mathbf{e}_j \omega_j = \sum_i \mathbf{e}_i L_i,$$

or

$$\sum_j (\mathbf{e}_i \cdot I\mathbf{e}_j)\omega_j = L_i.$$

If we call

$$(\mathbf{e}_i \cdot I\mathbf{e}_j) = I_{ij}, \tag{3.24}$$

we regain (3.11). The quantity $(\mathbf{e}_i \cdot I\mathbf{e}_j)$ does not depend on the vector $\boldsymbol{\omega}$ but only on the operator I and the particular choice of basis vectors $\mathbf{e}_i$ that has been made. The collection of quantities I_{ij} in (3.24) is called a *representation* of the operator I in the *basis* defined by the $\mathbf{e}_i$. We now ask how this representation changes when the basis is changed. From (3.21) and the definition (3.24) the representation of I in a different basis set $\mathbf{f}_\alpha$ is:

$$\begin{aligned} I'_{\alpha\beta} = (\mathbf{f}_\alpha \cdot I\mathbf{f}_\beta) &= \sum_{i,j} A_{\alpha i}(\mathbf{e}_i \cdot I\mathbf{e}_j)A_{\beta j} \\ &= \sum_{i,j} A_{\alpha i} I_{ij} A^T_{j\beta}, \end{aligned} \tag{3.25}$$

or

$$I' = AIA^T = AIA^{-1} \tag{3.26}$$

in a matrix notation.

In line with the interpretation of I as an operator, another interpretation to (3.13) can be given by regarding the matrix A as an *operator* which acts of $\mathbf{Q}$ and changes it into a new vector $\mathbf{q}$,

$$\mathbf{Q} \to \mathbf{q} = A\mathbf{Q} \tag{3.27}$$

without changing the basis. This is called the active interpretation of the operator A as opposed to the passive interpretation in (3.13) where A changed the basis, but left the vector alone. Equation (3.27) is still a rotation. The effect of A has simply been to point $\mathbf{Q}$ in a new direction without changing its length.

We can also express (3.27) as a relation between components of the vectors $\mathbf{q}$ and $\mathbf{Q}$ referred to the same basis, i.e.

$$\sum_i q_i\mathbf{e}_i = \sum_j A\mathbf{e}_j Q_j$$

or

$$q_i = \sum A_{ij}Q_j, \quad A_{ij} = (\mathbf{e}_i \cdot A\mathbf{e}_j). \tag{3.28}$$

For a *given* rotation matrix A (3.28) and (3.13) are the same, i.e. the components q_i of the vector $\mathbf{q}$ in the old basis are identical with the components Q'_k of the vector $\mathbf{Q}$ in the *new* basis. The point is that the components of $\mathbf{Q}$ in some basis only depend on the relative orientation of $\mathbf{Q}$ to that basis. We are therefore at liberty to rotate either the *basis* "under the vector", or rotate the vector "over the basis" *in the opposite* *sense* and still end up with the same components of the vector that did not move (that moved) with respect to the basis that moved (that did not move). We will adopt either point of view according to our needs. The important link to bear in mind is that A always rotates the vector in the opposite sense to that in which it rotates the basis. Fig. 3.4 will be helpful to the reader who is unaccustomed to the mental gymnastics that are necessary to translate easily from the one point of view to the other.

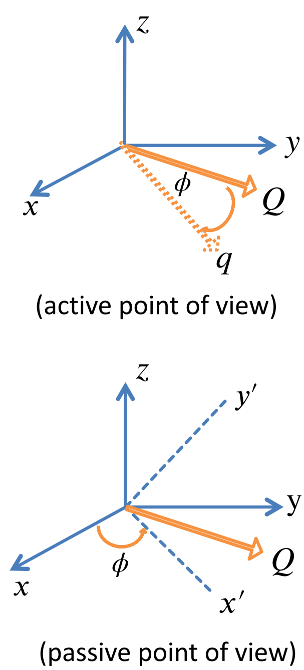

Figure 3.4: Upper panel: Active point of view. $A(\phi)$ rotates $\mathbf{Q}$ closer to the x axis, Lower panel: Passive point of view. $A(\phi)$ rotates the x axis closer to the vector $\mathbf{Q}$.

One important property of rotations is their *group* property, i.e. the combination of two successive rotations represented by matrices A and B can be described as a single rotation represented by a matrix $C = BA$, where the "law of combination" is matrix multiplication. The proof is simple. If the matrix A rotates O_{xyz} with unit vectors $\mathbf{e}_i$, to occupy the orientation $O_{x'y'z'}$ with unit vectors $\mathbf{f}_\alpha$, and B rotates $O_{x'y'z'}$ to occupy the orientation $O_{x''y''z''}$ with unit vectors $\mathbf{g}_\mu$, then we see that

$$g_\mu = \sum_\alpha B_{\mu\alpha}\mathbf{f}_\alpha = \sum_{\alpha,i} B_{\mu\alpha}A_{\alpha i}\mathbf{e}_i,$$

upon applying (3.21) twice, or that

$$\mathbf{g}_\mu = \sum_i C_{\mu i}\mathbf{e}_i, \quad C_{\mu i} = \sum_\alpha B_{\mu\alpha}A_{\alpha i}, \tag{3.29}$$

so that the matrix C is just the matrix product of A and B in a specified order. The rotational operation is in general a non-commutative one; rotations induced by BA and AB are in general distinct from each other.

3-5 *The Euler Angles*

We are now in a better position to answer the question of how to parametrize the orientation of the body-fixed axes $O_{x'y'z'}$ with respect to the space-fixed system O_{xyz}, i.e. how to choose the coordinates $q_l(t)$ that describe the motion.

Equation (3.21) connecting the unit vectors along O_{xyz} and $O_{x'y'z'}$ suggests one possibility, that of using the matrix elements A_{α_i} as coordinates specifying the orientation of the body-fixed axes. This simply amounts to specifying the direction cosines of each of the $\mathbf{f}_\alpha$ with respect to the fixed axes O_{xyz}. But this gives us 9 parameters for a system having only 3 degrees of freedom. The point is of course that the $A_{\alpha i}$ are not all linearly independent. The condition $AA^T = 1$ in (3.14) provides for 6 linearly dependent relations between the $A_{\alpha i}$, effectively reducing the number of independent parameters to 3. It would obviously be much simpler therefore to choose a set of independent parameters that automatically satisfy these conditions. One such set of parameters is provided by Euler's three angles (ϕ, θ, ψ) which we now proceed to define. The convention followed by Edmonds[42] will be used in defining ϕ, θ and ψ. However, the reader should beware of the fact that there is no general agreement in the literature of rigid body dynamics (or of quantum mechanics!) on a "best" definition of these angles. In Edmonds' convention the three angles ϕ, θ, ψ have the following significance: initially the body-fixed axes $O_{x'y'z'}$ coincide with the space-fixed axes O_{xyz}. Then the following rotations are performed on the body-fixed axes about the indicated directions (see Fig. 3.5):

[42] A.R. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton University Press, Princeton, New Jersey, 1957.

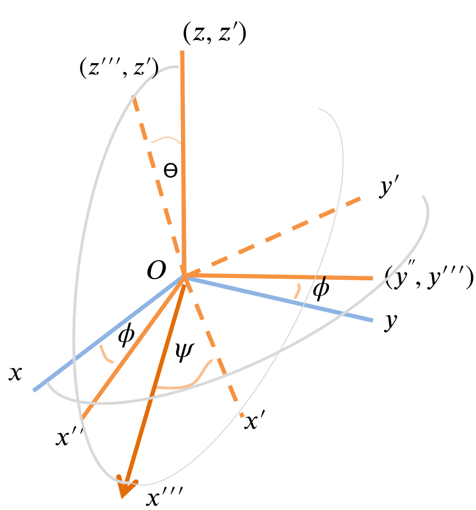

Figure 3.5: Euler angles.

(i) a rotation ϕ about the z axis, $0 \leq \phi \leq 2\pi$
(ii) a rotation θ about the *new* y axis (y''), $0 \leq \theta \leq \pi$

(iii) a rotation ψ about the *new* z axis (z'), $0 \le \psi \le 2\pi$.

This set of rotations, performed in the indicated manner, brings the body-fixed system to rest in the final orientation $O_{x'y'z'}$ shown in Fig. 3.5. The sense of rotation is such that the angles ϕ, θ, ψ are considered to be positive when the rotation would cause a right handed screw to advance along the axis of rotation.

Each rotation through an Euler angle has a rotation matrix A associated with it. We write

$A_z(\phi)$ for the rotation ϕ about z

$A_{y''}(\theta)$ for the rotation θ about y''

$A_{z'}(\psi)$ for the rotation ψ about z'.

From the group property of rotations displayed in (3.29) we know that we can combine these three rotations into a single rotation induced by the matrix

$$R(\phi\,\theta\,\psi) = A_{z'}(\psi)A_{y''}(\theta)A_z(\phi). \tag{3.30}$$

$R(\phi\,\theta\,\psi)$ is the matrix which will rotate the body-fixed axes directly over to their final orientation shown in Fig. 3.5. The individual rotation matrices in (3.30) are given by

$$A_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{for a rotation } \phi \text{ about the z axis,} \tag{3.31}$$

$$A_{y''} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \text{for a rotation } \theta \text{ about the } y''\text{axis,} \tag{3.32}$$

$$A_{z'} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{for a rotation } \psi \text{ about the } z' \text{ axis.} \tag{3.33}$$

The combined rotation matrix is obtained by multiplying these matrices together in the specified order (3.30). The result is

$$R(\phi\,\theta\,\psi) = \begin{pmatrix} \cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi & \cos\psi\cos\theta\sin\phi + \sin\phi\cos\phi & -\cos\psi\sin\theta \\ -\sin\psi\cos\theta\cos\phi - \cos\psi\sin\phi & -\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}. \tag{3.34}$$

3-6 Infinitesimal Rotations and Euler Angles

The three Euler angles $\vartheta_\lambda = (\phi, \theta, \psi)$, considered as functions of time, serve as a very convenient set of generalized cordinates $q_k(t)$ to describe the motion of a rigid body. We now ask for expressions for the angular velocity and angular momentum of a rigid body in these coordinates. To find the angular velocity, consider an infinitesimal rotation $\delta\varphi$ of the rigid body away from its position occupied at time t and given by Euler angles $\vartheta^\lambda(t)$. Writing $\delta\vartheta^\lambda = (\delta\phi, \delta\theta, \delta\psi)$ for the increments in the three Euler angles, we can make up this rotation as follows:

$$\delta\boldsymbol{\varphi} = \sum_\lambda \mathbf{g}_\lambda \delta\vartheta^\lambda, \tag{3.35}$$

where the $\mathbf{g}_\lambda \delta\vartheta^\lambda$, $\lambda = 1,2,3$ represent infinitesimal rotations about the three axes of rotation for the Euler angles. From Fig. 3.5 the unit vectors along these axes are

$$\begin{aligned} \mathbf{g}_\phi &= \mathbf{e}_z, \quad \text{the z axis,} \\ \mathbf{g}_\theta &= (\mathbf{e}_z \times \mathbf{f}_{z'})\frac{1}{\sin\theta}, \quad \text{the new } y \text{ axis } (y''), \\ \mathbf{g}_\psi &= \mathbf{f}_{z'}, \quad \text{the new } z \text{ axis } (z'). \end{aligned} \tag{3.36}$$

