Chapter 2 Particle Dynamics

2-1 Introduction

This chapter is devoted to a discussion of the motion of a single particle under specified forces. Its purpose is twofold. Firstly, the simplicity of such problems will allow us to become acquainted with the methods of chapter 1 in specific applications, without undue complication. Secondly, the physical discussion of the two most important particle systems in nature - the celestial system and the atomic system - require for their building blocks an understanding of single particle motion in a force field (the planets around the sun for the former, the electrons around the nucleus in the latter). Also, most of the information on atomic, nuclear and nucleon structure comes from devising *scattering experiments* in which the system under study is bombarded with a probe particle. The full interpretation of the results of such experiments lies outside the capabilities of classical mechanics; one has to appeal to quantum mechanics. However, most of the concepts and parameters of the classical description appear unscathed in the quantum mechanical version and it is therefore of utmost importance to study the classical one-body scattering problem in detail. This is done in Sec. 2-6.

One of many practical aspects of applied classical mechanics appears in everyday life in the form of satellite communications and weather systems, which depend for their successful operation on knowing quite precisely how such a satellite will orbit the earth. The actual calculation of satellite orbits is much more complex than the standard orbit problem we discuss in Sec. 2-4. But the complexity is in degree of sophistication and not in the principles involved. References are provided there to bring the curious reader up to date on what these complexities are. The classic problem of planetary motion around the sun falls in this category also and is of course indelibly woven into the historical development of mechanics.

2-2 Systems with One Degree of Freedom

Consider the motion of a particle, mass m, in a conservative force field $V(x)$, where x measures the displacement from some standard position.

The Lagrange function reads

$$L = \frac{1}{2}m\dot{x}^2 - V(x). \tag{2.1}$$

The equation of motion for x is

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\frac{\partial V}{\partial x} - m\ddot{x} = 0,$$

or

$$m\ddot{x} = F(x), \tag{2.2}$$

where $F(x) = -\partial V/\partial x$ is the force on the particle at position x. This equation is special only to the extent that $F(x)$ was derivable from a potential. Otherwise, we would use the general Lagrange equations (1.43); then F could have any form, e.g. $F = F(x, \dot{x}, t)$, that is not necessarily derivable from a potential function. However, if F is derivable from a potential as supposed above, we can integrate (2.2) immediately. Multiply by $\dot{x}$. Then,

$$m\dot{x}\ddot{x} = \frac{d}{dt}(\frac{1}{2}m\dot{x}^2) = F(x)\dot{x},$$

or

$$[\frac{1}{2}m\dot{x}^2]_{x_0}^{x} = \int_{x_0}^{x} f(x)\,dx = -V(x) + V(x_0),$$

after integrating both sides from x_0 to x. The last statement says

$$\frac{1}{2}m\dot{x}^2 + V(x) = \frac{1}{2}m\dot{x}_0^2 + V(x_0) = \text{a constant } E, \tag{2.3}$$

which just reproduces the law of energy conservation. We could have realized this directly from the form of L in (2.1): since L does not depend on time, the quantity

$$H = \frac{\partial L}{\partial \dot{x}}\dot{x} - L = \frac{1}{2}m\dot{x}^2 + V(x) = E \tag{2.4}$$

is conserved and equals the total energy, according to (1.77) and (1.81). A second integration of the energy equation gives the position at time t:

$$t - t_0 = \sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx}{\sqrt{E - V(x)}}, \tag{2.5}$$

where x_0 is the position of the particle at time t_0. We have chosen the positive square root, implying that the particle is moving from x_0 to x. Notice that *two* boundary conditions are required before this solution becomes unique: for example, the position and velocity at time t_0 are required to determine E in (2.5).

If the force F in (2.2) depends on the velocity of the particle, or the time, then the integral (2.5) is invalidated, and one has to seek other methods of integration. The variety of such problems is endless, depending on the form of F, and so the integration of various special cases seems pointless here. Rather we discuss some standard problems that appear frequently in physical applications, and that will furthermore serve as "guinea pigs" for the more exotic solution techniques that are developed in subsequent chapters.

(i) Free fall under gravity

Here the accelerating force is the gravitational pull of the earth. Near the earth's surface this is a constant force mg. The potential is therefore $-mgx$ if we measure x from the point of release. Then, the time to fall a distance x is given by (2.5):

$$t - t_0 = \sqrt{\frac{m}{2}} \int_0^x \frac{dx}{\sqrt{E + mgx}} = \frac{1}{g}\left[\sqrt{\frac{2}{m}(E + mgx)} - \sqrt{\frac{2E}{m}}\right]. \tag{2.6}$$

The value of E reflects with what velocity the particle was released at $t = t_0$. In free fall from rest, $E = 0$, and

$$t - t_0 = \sqrt{\frac{2x}{g}}, \quad or \quad x = \frac{1}{2}g(t - t_0)^2$$

gives the distance fallen in time $t - t_0$.

(ii) Harmonic oscillator in one dimension

A particle attracted to a fixed point by a force proportional to its displacement from that point performs an oscillatory motion. For if $F = -m\omega^2 x$, calling the constant of proportionality $m\omega^2$, then the potential is $V = \frac{1}{2}m\omega^2 x^2$ and the equation of motion and equation of energy read

$$\ddot{x} + \omega^2 x = 0 \tag{2.7}$$

and

$$\dot{x}^2 + \omega^2 x^2 = \frac{2E}{m}, \tag{2.8}$$

respectively.

Since $\dot{x}$ must be real, we find from the second equation that the particle is not found further from the origin than $\pm a = \pm\sqrt{2E/m\omega^2}$; the velocity vanishes at this value of x but not the force. The particle is pulled back towards the origin, so that the sense of its motion reverses. The positions $x = \pm a$ are called *turning points* of the motion. Obviously, the particle oscillates back and forth between a and $-a$ an infinite number of times. The quantity a is called the *amplitude* of oscillation. It is determined by the energy of the system, for a given force constant $m\omega^2$. Therefore, instead of specifying the energy at which the motion occurs, we can prescribe its amplitude. The position at any time t is then given by

$$t - t_0 = \frac{1}{\omega} \int_0^x \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{\omega} \sin^{-1}\left(\frac{x}{a}\right),$$

or[17]

$$x = a\sin(\omega t - \omega t_0) \tag{2.9}$$

if the particle passes the origin at $t = t_0$. The constant ωt_0 is called the phase[18]. The solution we have obtained for x confirms our picture of the motion. The particle oscillates between $+a$ and $-a$; the time taken for one complete oscillation (called the *period*) is

$$T = \frac{2\pi}{\omega}. \tag{2.10}$$

It is inversely proportional to ω, i.e. to the square root of the coupling strength per unit mass.

The problem we have just solved is an example of a *bounded* motion in mechanics. The particle never gets further than $\pm a$ from the origin. It is possible to decide quite generally from the energy equation when such motion occurs. Writing

$$\dot{x} = \pm\sqrt{\frac{2}{m}(E - V(x))}, \tag{2.11}$$

we see that $\dot{x}$ must vanish if $V(x)$ equals E so that the particle is turned back by the potential $V(x)$ at values of x that are roots of

$$V(x) = E. \tag{2.12}$$

If this equation has two *adjacent* roots at $x = a$ and $x = b$ respectively, (it may have more than two roots of course), we can get an oscillatory motion between a and b, the particle shuttling back and forth an infinite number of times. The motion is therefore bounded by the turning points a and b. Furthermore, (2.11) shows (the plus and minus sign) that the velocity is the same whether the particle is "coming or going" at x. Therefore, the transit time from a to b is the same as the transit time from b to a and stays constant for every subsequent traversal (provided the system does not lose energy). Hence, we can define a period T of the motion; T is taken as twice the time required to go from a to b. Obviously, the latter time equals the time taken for the particle to return to any position x on its path.

On the other hand, a single root means the particle just turns around once and proceeds back to infinity from whence it came. Hence, the shape of the potential $V(x)$ and the energy determine the type of motion. In Fig. 2.1 oscillatory motion occurs for all positive values of the energy, $E > 0$; in Fig. 2.2 oscillatory motion occurs for $E_1 > E > 0$, if the particle is located on the left of the origin, unbounded motion if the particle is located on the right. The "hump" in the potential in Fig. 2.2 is termed a *potential barrier*. For $E < E_1$ the particle cannot[19] penetrate such a barrier since that would require a negative kinetic energy. On the other hand, if

[17] Or $x = a\cos(\omega t + \alpha)$, where α is some other phase. Either form is a linear combination of the fundamental pair of solutions $x_\pm = \exp(\pm i\omega t)$ of (2.7) with a different boundary condition.

[18] Sometimes the negative of this quantity is called the phase.

[19] This restriction is lifted in quantum mechanics.

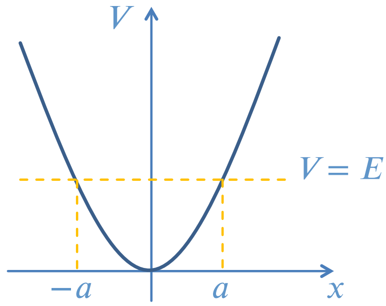

Figure 2.1: The potential V of a harmonic oscillator.

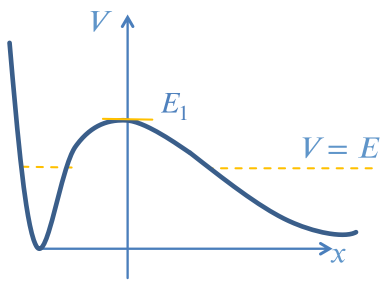

Figure 2.2: The potential V with a barrier.

$E > E_1$ the particle passes over the hump, and the motion is no longer oscillatory.

The period of the motion is easily determined in the general case. By definition, the period T is the time taken to go "there and back", starting at a particular turning point a, going to a further one b and returning to a. Thus,

$$T = \int_a^b \frac{dx}{\sqrt{\frac{2}{m}(E-V)}} - \int_b^a \frac{dx}{\sqrt{\frac{2}{m}(E-V)}},$$

using the minus sign on the second radical for the return journey. Mathematically speaking, both branches of the multivalued function $\sqrt{E-V}$ appear in T, if $V(x)$ is analytic in a suitable region in the complex plane, this facet allows one to express T as a contour integral[20] around the branch points (at a and b) of this function, in the complex plane $z = x + iy$,

$$T = \int_C \frac{dz}{\sqrt{\frac{2}{m}(E-V)}} = \frac{\partial}{\partial E}\int_C \sqrt{2m(E-V)}\,dz. \qquad (2.13)$$

[20] A reader unfamiliar with the elements of complex integration may consult H. Margenau and G.M. Murphy, *The Mathematics of Physics and Chemistry*, D. van Nostrand Company, Inc., New York, 1943.

The "dumbbell" contour C is shown in Fig. 2.3. We have to cut the $z-$plane from a to b along the real axis to make the integrand in (2.13) single-valued, after choosing the positive square root just below the cut, the negative square root just above. A word of caution is in order: since the turning points a and b are zeros of the integrand in (2.13), the first integral for T is actually an improper one. Physically, we want the oscillation to have a *finite* period. Therefore, we also stipulate that

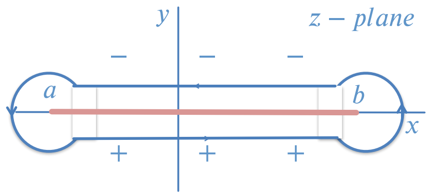

Figure 2.3: The contour of integration C.

the improper integral should converge. This is the case, provided the singularities are not too severe (see Prob. 2-1). Incidentally, the second form for T in (2.13) shows a premise of things to come.

Since

$$\int_C \sqrt{2m(E-V)}\,dx = S_0$$

is just the reduced action for one period of oscillation, one has $T = \partial S_0/\partial E$ as a special case of (1.109). If we can find the reduced action function S_0 over one period in a periodic motion by some means or other, the period of this motion is just the energy derivative of that S_0.

As an example of the application of (2.13), take the harmonic oscillator problem. Then $E = \frac{1}{2}m\omega^2 a^2$, $V(x) = \frac{1}{2}m\omega^2 x^2$, and

$$\omega T = \int_C \frac{dz}{\sqrt{a^2 - z^2}}.$$

We can deform the contour C into a large circle of radius R surrounding the origin without crossing any singularities of the integrand. Therefore

$$\omega T = -i\int_T \frac{dz}{z} = -i(2\pi i) = 2\pi$$

by Cauchy's theorem, in agreement with the result (2.10). Notice in passing that the period of oscillations in an oscillator potential is *independent* of the amplitude of oscillation. This is not generally true. Any deviation from the parabolic potential shape introduces an amplitude dependence into T. However, many potential fields in physics can be usefully replaced by a quadratic approximation near the equilibrium position of the system ($x = 0$ in Fig. 2.1) and therefore retain this simple property approximately. The entire subject of small oscillations that is fundamental to the understanding of molecular and crystal lattice vibrations is based on such a premise. We return to such matters in Chap. 4.