Notice that the unit vectors $\mathbf{g}_\lambda$ are *not* orthogonal. Correspondingly, we must distinguish between contravariant and covariant components of a vector[43] along these unit vectors. We see that the $\delta\vartheta^\lambda$ are contravariant components of $\delta\boldsymbol{\varphi}$. The magnitude of the infinitesimal rotation $\delta\phi$ defines a metric tensor for the $\delta\vartheta^\lambda$ "space" according to

[43] See for example B. Spain, *Tensor Calculus*, Oliver and Boyd, Edinburgh, 1953.

$$(\delta\varphi)^2 = \sum_{\lambda,\mu} g_{\lambda\mu}\delta\varphi^\lambda\delta\vartheta_\mu, \tag{3.37}$$

where $g_{\lambda\mu}$ is the metric tensor. From (3.35) and (3.36) we find that

$$(\delta\varphi)^2 = (\delta\phi)^2 + (\delta\theta)^2 + (\delta\psi)^2 + 2\cos\theta\,\delta\phi\,\delta\psi,$$

since only $\mathbf{g}_\phi$ and $\mathbf{g}_\psi$ are not orthogonal. Therefore

$$(g_{\lambda\mu}) = \begin{pmatrix} 1 & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & 1 \end{pmatrix}, \quad \text{Det }(g_{\lambda\mu}) = \sin^2\theta, \tag{3.38}$$

where Det $(g_{\lambda\mu})$ is the determinant of $g_{\lambda\mu}$ considered as a matrix. To complete the picture we also construct the inverse of $g_{\lambda\mu}$. This is

$$(g^{\lambda\mu}) = \frac{1}{\sin^2\theta}\begin{pmatrix} 1 & 0 & -\cos\theta \\ 0 & \sin^2\theta & 0 \\ -\cos\theta & 0 & 1 \end{pmatrix}, \tag{3.39}$$

provided that Det $(g_{\lambda\mu}) = \sin^2\theta \neq 0$, and allows us to pass from the covariant components $\delta\vartheta_\mu$ to the contravariant components $\delta\vartheta^\lambda = \sum_\mu g^{\lambda\mu}\delta\vartheta_\mu$ in the usual manner. We note in passing that the singularity in $g^{\lambda\mu}$ at $\theta = 0$ is related to the fact that the Euler angles are not everywhere unique.

We may also project $\delta\boldsymbol{\varphi}$ along the space-fixed or body-fixed axes. We have

$$\sum \mathbf{e}_i \delta\varphi_i = \sum_\lambda \mathbf{g}_\lambda \delta\vartheta^\lambda = \sum_\alpha \mathbf{f}_\alpha \delta\varphi'_\alpha, \tag{3.40}$$

where $\delta\varphi_i = (\delta\varphi_x, \delta\varphi_y, \delta\varphi_z)$ and $\delta\varphi'_\alpha = (\delta\varphi'_{x'}, \delta\varphi'_{y'}, \delta\varphi'_{z'})$ are the components of $\delta\boldsymbol{\varphi}$ along the space-fixed and body-fixed axes respectively. The transformation matrices from the set $\delta\vartheta^\lambda$ to the sets $\delta\varphi_i$ or $\delta\varphi'_\alpha$ are given by[44]

$$(P_{i\lambda}) = (\mathbf{e}_i \cdot \mathbf{g}_\lambda), \quad (Q_{\alpha\lambda}) = (\mathbf{f}_\alpha \cdot \mathbf{g}_\lambda), \tag{3.41}$$

so that we can write

$$\delta\varphi_i = \sum_\lambda P_{i\lambda}\delta\vartheta^\lambda \tag{3.42}$$

$$\delta\varphi'_\alpha = \sum Q_{\alpha\lambda}\delta\vartheta^\lambda. \tag{3.43}$$

[44] The matrices P and Q were introduced by H.B.G. Casimir in his doctoral dissertation, *Rotation of a Rigid body in Quantum Mechanics*, Wolter's Uitgevers-Maatschappij, Groningen, 1931.

It is clear that P and Q are *not* rotation matrices in the sense of (3.19), since $P^T \neq P^{-1}$. However, the inverse transformations

$$\delta\vartheta^\lambda = \sum_i (P^{-1})_{\lambda i}\delta\varphi_i = \sum_\alpha (Q^{-1})_{\lambda\alpha}\delta\varphi'_\alpha \tag{3.44}$$

are still possible if the matrices P and Q are non-singular.

To construct the matrices P and Q we require the projections of the $\mathbf{g}_\lambda$ along the space-fixed unit vectors $\mathbf{e}_i$, and the body-fixed unit vectors $\mathbf{f}_\alpha$. These projections are easy to compute. We observe from Fig. 3.5 that (θ, ϕ) are the polar angles for the direction of $\mathbf{f}_{z'}$, with respect to the space-fixed axes, while $(\theta, \pi - \psi)$ are the polar angles for the direction of $\mathbf{e}_z$ with respect to the body-fixed axes. Therefore,

$$\mathbf{g}_\phi = \mathbf{e}_z = (-\sin\theta\cos\psi)\mathbf{f}_{x'} + (\sin\theta\sin\psi)\mathbf{f}_{y'} + (\cos\theta)\mathbf{f}_{z'} \tag{3.45}$$

$$\mathbf{g}_\psi = \mathbf{f}_{z'} = (\sin\theta\cos\phi)\mathbf{e}_x + (\sin\theta\sin\phi)\mathbf{e}_y + (\cos\theta)\mathbf{e}_z, \tag{3.46}$$

and, combining these,

$$\begin{aligned}\mathbf{g}_\theta = \frac{1}{\sin\theta}(\mathbf{e}_z \times \mathbf{f}_{z'}) &= -\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y \\ &= \sin\psi\,\mathbf{f}_{x'} + \cos\psi\,\mathbf{f}_{y'}.\end{aligned} \tag{3.47}$$

We use the above relations to express the transformation matrices, (3.41), in terms of Euler angles and find

$$P = \begin{pmatrix} 0 & -\sin\phi & \cos\phi\sin\theta \\ 0 & \cos\phi & \sin\phi\sin\theta \\ 1 & 0 & \cos\theta \end{pmatrix}; \; Q = \begin{pmatrix} -\sin\theta\cos\psi & \sin\psi & 0 \\ \sin\theta\sin\psi & \cos\psi & 0 \\ \cos\theta & 0 & 1 \end{pmatrix}. \tag{3.48}$$

Clearly, Det P = Det $Q = -\sin\theta$, so that the transformations (3.42) and (3.43) are non-singular for $\sin\theta \neq 0$.

The components of the angular velocity vector $\boldsymbol{\omega} = \lim_{\delta t \to 0} \delta\boldsymbol{\phi}/\delta t$ follow immediately from (3.42) and (3.43). We have

$$\omega_i = \lim_{\delta t \to 0} \frac{\delta \phi_i}{\delta t} = \sum_\lambda P_{i\lambda} \dot{\theta}^\lambda \tag{3.49}$$

for the space-fixed components of $\boldsymbol{\omega}$, and

$$\omega'_\alpha = \lim_{\delta t \to 0} \frac{\delta \phi'_\alpha}{\delta t} = \sum_\lambda Q_{\alpha\lambda} \dot{\theta}^\lambda \tag{3.50}$$

for the body-fixed components of $\boldsymbol{\omega}$. Written out as matrix equations, (3.49) and (3.50) read

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}_{\text{space}} = \begin{pmatrix} 0 & -\sin\phi & \cos\phi\sin\theta \\ 0 & \cos\phi & \sin\phi\sin\theta \\ 1 & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} \tag{3.51}$$

and

$$\begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix}_{\text{body}} = \begin{pmatrix} -\sin\theta\cos\psi & -\sin\psi & 0 \\ \sin\theta\sin\psi & \cos\psi & 0 \\ \cos\theta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}. \tag{3.52}$$

We are now in a position to write down the angular velocity and angular momentum of a rigid body in terms of the Euler angles and their time derivatives. For convenience we choose the body-fixed axes $O_{x'y'z'}$ as principal axes of the rigid body in question and revert to our previous notation of $\omega_1, \omega_2, \omega_3$, etc. for the components of a vector along these axes.

The components of angular velocity along the principal axes are obtained from (3.52). The components of angular momentum along the principal axes are given by the relation (3.12), viz. $L_i = I_i \omega_i$, $i = 1, 2, 3$. We thus have

$$\begin{aligned} \omega_1 &= -\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi = L_1/I_1 \\ \omega_2 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi = L_2/I_2 \\ \omega_3 &= \dot{\phi}\cos\theta + \dot{\psi} = L_3/I_3. \end{aligned} \tag{3.53}$$

The kinetic energy is given by (3.7) with the angular velocity components given by (3.53). We write out the answer for the particular case where $I_1 = I_2 \neq I_3$: Such rigid bodies are called symmetric tops and will be of interest to us later on. For them,

$$T = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi}\cos\theta + \dot{\psi})^2. \tag{3.54}$$

We also observe from (3.7) and (3.12) that the kinetic energy may be written as the scalar product

$$2T = \sum_i \omega_i L_i = (\boldsymbol{\omega} \cdot \boldsymbol{L}). \tag{3.55}$$

Writing out this scalar product in terms of the components of $\boldsymbol{\omega}$ and $\boldsymbol{L}$ along the triad of unit vectors $\boldsymbol{g}_\lambda$ produces

$$2T = \sum_\lambda \dot{\theta}^\lambda L_\lambda = \text{scalar}. \tag{3.56}$$

If we multiply both sides of this equation by an infinitesimal time interval δt we get

$$2T\delta t = \sum_\lambda \delta\theta^\lambda L_\lambda, \tag{3.57}$$

where $\delta\theta^\lambda$ are the increments in the Euler angles θ^λ.

The L_λ are in fact the momentum variables canonical to the Euler angles θ^λ in the sense described by (1.72) of Chapter 1. From (3.54) the value of $2T\delta t$ is

$$\begin{aligned} 2T\delta t = {} & [I_1\dot{\phi}\sin^2\theta + I_3(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta]\delta\phi \\ & + (I_1\dot{\theta})\delta\theta + [I_3(\dot{\phi}\cos\theta + \dot{\psi})]\delta\psi. \end{aligned} \tag{3.58}$$

Comparison of the right-hand members of (3.57) and (3.58) yields the L_λ:

$$\begin{aligned} L_\phi &= I_1\dot{\phi}\sin^2\theta + I_3(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta \\ L_\theta &= I_1\dot{\theta} \\ L_\psi &= I_3(\dot{\phi}\cos\theta + \dot{\psi}). \end{aligned} \tag{3.59}$$

Since the L_λ appear as the components of a covariant vector in (3.57) (as they must), we know that the expressions (3.59) are also the physical components of the angular momentum $\mathbf{L}$ along the unit vectors $\mathbf{g}_\lambda$. By contrast the covariant components of $\mathbf{L}$ along these directions are $L^\lambda = \sum_\mu g^{\lambda\mu} L_\mu$, or

$$L^\phi = I_1\dot{\phi};\ L^\theta = I_1\dot{\theta};\ L^\psi = I_3\dot{\psi} + (I_3 - I_1)\dot{\psi}\cos\theta \tag{3.60}$$

using an obvious notation. Equation (3.60) answers the question that must have occurred to the reader by now, viz. what meaning can one attach to a quantity *moment of inertia* $\times$ *rate of change of an Euler angle*?

We are now in possession of all of the ingredients to construct the Lagrange function of a rigid body with $I_1 = I_2 \neq I_3$ moving about a fixed point. If the forces are derivable from a potential function $V(\theta^\lambda)$, we have

$$L = \frac{1}{2}I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - V(\phi,\theta,\psi) \tag{3.61}$$

as a standard form for the Lagrange function.

Let us first verify that L_λ in (3.59) are indeed the canonical momenta.

We have

$$\begin{aligned}\frac{\partial L}{\partial \dot{\phi}} &= I_1\dot{\phi}\sin^2\theta + I_3(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta = L_\phi \\ \frac{\partial L}{\partial \dot{\theta}} &= I_1\dot{\theta} = L_\theta \\ \frac{\partial L}{\partial \dot{\psi}} &= I_3(\dot{\phi}\cos\theta + \dot{\psi}) = L_\psi,\end{aligned} \tag{3.62}$$

which proves the assertion. The Lagrange function L may be used to study the motion of a rigid body about a fixed point once the form of $V(\theta^\lambda)$ is known.

3.7 Applications

The applications of the Lagrange function in (3.61) to the motion of rigid bodies with an axis of symmetry are manifold. However, we will restrict the discussion to two specific examples. As a first application, look at the motion of an ordinary top spinning on a rough surface so that the peg of the top does not wander around. Alternatively, imagine the peg of the top to be attached to a fixed point by a "universal joint", so that the top can move freely about the point of attachment. In either event we are dealing with the action of a rigid body about a fixed point, and if the motion is due to conservative forces only, (3.61) for the Lagrange function will suffice to study the motion.

Anyone who has watched an ordinary top in motion must have been struck by how peacefully the top moves: spinning very rapidly around its symmetry axis, while turning as a whole rather slowly about a vertical axis with the symmetry axis held at an angle to the vertical. This motion is referred to as a steady, or regular precession and is the easiest type of motion to discuss analytically. Often such more complicated motions occur. The most common of these is produced if the top is set spinning rapidly about its symmetry axis and then set down gently on its peg with this axis inclined to the vertical. Initially the top starts to fall under the action of gravity but, remarkably, catches itself and climbs back to (almost) its initial inclination to the vertical while performing a precessional motion about the vertical at the same time. This is called pseudo-regular precession since it differs from the pure precession by the tops performing an additional nodding (nutational) motion during precession. The ability of such spinning bodies to stabilize themselves against the force of gravity has fascinated scientists and mystics alike throughout the history of mechanics. It will be an interesting task to understand the reasons behind these rather surprising motions in terms of the dynamics governing the motion of a rigid body.

We take an ordinary top spinning on a horizontal surface as a prototype example of a rigid body moving about a fixed point. Thus, we assume either that the surface is rough enough to prevent the peg of the top from wandering around on it, or that the peg has been attached to a fixed point by some device (like a universal joint) that does not interfere with the orientation of the top in any way. We shall also assume that the top is symmetric, i.e. it possesses an axis of symmetry which passes through its center of gravity G and the point of support O. This means that OG is a principal axis of inertia, that *any* pair of orthogonal axes in the plane normal to OG will serve as principal axes, and that the moments of inertia about these two axes are equal. Therefore, in treating the motion of a symmetrical body we are at liberty to choose axes that are only partially "frozen" into the body, without destroying the validity of relations like (3.7) and (3.12). In particular, this means that the pair of principal axes normal to OG need not rotate around OG with the top.

The top is positioned in space by rotating it through Euler angles ϕ, θ and ψ relative to the space-fixed axes O_{xyz} as prescribed by (3.30). Fig. 3.6 shows the final position of the top after such a rotation has been performed. In the present context the angles θ and ϕ give the polar angles of the symmetry axis OG, while ψ measures the amount of rotation of the top around OG. The unit vectors $\mathbf{g}_\phi$, $\mathbf{g}_\theta$ and $\mathbf{g}_\psi$ that are associated with these angles are also shown. When the top moves the unit vectors $\mathbf{g}_\theta$ and $\mathbf{g}_\psi$ move with it, while $\mathbf{g}_\phi$ always maintains its direction along O_z which we choose as the vertical direction. Changes in ϕ describe the precession, θ the nutation (or nodding) of the top, while ψ measures the rotation about OG.

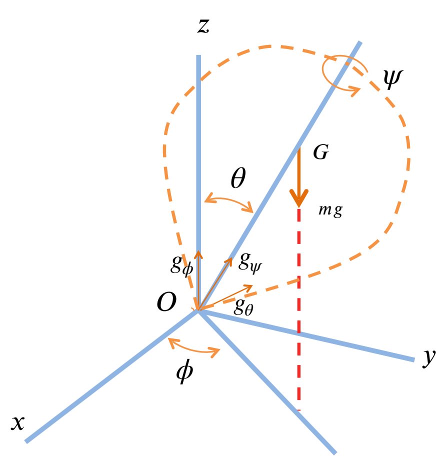

Figure 3.6: Rotation of the top with space-fixed axes.

Now, we must choose our principal axes at O: $\mathbf{g}_\psi = \overrightarrow{OG}$ is already a principal axis. Also we have remarked earlier that any axis in the plane normal to OG is a principal axis. Therefore, $\mathbf{g}_\theta$ also lies along a principal axis. The remaining unit vector

$$\mathbf{f} = (\mathbf{g}_\psi \cos\theta - \mathbf{g}_\phi) \operatorname{cosec}\theta$$

perpendicular to $\mathbf{g}_\theta$ and $\mathbf{g}_\phi$ and in the plane zOG is our other principal axis. Notice that the triad of partially frozen unit vectors $(\mathbf{f}, \mathbf{g}_\theta, \mathbf{g}_\psi)$ differs from the body-fixed triad $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ in that it ignores the rotation ψ about $\mathbf{g}_\psi$. Therefore, the components of the angular velocity and angular momentum along *these* axes follow from (3.53) on setting $\psi = 0$ (but *not* $\dot\psi = 0$!) in those equations:

$$\begin{aligned}
\omega_1 &= -\dot\phi \sin\theta = L_1/A \quad \text{along} \quad \mathbf{f} \\
\omega_2 &= \dot\theta = L_2/A \quad \text{along} \quad \mathbf{g}_\theta \\
\omega_3 &= \dot\phi\cos\theta + \dot\psi = L_3/C \quad \text{along} \quad \mathbf{g}_\psi,
\end{aligned} \tag{3.63}$$

if we denote the equal moments of inertia at O by $I_1 = I_2 = A$ and the unequal moment about OG by C.

Call $OG = h$; then, if the top moves in a constant gravitational field g that acts downward along $-z$, the potential energy is $V = mgh\cos\theta$, $m =$ mass of the top. Then, (3.61) reads

$$L = \frac{1}{2}A(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}C(\dot\phi\cos\theta + \dot\psi)^2 - mgh\cos\theta. \tag{3.64}$$

We observe immediately that both angles ϕ and ψ are absent from L since the interaction energy with gravity is insensitive to changes in these angles. Thus, ϕ and ψ are cyclic and the corresponding canonical momenta are constants of motion. These momenta are given in the first and last lines of (3.62). Also, L does not depend on time explicitly. Therefore, the total energy is conserved during the motion. Notice, that by just examining the invariance properties of L we have discovered three constants of motion. They are

$$A\dot\phi\sin^2\theta + C(\dot\phi\cos\theta + \dot\psi)\cos\theta = M \tag{3.65}$$

$$C(\dot\phi\cos\theta + \dot\psi) = K \tag{3.66}$$

$$\frac{1}{2}A(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}C(\dot\phi\cos\theta + \dot\psi)^2 + mgh\cos\theta = E, \tag{3.67}$$

calling E the total energy and M and K the constant angular momentum projections along space axis O_z and symmetry axis OG respectively.

The above set of equations determines the top motion completely in terms of these constants, which are fixed by the initial conditions[45].