2-3 *Systems with Two or More Degrees of Freedom*

An important example of a system with two degrees of freedom consists of a mass m moving in a potential field V. The Lagrange function in any system of coordinates $q_1 = u$, $q_2 = v$ is

$$L = \frac{1}{2}m(\frac{ds}{dt})^2 - V(u,v) \tag{2.14}$$

where ds is the line element given by (1.101). The general problem posed by (2.14) does not have either u or v as cyclic coordinates. Therefore there are no conserved canonical momenta. However, the energy E is conserved,

$$E = \frac{1}{2}m(\frac{ds}{dt})^2 + V(u,v), \tag{2.15}$$

since L does not contain the time explicitly. We discuss two special cases of (2.14):

(i) Projectile motion near the surface of the earth

We position the projectile in cartesian coordinates $u = x, v = y$ with respect to the point of projection. If the $x-$axis points vertically downwards, the potential energy is just a function of x and is given by $V = -mgx$, where as before g is the acceleration due to gravity. The line element is, trivially,

$$(ds)^2 = (dx)^2 + (dy)^2$$

making $E_u = 1$, $F_{uv} = 0$ and $G_V = 1$. Therefore

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgx.$$

Since the coordinate y is cyclic, the linear momentum

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

is conserved. This relation allows us to pass from derivatives in t to derivatives in y:

$$\frac{d}{dt} = \dot{y}\frac{d}{dy} = \frac{p_y}{m}\frac{d}{dy},$$

and, together with the energy equation,

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgx = \frac{1}{2}m\dot{x}^2 + \frac{p_y^2}{2m} - mgx$$

determines the motion completely. We write $\dot{x}$ in the alternative forms

$$\dot{x} = \pm\sqrt{\frac{2}{m}(E - \frac{p_y^2}{2m} + mgx)} = \frac{p_y}{m}\frac{dx}{dy}$$

and find

$$t = \int_0^x \frac{dx}{\sqrt{\frac{2}{m}(E - \frac{p_y^2}{2m} + mgx)}}; \quad y = \int_0^x \frac{p_y dx}{\sqrt{2m(E - \frac{p_y^2}{2m} + mgx)}}$$

if the projectile leaves the origin at $t = 0$. The integration is elementary and we find

$$x = v_x t + \frac{1}{2}gt^2; \quad \text{or} \quad x = \frac{v_x}{v_y}y + \frac{g}{2v_y^2}y^2$$

for the motion in time, and the path (a parabola). Here, v_x and v_y are the velocity components at the projection point. The ambiguity in sign of the radicals is settled by knowing the direction of v_x. Alternatively we could have found the path directly from the variational principle [(1.102) and its sister equation for v].

(ii) Motion in a central field

A problem of particular physical interest arises in (2.14) when V is spherically symmetric, i.e. a *central potential*.

This means that V only depends on the distance $r = OP$ of the particle P from some fixed point O (the force center, see Fig. 2.4). The force $F(r)$ that this potential gives rise to points along the radius vector OP, i.e. it is a *central force*.

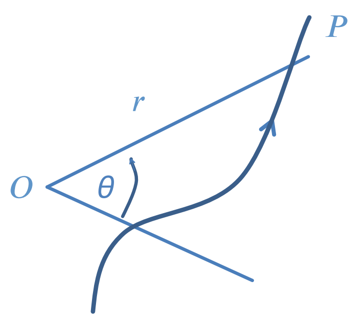

Figure 2.4: A particle moving in a trajectory measured from the fixed point O.

Polar coordinates are indicated: setting $u = r$ and $v = \theta$ the line element becomes

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2,$$

making $E_u = 1$, $F_{uv} = 0$ and $G_v = r^2$. The Lagrange function is therefore

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{2.16}$$

The angle θ is cyclic. Therefore, the canonical momentum

$$p_\theta = \frac{\partial l}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{2.17}$$

is constant in a central field. Physically, p_θ is the angular momentum of the particle about 0. It is conserved because the radial force $F(r)$ has no moment about 0. As before, (2.17) allows us to pass from derivatives in t to derivatives in θ:

$$\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta} = \frac{p_\theta}{mr^2}\frac{d}{d\theta}. \tag{2.18}$$

Taken together with the angular momentum equation, the equation of energy

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{p_\theta^2}{2mr^2} + V(r)$$

provides us with a complete picture of the motion again. Introduce the *effective potential*

$$U(r) = \frac{p_\theta^2}{2mr^2} + V(r) \tag{2.19}$$

consisting of the original potential V and the *centrifugal potential*[21] $p_\theta^2/2mr^2$. A knowledge of U and E determines the motion completely. To see this, write the energy equation as

[21] See Sec. 2-9.

$$\dot{r} = \pm\sqrt{\frac{2}{m}(E-U)}. \tag{2.20}$$

This is now a one-dimensional problem as in Sec. 2-2. We ask: Where does the radial motion cease? This is determined by the roots of

$$U(r) = E.$$

A single root means that the motion is unbounded. The particle comes in from infinity, is turned around at the *closest distance of approach* to 0 and is pushed out to infinity again. On the other hand, two roots at r_{min} and r_{max},

$$U(r_{min}) = U(r_{max}) = E$$

mean that $\dot{r}$ vanishes twice. If r_{min} and r_{max} are separated by a region of potential that is energetically accessible to the particle, then the motion can be bounded: r oscillates between circles of radii r_{min} and r_{max}. This does not mean, however, that the motion ceases at these points or that the particle returns to its original position when r returns to its original value. To find out what angle OP swings through when r changes, i.e. to find the *orbit*, we write (2.20) in the form

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}(E-U)} = \frac{p_\theta}{mr^2}\frac{dr}{d\theta}, \tag{2.21}$$

and calculate the change in θ from

$$d\theta = \pm\frac{1}{r^2}\frac{p_\theta dr}{\sqrt{2m(E-U)}}.$$

A single integration gives

$$\theta - \theta_0 = \int_{r_{min}}^{r} \frac{p_\theta}{\sqrt{2m(E-U)}}\frac{dr}{r^2}, \tag{2.22}$$

if we set $\theta = \theta_0$ when $r = r_{min}$. Since r must increase with increasing θ after this point the sign on the radical is determined. The time taken for OP to reach the position θ is given by the first form in (2.21):

$$t - t_0 = \int_{r_{min}}^{r} \frac{dr}{\sqrt{\frac{2}{m}(E-U)}}, \tag{2.23}$$

where t_0 is the time at $\theta = \theta_0$. As OP changes from r_{min} to r_{max} and back, θ changes by

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{p_\theta}{\sqrt{2m(E-U)}}\frac{dr}{r^2} = -\frac{\partial}{\partial p_\theta}\int_C \sqrt{2m(E-U)}\,dz \qquad (2.24)$$

in time

$$\Delta t = 2\int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{2m(E-U)}} = \frac{\partial}{\partial E}\int_C \sqrt{2m(E-U)}\,dz \qquad (2.25)$$

if we use the same trick as before to express the integrals on r as contour integrals in the complex plane $z = r + i\zeta$. The contour C is the same as that depicted in Fig. 2.3 after replacing a and b by r_{min} and r_{max}. The surprising symmetry in the expressions for $\Delta\theta$ and Δt (and θ and t for that matter) as derivatives of the same integral with respect to the constants of motion p_θ and E is no accident. Its appreciation must, however, wait for the developments of Chap. 6. Equation (2.25) is once more a special case of (1.109) with the reduced action calculated over a full period of the variable r.

Equations (2.22) and (2.23) determine the motion completely. As far as the orbit is concerned, we realize from the plus and minus signs on the radial velocity in (2.21) that the turning points can be approached from either side with the same radial velocity, i.e. the orbit is *symmetric* about the line $\theta = \theta_0$. The same is true at the other turning point. Therefore, the entire orbit can be constructed once the segment from r_{min} to r_{max} is known, by successively "folding over" this segment about the lines $\theta = \theta_0$ and $\theta_0 + \Delta\theta/2$, respectively.

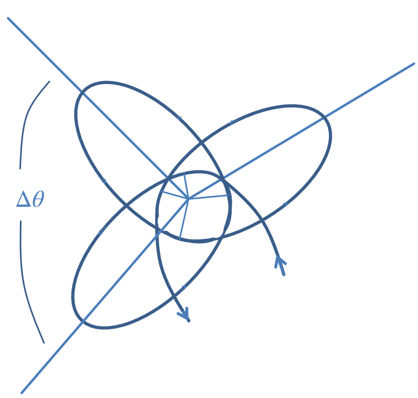

Figure 2.5: A possible central orbit.

A typical orbit is shown in Fig. 2.5. This figure illustrates an important fact: the orbit will not be a *closed* one unless $\Delta\theta$ is some rational fraction of 2π. Otherwise, the particle never returns to its original position for any θ and the orbit will eventually fill out the entire annulus between the two circles $r = r_{min}$ and $r = r_{max}$ in Fig. 2.5. It is also clear that the orbit in a central field always lies in a plane, the orientation of this plane being determined by the direction of the angular momentum $\mathbf{p}_\theta$. Vectorially, we have

$$\mathbf{p}_\theta = \mathbf{r} \times \mathbf{p},$$

and since the field $V(r)$ is independent of the orientation of $\mathbf{r}$ in space, $\mathbf{p}_\theta$ will clearly maintain its original *direction* also (we have already seen that its magnitude is constant). This means that the orbit lies in the plane perpendicular to $\mathbf{p}_\theta$, passing through the center of force.

We stress the fact: all the information gained up to this point (and it is a great deal) has not required us to specify $V(r)$. Of course the actual shape of the orbit does depend on $V(r)$ and we have to know it before the orbit can be obtained. We will do such a calculation in the next section for the important cases of inverse ($\sim 1/r$) and quadratic ($\sim r^2$) central fields. However, two general comments on the orbit problem are in order here: often the problem is not to find the orbit but to find the force (or potential) giving rise to a given orbit, i.e. to turn (2.22) "inside out" for the function U. This is easily accomplished by obtaining instead a *differential equation* for the orbit. Instead of integrating (2.21) to find the orbit, we differentiate it to find the force (no, $dU/d\theta$ is *not* zero, because a total derivative is being formed):

$$p_\theta \frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) = \frac{-\frac{dU}{d\theta}}{\sqrt{\frac{2}{m}(E-U)}} = \frac{-\frac{\partial V}{\partial r}\frac{dr}{d\theta}}{\frac{p_\theta}{mr^2}\frac{dr}{d\theta}} = -\frac{mr^2}{p_\theta}\frac{\partial U}{\partial r},$$

or

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{mr^2}{p_\theta^2}\frac{\partial U}{\partial r}.$$

Entering the value of U from (2.19), writing $F = -\partial V/\partial r$ and setting $u = 1/r$, we find the more usual form of this equation (see also Prob. 2-3),

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{p_\theta^2 u^2} F\left(\frac{1}{u}\right). \tag{2.26}$$

Knowing the orbits, the force can therefore be determined up to a constant (m/p_θ^2). No further information is necessary.

The other comment concerns calculating the orbits for a given force law. The construction of (2.22) has reduced this problem "to quadratures"; mathematically the problem has been solved. The integral on the right of this equation can often be expressed in terms of elementary functions. More often not so elementary functions appear (elliptic integrals) depending on the form of $V(r)$. Whittaker ("Analytical Dynamics", p81, fourth edition, Cambridge University Press, London, 1936) gives a rather complete list of integrable potentials in this sense. Of course there is nothing to prevent a numerical evaluation of the integral if it converges. Furthermore, as we have seen, a knowledge of the segment of the path from r_{min} to r_{max} determines the entire orbit, reducing such numerical work considerably.

2-4 Kepler's Problem

A problem of fundamental significance, both historically and scientifically[22], concerns the motion of a particle in an attractive inverse square central force field[23]. This force field is generated by the potential $(\alpha > 0)$

$$V = -\frac{\alpha}{r}, \tag{2.27}$$

which at the same time represents the potential energy of a planet around the sun, or an electron around an atomic nucleus. The effective potential for a particle of angular momentum p_θ is

$$U = \frac{p_\theta^2}{2mr^2} - \frac{\alpha}{r} \tag{2.28}$$

in this field. The function U is seen to approach $+\infty$ at the origin, and zero from the negative side as $r \to \infty$. Therefore, two turning points are possible if $E < 0$, but only one for $E \geq 0$. Hence the orbits in the potential (2.27) are uniquely classified as bounded or unbounded accordingly, as $E < 0$ or $E \geq 0$, see Figs. 2.6 and 2.7.

[22] The science of mechanics may be said to have begun seriously with the explanation of Kepler's laws of planetary motion by Newton (1687).

[23] For the reader curious enough to want to know how planetary (or satellite) orbits are actually calculated we recommend D. Brouwer and G.M. Clemence, "Methods of Celestial Mechanics", Academic Press Inc., London and New York, 1961.

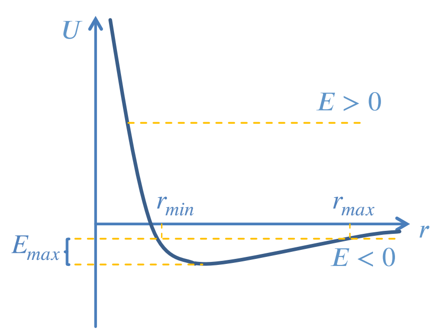

Figure 2.6: The potential U showing positive and negative energies.

We first assume E is negative, $E = -|E|$. The orbit is a bound one and its shape is given by (2.22):

$$\begin{aligned}
\theta - \theta_0 &= \int_{r_{min}}^{r} \frac{p_\theta}{\sqrt{2m(-|E| - \frac{p_\theta^2}{2mr^2} + \frac{\alpha}{r})}} \frac{dr}{r^2} \\
&= \left[\cos^{-1}\left[\frac{\frac{p_\theta^2}{m\alpha}\frac{1}{r} - 1}{\sqrt{1 - \frac{2|E|p_\theta^2}{m\alpha^2}}}\right]\right]_{r_{min}}^{r},
\end{aligned}$$

or (if we measure θ from the point of closest approach r_{min}),

$$\frac{l}{r} = 1 + e\cos\theta \tag{2.29}$$

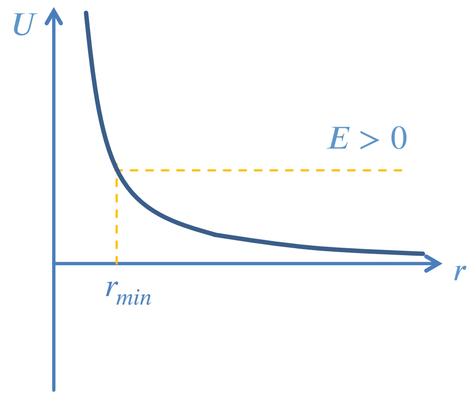

Figure 2.7: A potential $U \sim 1/r^2$ that allows for a positive energy only.

since the ratio in brackets of the cosine bracket is unity when $r = r_{min}$. The constants l and e are

$$l = \frac{p_\theta^2}{m\alpha}; \quad e = \sqrt{1 - \frac{2p_\theta^2}{m\alpha^2}|E|}. \tag{2.30}$$

The reader will recognise (2.29) as the equation for a conic section with semi-*latus rectum* l and eccentricity e, referred to polar coordinates with origin at one focus. The constants l and e are called the *geometrical* constants of the orbit as opposed to the *dynamical* constants E and p_θ. Clearly $e < 1$ from (2.30) so the bound orbit is an (*ellipse* [24]). Its semi-major and -minor axes are

$$a = \frac{l}{1-e^2} = \frac{\alpha}{2|E|}; \quad b = \frac{l}{\sqrt{1-e^2}} = \frac{p_\theta}{\sqrt{2m|E|}} \quad \text{elliptic orbits.} \tag{2.31}$$

[24] There are two limiting cases: if $|E|$ attains its largest physically allowable value $E_{max} = p_\theta^2/2mr^2$, the radii r_{min} and r_{max} coalesce into a single radius r_0 and a circular orbit of this radius is performed. The other limiting case, $|E| = 0$ gives rise to a *parabola*.

The geometrical significance of these various constants is shown in Fig. 2.8. The force center is at S, the one focus of the ellipse (we use

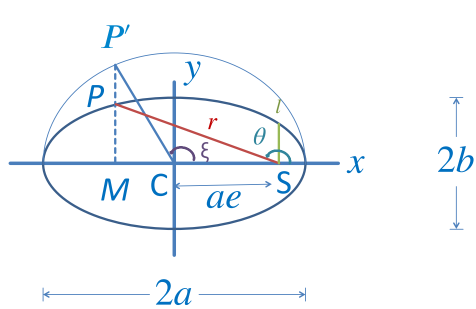

Figure 2.8: The particle of mass m moving in a vertical circular path, and acted on by gravity.

S instead of 0 in Fig. 2.8 to recall the role of the sun as the center of attraction for the planetary system). The orbit is a closed one, the angle $\Delta\theta$ of (2.24) equalling 2π in this case. The least and greatest distances from S are

$$r_{min} = \frac{l}{1+e} = a(1-e); \quad r_{max} = \frac{l}{1-e} = a(1+e). \tag{2.32}$$

The time required to reach any point in the orbit is given by (2.23). In the present example

$$t - t_0 = \sqrt{\frac{m}{2|E|}} \int_{r_{min}}^{r} \frac{r\,dr}{\sqrt{-\frac{p_\theta^2}{2m|E|} + \frac{\alpha}{|E|} r - r^2}}.$$

But

$$-\frac{p_\theta^2}{2m|E|} + \frac{\alpha}{|E|} r - r^2 = -la + 2ar - r^2 = (ae)^2 - (a-r)^2,$$

using the relations (2.30) and (2.31) to introduce the geometrical constants of the elliptic orbit. The integral can be evaluated by introducing the angular parameter ξ, where

$$a - r = ae\cos\xi. \tag{2.33}$$

[The value $\xi = 0$ puts r at its minimum value $r_{min} = a(1-e)$]. Then,

$$t = \sqrt{\frac{m}{2|E|}} \int_0^{\xi} a(1 - e\cos\xi)\,d\xi = a\sqrt{\frac{m}{2|E|}}(\xi - e\sin\xi)$$

if t is also measured from the closest distance of approach. Thus,

$$r = a(1 - e\cos\xi); \quad t = a\sqrt{\frac{m}{2|E|}}(\xi - e\sin\xi) \tag{2.34}$$

provide parametric equations for the radial position at time t. Finally, (2.29) gives the corresponding angular position and thus provides us with a complete picture of the motion.

In astronomy, the angles ξ and θ denote the *eccentric anomaly* and *true anomaly* of a planet when, as here, they are measured from the closest point of approach (the *perihelion*) to the sun. The geometrical meaning of θ is clear but what is ξ? [25] Its meaning can be ascertained by inspection from Fig. 2.8. If we draw a circle of radius a centered at the center C of the ellipse and extend the perpendicular PM through P on the major axis to intersect this circle at P', then the position (a, ξ) of P' is related to the position (r, θ) of P by

$$ae + a\cos(\pi - \xi) = r\cos(\pi - \theta).$$

This equation expresses the length SM in two ways. Substituting for $\cos\theta$ from the orbit equation (2.29) one has

$$ae - a\cos\xi = -\frac{l}{e} + \frac{r}{e} = -\frac{a}{e}(1 - e^2) + \frac{r}{e} \tag{2.37}$$

[25] The inverse problem of finding ξ at a given time is interesting in that it gave rise to the development of what we now know as Bessel functions. Since $\xi(t)$ must be an odd function of nt in the interval $(-\pi, \pi)$ it will bear expansion in a Fourier sine series. Consequently, its derivative $d\xi/d(nt)$ has a cosine expansion

$$\frac{d\xi}{d(nt)} = \frac{1}{2}b_0 + \sum_{m=1}^{\infty} b_m \cos mt, \tag{2.35}$$

with $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{d\xi}{d(nt)} \times \cos mnt\, d(nt) = 2J_m(me)$,

where $J_m(z)$ is a *Bessel function* of order m (see for example W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics*, Chelsea Publishing Company, New York, 1949). Therefore, upon integration,

$$\xi(nt) = nt + 2\sum_{m=1}^{\infty} \frac{1}{m} J_m(me) \times \sin mnt. \tag{2.36}$$

This expression is originally due to Lagrange.

or $r = a(1 - e\cos\xi)$, which confirms the first member of (2.34). The second member of (2.34) shows that the time taken to complete one orbit is

$$T = 2\pi\sqrt{\frac{ma^3}{\alpha}} = \pi\alpha\sqrt{\frac{m}{2|E|^3}}, \tag{2.38}$$

that is, the periodic time only depends on the major axis (equivalently, total energy) of the orbit. The proportionality $T^2 \sim a^3$ is one of the laws of planetary motion discovered empirically by Kepler. Actually Kepler's work went a lot further than just this statement. Since the ratio $2\pi/T = n$ is the mean angular velocity of the planet over one orbit, the second member of (2.34) is equivalent to

$$nt = \xi - e\sin\xi. \tag{2.39}$$

This is Kepler's equation expressing the *mean anomaly* nt in terms of the eccentric anomaly. [26]. To express nt in terms of the true anomaly θ, we multiply r as given by the first of equations (2.34) together with (2.29) to find

[26] Kepler published his laws of planetary motion in 1609.

$$(1 - e\cos\xi)(1 + e\cos\theta) = 1 - e^2, \quad \text{or} \quad \sin\xi = \sqrt{1-e^2}\frac{\sin\theta}{1+e\cos\theta}. \tag{2.40}$$

Consequently, the mean anomaly is connected to the true anomaly through

$$nt = \sin^{-1}\{\sqrt{1-e^2}\frac{\sin\theta}{1+e\cos\theta}\} - e\sqrt{1-e^2}\frac{\sin\theta}{1+e\cos\theta}. \tag{2.41}$$

This equation gives the time required to reach a specified angle θ in the elliptic orbit. Knowing ξ and its relation to θ also allows us to write down the cartesian coordinates (x, y) of the particle with respect to the center C of the orbit:

$$\begin{aligned} x = a\cos\xi &= a\frac{e+\cos\theta}{1+e\cos\theta} \\ y = a\sqrt{1-e^2}\sin\xi &= a(1-e^2)\frac{\sin\theta}{1+e\cos\theta} \end{aligned} \tag{2.42}$$

so that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \quad b = a\sqrt{1-e^2} \tag{2.43}$$

as expected.

Now, suppose the energy is positive, $E > 0$. Then, the orbit becomes unbounded. The shape is given by (2.29) but with E replacing $-|E|$ in the formula (2.30) for the eccentricity e. Thus e becomes greater than unity: *hyperbolic* orbits are described for positive energies. The relation between the semi-major axis, the semi-latus rectum l and the energy E is changed to

$$a = \frac{l}{e^2-1} = \frac{\alpha}{2E} \quad \text{hyperbolic orbits} \tag{2.44}$$

instead of the first member of (2.31); the minor axis b becomes imaginary as it should and the closest distance of approach is

$$r_{min} = \frac{l}{e+1} = a(e-1). \tag{2.45}$$

The relations (2.34) are also untenable. Instead, the time to reach a given radial distance r is now given by exactly similar methods as

$$r = (e\cosh\xi - 1); \quad t = a\sqrt{\frac{m}{2E}}(e\sinh\xi - \xi), \tag{2.46}$$

where the parameter ξ now varies from $-\infty$ to $+\infty$. The orbit itself is shown in Fig. 2.9.

Finally, we consider the repulsive potential $V = \alpha/r$. Then, the effective potential U is always positive, and only positive values of E are physically permissable: the motion is always unbounded, see Fig. 2.7. The shape of the orbit is hyperbolic,

$$\frac{l}{r} = -1 + e\cos\theta, \tag{2.47}$$

where l and e are again given by (2.30) with E replacing $-|E|$. The time-dependence is given by the parametric equations

$$r = a(a\cosh\xi + 1), \quad t = a\sqrt{\frac{m}{2E}}(e\sinh\xi + \xi) \tag{2.48}$$

instead of (2.46).

From (2.47) we see that the closest distance of approach to the force center is

$$r_{min} = \frac{l}{e-1} = a(e+1), \tag{2.49}$$

instead of the value (2.45). The orbit is a hyperbola with the force center at the *external* focus, contrary to the case of an attractive field [(2.29) for $e > 1$], where the force center lies at the *internal* focus, cf. Fig. 2.9.

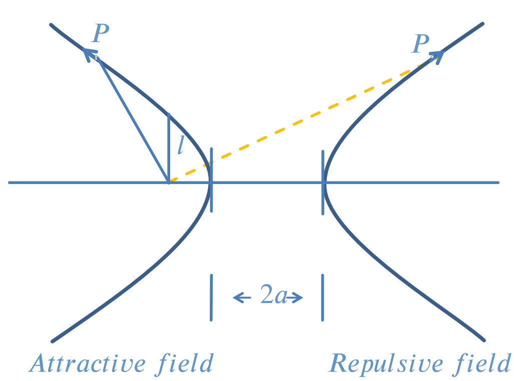

Figure 2.9: Attractive and repulsive motion.

2-5 *The Space Oscillator*

A particle moving subject to the potential

$$V(r) = \frac{1}{2} m\omega^2 r^2 \tag{2.50}$$

($\omega =$ a constant) is called an *isotropic space oscillator*. As before the potential is a central one, so the orbit always lies in a plane. But contrary to our previous example, all orbits are bounded, the potential V increasing without limit as $r \to \infty$.

The effective potential is sketched in Fig. 2.10. Any energy $> E_{min}$ gives a bound orbit. Its shape can be computed from (2.22). However, it is much easier to proceed as follows: since $V(r)$ can be written as

$$V(r) = \frac{1}{2} m\omega^2 (x^2 + y^2) \tag{2.51}$$

in terms of cartesian coordinates (x, y) through the center of force, we simply obtain two one-dimensional oscillators along x and y

$$\begin{aligned} \ddot{x} + \omega^2 x &= 0 \\ \ddot{y} + \omega^2 y &= 0, \end{aligned} \tag{2.52}$$

that are not coupled to each other.

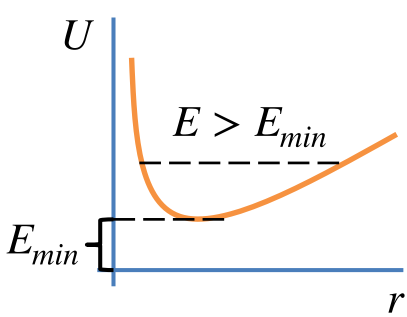

Figure 2.10: Bound state in the effective potential.

Therefore, the motion in time is given essentially by

$$x = a \cos \omega t, \quad y = b \sin \omega t, \tag{2.53}$$

where a and b are the maximum amplitudes of oscillation in the x and y directions. Consequently, the orbit is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.54}$$

about the center of force which lies at the center of this ellipse (not the focus). As in the case of the $1/r$ field, the orbit is also closed. The connection between the geometrical constants of this orbit and the dynamical constants p_θ and E follow from (2.52):

$$\begin{aligned} p_\theta &= m(x\dot{y} - \dot{x}y) = m\omega(ab) \\ E &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m\omega^2(x^2 + y^2) = \frac{1}{2}m\omega^2(a^2 + b^2). \end{aligned}$$

Therefore, both axes determine the energy (or vice versa) in this case, contrary to the situation in the $1/r$ field. However, the period of motion in the oscillator potential only depends on the force constant $T = 2\pi/\omega$, independent of the size of the orbit.

2-6 Scattering by a Central Force

We mentioned the importance of scattering experiments in physics at the beginning of this chapter. Let us now study in detail what such an experiment entails within the domain of classical mechanics.

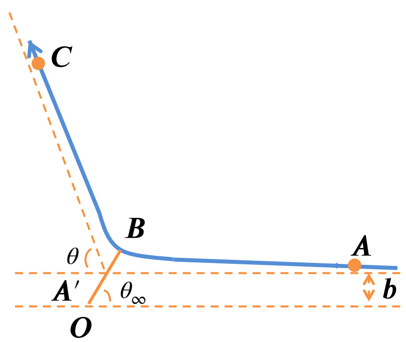

Figure 2.11: The geometry of a scattering problem.