Let us examine their contents. Equation (3.66) insists that the angular momentum $L_3 = C\omega_3$ and therefore the angular velocity $\omega_3 = (\dot\phi\cos\theta + \dot\psi)$ of the top along the symmetry axis is constant. Let us call this constant n, so that

$$n = \dot\phi\cos\theta + \dot\psi = K/C \tag{3.68}$$

can be used interchangeably with the constant angular momentum K if desired. The quantity n is called the *spin* of the top. Notice that it

[45] As an aside, notice that precisely the same set of constants E, M and K would be subject to quantization conditions if the top were considered to be a quantum mechanical system.

has contributions from both $\dot{\phi}$ and $\dot{\psi}$. Obviously, it will prove useful to eliminate the combination $\dot{\phi}\cos\theta + \dot{\psi}$ in equations (3.65) through (3.67); we find

$$Cn = K; \quad \frac{d\phi}{dt} = \frac{M - K\cos\theta}{A\sin^2\theta}, \tag{3.69}$$

which combines with the energy (3.67) to give

$$\frac{1}{2}A(\frac{d\theta}{dt})^2 = (E - \frac{K^2}{2C} - mgh\cos\theta) - \frac{1}{2A}(\frac{M - K\cos\theta}{\sin\theta})^2. \tag{3.70}$$

The second of (3.69) and (3.70) express the precession $\dot{\phi}$ and nutation $\dot{\theta}$ in terms of the angular inclination of the top's symmetry axis; we have written these angular velocities as $d\phi/dt$ and $d\theta/dt$ respectively to emphasize the fact that the problem of finding $\phi(t)$ and $\theta(t)$ has been "reduced to quadratures" which is a formal way of saying that you know the answer if you can do the integrals. In fact (3.69) and (3.70) can be integrated in terms of elliptic functions[46], but the answer one gets is not particularly illuminating. Instead of carrying through the solution in general let us examine these equations for the particular example where the top starts off with an initial spin n about the symmetry axis that is inclined at an angle α to the vertical. The initial conditions for this motion are

[46] A. Sommerfeld, *Lectures on Theoretical Physics, Vol 1, Mechanics*, Academic Bress Inc., New York, 1952.

$$\begin{aligned} \phi = 0, \quad \theta = \alpha, \quad \psi = 0 \\ \omega_1 = 0, \quad \omega_2 = 0, \quad \omega_3 = n \end{aligned} \tag{3.71}$$

at $t = 0$. The constants of motion therefore take on the values

$$K = Cn, \quad M = Cn\cos\alpha, \quad E = \frac{1}{2}Cn^2 + mgh\cos\alpha.$$

Entering this information into (3.69) and (3.70) we find that

$$\frac{d\phi}{dt} = n\frac{C}{A}(\frac{\cos\alpha - \cos\theta}{1 - \cos^2\theta}) \tag{3.72}$$

and

$$\begin{aligned} (A\sin\theta\frac{d\theta}{dt})^2 = (\cos\alpha - \cos\theta) \\ \times [2Amgh(1 - \cos^2\theta) - C^2n^2(\cos\alpha - \cos\theta)] \end{aligned} \tag{3.73}$$

after a slight rearrangement of terms. Let us supplement these equations with the Lagrange equation for θ; this is

$$A\ddot{\theta} = \frac{\partial L}{\partial\theta} = \sin\theta[A\dot{\phi}^2\cos\theta - Cn\dot{\phi} + mgh], \tag{3.74}$$

where we calculate $\partial L/\partial\theta$ from (3.64). Alternatively, we could obtain this relation by differentiating (3.73) with respect to time. We require it to establish the initial motion of the top. Since $\dot{\phi} = 0$ at $t = 0$ when $\theta = \alpha$ we see that $\ddot{\theta}(t = 0) = (mgh \sin \alpha)/A$ is positive $(0 < \alpha < \pi)$, i.e. initially the top begins to fall to the ground as one might intuitively expect it to do. However, this downward motion of the symmetry axis ceases when the nutational angular velocity $\dot{\theta}$ vanishes. Equation (3.73) provides us with information on this aspect of the motion: $\dot{\theta}$ vanishes when $\cos \theta = \omega$ reaches one of the roots of the cubic equation

$$(\omega_0 - \omega)[2Amgh(1 - \omega^2) - C^2 n^2 (\omega_0 - \omega)] = 0$$
$$\omega = \cos \theta, \quad \omega_0 = \cos \alpha, \quad |\omega_0| < 1.$$

The first factor $(\omega_0 - \omega) = 0$ just assures us that the axis will return to its initial inclination $\omega_0 = \cos \alpha$ during the subsequent motion. The other orientation of the symmetry axis for which $\dot{\theta} = 0$ is supplied by the real roots of the quadratic

$$f(\omega) = 2Amgh(1 - \omega^2) - C^2 n^2 (\omega_0 - \omega) = 0 \tag{3.75}$$

that satisfy $-1 < \omega < 1$ (θ must be real). We observe that $f(-1)$ is negative, while both $f(\omega_0)$ and $f(+1)$ are positive. Hence, $f(\omega)$ has one real root between -1 and ω_0. Therefore, the symmetry axis of a top started off with the initial conditions (3.71) will swing between the angle α and the larger angle β that satisfies $f(\cos \beta) = 0$. If the initial spin is large we have

$$\cos \beta - \cos \alpha \simeq -2 \frac{Amgh}{(Cn)^2} \sin^2 \alpha. \tag{3.76}$$

Thus, by making the initial spin n (more correctly the angular momentum $K = Cn$) large enough we can effectively suppress the nutational motion of the top. Thus, θ is forced to vary over the small angular range $\alpha \leq \theta \leq \beta$ and we can therefore replace θ by its average value over this interval. From (3.76) we find that this angular interval is

$$\delta\alpha = (\beta - \alpha) \simeq 2 \frac{Amgh}{(Cn)^2} \sin \alpha$$

approximately, so that we can replace θ by the average value

$$\langle \theta \rangle = \alpha + \langle \delta\alpha \rangle \simeq \alpha + \frac{Amgh}{(Cn)^2} \sin \alpha \tag{3.77}$$

in our equations. Thus, the average precession rate is given by (3.72) with θ replaced by $\langle \theta \rangle$. We get

$$\langle \dot{\phi} \rangle \simeq \frac{Cn}{A} \frac{\langle \delta\alpha \rangle}{\sin \alpha} = \frac{mgh}{Cn} \tag{3.78}$$

if Cn is large. This equation shows that the precession about the vertical axis is in a positive sense under these circumstances and independent of the angle of inclination α.

Our picture of the top motion is now complete: the top precesses slowly around the vertical with an average precession given by (3.78) while spinning around its symmetry axis which nods up and down between the angles α and $\beta = \alpha + \delta\alpha$. The motion is perhaps best visualized by following the path that the extremity of the symmetry axis OG would trace out on a sphere centered at O. This path will be bounded by two circles on the sphere corresponding to the limiting orientations $\theta = \alpha$ and $\theta = \beta$ at OG. The exact nature of this path depends on the starting conditions. For the particular starting conditions we have been discussing it is clear from (3.72) that the precession $\dot{\phi}$ vanishes whenever the symmetry axis returns to its initial orientation $\theta = \alpha$ (where $\dot{\theta} = 0$ also). The symmetry axis therefore periodically comes to rest momentarily in this position. The path that its extremity traces out on a sphere centered at O is characterized by cusps at the points $\theta = \alpha$, $\dot{\phi}$, $\dot{\theta} = 0$ as shown in Fig. 3.7.

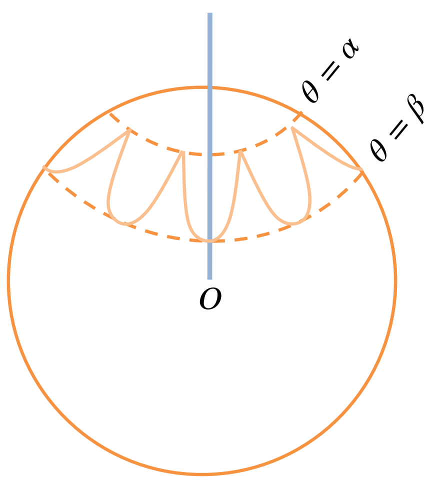

Figure 3.7: Cuspidal motion of top.

We can probe the problem for more detail by asking for example what the nature of the nutational motion is like. To investigate this point, we appeal to the equation of motion (3.74) for θ, set $\theta = \alpha + \varphi$, and use the unaveraged version of (3.78) with $\langle\delta\alpha\rangle$ replaced by φ for the precession, i.e.

$$\dot{\phi} = n\frac{C}{A}\frac{\varphi}{\sin\alpha}.$$

We substitute this value of $\dot{\phi}$ in (3.74) and find to first order in the small angle φ that

$$\ddot{\varphi} \simeq -(\frac{C}{A}n)^2\phi + \frac{mgh}{A}\sin\alpha,$$

which has the solution

$$\varphi(t) = \frac{Amgh}{(Cn)^2}\sin\alpha[1-\cos(\frac{C}{A}nt)], \tag{3.79}$$

if we incorporate the boundary conditions that φ must lie between O and the maximum value $\delta\alpha = [\frac{2Amgh}{C^2n^2}]\sin\alpha$. The nutation of the top is therefore a pure harmonic oscillation in the angle θ about the average orientation, (3.77), with the period

$$T_n = \frac{2\pi}{n}\frac{A}{C}.$$

If we compare this with the period of average precession,

$$T_p = \frac{2\pi}{\langle\dot{\phi}\rangle} = 2\pi\frac{Cn}{mgh},$$

we find

$$\frac{T_n}{T_p} = \frac{Amgh}{(Cn)^2} \ll 1, \tag{3.80}$$

which is a precise physical statement of when our approximate treatment is valid: the precessional motion must be much slower than the nutational motion. Equation (3.80) also identifies $(Amgh)/(Cn)^2$ as the appropriate expansion parameter for this case. We note in passing that the two periods in (3.80) are incommensurate in general.

Equation (3.79) also allows us to compute the instantaneous precession to the same order of approximation. We have

$$\dot{\phi} \simeq \langle\dot{\phi}\rangle[1 - \cos(\frac{C}{A}nt)]$$

from (3.72), and integration with respect to time gives us ϕ:

$$\phi(t) = \langle\dot{\phi}\rangle t - \frac{Amgh}{(Cn)^2}\sin(\frac{C}{A}nt), \tag{3.81}$$

showing typical "drift" and oscillatory terms. Equations (3.79) and (3.81) specify the angles θ and ϕ as a function of time; we can get $\psi(t)$ from (3.68). Hence, our approximate solution is complete.

The motion we have just been discussing is called *pseudoregular* precession of a top because it differs from the true regular precession $(\theta = \alpha, \dot{\theta} = 0)$ by terms of order T_n/T_p in the angular velocity of nutation, $\dot{\theta}$. Consequently, these small nutations are not visible to the casual observer and the top *appears* to perform a regular precession.

These observations naturally lead us to ask for the conditions under which a true regular precession is possible. Again, the equation of motion (3.74) provides the answer: for regular precession, both $\dot{\theta}$ and $\ddot{\theta}$ must vanish. This means $\partial L/\partial\theta = 0$, or that

$$\sin\alpha(Ap^2\cos\alpha - Cnp + mgh) = 0 \tag{3.82}$$

for a regular precession with $\theta = \alpha$. We use p to denote the constant value of $\dot{\phi}$ that is allowed. If we exclude the special case $\sin\alpha = 0$, there are two allowed values of $\dot{\phi} = p$ in regular precession,

$$p = \left(Cn \pm \sqrt{(Cn)^2 - 4Amgh\cos\alpha}\right)/2A\cos\alpha.$$

We notice that regular precession is possible only if $(Cn)^2 > 4Amgh\cos\alpha$; there is a minimum spin angular momentum required to stabilize the top.

Let us suppose again that Cn is large. Then, we find two physically distinct roots for p that can be classified as a fast and slow precession, i.e.

$$p \simeq \begin{cases} \frac{C}{A\cos\alpha}n, & \text{fast precession} \\ \frac{mgh}{Cn}, & \text{slow precession.} \end{cases} \tag{3.83}$$

Notice that both these precessions are in the same direction, and that the slow one coincides with the average precession of our first top, as given by (3.78). So the top we started off with initial conditions (3.71) nearly performs a regular precession. If we had provided for an initial precessional velocity $p = mgh/Cn$, it would have done so exactly. Clearly, other varieties of starting conditions that favor other types of motion that can be analysed in the same way as the examples we have discussed.

We have discussed the ordinary top in some detail because it forms a prototype example of how the motion of a rigid body may be investigated. We have also used Lagrange equations (and their consequences) throughout the discussion in order to stress once more the power and uniformity of such an approach. But having once obtained the answer it is always essential to try to understand the reasons behind our results. Perhaps, the most surprising aspect of the top motion is the way in which it sidesteps gravity instead of falling down in both its regular and pseudo-regular modes of precession. The reason for this lies *not* in the conservation of angular momentum (for there are torques present) but rather in the law of its change, as the following simple argument shows. We start by borrowing (1.14) from Chapter 1:

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}; \quad d\mathbf{L} = \mathbf{N}dt, \tag{3.84}$$

where $\mathbf{N}$ is the total torque that is responsible for changes in the angular momentum vector $\mathbf{L}$. From Fig. 3.6 we see that the torque $\mathbf{N}$ for our ordinary top about the peg O is provided by the gravitational couple $\mathbf{N} = mgh\sin\theta\mathbf{g}_\theta$ that is always normal to the plane OzG. Also, in steady precession we have the condition that $\dot\theta = 0$: hence the only change in $\mathbf{L}$ that can be brought about by $\mathbf{N}$ is one of direction (the length of $\mathbf{L}$ cannot change since it lies in the OzG plane and we know its projections along Oz and OG must be constant). The state of affairs is depicted in Fig. 3.8, which is the "skeleton" of Fig. 3.6. The angular momentum components are drawn in to show how $\mathbf{L}$ is made up of the components Cn and $A\dot\phi\sin\alpha$ along and perpendicular to the symmetry axis of the top; α is the value of θ in steady precession as before and γ is the angle between the angular momentum $\mathbf{L}$ and the symmetry axis. Since this entire

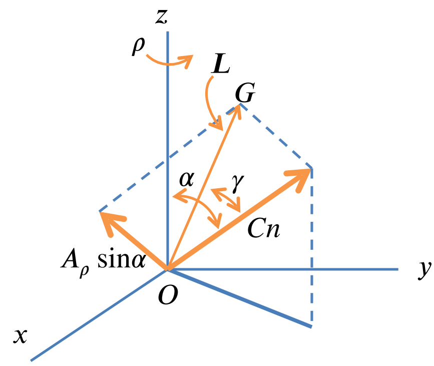

Figure 3.8: Directional change of angular momentum.

pattern precesses around the Oz axis with constant angular velocity p, we have immediately that $|d\mathbf{L}| = |\mathbf{L}|\sin(\alpha-\gamma)pdt$ and $d\mathbf{L}$ points along $\mathbf{g}_\theta$. But

$$\begin{aligned} |\mathbf{L}|\sin(\alpha-\gamma) &= (|\mathbf{L}|\cos\gamma)\sin\alpha - (|\mathbf{L}|\sin\gamma)\cos\alpha \\ &= Cn\sin\alpha - Ap\sin\alpha\cos\alpha \end{aligned}$$

from the geometry of Fig. 3.8 so we have finally, using (3.84),

$$(Cn - Ap\cos\alpha)\sin\alpha pdt = mgh\sin\alpha dt,$$

which is the same condition as (3.82).

The magnitude and sign (does the angular momentum axis lie above or below the symmetry axis?) of the angle γ can also be read off from Fig. 3.8. Since $|\mathbf{L}|\sin\gamma = Ap\sin\alpha$, $|\mathbf{L}|\cos\gamma = Cn$, we have

$$\tan\gamma = \frac{Ap}{Cn}\sin\alpha, \tag{3.85}$$

which shows that Ap/Cn is the controlling ratio. Since we are dealing with steady precession, p and n are connected by (3.82) with the limiting values of p for fast and slow modes of precession given by (3.83). The latter equation and (3.85) combine to show that $\gamma \simeq \alpha$ in fast precession, while $\gamma \simeq 0$ in slow precession, i.e. the angular momentum vector aligns itself along the vertical axis in fast precession, and aligns itself along the spin, or symmetry axis, in slow precession.

If we are dealing with non-steady motions of the top, the above discussion is not complete. Changes in $\dot{\theta}$ away from zero introduce a component $A\dot{\theta}$ of $\mathbf{L}$ along the unit vector $\mathbf{g}_\theta$, so that both the direction and magitude of $\mathbf{L}$ change, subject always of course to the conditions that the projections M and K are constant. These restrictions mean that $\mathbf{L}$ can at most "wave" back and forth through the plane OzG alternately leading and lagging the spin axis while keeping M and K constant. If the motion considered is pseudo-regular precession, (3.79) tells us that this fluctuation in length and direction (out of the OzG plane) of $\mathbf{L}$ is periodic with period T_n. Consequently $\tan\gamma(t)$ will vary with time like

$$\tan\gamma(t) \simeq \frac{A\langle\dot{\phi}\rangle \sin\alpha}{Cn}[1 - \cos(\frac{C}{A}nt)],$$

with an average value of $(A\langle\dot{\phi}\rangle \sin\alpha)/Cn$, i.e. just the value (3.85) with p replaced by the average precession.

3-8 General Motion of a Rigid Body

So far we have restricted the discussion to the case of a rigid body moving about a fixed point, since this had the advantage of focussing our attention on the pure rotational aspects of the motion and how these may be handled within a Lagrange framework. With this experience behind us, let us remove this restriction and consider the general case of unrestricted motion of a rigid body. In order to get a different perspective of the problem let us develop the general case starting afresh from first principles instead of using a Lagrange equation approach. At the end of our discussion we can then investigate to what extent Lagrange equations can be applied to the general problem of rigid body motion.

We have already set down the general equations of motion for any many-body system in Chapter 1. We recall for convenience the principles of linear and angular momentum:

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}; \quad \frac{d\mathbf{L}}{dt} = \mathbf{N}, \tag{3.86}$$

which relate the rates of change of total linear and angular momentum to the total force $(\mathbf{F})$ and the total moment or torque $(\mathbf{N})$ that are responsible for these changes. At first sight, (3.86) appear to be very innocent. The trouble is of course that they only hold with respect to an inertial frame of reference (for our purposes, one fixed in space), while, for example, our expressions for the angular momentum were only simply related to angular velocity in a body-fixed frame which is *not* an inertial frame of reference. The calculation we must do is obvious: retain the simple relations between $\mathbf{L}$ and $\boldsymbol{\omega}$ and modify (3.86) so that they hold in the body-fixed frame. (Note in passing that we have to do this for both equations, since they are coupled to each other by the condition that $\mathbf{F}$ also provides the torque $\mathbf{N}$ and consequently it would be extremely awkward to have the first equation expressed in an inertial frame and the second in a moving, non-inertial frame). But this transformation is simple to perform. We attach a set of axes to the center-of-mass G of the body: call $\boldsymbol{\Omega}$ the angular velocity of this set of axes. If the body has an arbitrary shape, these axes will also only be principal axes if they are rigidly "frozen" into the body; the angular velocity of the body will then also be $\boldsymbol{\omega} = \boldsymbol{\Omega}$. However, if the rigid body has an axis of symmetry as our top had, we see that the principal axes need only be partially "frozen" into the body, in which case $\boldsymbol{\omega} \neq \boldsymbol{\Omega}$.

Consider now, how the effect of moving axes enters into the expression for the rate of change of a vector like $\mathbf{L}$. This has a time derivative $d\mathbf{L}/dt$ with respect to fixed axes. Suppose now that $\mathbf{L}$ was rigidly attached to the moving axes that are rotating with angular velocity $\boldsymbol{\Omega}$; its rate of change would be zero in the moving system. In the fixed system, however, $\mathbf{L}$ changes by $\delta\mathbf{L} = \delta t(\boldsymbol{\Omega} \times \mathbf{L})$ in time δt, see (3.2). In general $\mathbf{L}$ will also change with respect to the moving axes. Calling this rate of change $\partial\mathbf{L}/\partial t$ we have

$$\frac{d\mathbf{L}}{dt} = (\frac{\partial\mathbf{L}}{\partial t})_{\text{moving}} + (\boldsymbol{\Omega} \times \mathbf{L}), \tag{3.87}$$

where the differentiation on the right is now performed with respect to the moving axes. Equation (3.87) holds the key to our problem. We can now refer both sides of our basic equation (3.86) to moving axes. If these axes are principal axes (as they will be from now on) we possess simple expressions for the components of $\mathbf{L}$. In moving axes then we have

$$\begin{aligned} (\frac{\partial\mathbf{L}}{\partial t})_{\text{moving}} + \boldsymbol{\Omega} \times \mathbf{L} = \mathbf{N} \\ (\frac{\partial\mathbf{P}}{\partial t})_{\text{moving}} + \boldsymbol{\Omega} \times \mathbf{P} = \mathbf{F}. \end{aligned} \tag{3.88}$$

Let us break up each vector into its components along the principal axes and draw on the expressions (3.12) for the components of angular momentum. Then, we have

$$\begin{aligned} I_1\dot{\omega}_1 + \Omega_2 I_3\omega_3 - \Omega_3 I_2\omega_2 &= N_1 \\ I_2\dot{\omega}_2 + \Omega_3 I_1\omega_1 - \Omega_1 I_3\omega_3 &= N_2 \\ I_3\dot{\omega}_3 + \Omega_1 I_2\omega_2 - \Omega_2 I_1\omega_1 &= N_3 \end{aligned} \tag{3.89}$$

for the motion about the center-of-mass, and

$$\begin{aligned} \dot{P}_1 + \Omega_2 P_3 - \Omega_3 P_2 &= F_1 \\ \dot{P}_2 + \Omega_3 P_1 - \Omega_1 P_3 &= F_2 \\ \dot{P}_3 + \Omega_1 P_2 - \Omega_2 P_1 &= F_3 \end{aligned} \tag{3.90}$$

for the motion of the center-of-mass. We are accordingly treated to the full content of (3.88) which we see is not so simple after all. Equations (3.88), or in their component form above, are called Euler's equations. They were first constructed by him in 1758. Apparently, Euler was the first to recognize the importance of using moving axes in mechanics.