A beam of particles is incident on a scattering center at O, which repels each particle according to some force law $F(r)$. A particular incident particle at A that is initially aimed to miss the scattering center by a distance b will be deflected along the orbit ABC, and be turned through an angle ϑ from its initial direction (Fig. 2.11). This angle is called the *scattering angle*. We know from our previous work that the orbit is symmetrical about the closest distance of approach, line OB. Consequently, the scattering angle is related to the total angle turned through by the radius vector OP as the particle moves in from infinity to its closest distance of approach. Calling this angle θ_∞, we have

$$\vartheta = |\pi - 2\theta_\infty|. \tag{2.55}$$

(The modulus sign is necessary since ϑ is considered to lie between 0 and π by convention). Now, θ_∞ is given by (2.22) if we set the upper limit $r \to \infty$. Before using this equation, however, let us insert for p_θ its value in terms of the *impact parameter* b:

$$p_\theta = \sqrt{2mE}b,$$

for a particle of energy E.

The parameters E and b are the natural parameters for describing the scattering of the particle. In terms of them we have

$$\theta_\infty = \int_{r_{min}}^{\infty} \frac{b}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \frac{dr}{r^2}, \qquad (2.56)$$

where $V(r)$ is the potential energy. Equations (2.55) and (2.56) uniquely determine the angle through which a particle of energy E and impact parameter b is deflected. However, this is not what we want. Under normal experimental conditions, a uniform beam of n particles per unit area impinges on the scattering center with energy E, but with *different* impact parameters b. Therefore, the interesting quantity is the number of particles that are scattered per second into the solid angle $d\Omega = \sin\vartheta d\vartheta d\phi$ subtended at (ϑ, φ) on a large sphere centered at the scattering center O (see Fig. 2.12).

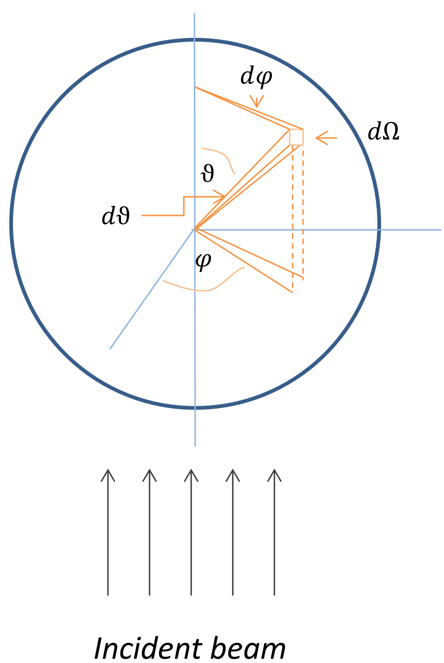

Figure 2.12: Geometry of a scattering center.

The number of particles dN passing into $d\Omega$ per second (the particle *flux*) clearly equals the number passing through the area segment $b\,db\,d\varphi$ between circles b and $b + db$ on a plane held perpendicular to the incident beam at infinity. Hence, $dN = nb\,db\,d\varphi$ where n is the incident flux. Since dN is proportional to the incident flux n, not dN but the ratio $d\sigma = dN/n$ gives a physically useful description of the effect of the scattering center; this ratio is called the *differential cross section*. We have

$$d\sigma = \frac{dN}{n} = b\,db\,d\varphi = \frac{b}{\sin\varphi} \frac{db}{d\vartheta} \sin\vartheta\, d\vartheta\, d\varphi,$$

or

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\vartheta}|\frac{db}{d\vartheta}|, \tag{2.57}$$

if we introduce the solid angle element $d\Omega = \sin\vartheta\, d\vartheta\, d\varphi$. The modulus keeps the intrinsically positive quantity $d\sigma/d\Omega$ positive. The derivation leading up to this formula has assumed a one-to-one correspondence between a given impact parameter b and angle of scattering ϑ. If this is not so, i.e. if the function $b = b(\vartheta)$ is multivalued, the formula (2.57) must include a sum over all these values. Finally, if convergent, the integral of (2.57) over all angles gives the *total cross section*

$$\sigma = \int \frac{d\sigma}{d\Omega} = \int \frac{b}{\sin\vartheta}|\frac{db}{d\vartheta}|d\Omega.$$

The scattering cross section is thus determined classically by knowing b as a function of ϑ (or θ_∞). This information is supplied by (2.56) for any central field $V(r)$. We calculate $d\sigma/d\Omega$ for the important case that $V(r) = \alpha/r$. Then

$$\theta_\infty = \cos^{-1}\{\frac{1}{\sqrt{1+4E^2b^2/\alpha^2}}\}$$

by elementary integration (or simply setting $r \to \infty$ and $p_\theta = \sqrt{2mE}b$ in (2.47)). Inverting, we find

$$1 + \frac{4E^2b^2}{\alpha^2} = \frac{1}{\sin^2(\frac{\vartheta}{2})}; \quad \frac{8E^2}{\alpha^2}b\frac{db}{d\vartheta} = -\frac{\cos(\frac{\vartheta}{2})}{\sin^2(\frac{\vartheta}{2})}$$

and

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2}\frac{\cos(\frac{\vartheta}{2})}{\sin^2(\frac{\vartheta}{2})}\frac{1}{2\sin\frac{\vartheta}{2}\cos(\frac{\vartheta}{2})} = (\frac{\alpha}{4E})^2\frac{1}{\sin^4(\frac{\vartheta}{2})}. \tag{2.58}$$

This is the famous *Rutherford scattering law* [27] obeyed by charged particles scattered by a center of charge. This formula played a fundamental role in paving the way to what we now know about the structure of atoms. Notice that this formula is insensitive to the *sign* of the interaction. An attractive inverse field gives rise to exactly the same scattering cross section. Notice also that the cross sections are infinite in this case (why?).

[27] Phil. Mag,. bf 21, 669 (1911).

A second example of scattering by a central force that is in many ways the opposite of the Rutherford scattering law is provided by the square well potential,

$$\begin{aligned} V &= -V_0, \quad r < a \\ &= 0, \quad r \geq a \end{aligned}$$

of range a and depth V_0. This potential has a finite range and consequently gives rise to a finite total cross section, contrary to the case of the inverse field that has an infinite range.

The particle receives an inward radial impulse as it crosses the spherical surface $r = a$; we saw in Chap.1, Sec. 1-8, that a particle of energy E is refracted at such a potential discontinuity according to the relation

$$\frac{\sin\alpha}{\sin\beta} = n = \sqrt{1+\frac{V_0}{E}}, \qquad (2.59)$$

where α and β are the angle of incidence and angle of refraction respectively. Applying this relation at the spherical surface $r = a$ where the particle enters and leaves, we get

$$\frac{1}{2}\vartheta = \alpha - \beta$$

from the geometry of Fig. 2.13.

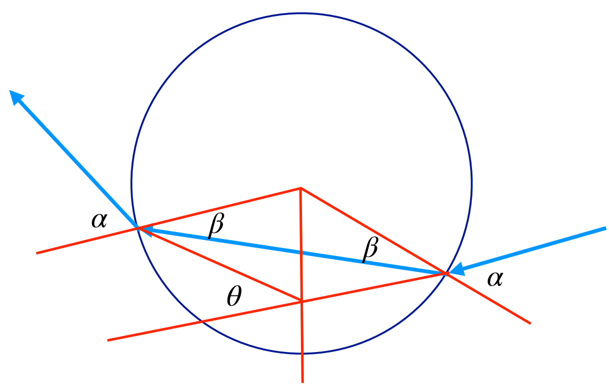

Figure 2.13: Geometry for scattering by a square well.

More fomally, we can proceed from (2.56) after establishing that the effective U/E potential and distance of closest approach are given by

$$\frac{U}{E} = \frac{b^2}{r^2} - n^2 + 1 \quad \text{for} \quad r < a, \quad \frac{b^2}{r^2} + 1 \quad \text{for} \quad r \geq a$$

and $r_{min} = b/n$. Then, for $b \leq a$ (the particle must hit the potential to scatter),

$$\begin{aligned} \theta_\infty &= \int_{b/a}^{a} \frac{b}{\sqrt{n^2 - b^2/r^2}}\frac{dr}{r^2} + \int_a^\infty \frac{b}{\sqrt{1-b^2/r^2}}\frac{dr}{r^2} \\ &= \cos^{-1}(\frac{b}{na}) + \frac{\pi}{2} - \cos^{-1}(\frac{b}{a}) \\ &= (\frac{\pi}{2} - \beta) + \frac{\pi}{2} - (\frac{\pi}{2} - \alpha) \end{aligned} \qquad (2.60)$$

using Fig. 2.13 again to identify the angles $\cos^{-1}(b/na)$ and $\cos^{-1}(b/a)$. Finally, $\vartheta = (\pi - 2\theta_\infty)$, so we regain (2.59). Writing the refraction law as

$$\frac{\sin\beta}{\sin\alpha} = \frac{\sin(\alpha - \vartheta/2)}{\sin\alpha} = \frac{1}{n},$$

and noting that $b/a = \sin\alpha$, one finds the impact parameter b as a function of ϑ. The differential cross-section follows immediately from (2.57):

$$\begin{aligned} \frac{b^2}{a^2} &= \frac{\sin^2\frac{\vartheta}{2}}{1-\frac{2}{n}\cos\frac{\vartheta}{2}+\frac{1}{n^2}} \qquad (b \le a) \\ \frac{d\sigma}{d\Omega} &= \frac{a^2}{4\cos\frac{\vartheta}{2}} \frac{(\cos\frac{\vartheta}{2}-\frac{1}{n})(1-\frac{1}{n}\cos\frac{\vartheta}{2})}{(1-\frac{2}{n}\cos\frac{\vartheta}{2}+\frac{1}{n^2})^2}, \qquad (\frac{1}{n} \le \cos\frac{\vartheta}{2} \le n). \end{aligned} \tag{2.61}$$

The total cross-section is finite: $\sigma = \pi a^2$.

2-7 *The Two-Body Problem*

So far, we have pretended that the center of force in both the bound and scattering problems of the previous two sections was fixed. This is not the case in the physical world. The naturally occurring central force systems are interactions *between* two masses (the planet-sun system in astronomy, the electron-proton system in atomic hydrogen, the neutron-proton system in the deuteron). We are dealing with a *two-body* problem, and, since the forces acting on each are equal and opposite by Newton's third law, the motion of one influences the motion of the other. We now show how this problem can be solved exactly in terms of the theory of central motion developed in Sec. 2-3.

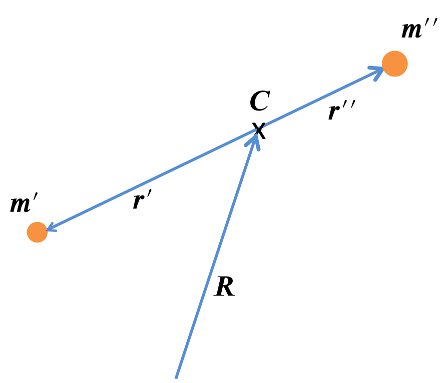

Figure 2.14: Relative and center-of-mass coordinates for the two-body problem.

Suppose then that m' and m'' are two interacting particles. We locate them at $\mathbf{r}'$ and $\mathbf{r}''$ with respect to their center of mass C which lies at $\mathbf{R}$ say, see Fig. 2.14. Then, the Lagrange function is

$$L = \frac{1}{2}(m'+m'')\dot{R}^2 + \frac{1}{2}m'\dot{r}'^2 + \frac{1}{2}m''\dot{r}''^2 - V(|\mathbf{r}'-\mathbf{r}''|), \tag{2.62}$$

since the interaction potential can only depend on the relative separation of the two particles. We see immediately that $\mathbf{R}$ is cyclic so that the center of mass moves with a constant momentum

$$\mathbf{P} = \nabla_R L = (m' + m'')\dot{\mathbf{R}}.$$

This result is also obvious from (1.7), for our system has no external forces acting on it. Our problem is therefore to determine the motion of the two particles relative to their center of mass, i.e. in the center-of-mass system. We can also do this from L in (2.62). But there is one difficulty: the vectors $\mathbf{r}'$ and $\mathbf{r}''$ are not independent and the system is overspecified. Indeed,

$$m'\mathbf{r}' + m''\mathbf{r}'' = 0 \tag{2.63}$$

just gives the center-of-mass position in the center-of-mass system and is therefore identically zero. So we can use either $\mathbf{r}'$ or $\mathbf{r}''$ as the other vector to describe the motion. However, the form of the potential energy in L suggests that neither $\mathbf{r}'$ nor $\mathbf{r}''$, but rather their difference

$$\mathbf{r} = \mathbf{r}' - \mathbf{r}''$$

is the appropriate coordinate to use. We have

$$\mathbf{r}' = \frac{m''}{m' + m''}\mathbf{r}; \quad \mathbf{r}'' = -\frac{m'}{m' + m''}\mathbf{r}, \tag{2.64}$$

so that L reads

$$L = \frac{1}{2}(m' + m'')\dot{R}^2 + \frac{1}{2}\frac{m'm''}{m' + m''}\dot{r}^2 - V(r), \tag{2.65}$$

in the independent position coordinates $\mathbf{R}$ and $\mathbf{r}$. The motion of the center-of-mass is given by the first term as we have already seen. The remaining part

$$L' = \frac{1}{2}\mu(\frac{d\mathbf{r}}{dt})^2 - V(r); \quad \mu = \frac{m'm''}{m' + m''} \tag{2.66}$$

governs the relative motion. It is *identical* with the problem of a particle of *reduced mass* μ moving around a *fixed* force center with velocity $d\mathbf{r}/dt$, as comparison with (2.14) will show. Thus, the two-body problem can be solved completely by considering an appropriate *one-body* problem. [It was one of the great regrets and headaches of the founders of analytic mechanics[28] that systems of three or more particles do not submit to such simplification]. Once the solutions for $\mathbf{r}$ are known we can calculate the motion of the individual masses from the relations (2.64). Notice that the shapes of their orbits are identical (since $\mathbf{r}'$ and $\mathbf{r}''$ are proportional to $\mathbf{r}$) apart from a scaling factor due to the mass difference. Thus, if the orbit for relative motion is an ellipse, the orbits of m' and m'' will likewise be ellipses described about their common center of mass, and these ellipses share a common focus (see Prob. 2-14).

[28] The history of the Problem of Three Bodies is given by A. Gautier, *Essai historique sur le problème des trois corps* (Paris, 1817).

The scattering problem of one particle to another is reduced in exactly the same manner, if we introduce the reduced mass μ and interpret E in (2.56) (and that which follows) as the energy in the center-of-mass system. With these modifications, (2.57) gives the scattering cross-section in the center-of-mass system, the angle ϑ now measuring the scattering angle in this system.

However, measurements are always done in the laboratory system. The connection between the scattering angles and hence the cross-section in the two systems can be established as follows: in the center-of-mass system the momenta of the colliding particles are always equal and opposite [Differentiate (2.63) with respect to time]. Call this momentum $\mathbf{p}$. The magnitude of $\mathbf{p}$ is not changed after the collision, only its direction, because of energy conservation[29]. Thus, all the collision does, is to swing the direction of $\mathbf{p}$ through the angle ϑ without changing its magnitude. When viewed in the laboratory system, this state of affairs appears as follows: before or after the collision m' and m'' have momenta given by the vectors

$$\mathbf{p} + \frac{m'}{m'+m''}\mathbf{P}, \quad -\mathbf{p} + \frac{m''}{m'+m''}\mathbf{P} \tag{2.67}$$

with only the direction of $\mathbf{p}$ changed; $\mathbf{P}$ is the momentum of the center-of-mass. These vectors are displayed in Fig. 2.15. The vector $\mathbf{P}$ is fixed in magnitude and direction throughout the collision by the conservation of total momentum. The vector $\mathbf{p}$ is fixed in magnitude but not in direction by the conservation of energy. Therefore, the two elements, length and direction of AB and the radius OC of the circle, are determined. The mass ratio m'/m'' determines where the point O falls along AB. Thus, all a scattering event accomplishes is to push the point C around the circle to C' *through the center-of-mass scattering angle* ϑ. The change in direction of each particle in the laboratory is given by the angles χ and χ', Most commonly the particle with mass m'' (the *target*) is at rest initially ($\mathbf{p}'' = 0$), so that $\mathbf{P}$ equals the initial momentum $\mathbf{p}'$ of m' (the *projectile*).

[29] Assuming that the collision is *elastic*, i.e. that no energy is deposited in the target or projectile system as internal energy

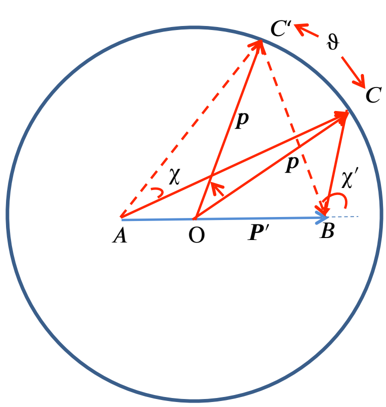

Figure 2.15: Vectors in the center-of-mass system.

Then B moves onto the circle at C (Fig. 2.16), the length AB now equalling $|\mathbf{p}'|$. The scattering angle χ of the projectile relative to its initial direction $\mathbf{p}'$ is evidently given by

$$\tan\chi = \frac{C'M}{AM} = \frac{OC'\sin\vartheta}{AO + OC'\cos\vartheta} = \frac{\sin\vartheta}{\frac{m'}{m''} + \cos\vartheta}, \tag{2.68}$$

since $OA/OC' = m'/m''$ in this case. This result goes over into $\tan\chi = \tan\vartheta$, or $\chi = \vartheta$, if the target is infinitely heavy $(m'' \to \infty)$ as is to be expected. Since the target particle is at rest, the angle χ' is now undefined. However, we can measure the direction of travel of this particle relative to the direction of $\mathbf{p}'$, i.e. by the angle ABC$'$; this is given by $(\pi - \vartheta)/2$.

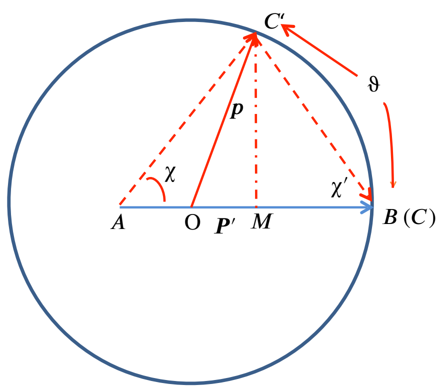

Figure 2.16: Vectors in the center-of-mass system: B moves to C.

The scattering cross-section in the laboratory system can now be calculated by using (2.68) to express the solid angle $d\omega$ in the laboratory system as

$$\begin{aligned} d\omega &= \sin\chi\, d\chi\, d\varphi = (\sin\vartheta\, d\vartheta\, d\varphi)\frac{\sin\chi\, d\chi}{\sin\vartheta\, d\vartheta} \\ &= d\Omega \frac{|1 + \frac{m'}{m''}\cos\vartheta|}{[1 + \frac{2m'}{m''}\cos\vartheta + (\frac{m'}{m''})^2]^{\frac{3}{2}}}, \end{aligned}$$

so that

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma'}{d\omega}\frac{d\omega}{d\Omega} = \frac{|1 + \frac{m'}{m''}\cos\vartheta|}{[1 + \frac{2m'}{m''}\cos\vartheta + (\frac{m'}{m''})^2]^{\frac{3}{2}}}\frac{d\sigma'}{d\omega}, \tag{2.69}$$

where $d\sigma'/d\omega$ is the differential cross-section as measured in the laboratory. This formula is rather complicated. On the other hand, the scattering cross-section of the target particle that was initially at rest is especially simple. Its scattering angle in the laboratory is just $\chi'' = (\pi - \vartheta)/2$, as we have just seen. Therefore, the differential cross-section is

$$\frac{d\sigma''}{d\omega''} = \frac{d\sigma}{d\Omega}(\vartheta = \pi - 2\chi'')\cdot(4\cos\chi'')$$

for scattering into the solid angle $d\omega'' = \sin\chi''\, d\chi''\, d\varphi$. For example, the recoiling particle scatters according to the law

$$\frac{d\sigma''}{d\omega''} = (\frac{\alpha}{2E})^2 \frac{1}{\cos^3\chi''}$$

for Rutherford scattering of the incident particle, (2.58). Here E is still the total energy in the center-of-mass system.

2-8 Motion in an Electromagnetic Field

We have already discussed the Rutherford scattering law that has important applications in atomic and nuclear physics. Another important example of single particle motion in physics is the motion of a charged particle in an electromagnetic field. Incidentally, this problem will afford us with our first example where the Lagrange function is *not* simply $T - V$.

Consider then a particle of mass m and charge e moving in an electromagnetic field characterized by electric and magnetic vectors $\mathbf{E}$ and $\mathbf{B}$. The interaction of a point charge with such a field is given by the famous *Lorentz force law*

$$\mathbf{F} = e\mathbf{E} + e(\mathbf{v} \times \mathbf{B}), \tag{2.70}$$

as determined by experiment; $\mathbf{v}$ is the velocity of the charge. How to fit this into a Lagrangian formulation for the motion of e? We *seek* a suitable Lagrange function, guided by the following observations: since the action integral S in (1.20) is a scalar under point transformations of coordinates, the only permissible forms of L in its integrand must likewise be scalars. Furthermore, the part of L (call it L_{int}) which refers to the interaction between the charge and the electromagnetic field must have "one foot on the charge and the other in the electromagnetic field", otherwise change in the latter could not affect the motion of the former. We know that the electromagnetic field can be conveniently described[30] in terms of its *scalar* and *vector potentials* $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$, where

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}; \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{2.71}$$

Therefore, at least two scalars are available that refer jointly to the charge and the field. They are[31]

$$e\phi(\mathbf{x}, t) \quad \text{and} \quad e\mathbf{A}(\mathbf{x}, t) \cdot \mathbf{v}, \tag{2.72}$$

where $\mathbf{v}$ is the velocity of the charge. Notice the fact that the particle *is* charged must always appear through the *coupling constant* e in these expressions, because this is how the electromagnetic field distinguishes the attribute "charge" of the particle.

[30] See, for example L. Landau and E. Lifshitz, *The Classical Theory of Fields*, Adiison-Wesley Press Inc., Cambridge, Mass., 1951 (translated from the Russian by M. Hamermesh).

[31] The reader may well wonder why one does not form scalar products directly from $\mathbf{E}$ and $\mathbf{B}$ and some vector associated with the particle. The reason for not doing so will become clear in Chap. 6.

The other ingredient we need for our Lagrange function is L_0, describing the free motion of the charge under no forces (since it has mass). This is just given by the kinetic energy

$$L_0 = T = \frac{1}{2}mv^2.$$

We now show that the combination (the units are all MKS)

$$L = L_0 + L_{int} = \frac{1}{2}mv^2 - e\phi + e\mathbf{A}\cdot\mathbf{v} \tag{2.73}$$

leads to the correct equation of motion, i.e. gives the Lorentz force (2.70) correctly. We work in cartesian coordinates $x_k = (x, y, z)$ for the particle. Then,

$$\begin{aligned} L &= \sum_l \frac{1}{2}m\dot{x}_l^2 - e\phi(\mathbf{x}, t) + e\sum_l A_l(\mathbf{x}, t)\dot{x}_l \\ \frac{\partial L}{\partial x_k} &= -e\frac{\partial \phi}{\partial x_k} + \sum_l e\frac{\partial A_l}{\partial x_k}\dot{x}_l \\ \frac{\partial L}{\partial \dot{x}_k} &= m\dot{x}_k + eA_k = p_k, \end{aligned} \tag{2.74}$$

so that the Lagrange equation of motion for x_k is

$$\dot{p}_k = \frac{d}{dt}(m\dot{x}_k + eA_k) = -e\frac{\partial \phi}{\partial x_k} + \sum_l e\frac{\partial A_l}{\partial x_k}\dot{x}_l. \tag{2.75}$$

The canonical momentum p_k is *not* the mechanical momentum $m\dot{x}_k$ in this case; it is supplemented by a contribution eA_k from the field. However, Newton's law of motion refers only to the rate of change of the mechanical momentum. Therefore, we recast (2.75) as

$$\frac{d}{dt}(m\dot{x}_k) = -e\frac{\partial \phi}{\partial x_k} + \sum_l e(\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l})\dot{x}_l - e\frac{\partial A_k}{\partial t} \tag{2.76}$$

after evaluating the total time derivative of $A_k(\mathbf{x}, t)$ explicitly. Now

$$\begin{aligned} \sum_l(\frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l})\dot{x}_l &= (\frac{\partial A_x}{\partial x} - \frac{\partial A_x}{\partial x})\dot{x} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\dot{y} + (\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z})\dot{z} \\ &= \dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y \\ &= [\mathbf{v} \times \mathbf{A}]_x \end{aligned}$$

for the $x-$component ($k = 1$) of this sum. By symmetry the $y-$ and $z-$ components will lead to the corresponding components of $\mathbf{v} \times (\nabla \times \mathbf{A})$. Thus, we obtain the single vector equation

$$\frac{d}{dt}(m\mathbf{v}) = -e(\nabla\phi + \frac{\partial \mathbf{A}}{\partial t}) + e\mathbf{v} \times (\nabla \times \mathbf{A}),$$

or

$$\frac{d}{dt}(m\mathbf{v}) = e\mathbf{E} + e(\mathbf{v} \times \mathbf{B}), \tag{2.77}$$

on account of (2.71). The force on the right side of this equation therefore agrees with (2.70) with the choice (2.73) for L. Observe that for (2.77) to be an acceptable vector equation, the quantity $\mathbf{B}$ must *not* change sign under reflections of the coordinate directions (since $\mathbf{v}$ does), i.e. $\mathbf{B}$ must be a *pseudovector*. It is. This can be seen directly from its construction in terms of $\mathbf{A}$ in (2.71).

The details of the motion described by (2.77) of course depends on how $\mathbf{E}$ and $\mathbf{B}$ behave. Nevertheless, we can obtain some general information by going back to L and examining it for (i) cyclic coordinates, (ii) time-dependence and (iii) uniqueness. Item (i) is a specific detail influenced by how the electromagnetic fields vary in space. Without this information, we cannot say anything more. For item (ii) we calculate

$$dH = -dt(-e\frac{\partial \phi}{\partial t} + \sum_k e\frac{\partial A_k}{\partial t}\dot{x}_k),$$

where

$$H = \sum_k (m\dot{x}_k + eA_k)\dot{x}_k - \sum_k \frac{1}{2}m\dot{x}_k^2 + e\phi - \sum_k eA_k\dot{x}_k = T + e\phi \qquad (2.78)$$

according to the instructions of (1.77) and (1.78). Therefore, the change in kinetic energy, dT, is

$$dT = -dT = -d(e\phi) + e\frac{\partial \phi}{\partial t}dt - \sum_k e\frac{\partial A_k}{\partial t}dx_k = \sum_k e(\frac{\partial \phi}{\partial x_k} + \frac{\partial A_k}{\partial t})dx_k$$

or

$$dT = \sum_k eE_k\,dx_k = e\mathbf{E}\cdot d\mathbf{r}. \qquad (2.79)$$

Change in particle energy is wrought by the electric field $\mathbf{E}$ only; the magnetic vector $\mathbf{B}$ does no work on the charge in motion and consequently does not contribute to dT. This can be seen directly from the form of the Lorentz force (2.70): the magnetic interaction $\mathbf{v}\times\mathbf{B}$ is always perpendicular to the displacement $d\mathbf{r} = \mathbf{v}dt$ of the particle in time dt. Therefore, this contribution to the Lorentz force is workless. (iii) It is important to observe that only the field strengths $\mathbf{E}$ and $\mathbf{B}$ appear in dynamical relations like (2.77) or (2.79). The scalar or vector potentials ϕ or $\mathbf{A}$ do not appear. Indeed, they cannot, since one knows that these potentials are not unique. A transformation (called a *gauge transformation* in electrodynamics),

$$\begin{aligned}\phi \to \phi' &= \phi - \frac{\partial f}{\partial t}\\ \mathbf{A} \to \mathbf{A}' &= \mathbf{A} + \nabla f,\end{aligned} \qquad (2.80)$$

where f is an arbitrary function of position and time, renders $\mathbf{E}$ and $\mathbf{B}$ in (2.71) unchanged in value. The equation of motion (2.77), should likewise be immune to a gauge transformation on the Lagrange function. This is indeed so, since L goes into

$$\begin{aligned} L \to L' &= \frac{1}{2}mv^2 - e(\phi - \frac{\partial f}{\partial t}) + e(\mathbf{A} + \nabla f) \cdot \mathbf{v} \\ &= \frac{1}{2}mv^2 - e\phi + e\mathbf{A} \cdot \mathbf{v} + e\frac{df}{dt}, \end{aligned} \tag{2.81}$$

and therefore only differs from L in (2.73) by the total time derivative of f. We saw in Chap. 1, Sec. 1-4, that the addition of such a term to the Lagrange function does not alter the equations of motion. The relation (2.79) is also left unchanged, but the value of the function H itself is changed of course under a gauge transformation:

$$H \to H' = T + e(\phi - \frac{\partial f}{\partial t})$$

instead of the value (2.78).