Before going on to consider applications of these equations to problems that merit them let us remark that we could discuss the ordinary top starting from these equations just as well. However, in the case of the motion of a rigid body about a fixed point the reactions at that point are unknown in general and therefore also some of the force and torque components appearing on the right hand sides of (3.89) and (3.90) are unknown. Thus, it is much more practical to take moments about the fixed point. We introduce principal axes at this point (call it O as before) that rotate with angular velocity $\mathbf{\Omega}$, the unknown reactions have zero moments about O and we get

$$(\frac{\partial \mathbf{L}}{\partial t})_{\text{moving}} + \mathbf{\Omega} \times \mathbf{L} = \mathbf{N}, \tag{3.91}$$

which is *not* the same as the first of (3.88) because $\mathbf{L}$ and $\mathbf{N}$ refer to the fixed point O and not to the center-of-mass G.

3-9 The Tippe-top

If a top like the one in Fig. 3.9(a) is set spinning rapidly on its broad end, it acts rather surprisingly. The thin peg describes a downward spiral and when it touches the floor the top rises rapidly on the peg so that it is spinning as in Fig. 3.9(b). Similarly if one takes an egg-shaped object as in Fig. 3.9(c) one discovers that it will only spin stably on its thin end as shown in Fig. 3.9(d). If set spinning as in Fig. 3.9(c) it also rapidly rises on to its thin end.

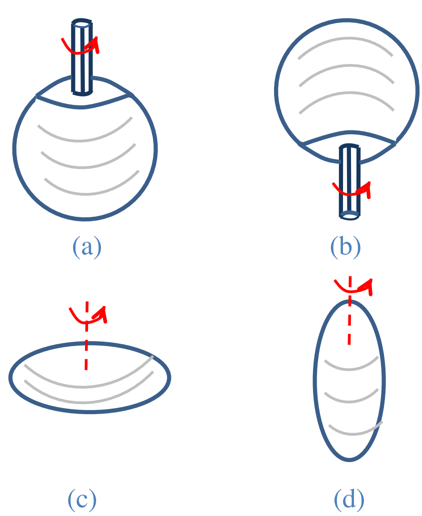

Figure 3.9: Rotating top ((*a*) and (*b*)) and rotating egg ((*c*) and (*d*)).

Quite a literature has grown up around problems of this sort[47] which concerns itself with the problem of the inverting top. The tippe-top provides us with an excellent system to study as an example of general rigid body motion. Moreover, there is the interesting question of whether we can understand the reasons for the peculiar behavior of the top, i.e. we have an experimental result that requires an explanation (a rather

[47] E.G. Gallop, Cambridge Phil. Soc. Transactions **19**, 356 (1904); M.M. Hugenholtz, Physica **18**, 515 (1952); D.G. Parkyn Math. Gazette XL, 260 (1956); Physica **24**, 313 (1958).

rare bird in Classical Mechanics!). We will in fact find that the problem of inverting tops yields to analysis in a straightforward manner.

Fig. 3.10 shows a tippe-top resting in an arbitrary orientation on a horizontal surface (a table).

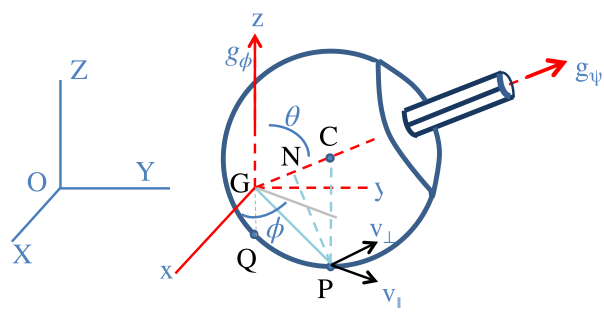

Figure 3.10: The tippe top.

$CP = a$
$CG = h$
$QG = a\text{-}h\cos\theta$
$NP = a\sin\theta$
$QP = h\sin\theta$
$NG = h - a\cos\theta$

We start the problem again by considering the degrees of freedom involved. If the top is required to be in contact with the table at all times, we introduce a constraint on the six degrees of freedom (three for translational motion plus three for rotational motion) that the rigid body has in arbitrary motion. Hence, we have five degrees of freedom and must therefore pick five independent coordinates. We do this in the following manner. Assume that the lower end of the tippe-top is spherical with radius a. The center of this sphere is at C on the symmetry axis of the top passing through its center-of-mass G and along the thin peg as shown. C will always be vertically above the point P where the top touches the table. We set up the usual unit vectors $\mathbf{g}_\phi$, $\mathbf{g}_\theta$ $\mathbf{g}_\psi$ attached to the center-of-mass G in order to define the Euler angles ϕ, θ, ψ, which orient the top with respect to axes G_{xyz} that move with G but keep their orientation in space. As in the case of the ordinary top $\mathbf{g}_\phi$ is vertical, $\mathbf{g}_\psi$ points along the symmetry axis and $\mathbf{g}_\theta = (\mathbf{g}_\phi \times \mathbf{g}_\psi)/\sin\theta$ is normal to the vertical plane containing $\mathbf{g}_\phi$ and the symmetry axis. Since the top is symmetric about $\mathbf{g}_\psi$, the orthogonal triad $(\mathbf{f}, \mathbf{g}_\theta, \mathbf{g}_\psi)$ at G where

$$\mathbf{f} = \mathbf{g}_\psi \cot\theta - \frac{1}{\sin\theta}\mathbf{g}_\phi \tag{3.92}$$

lies in the vertical plane containing the symmetry axis, again form a set of principal axes. We call the two equal moments of inertia $I_1 = I_2 = A$ and $I_3 = C$ is the moment of inertia about the symmetry axis.

Notice that A and C now refer to the center-of-mass and not the point of support as was the case previously. The orientation of the tippe-top with respect to axes G_{xyz} is thus fully specified. Finally, we locate G with respect to the table top using axes O_{XYZ}. However, the vertical distance $Z = QG$ of G is not independent of θ because of our constraint on the top. Calling $h = CG$ the center-of-mass - center-of-curvature separation, we see from Fig. 3.10 that

$$Z = QG = (a - h\cos\theta) \tag{3.93}$$

which expresses our constraint equation on Z.

The system of forces acting on the top are, in addition to gravity at G and the normal reaction $\mathbf{R} = R\mathbf{g}_\phi$ at P, also forces of friction at P if the table is not perfectly smooth. The direction of the friction force will always oppose the motion of the contact point P on the spherical surface of the top. Let us break up the displacement of P into components parallel and perpendicular to the line QP. Then, to find these displacements we recall that the triad $(\mathbf{f}, \mathbf{g}_\theta, \mathbf{g}_\psi)$ moves with the top but does not share its rotation about $\mathbf{g}_\psi$. Hence, infinitesimal increments $\delta\theta^\lambda$ in the Euler angles (ϕ, θ, ψ) will generate the infinitesimal rotations

$$\delta\phi_1 = -\sin\theta\delta\phi, \quad \delta\phi_2 = \delta\theta, \quad \delta\phi_3 = \delta\psi + \delta\phi\cos\theta$$

along this triad. This result is geometrically obvious; alternatively we can use (3.43) with the Q-matrix given by (3.48) with ψ set equal to zero. The displacements of P along QP and normal to QP are thus, from Fig. 3.10

$$\begin{aligned} \delta G_{||} - \delta\phi_2(QG) &= \delta G_{||} - \delta\theta(a - h\cos\theta) \\ \delta G_\perp + \delta\phi_3(NP) - \delta\phi_1(NG) &= \delta G_\perp + [a\delta\phi_3 + \delta\phi(h - a\cos\theta)]\sin\theta, \end{aligned} \tag{3.94}$$

where $\delta G_{||}$ and $\delta G_\perp$ symbolize the displacements of G itself along and normal to QP (such displacements are not governed by changes in the Euler angles). Dividing these expressions by the time interval δt during which they occurred, we get the velocity components of P,

$$\begin{aligned} v_{||} &= u - \dot\theta(a - h\cos\theta) \\ v_\perp &= v + [a\omega_3 + \dot\phi(h - a\cos\theta)]\sin\theta, \end{aligned} \tag{3.95}$$

where ω_3 is the angular velocity of the top along g_ψ. We have called the velocity components of G along and normal to QP, u and v. The friction force can now be displayed as the components

$$\begin{aligned} F_{||} &= -\mu R \frac{v_{||}}{|v_p|} \\ F_\perp &= -\mu R \frac{v_\perp}{|v_p|} \end{aligned} \tag{3.96}$$

with μ the coefficient of friction, R the normal reaction and $|v_p| = \sqrt{v_{||}^2 + v_{\perp}^2}$ the speed of P.

The next step is to fill in (3.88) for our particular problem. We take moments about G. Then the force of gravity drops out, having no moment about G, and the total moment along $\mathbf{g}_\phi$ and $\mathbf{g}_\psi$ arise solely from the frictional component $F_\perp$, while R and $F_{||}$ provide for moments along $\mathbf{g}_\theta$. We get from Fig. 3.10

$$\begin{aligned} N_\phi &= F_\perp(QP) = F_\perp h \sin\theta \quad \text{(a)} \\ N_\theta &= -R(QP) - F_{||}(QG) \quad \text{(b)} \\ N_\psi &= F_\perp(NP) = F_\perp a \sin\theta \quad \text{(c)} \end{aligned} \tag{3.97}$$

for the moments along $\mathbf{g}_\phi$, $\mathbf{g}_\theta$ and $\mathbf{g}_\psi$ respectively.