The Lagrange function (2.73) has many applications in pure and applied physics, governing at the same time the motion of an electron in an atom and the motion of a charged particle in a cyclotron[32]. Indeed, the motion of charged particles in electromagnetic fields constitute the only actual example of "particle" motion in Newtonian mechanics. We discuss two examples:

[32] A device for producing high energy beams of charged particles.

(i) Motion in a constant magnetic field. In this case

$$\phi = 0, \quad \text{and} \quad \mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r}),$$

so that

$$L = \frac{1}{2}mv^2 + \frac{1}{2}e(\mathbf{B} \times \mathbf{r}) \cdot \mathbf{v} = \frac{1}{2}mv^2 + \frac{e}{2m}(\mathbf{B} \cdot \mathbf{L}). \tag{2.82}$$

$\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum of the particle. We see that the charge in motion interacts with external field $\mathbf{B}$ as though it possesses a *magnetic moment*[33] $\mu = e\mathbf{L}/2m$ in its interaction with the magnetic field. The Lorentz force arising from a pure magnetic field is seen from (2.70) to be perpendicular to both $\mathbf{B}$ and the velocity $\mathbf{v}$. Furthermore, the kinetic energy $\frac{1}{2}mv^2$ of the motion is constant, since $\mathbf{B}$ does no work while moving the charge [see (2.79)]. Therefore, only the direction of $\mathbf{v}$ is changed by the magnetic field, resulting in a *circular* orbit about the direction of $\mathbf{B}$ (counterclockwise if the charge is positive, clockwise if the charge is negative).

[33] See, for example, L. Landau and E. Lifschitz, *ibid.*

The reaction force needed to keep a particle having speed v moving in a circle of radius ρ was calculated in Chap. 1, Sec. 1-6, to be $R = mv^2/\rho$. In the present circumstance, $R = eBv_\perp$, ($v_\perp$ is the velocity component in the plane perpendicular to $\mathbf{B}$), so that

$$B\rho = \frac{mv_\perp}{e}.$$

The product $B\rho$ is sometimes called the magnetic rigidity of a charged particle. It measures the mechanical momentum to charge ratio of the particle.

The calculation of this orbit goes as follows: choose $\mathbf{B}$ along the z axis of a cartesian coordinate system and locate the particle at (x, y, z). Then

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{eB}{2}(x\dot{y} - \dot{x}y)$$

so that

$$\dot{p}_x = \frac{eB}{2}\dot{y}; \quad \dot{p}_y = -\frac{eB}{2}\dot{x}, \quad \dot{p}_z = 0,$$

with the canonical momenta themselves given by

$$p_x = m\dot{x} - \frac{eB}{2}y, \quad p_y = m\dot{y} + \frac{eB}{2}x, \quad p_z = m\dot{z}. \tag{2.83}$$

Since $\dot{p}_z = 0$ the motion along the direction of $\mathbf{B}$ is always uniform. The projection of motion in a plane perpendicular to $\mathbf{B}$ (the $x - y$ plane in our example) is given by

$$\ddot{x} = \frac{eB}{m}\dot{y}, \qquad \ddot{y} = -\frac{eB}{m}\dot{x}$$

or

$$\ddot{\zeta} + i\frac{eB}{m}\dot{\zeta} = 0, \tag{2.84}$$

where $\zeta = x + iy$. Trying an oscillatory solution of the form $\zeta \sim \exp -i\omega t$, one finds that

$$\omega(\omega - \frac{eB}{m}) = 0, \quad \text{or} \quad \omega = 0, \quad \text{or} \quad \omega = \frac{eB}{m}.$$

Therefore,

$$\zeta = A + A' e^{-i\frac{eB}{m}t} \tag{2.85}$$

is the general solution, where A and A' are (complex) constants. They are connected to the position $\zeta_0 = x_0 + iy_0$ and velocity $\dot{\zeta}_0 = v_x + iv_y$ at $t = 0$ by

$$A = \zeta_0 - i\frac{m\dot{\zeta}}{eB}; \quad A' = i\frac{m\dot{\zeta}}{eB}. \tag{2.86}$$

Consequently, the charge travels in a *circle* of radius

$$|A'| = \frac{m|\dot{\zeta}|}{eB} = \frac{mv_\perp}{eB},$$

with angular frequency eB/m. The coordinates of the center of this circle are given by the real and imaginary parts of A in (2.86). The speed $v_\perp = |v_x + iv_y|$ in the $x - y$ plane remains constant of course: from (2.85) and (2.86)

$$\dot{\zeta} = -i\frac{eB}{m}A' e^{-i\frac{eB}{m}t} = \dot{\zeta}_0 e^{-i\frac{eB}{m}t}$$

or

$$|\dot{\zeta}| = |\dot{\zeta}_0| = v_\perp, \quad \text{a constant.}$$

Thus, we confirm our previous picture of the motion in a constant magnetic field.

The Lagrange function (2.82) is of special interest in the case that it contains in addition a restoring potential $\frac{1}{2}\omega_0^2(|\zeta|^2 + z^2)$ that binds the charge to some center of force:

$$L = \frac{1}{2}mv^2 - \frac{1}{2}m\omega_0^2(x^2 + y^2 + z^2) + \frac{e}{2m}\mathbf{B} \cdot \mathbf{l}.$$

This is the *Lorentz oscillator* model of a light source in an external magnetic field. The equations of motion for a charge under such circumstances are

$$\ddot{\zeta} + \omega_0^2\zeta + i\frac{eB}{m}\dot{\zeta} = 0; \quad \ddot{z} + \omega_0^2 z = 0. \tag{2.87}$$

We insert $\zeta \sim e^{-i\omega t}$ once more and find the allowed frequencies in the $x - y$ plane to be

$$\omega^2 - 2(\frac{eB}{2m})\omega - \omega_0^2 = 0, \tag{2.88}$$

or

$$\begin{aligned} \omega_1 &= \Omega + \sqrt{\omega_0^2 + \Omega^2} \simeq \Omega + \omega_0 \\ \omega_2 &= \Omega - \sqrt{\omega_0^2 + \Omega^2} \simeq \Omega - \omega_0 \end{aligned} \tag{2.89}$$

where $\Omega = eB/2m$. The approximate expressions of ω_1 and ω_2 are valid if $\Omega \ll \omega_0$ (weak field case). The solutions of (2.87) are

$$x + iy = \zeta = Ae^{-i\omega_1 t} + A'e^{-i\omega_2 t}; \quad z = A''(\cos\omega_0 t + \alpha), \tag{2.90}$$

where the A's and α are constants. Thus, the motion of the charge in space can be broken down into a linear oscillation along z of angular frequency ω_0, and a *superposition* of circular motions $\zeta_1 = Ae^{-i\omega_1 t}$ and $\zeta_2 = A'e^{-i\omega_2 t}$ with angular frequencies ω_1 and ω_2. For *weak* fields ($\Omega \ll \omega_0$) the motion in the $x - y$ plane is especially simple to describe. We insert the approximate values of ω_1 and ω_2 in the equation for $\zeta = x + iy$ and find

$$\zeta \simeq Ae^{-i(\omega_0+\Omega)t} + A'e^{i(\omega_0-\Omega)t} = \zeta' e^{-i\Omega t}, \tag{2.91}$$

where

$$\zeta' = Ae^{-i\omega_0 t} + A'e^{i\omega_0 t} \tag{2.92}$$

is the motion in the *absence* of the magnetic field. Thus, all a weak field does is to *rotate* the pattern of motion (2.92) around the direction of **B** (in a negative sense for a positive charge, a positive sense for a negative charge) without changing the amplitudes A and A'. This is called *Larmor's theorem*, Ω the Larmor (angular) frequency. We discuss this theorem generally in a moment. But before doing so, let us trespass into the classical theory of radiation from moving charges[34] and ask, with

[34] L. Landau and E. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley Press Inc., Cambridge, Mass.; 1951 (tranlated from the Russian by M. Hamermesh.

Lorentz, how the pattern of radiation from the oscillator (2.92) changes in an external magnetic field (classical theory of the *Zeeman effect*).

Now, the motion (2.92) consists of a superposition of two circular motions in the $x-y$ plane with the same angular frequency ω_0. When viewed along the z-axis (**B** points at us) these motions give rise to separate radiation components of angular frequency ω_0, but which are circularly polarized in opposite senses and have different amplitudes. These two components add coherently and the resultant radiation is elliptically polarized in general. Actually, the radiation from a collection of such oscillators is unpolarized since there is no preferred direction. The magnetic field changes this picture. From (2.91) we see that the positive frequency component speeds up to $(\omega_0 + |\Omega|)$ $(\Omega = -|\Omega|)$ in the magnetic field. Therefore, circularly polarized radiation of *two* frequencies is now expected along the z-axis:

$$\begin{aligned} &\omega_0 + |\frac{eB}{2m}| \quad \text{left circular polarized along } \mathbf{B} \\ &\omega_0 - |\frac{eB}{2m}| \quad \text{right circular polarized along } \mathbf{B}. \end{aligned}$$

No radiation with frequency ω_0 will be observed along the z axis since the z motion is not "visible" in this direction. However, radiation in the $x-y$ plane perpendicular to **B** will contain three frequencies

$$\omega_0, \quad \omega_0 \pm |\frac{eB}{2m}|$$

all linearly polarized (the first along **B**, the other two perpendicular to **B**). These predictions are in full accord with experiment and constitute the *normal Zeeman effect* in atomic spectra. Actually, the normal Zeeman effect is less normal than the so-called *anomalous Zeeman effect* which does not show the simple separation of a spectral line into only three components. The explanation of the anomalous Zeeman effect lies outside the domain of classical mechanics.

(ii) Larmor's Theorem. We saw in the previous problem (2.91) in particular, that a weak magnetic field simply rotates the unperturbed pattern of motion around the direction of the field. One may obtain this result directly from the Lagrange function as we now show for any atomic system. Suppose an atom containing Z electrons is placed in an external magnetic field **B**. Introduce cartesian coordinate axes with z pointing along B and the origin at the atomic nucleus. Then, if (x_k, y_k, z_k) locates the k^{th} electron,

$$L = \sum_k \frac{1}{2} m(\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2) - V(x_1, y_1, z_1; x_2, y_2, z_2; ...) - |\Omega| \sum_k l_k,$$

where

$$l_k = m(x_k \dot{y}_k - y_k \dot{x}_k)$$

is the angular momentum of the k^{th} electron along B; V is the total electrostatic interaction potential. Now, introduce the *cylindrical polar coordinates* (r_k, φ_k, z_k) for the k^{th} electron [see Fig. 2.17].

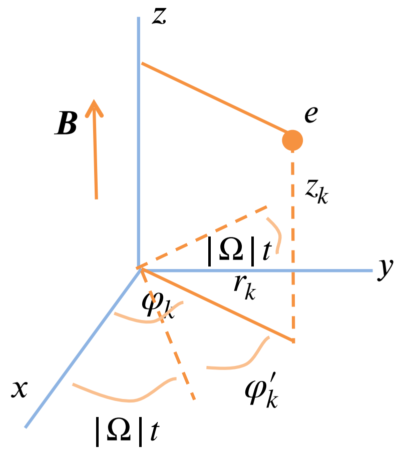

Figure 2.17: Cylindrical coordinate system with external magnetic field **B** indicated.

Then, the expression for $\sum l_k = \sum_k m r_k^2 \dot{\varphi}_k$, so that

$$L = \sum_k \frac{1}{2} m (\dot{r}_k^2 + r_k^2[(\dot{\varphi}_k - |\Omega|)^2 - \Omega^2] + \dot{z}_k^2) - V \tag{2.93}$$

after regrouping the terms involving the $\dot{\varphi}_k$. We now drop the term $\sim \Omega^2$ and introduce a moving coordinate system that rotates about z in Fig. 2.17 in the positive sense with angular frequency $|\Omega|$, then

$$\varphi_k = |\Omega| t + \varphi'_k \tag{2.94}$$

while r_k and z_k are not affected. Then, L reads (note that V does not depend on φ_k)

$$L = \sum_k \frac{1}{2} m (\dot{r}_k^2 + r_k^2 \dot{\varphi}_k'^2 + \dot{z}_k^2) - V$$

in the moving frame, which is identical with the Lagrange function in the fixed frame before the magnetic field was turned on. Therefore, the unperturbed pattern of motion of the electrons is rotated bodily by the magnetic field, so that their total angular momentum vector

$$\mathbf{l} = \sum_k (\mathbf{r}_k \times \mathbf{p}_k) \tag{2.95}$$

precesses about the direction of **B** with angular frequency $|\Omega|$ (see Fig. 2.18).