Let us calculate the rate of change $d\mathbf{L}/dt$ along the axes $\mathbf{g}_\phi$, $\mathbf{g}_\theta$ and $\mathbf{g}_\psi$ also. The angular velocity of the principal axes $(\mathbf{f}, \mathbf{g}_\theta, \mathbf{g}_\psi)$ is

$$\mathbf{\Omega} = \mathbf{g}_\phi \dot{\phi} + \mathbf{g}_\theta \dot{\theta},$$

written in terms of its contravariant components $\dot{\phi}$ and $\dot{\theta}$. We write $\mathbf{L}$ the same way, i.e.

$$\mathbf{L} = \sum_\lambda \mathbf{g}_\lambda L^\lambda = \mathbf{g}_\phi L^\phi + \mathbf{g}_\theta L^\theta + \mathbf{g}_\psi L^\psi$$

and find

$$\frac{d\mathbf{L}}{dt} = \sum_\lambda \mathbf{g}_\lambda \dot{L}^\lambda + \sum_\lambda \dot{\mathbf{g}}_\lambda L^\lambda. \tag{3.98}$$

The time derivatives of $\mathbf{g}_\lambda$ are easy to obtain. We observe that $\mathbf{g}_\phi$ does not rotate at all while $\mathbf{g}_\theta$ and $\mathbf{g}_\psi$ rotate with angular velocity $\mathbf{\Omega}$. Noting that

$$\begin{aligned} \mathbf{g}_\phi \times \mathbf{g}_\theta &= \mathbf{g}_\phi \cot\theta - \mathbf{g}_\psi \frac{1}{\sin\theta} \\ \mathbf{g}_\theta \times \mathbf{g}_\psi &= -\mathbf{g}_\phi \frac{1}{\sin\theta} + \mathbf{g}_\psi \cot\theta \\ \mathbf{g}_\psi \times \mathbf{g}_\phi &= -\mathbf{g}_\theta \sin\theta \end{aligned} \tag{3.99}$$

we have immediately that

$$\begin{aligned} \dot{\mathbf{g}}_\phi &= 0 \\ \dot{\mathbf{g}}_\theta &= \mathbf{\Omega} \times \mathbf{g}_\theta = \mathbf{g}_\phi(\dot{\phi}\cot\theta) - \mathbf{g}_\psi(\dot{\phi}\frac{1}{\sin\theta}) \\ \dot{\mathbf{g}}_\psi &= \mathbf{\Omega} \times \mathbf{g}_\psi = -\mathbf{g}_\phi(\dot{\theta}\frac{1}{\sin\theta}) + \mathbf{g}_\theta(\dot{\phi}\sin\theta) + \mathbf{g}_\psi(\dot{\theta}\cot\theta). \end{aligned} \tag{3.100}$$

Upon substitution for $\dot{\mathbf{g}}_\lambda$ in (3.98) and regrouping we find $d\mathbf{L}/dt$ to be given by

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \mathbf{g}_\phi\{\frac{1}{\sin\theta}[\frac{d}{dt}(L^\phi \sin\theta) - \dot{\theta}L^\psi]\} \\ &+ \mathbf{g}_\theta\{\frac{dL^\theta}{dt} + \dot{\phi}L\sin\theta\} \\ &+ \mathbf{g}_\psi\{\frac{1}{\sin\theta}[\frac{d}{dt}(L^\psi \sin\theta) - \dot{\phi}L^\theta]\},\end{aligned} \tag{3.101}$$

where the L^λ are given by (3.60) with $I_1 = A$ and $I_3 = C$, i.e.

$$L^\phi = A\dot{\phi}, \quad L^\theta = A\dot{\theta}, \quad L^\psi = C\omega_3 - A\dot{\phi}\cos\theta, \tag{3.102}$$

where ω_3 is again the angular velocity component of the top along $\mathbf{g}_\psi$. The expressions in curly brackets are of course contravariant components. The covariant components along the $\mathbf{g}_\lambda$ follow with the help of (3.38). We equate these covariant components to the corresponding moments N_λ given by (3.97). Our basic equations for the motion about G then read

$$\begin{aligned}\frac{dL_\phi}{dt} &= F_\perp h\sin\theta \quad \text{(a)} \\ \frac{d}{dt}(A\dot{\theta}) + \dot{\phi}L^\psi \sin\theta &= -Rh\sin\theta - F_{||}(QG) \quad \text{(b)} \\ \frac{d}{dt}(A\omega_3) &= F_\perp a\sin\theta, \quad \text{(c)}\end{aligned} \tag{3.103}$$

where

$$L_\phi = A\dot{\phi} + L^\psi\cos\theta = A\dot{\phi}\sin^2\theta + C\omega_3\cos\theta$$

and

$$L_\psi = L^\psi + A\dot{\phi}\cos\theta = C\omega_3$$

are the covariant components of the angular momentum along the vertical and along the symmetry axis as we already discovered in (3.59).

We supplement these with the equations of motion for the center-of-mass G. Since the velocity components u, v of G that appear in (3.95) are referred to the axis QP which is rotating with angular velocity $\dot{\phi}\mathbf{g}_\phi$ about the vertical, we must use (3.90) with $\mathbf{\Omega} = \dot{\phi}\mathbf{g}_\phi$. Then G moves according to

$$\begin{aligned}m(\dot{u} - v\dot{\phi}) &= F_{||} \quad \text{(a)} \\ m(\dot{v} + u\dot{\phi}) &= F_\perp \quad \text{(b)} \\ m\frac{d^2}{dt^2}(a - h\cos\theta) &= R - mg \quad \text{(c)}\end{aligned} \tag{3.104}$$

in terms of acceleration components along and normal to QP and the vertical.

The equations (3.103) and (3.104) together with the information given by (3.95) and (3.96) define our problem completely. We observe immediately that the "fly in the ointment" is the force of friction. Without it, we would regain our two previous angular momentum constants $L_\phi = M$ and $L_\psi = K$. Moreover, the total energy would also be conserved (the top would not scrape on the table) and our problem for the tippe-top would reduce in all respects to that of the ordinary top. However, let us first see how far we can go without making specific assumptions about the nature of the motion. The first point to notice is, that while L_ϕ and $C\omega_3$ are no longer conserved quantities the linear combination $(aL_\phi - hC\omega_3)$ is. This follows from (3.103a) and (3.103c) upon multiplication by a and h respectively and subtracting. We have

$$\frac{d}{dt}[aA\dot{\phi}\sin^2\theta + aC\omega_3\cos\theta - hC\omega_3] = 0 \tag{3.105}$$

or

$$aA\dot{\phi}\sin^2\theta + (a\cos\theta - h)C\omega_3 = \text{constant} \tag{3.106}$$

after inserting the value of L_ϕ given below (3.103). We have thus found one integral of motion even in the presence of friction. The latter relation, (3.106), is known as Jellett's integral. It seems first to have been discovered by him and is quoted in his book on friction[48]. As far as is known, this is the only first integral of motion that exists for a top spinning on a rough surface.

[48] Jellet, *Theory of friction*, Chap. viii, 1872.

To make further progress we observe that (3.104) allow us to replace the unknown friction forces and normal reaction on the right of (3.103) by the accelerations they produce for G. In the case of R this is very helpful for we have *prescribed* how G must move vertically through the constraint equation (3.93). We eliminate R, $F_\perp$ and $F_{||}$ from (3.103) in this way and find

$$\frac{dL_\phi}{dt} = m(\dot{v} + u\dot{\phi})h\sin\theta \qquad \text{(a)}$$

$$\frac{d}{dt}[(A + mh^2\sin^2\theta)\dot{\theta}] - mh^2\sin\theta\cos\theta\dot{\theta}^2 + \dot{\phi}L^\psi\sin\theta$$
$$= -mgh\sin\theta - m(\dot{u} - v\dot{\phi})(a - h\cos\theta) \qquad \text{(b)}$$

$$\frac{d}{dt}(C\omega_3) = m(\dot{v} + u\dot{\phi})a\sin\theta. \qquad \text{(c)}$$

$$\tag{3.107}$$

These equations still contain the unknown acceleration and velocity components of the center-of-mass G and so are not too much help as they stand. We have to provide information about the physical conditions at the point of contact as the following limiting cases will illustrate.

(a) *Pure sliding*. We assume the table is perfectly smooth so that the top slides freely. Then, the contact point P *slides* without friction on the table and the center-of-mass only has an acceleration in the vertical direction. The horizontal motion of G is one of constant velocity in a straight line. Steady precession in the pure sliding mode takes place with G at rest and P moving in a circle. Since $\theta = \alpha$, $\dot{\phi} = p$, $\omega_3 = n$ in addition to $u = 0$, $v = 0$, $\dot{u} = 0$, $\dot{v} = 0$ in steady precession, (3.107b) becomes

$$pL^{\psi} \sin\alpha = -mgh \sin\alpha, \tag{3.108}$$

which is just (3.82) again, but in a compact form. Except for one point: the sign of h is different. Setting $L^{\psi} = Cn - Ap\cos\alpha$, its steady precession value, we find that (3.108) has the two roots for $p(\sin\alpha \neq 0)$,

$$Cn \simeq Ap\cos\alpha, \quad \text{or} \quad Cn \simeq -\frac{mgh}{p}, \tag{3.109}$$

connecting n and p in the fast and slow precession respectively, if Cn is large. The sign for the slow precession is different, indicating that if we started off the tippe-top like we did the ordinary top by spinning it and setting it down gently it would choose the *fast* precession root $Cn \simeq Ap\cos\alpha$, and the nutational motion consists of the top nodding up and down while G moves up and down vertically. The rest of the analysis proceeds as in the case of the ordinary top. As we have already mentioned, the three first integrals of motion, conservation of angular momentum along the vertical and symmetry axis, and conservation of energy, exist and allow us to duplicate the ordinary top discussion.

(b) *Pure rolling*. The top is said to roll if the point of contact P is always instantaneously at rest so that there is no relative motion between the surface of the top and the table at their common point of contact. The condition for pure rolling can be expressed as conditions on the angular velocity and orientation of the top and the velocity of G. From (3.95) the point P is instantaneously at rest if $v_{||}$ and $v_{\perp}$ are zero, i.e.

$$\begin{aligned} u &= (a - h\cos\theta)\dot{\theta} \\ v &= -[a\omega_3 + (h - a\cos\theta)\dot{\phi}]\sin\theta, \end{aligned} \tag{3.110}$$

which tell us what the velocity components of G *must be* if the top rolls. This in turn means that the table must be rough enough to sustain G in a motion with u and v given by (3.110). Notice in passing that the pure rolling conditions are *non-holonomic*. We cannot integrate (3.110) in time until we know the motion of the top.

The usefulness of these constraints on the motion of G is that they allow us to eliminate friction entirely in terms of the motion it causes, i.e. tolling. The right sides of (3.107) then become *prescribed* functions of the Euler angles and their time derivatives. Let us look at (3.107c) in particular. We obtain the combination

$$(\dot{v} + u\dot{\phi})a\sin\theta = -a\sin\theta\frac{d}{dt}(a\omega_3\sin\theta) - (h - a\cos\theta)\frac{d}{dt}(a\dot{\phi}\sin^2\theta)$$

from (3.110) and insert it into the right side of (3.107c) after employing Jellett's relation, (3.106), to find $d/dt(a\dot{\phi}\sin^2\theta)$. The result can be expressed as

$$\frac{1}{2}\frac{d}{dt}\{\omega_3^2[C + ma^2\sin^2\theta + m\frac{C}{A}(h - a\cos\theta)^2]\} = 0,$$

or that

$$\omega_3^2[C + ma^2\sin^2\theta + m\frac{C}{A}(h - a\cos\theta)^2] = \text{constant} \tag{3.111}$$

is a constant of the motion. We have thus discovered another constant of motion for pure rolling, in addition to the conservation of energy (friction forces do no work in pure rolling motion) and Jellett's integral (which is always valid). Thus, we are once more armed with three first integrals of motion and we can discuss precession, nutation and the stability thereof as we did before for the ordinary top. We note one difference from case (a): In pure rolling the steady precession motion has both G and P going around in circles about each other while G still stays at a fixed height about the table (P instantaneously at rest does *not* mean P stays in the same spot!). The equation relating α, p, and n in steady precession follows from (3.107b) with $\dot{u}$ set to zero, v given by (3.110):

$$pL^{\psi}\sin\alpha = -mgh\sin\alpha - m[an + (h - a\cos\alpha)p]\sin\alpha, \tag{3.112}$$

which differs in detail from the condition when pure slipping is present, but is still a quadratic equation for p and we expect slow and fast precession modes for large Cn as before.

(c) *Quasi steady motion*. If we assume that the friction forces and moments are small the motion of the top will be a quasi steady version of the precessional motion in pure sliding. For we may argue that, if the friction is small it acts like a disturbance of the steady motion which, as we know, produces a small oscillatory motion of the top about its steady precession orientation $\theta = \alpha$. (See problems). With one difference: The "small disturbance" one always invokes to perturb a system in order to ascertain its stability does just that and then goes away. This is not true for friction, which of course continues to act so that we expect a steady "drift" away from steady precession in addition to the oscillations we just mentioned. Now, we argue that the oscillations in θ around $\theta = \alpha$ are very rapid for large spin and hence will average out over times in excess of the nutation period T_n. Therefore, if we look only at the time-averaged motion for times long compared with T_n, but short compared with the precession period T_p (so that the motion does not depart substantially from steady precession), we should be able to estimate the effects introduced by friction. However, we can see intuitively from (3.107a) what is going to happen: the frictional moment $F_\perp a$ tends to increase the projection of $\mathbf{L}$ along the z-axis if $F_\perp$ is positive, and decrease it if $F_\perp$ is negative. A decreasing L_ϕ means that the angular momentum vector is falling and so too the symmetry axis, since from (3.107c) $C\omega_3 = L_\psi$ must also decrease. The top therefore begins to fall if $F_\perp < 0$. But the direction of $F_\perp$ is determined by the velocity direction $v_\perp$ of the contact point on the top ($F_\perp$ is always opposite to $v_\perp$). Hence a falling top is characterized by $v_\perp > 0$, or

$$[an + (h - a\cos\alpha)p]\sin\alpha > 0, \tag{3.113}$$

(the velocity v of G is near zero) which for the tippe-top, where $Cn \simeq Ap\cos\alpha$, becomes[49]

$$h + \frac{A - C}{C} a\cos\alpha > 0 \tag{3.114}$$

a purely geometrical condition for a given top. So, when the tippe-top spins on its thin peg we can again use (3.114) to ascertain whether the top falls or rises. Assuming that the tip of the peg is spherical, radius a, and setting $h \to -h$ (for the center-of-mass G now lies on the other side of C) we note $a \ll |h|$ so that (3.114) is violated: the top climbs up on its peg instead of falling.

[49] This condition was first given by D.G. Parkyn, *Math. Gazette XL*, p. 260 (1956).

For the more curious reader we present a more motivated derivation of the condition (3.113). The discussion starts with (3.103c); we assume that deviations away from steady precession are small so the $F_\perp$ may be replaced by a constant value $F_\perp^{(0)}$ for quasi steady precession. The average change in the angular mometum Cn is thus (bars denote a time average)

$$C\delta n \simeq F_\perp^{(0)} t a \sin\alpha \quad (T_n \ll t \ll T_p), \quad (i)$$

where T_n, T_p are the nutation and precession periods. The change (i) induces a corresponding change in p which, from Jellett's relation is

$$aA\sin^2\alpha\bar{\delta p} \simeq (h - a\cos\alpha)C\bar{\delta n}. \quad (ii)$$

In the absence of friction, changes in θ respond to an outside disturbance according to

$$\ddot{\zeta} + \Omega_n^2\zeta = 0, \quad (iii)$$

where $\Omega_n^2 = (2\pi)^2/T_n^2 = [(Cn - 2Ap\cos\alpha)^2 + (Ap\sin\alpha)^2]/(AA')$ and $A' = A + mh^2\sin^2\alpha$.

The result (iii) follows from (3.107b) of the text after setting $\theta = \alpha + \zeta$ and ignoring the forms of friction $F_{||} \sim (\dot{u} - v\dot{\phi})$. The additional terms (i) and (ii) appear on the right side when friction (the $F_\perp^{(0)}$ dominantly) is present. Then, (iii) is replaced by

$$A'\ddot{\zeta} + A'\Omega_n^2\zeta = -\kappa C\bar{\delta n} \quad (iv)$$

with $\kappa = [aAp\sin^2\alpha + (h - a\cos\alpha)(Cn - 2Ap\cos\alpha)]\text{cosec}^2\alpha/(aA)$ if we continue to omit $F_{||}^{(0)}$. The particular integral of (iv) gives the forced motion:

$$\bar{\zeta}(t) = -\frac{\kappa C}{A'\Omega_h^2}\bar{\delta n} = -\frac{1}{A'\Omega_n^2}\kappa a\sin\alpha F_{\perp}^{(0)}t. \quad (v)$$

We readily verify that κ is positive when $Cn \simeq Ap\cos\alpha$; hence $\zeta(t)$ increases (the top falls) when $F_{\perp}^{(0)} < 0$, or $v_{\perp}^{(0)} > 0$, which is the condition stated in (3.113). More generally, we point out that for an ordinary top where $h \to -h$ in (3.113) and $a \ll h$ we have the condition $(an - hp) > 0$ independent of the angle α, giving a critial precession beyond which the effect is reversed. D.G. Parkyn [*Physica* XXIV, p. 313 (1958)] has analysed the motion of a top with a rounded peg both theoretically and experimentally. He finds that an ordinary top in motion does effectively remain in the rolling mode, but that the motion is not in general one of pure rolling due to periodic sliding-rolling motions of the peg.

Recapitulation

If it seemed to the reader that the discussion of just two problems in rigid body mechanics took up far too much space, let him be mindful of the fact that (i) rigid body problems *are* difficult in general and (ii) the two problems we have discussed in detail serve as prototypes to illustrate the methods that one would employ in tackling any problem involving rigid body motion under given forces. Such methods are important and reach a high degree of sophistication in applied problems such as rocket guidance systems, gyrocompass design, servomechanisms etc. We do not discuss such applications here. On the more academic side, we have also avoided discussing the "egg rolling on a rotating paraboloid" types of problem, which tend to become rather devoid of physical content. However, such "academic" rigid body systems are indelibly woven into the historical fabric of mechanics and, like chess, they have their fascination. The reader who wishes to explore such topics further will do well to consult Routh's[50] Advanced Rigid Dynamics, or in a more modern (1965) vein, the treatise of Pars[51].

[50] E.J. Routh, *Advanced Rigid Dynamics*, Sixth Edition, MacMillan and Co., Ltd., London, 1905.

[51] L.A. Pars, *A Treatise on Analytical Dynamics*, John Wiley and Sons, Inc., New York, 1965.

Problems

3-1. Consider a rigid body in general motion about a fixed point O. Show that an instantaneous axis of rotation (i) exists and (ii) that this axis passes through O.

3-2. Find the principal moments of inertia and principal axes for the following rigid bodies at the points indicated. The mass of each body is M.

(i) a square plate, side a, at one corner
(ii) a rectangular plate, sides a and b, at one corner
(iii) a semi-circular plate of radius a, at one of the "corners"
(iv) a plate shaped like an equilateral triangle of side a, at any vertex

(v) a cube, side a, at the center-of-mass

3-3. Discuss the motion of a rigid body having an axis of symmetry and supported freely at its center-of-mass using the conservation laws only. Indicate how the direction of the angular velocity vector $\boldsymbol{\omega}$ behaves with respect to the direction of the angular momentum vector $\mathbf{M}$.

3-4. A uniform solid circular cylinder of radius a rests on a horizontal plane, and an identical cylinder rests on it, touching it along the highest generator. If no slipping occurs, show that as long as the cylinders remain in contact,

$$\dot{\theta}^2 = \frac{12g(1-\cos\theta)}{a(17+4\cos\theta-4\cos^2\theta)}$$

where θ is the angle which the plane containing the cylinder axes makes with the vertical (see Fig. 3.11).

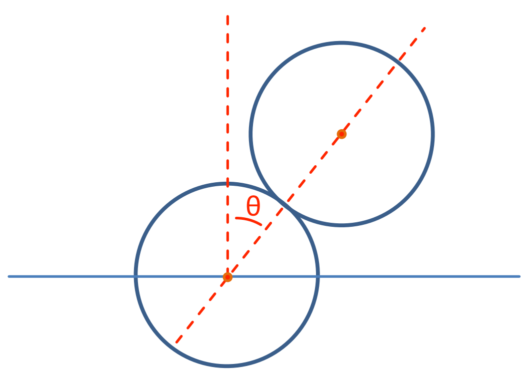

Figure 3.11: Sketch of two cylinders touching at a tangent point.

Show that the path described by the axis of the upper cylinder is

$$x = \frac{1}{3}a(\theta + 4\sin\theta), \quad y = 2a(1-\cos\theta)$$

relative to horizontal (x) and vertical (y) axes through the initial position of this axis.

3-5. A uniform right circular cone of semivertical angle α rolls *without slipping* (what is the condition for this?) on a plane inclined at an angle β with the horizontal, and is released from rest with the line of contact horizontal. Prove that the cone will remain in contact with the plane if

$$9\tan\beta < \cot\alpha + 4\tan\alpha.$$

3-6. An asymmetrical top rotates about its center of mass under no forces. The principal moments of inertia at the center-of-mass are $I_1, I_2, I_3, (I_1 > I_2 > I_3)$. If $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ give the angular velocity components along the principal axes, show that they have the form

$$\omega_1(t) = -A\text{sech}\,\tau(t); \quad \omega_2(t) = B\tanh\tau(t); \quad \omega_3(t) = C\text{sech}\,\tau(t),$$

if $M^2 = 2I_2T$ (M = angular momentum, T = kinetic energy) and $\omega_3 > 0, \omega_1 < 0$ at $t = 0$. Find A, B, C and $\tau(t)$. What happens as the time t increases indefinitely?

3-7. A gyro is set spinning with angular momentum **m** along a direction making an angle α with the symmetry axis. Show that the symmetry axis precesses around the vector **m** with angular velocity $|m|/I_1$. Show that this can also be written as $I_3\omega_3/I_1 \cos\alpha$, where ω_3 is the angular velocity along the symmetry axis.

3-8. A top is started spinning vertically so that θ and $\dot{\theta}$ are zero initially. Show that if $\omega_3^2 > 4mgh(I_1/I_3^2)$, the angle θ remains zero; if ω_3^2 is smaller than this the symmetry axis of the top will oscillate between O and an angle α. Find α.

3-9. Show that a steady precession of a top about the vertical is possible with its symmetry axis *horizontal* and find the relation between the precessional angular velocity and the "spin" of the top about its symmetry axis.