It is clear that the result (2.91) is a special case of this theorem, since a rotation of coordinates about the z axis through the angle $|\Omega| t = -\Omega t$ is expressed by

$$x + iy = (x' \cos \omega t + y' \sin \omega t) + i(y' \cos \Omega t - x' \sin \Omega t) = (x' + iy')e^{-i\Omega t}. \tag{2.96}$$

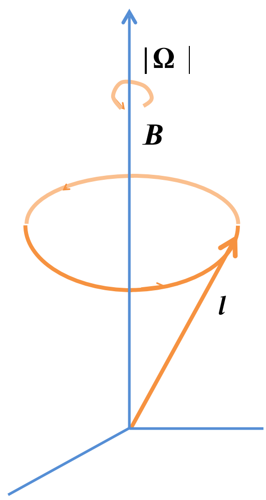

Figure 2.18: The angular momentum vector **l** precesses about the direction of the field **B**.

2-9 *Non-Inertial Frames*

We saw in the discussion of Larmor's theorem that it can be advantageous to introduce moving frames of reference for certain problems. Such moving frames occur naturally in the description of the motion of a particle near the earth's surface (which is rotating) and for the motion of rigid bodies (Chap. 3). We study their properties in this section. From the discussion in Chap. 1, Sec. 1-2., we know that a frame Σ' moving in an arbitrary manner with respect to an inertial frame Σ is *not* and inertial frame. Suppose however, that we deem a description of the motion referred to Σ' as convenient. Consider a particle P moving in Σ; its position vector and velocity in Σ are $\mathbf{r}$ and $\mathbf{v} = d\mathbf{r}/dt$, so that

$$L = \frac{1}{2}mv^2 - V(\mathbf{r}), \tag{2.97}$$

where V is the potential energy. Now suppose Σ' shares a common origin with Σ but rotates with angular velocity $\boldsymbol{\Omega}(t)$ with respect to Σ. The position vector of P as seen from Σ' and Σ is the same, $\mathbf{r}' = \mathbf{r}$; however, the velocities differ. If P has no velocity in Σ', then it is still displaced to P' in time dt, where (see Fig. 2.19)

$$PP' = (\Omega\, dt)(r \sin\vartheta) = |\boldsymbol{\Omega} \times \mathbf{r}|dt$$

and therefore moves with velocity $\boldsymbol{\Omega} \times \mathbf{r}$ in Σ. If P also has a velocity $\mathbf{v}'$ in Σ' the total velocity as seen from Σ is

$$\mathbf{v} = \mathbf{v}' + (\boldsymbol{\Omega} \times \mathbf{r}'). \tag{2.98}$$

Now, the equations of motion in Σ' are still given by the Lagrange equations (1.39), since the action principle is independent of the frame of

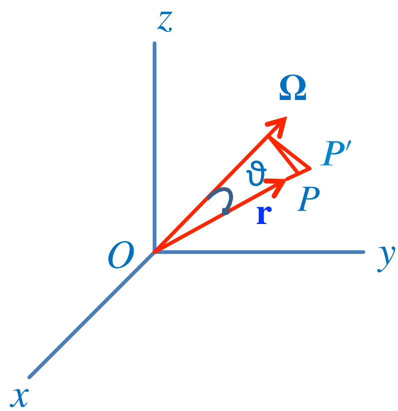

Figure 2.19: Coordinate vectors.

reference. However, L is not given by (2.97); we have to find it. But the value of L is known in the inertial frame from (2.97).
Therefore we only have to express this equation in the moving system: using (2.98) and the fact that $\mathbf{r} = \mathbf{r}'$; we have

$$L' = \frac{1}{2}mv'^2 - V(\mathbf{r}') + \mathbf{\Omega} \cdot \mathbf{l}' + \frac{1}{2}m(\mathbf{r}' \times \mathbf{\Omega})^2 \tag{2.99}$$

in the moving frame, where $\mathbf{l}' = \mathbf{r}' \times m\mathbf{v}'$. This is our second example of a Lagrange function that is not simply $T - V$. The modifications to L in the moving frame are seen to depend both on the angular velocity of the frame and the velocity (more precisely: angular momentum) of the particle in that frame.

The construction of the Lagrange equations requires the derivatives $\partial L'/\partial x'$, $\partial L'/\partial \dot{x}'$, etc. We can get these all at once by subjecting L' to a virtual displacement in the moving frame, $x' \to x' + \delta x'$, $\dot{x}' \to \dot{x}' + \delta \dot{x}'$, etc. Then

$$\begin{aligned} \delta L' &= \nabla L' \cdot \delta \mathbf{r}' + \nabla_{v'} L' \cdot \mathbf{v}' \\ &= (-\nabla V + (\mathbf{\Omega} \times m\mathbf{v}') + \mathbf{\Omega} \times (m\mathbf{r}' \times \mathbf{\Omega})) \cdot \delta \mathbf{r}' \\ &+ (m\mathbf{v}' + (\mathbf{\Omega} \times m\mathbf{r}')) \cdot \delta \mathbf{r}', \end{aligned}$$

in an obvious notation. The partial derivatives $\nabla L'$ and $\nabla_{v'} L'$ can be read off directly from this display, leading to the Lagrange equations of motion

$$\frac{d}{dt}(m\mathbf{v}') = \mathbf{F}' + 2(m\mathbf{v}' \times \mathbf{\Omega}) + \mathbf{\Omega} \times (m\mathbf{r}' \times \mathbf{\Omega}) + (m\mathbf{r}' \times \dot{\mathbf{\Omega}}) \tag{2.100}$$

in the rotating frame; $\mathbf{F}' = -\nabla V$ is the force on the particle in this frame. Equation (2.100) shows how Newton's second law of motion has to be amended *if we insist* on retaining the form $\dot{\mathbf{p}} = \mathbf{F}$ in a non-inertial frame also. We see that the actual force $\mathbf{F}'$ has to be supplemented by *fictitious forces*

$$2(m\mathbf{v}' \times \mathbf{\Omega}), \quad \mathbf{\Omega} \times (m\mathbf{r}' \times \mathbf{\Omega}) \text{ and } (m\mathbf{r}' \times \dot{\mathbf{\Omega}}). \tag{2.101}$$

The first and second expressions are referred to as the *Coriolis*[35] and *centrifugal forces* respectively. The third one is only present if the rotation is non-uniform, and is known as the Euler force.[36] They are all proportional to the mass of the particle and betray their origin in the non-inertial nature of Σ' by depending on $\mathbf{\Omega}$. Much of the confusion caused by naming the last three members in (2.100) forces (without the qualifying term fictitious) can be avoided by recalling their origin and consequently the fact that they can be transformed away by returning to an inertial frame.

Returning to the Lagrange function L in the inertial frame, we know that the total energy $E = \frac{1}{2}mv^2 + V$ is conserved. The corresponding conservation law in the rotating frame is found by calculating the function H of (1.78):

$$\begin{aligned} H' &= (m\mathbf{v}' + \mathbf{\Omega} \times m\mathbf{r}') \cdot \mathbf{v}' - L' \\ &= \frac{1}{2}m\mathbf{v}'^2 + V(\mathbf{r}') - \frac{1}{2}m(\mathbf{r}' \times \mathbf{\Omega})^2. \end{aligned} \tag{2.102}$$

H' will be conserved if L' does not have an explicit time-dependence, which means the rotation must be *uniform*. The rotation thus adds a term proportional to Ω^2 to the "energy" $\frac{1}{2}mv'^2 + V(\mathbf{r}')$ in the rotating frame. However, the latter quantity is not constant, only the sum H'. The term $-\frac{1}{2}m(\mathbf{r}' \times \mathbf{\Omega})^2$ is called the *centrifugal potential*, because it gives reise to the centrifugal force term in (2.78). The value of the constant H' is

$$H' = m\mathbf{v} \cdot \mathbf{v}' - L = E - \mathbf{\Omega} \cdot \mathbf{l}$$

(using (2.98) and (2.97)), where $\mathbf{l} = \mathbf{r} \times m\mathbf{v}$ is the angular momentum of the particle in *either* frame, since $\mathbf{l} = \mathbf{l}'$.

(i) Again, let us look at the motion of a charged particle in a constant magnetic field. Let us view the motion of the charge in a *slowly* rotating reference frame. Then, this Lagrange function is given by

$$L' = \frac{1}{2}mv'^2 - V(\mathbf{r}') + (\frac{e}{2m}\mathbf{B} + \mathbf{\Omega}) \cdot \mathbf{l}'$$

in this frame. The choice

$$\mathbf{\Omega} = -\frac{e}{2m}\mathbf{B}$$

eliminates the magnetic field and gives an almost trivial proof of Larmor's theorem. Alternatively, we may interpret Larmor's theorem to be the consequence of choosing the rotational angular velocity of the moving frame so that the Coriolis force just cancels the contribution from the magnetic part of the Lorentz force (2.70).

(ii) Motion relative to a rotating earth. In Sec. 2.3 we considered the motion of a projectile near the surface of the earth. Our treatment involved the tacit assumption that axes attached to the earth form an inertial frame. This is not exactly true of course, due to the rotation of

[35] Introduced by G. Coriolis, J. de l'ecole Polytechnique, Cahier **24**, 142 (1835).

[36] For the purists, we admit that for forces proportional to the mass of the particle, one cannot in fact distinguish between the "actual" and "fictitious" attributes insofar as the action of such forces can always be ascribed to the non-inertial nature of the coordinate system of the observer. In fact, this is one of the theses (*the principle of equivalence*) upon which Einstein's theory of gravitational attraction is based. Such matters lie outside the scope of these notes. The interested reader should consult L. Landau and E. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley Press Inc., Cambridge, Mass. 1951, Chap. 11.

the earth about its own axis and its motion around the sun. Let us study the effect of this rotation.

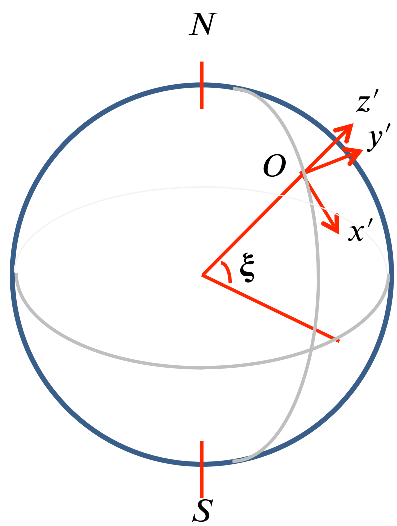

Figure 2.20: Coordinate axes x', y', z' for describing motion relative to the earth's surface.

Suppose $O_{x',y',z'}$ in (2.21) represents a rectangular frame of reference rigidly attached to the surface of the earth at O, see Fig. 2.20. We choose z' "vertical" and x' in the plane of the great circle passing through the poles. The point O is located by its angle of *latitude* ξ relative to the equatorial plane that lies at $\xi = 0$. Strictly speaking, the theory developed in this section does not apply to the $O_{x'y'z'}$. These axes have an additional acceleration due to the motion of the point O in space, and the Lagrange function (2.99) should be modified accordingly (see Prob. 2-15). However, this acceleration is proportional to Ω^2, where Ω is the angular velocity of rotation of the earth and, together with the centrifugal potential, will be neglected in this discussion. Then, the Lagrange function describing the motion of a particle near the surface of a rotating earth is given by (2.99).

$$L' = \frac{1}{2}mv'^2 - V(\mathbf{r}') + \mathbf{\Omega} \cdot \mathbf{l}',$$

where $V(\mathbf{r}') = V(x', y', z')$ is the potential energy of the particle of mass m under study. Since $\mathbf{\Omega}$ lies along the North-South axis of the earth, we have

$$L' = \frac{1}{2}(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2) - V(x', x', z') + \Omega_{x'} l_{x'} + \Omega_{z'} l_{z'}. \tag{2.103}$$

The rotation of the earth about its own axis is revealed by the last two terms; indeed, one can use this equation to demonstrate the earth's rotation. Consider a simple pendulum $O'P$ of length l, suspended directly above O from a point O' on the z' axis (Fig. 2.21). For small amplitude oscillations the motion of the pendulum is effectively restricted to the plane $z' = l$ parallel to the $x' - y'$ plane. The restoring force components parallel to x' and y' are $-m\omega_0^2 x'$ and $-m\omega_0^2 y'$ respectively, where $\omega_0 = \sqrt{g/l}$ is the pendulum frequency relative to a fixed earth. Therefore, the potential energy in (2.103) is $V = \frac{1}{2}m\omega_0^2(x'^2 + y'^2)$ and L' becomes

$$L' = \frac{1}{2}m(\dot{x}'^2 + \dot{y}'^2) - \frac{1}{2}m\omega_0^2(x'^2 + y'^2) + \Omega_{z'} l_{z'}. \qquad (2.104)$$

Introducing polar coordinates r', θ' in the $x' - y'$ plane, we find

$$\begin{aligned} L' &= \frac{1}{2}m(\dot{r}'^2 + r'^2\dot{\theta}'^2) - \frac{1}{2}m\omega_0^2 r'^2 + \Omega_{z'} m r'^2 \dot{\theta}', \\ &\simeq \frac{1}{2}m\dot{r}'^2 + \frac{1}{2}mr'^2(\dot{\theta}' + \Omega_{z'})^2 - \frac{1}{2}m\omega_0 r'^2 \end{aligned}$$

to first order in $\Omega_{z'}$. The situation is the reverse of the case of a

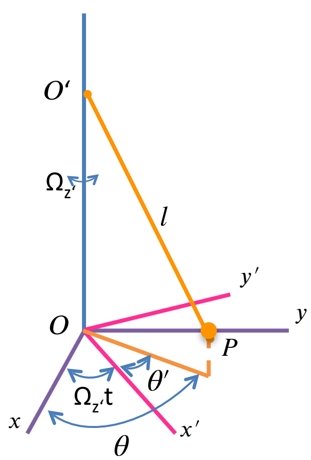

Figure 2.21: Foucault's pendulum.

charge in a magnetic field: here the effect of the motion of the earth is removed by going into an *inertial* frame: $\theta = \theta' + \Omega_{z'} t$. The analysis is in all respects similar to that leading up to (2.90); we simply give the answer

$$(x' + iy') = e^{-i\Omega_{z'} t}(x + iy),$$

where $x + iy$ gives the motion of the pendulum with respect to a fixed earth. Fig. 2.21 illustrates the motion.

The pendulum oscillates in the plane OPO' that retains its orientation in space, the "drift" of the pendulum away from a straight line in the $x' - y'$ plane being due, quite literally, to the earth turning out from under the pendulum. But, as observers on the earth, we turn with it and so record the path in the $x' - y'$ plane. Suppose then that the pendulum is pulled aside a distance **a** along the x' axis and released from rest. The motion we see as earth-dwellers is

$$x' + iy' = e^{-i\Omega_z t}[a\cos\omega_0 t + \frac{a\Omega_{z'}}{\omega_0}\sin\omega_0 t],$$

since the initial velocity $\dot{x}' + i\dot{y}'$ is zero. The velocity at later times is

$$\dot{x}' + i\dot{y}' = -e^{-i\Omega_{z'}t} a\omega_0 (1 - \frac{\Omega_{z'}^2}{\omega_0^2}) \sin \omega_0 t.$$

The sine factor show that the bob repeatedly comes to rest at times $\omega_0 t = 0, \pi, 2\pi$, etc., and is at the same distance $|x' + iy'| = a$ from the origin as initially at these times. How then does the projection of the bob on the $x' - y'$ plane move? The positions at which $\dot{x}' + i\dot{y}' = 0$ are clearly *cusps* of the path. The bob moves from one cusp to the next in a path that continually bends out of the $Ox'z'$ plane, the plane of oscillation rotating clockwise as seen from above (see Fig. 2.22). The cusp positions advance through an angle $2\pi\Omega_{z'}/\omega_0 = 2\pi(\Omega/\omega_0)\cos\xi$ along the circumference of a circle radius **a** per oscillation period of the pendulum. (The Kirchhoff Institute of Physics in Heidelberg has a beautiful example of such a pendulum on display). The device is known as *Foucault's pendulum* after G. Foucault, who used it in 1851 to demonstrate the rotation of the earth[37].

[37] It was fortunate that Foucault lived in the 19th century. Demonstration of such facts was not to be recommended during the 17th century (Galileo's century), being contrary to the theological beliefs of the time.

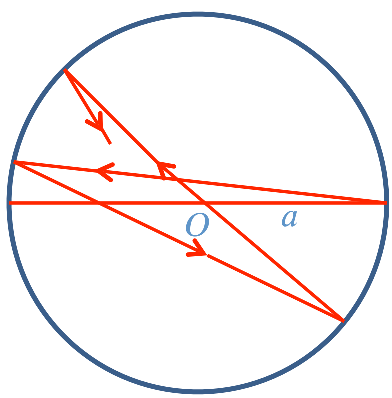

Figure 2.22: Projected path of the bob of Foucault's pendulum.

2-10 Systems of Interacting Particles

We finally consider a system consisting of N *interacting* particles. Such a system constitutes the fundamental many-body problem that is of primary interest in fields as far apart as celestial mechanics and atomic physics. As before, we assign cartesian coordinates x_k to each particle. The Lagrange function is then given by (1.37), i.e.

$$L = \sum_k \frac{1}{2} m_k \dot{x}_k^2 - V(x_1, x_2, \dots). \tag{2.105}$$

We will be primarily interested in systems where the potential field $V(x_1, x_2, \dots)$ is due to mutual interactions between the particles only. Such systems are termed *closed systems*. Notice in passing that the mere introduction of a potential function $V(x_1, x_2, \dots)$ in the Lagrange function (2.105) implies that the change in position of any particle in the system *instantaneously* affects the motion of all other particles, i.e. the interaction between particles propagates with infinite velocity. Thus, (2.105) violates one of the basic tenets of the Special Theory of Relativity (see Chap. 6).

For N interacting particles, L in (2.105) leads to $3N$ coupled Lagrange equations of motion. To progress anywhere beyond the formality of just writing down this set of equations, one usually has to introduce specific additional assumptions about the many-body system. Such special many-body systems are the subject of the next two chapters. Nevertheless, we can make some general statements about the system described by (2.105) by examining the invariance properties of L itself.

We are assuming the system is a closed one. Therefore, $V(x_1, x_2, \dots)$ is independent of the coordinate origin chosen for the x_k, i.e. the system is translationally invariant. In symbols, this means for example that displacing the x coordinates of all particles by a common amount Δx along the x axis[38] does not change V:

$$\Delta V = \left(\frac{\partial V}{\partial x} + \frac{\partial V}{\partial x'} + \dots\right)\Delta x = 0, \tag{2.106}$$

[38] Observe again that the translation Δx does not constitute a virtual displacement, see remarks after (1.75).

[we have reverted to $(x_1, x_2, x_3) = (x, y, z)$, $(x_4, x_5, x_6) = (x', y', z')$, etc. for greater clarity]. On the other hand, the equations of motion satisfied by the x, x', etc. are

$$\frac{\partial V}{\partial x} - \dot{p}_x = 0, \quad \frac{\partial V}{\partial x'} - \dot{p}_{x'} = 0, \quad \dots. \tag{2.107}$$

Adding these equations together and using (2.106) we conclude that

$$\frac{d}{dt}(p_x + p'_x + \dots) = 0$$

or that the total linear momentum, $P_x = p_x + p'_x + \dots$ in the x direction is conserved. Likewise, the components P_y and P_z are conserved. Thus, the conservation of the total linear momentum reflects the *homogeneity* of the space in which the many-body system is moving.

If L is invariant under rotations we get another conservation law. A rotation of the coordinate axes around the z direction by a small angle $\Delta\varphi$ replaces the x and y coordinates of each particle by

$$x \to x + y\Delta\varphi; \quad y \to y - x\Delta\varphi,$$

[see (3.31)]. The invariance of $V(x_1, x_2, \dots)$ under such rotations means that the change

$$\begin{aligned}\Delta V = \quad & V(x + y\Delta\varphi, y - x\Delta\varphi, z; x' + y'\Delta\varphi, y' - x'\Delta\varphi, z'; \dots) \\ & - V(x, y, z; x', y', z'; \dots)\end{aligned}$$

is zero, or that

$$(y\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial y} + y'\frac{\partial V}{\partial x'} - x'\frac{\partial V}{\partial y'} + \dots)\Delta\varphi = 0. \tag{2.108}$$

Substituting for the space derivatives of V from (2.107) in this equation, we find that

$$\frac{d}{dt}\{(xp_y - yp_x) + (x'p_{y'} - y'p_{x'}) + \dots\} = 0,$$

or that the total angular momentum along the z direction, $L_z = (xp_y - yp_x) + (x'p_{y'} - y'p_{x'}) + \dots$ is conserved. Invariance under rotations about x and y likewise show that L_x and L_y are conserved. Thus, the *isotropy* of space and angular momentum conservation go together. Finally, the lack of any explicit time dependence in L (homogeneity in time) shows that the total energy of the system is conserved.

Two further remarks about (2.105) are important. First, the lack of dependence of V on the coordinate origin can be made explicit by going over into center of mass and relative coordinates for the system. Changing notation once more, we define the position vectors $\mathbf{r}_1 = (x_1, x_2, x_3)$, $\mathbf{r}_2 = (x_4, x_5, x_6)$ for each particle as well as their center of mass coordinate

$$M\mathbf{R} = \sum_{i=1}^{N} m_i \mathbf{r}_i, \quad M = \sum_{i=1}^{N} m_i. \tag{2.109}$$

Now, the translational and rotational invariance of $V(x_1, x_2, \dots)$ shows it can only depend on the coordinate differences $(\mathbf{r}_i - \mathbf{r}_j)$, or only on the coordinate differences $(\mathbf{r}_i' - \mathbf{r}_j')$, where the coordinate $\mathbf{r}_i'$ are measured relative to the center-of-mass according to

$$\mathbf{r}_i' = \mathbf{r}_i - \mathbf{R}. \tag{2.110}$$

Introduction of these coordinates into L in (2.105) leads to a Lagrange function of the form

$$L = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i'^2 + \frac{1}{2} M\dot{\mathbf{R}}^2 - V(\mathbf{r}_1', \mathbf{r}_2', \dots) \tag{2.111}$$

The center-of-mass coordinate $\mathbf{R}$ is explicitly cyclic; thus its canonical momentum

$$\mathbf{P} = \nabla_R L = M\dot{\mathbf{R}}$$

is conserved. This is exactly the conservation of total momentum again as can be seen by differentiating the first member of (2.109) with respect to time to identify $M\dot{\mathbf{R}}$.

The other remark concerns the structure of $V(\mathbf{r}_1', \mathbf{r}_2', \dots)$ itself. As it stands in (2.111) V gives rise to many-body forces acting between particles (interaction between any pair of particles depends also on where all the other particles are). This is not the case if V can be displayed as a sum of two-body potentials $V_{ij}(|\mathbf{r}'_i - \mathbf{r}'_j|)$:

$$V_{ij}(|\mathbf{r}'_i - \mathbf{r}'_j|) = \sum_{i<i} V_{ij}(|\mathbf{r}'_i - \mathbf{r}'_j|). \tag{2.112}$$

Then, the interaction between the i, j^{th} pair only depends on their relative separation, and is quite independent of where the remaining particles are. The forces acting in atomic systems are of this type for example.

The remaining task is to discuss the motion of a many-body system described by the Lagrange function (2.105). The mathematical complexities of such a program are enormous in general and one has to lean heavily on specific properties of the system as an aid to its solution. For example, in *rigid bodies* the interparticle forces are strong enough to effective "freeze" the constituent particles into a rigid lattice. The interparticle spacing therefore does not change during the motion of such a system, so the potential energy-term in (2.105) remains constant and we can simply drop it. Thus

$$L = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 + \frac{1}{2} M \dot{\mathbf{R}}^2 \tag{2.113}$$

for a rigid body moving under no external forces. However, the conditions of rigidity introduce constraints between the coordinates r'_i, leaving the system with only six degrees of freedom (3 translational plus 3 rotational), instead of $3N$. Such systems are discussed in detail in the next chapter.

Relaxing the condition of perfect rigidity slightly allows the constituent particles of the system to indulge in small displacements about the equilibrium positions they held in the completely rigid body. Now, the potential term in (2.105) enters once more, but this complication is offset by the assumption of "small displacements". Perturbation methods can be applied to the system to study the oscillatory motion of all the $\mathbf{r}'_i$ about their equilibrium values. A systematic study of such small oscillations will be found in Chapter 4.

Finally we may relax the rigidity conditions completely and "let the system take care of itself". This involves us in the full complications of the many-body problem as it presents itself in atomic and nuclear physics. Furthermore, the number of degrees of freedom become infinite in the limit of a large number of interacting particles, $N \to \infty$. The "particle character" of the system is effectively overwhelmed by their mere number in this limit and we find ourselves discussing a "fluid", i.e. the dynamics of a continuous (on a macroscopic) medium. The dynamics of continuous media is taken up briefly in Chapter 5.

Problems

2-1. Discuss the convergence of the integral expression given in (2.13) for the period of a one-dimensional oscillation. Evaluate T for the periodic time of a planet [see (2.25)] by the method of contour integration.

2-2 Obtain the path of a projectile moving in a constant gravitational field directly from Jacobi's principle, (1.99).

2-3 Obtain the differential equation (2.26) for the orbit in a central field directly from Jacobi's principle, (1.99).

2-4 Show that the true anomaly of a planet may be expressed as the series

$$\theta = nt + 2e \sin nt + \frac{5}{4} e^2 \sin 2nt + \dots$$

in terms of the mean anomaly nt (e is the eccentricity of the elliptic orbit).

2-5 Find the equation expressing the time t in terms of the angular position θ for hyperbolic motion of a planetary object under an inverse square force law.

2-6 Show that

$$\sqrt{\frac{\alpha}{2p^3}}\, t = \tan \frac{\theta}{2} + \frac{1}{3} \tan^3 \frac{\theta}{3}$$

for parabolic motion in the potential field (2.27); p is the distance from the focus to the vertex of the parabola.

2-7 Two space capsules are in common circular orbit of radius r_0 around the earth, and have an angular separation β. The rear space capsule is desirous of transmitting a *small* metal object to the lead capsule. In what direction and with what velocity must this object be ejected? In particular, explain what sort of orbits the object follows when ejected (i) tangentially forwards, (ii) tangentially backwards, to the circular orbit.

2-8 A space capsule is in a polar orbit around the earth that is a perfect circle. Why are retro-rockets necessary to bring the capsule down again? If the capsule has to be brought down so that it is moving tangential to the earth's surface at the equator, calculate the energy change, ΔE, that the rockets have to cause. Assume that the rockets fire when the capsule is a height h above one pole and that they fire instantaneously. Must the capsule lose or gain energy in order to return to the earth? Is the direction the rockets fire in important?

2-9 Investigate the motion of a satellite in the gravitational field of a deformed earth. Assume that the earth has the shape of an oblate spheroid, i.e. flattening at the poles and bulging at the equator. Calculate the gravitational potential of such a mass distribution assuming the deviation from a sphere to be small. Hence calculate approximately the main effects of such deviations in the gravitational potential away from a central one on satellite motion. (A systematic way of handling such problems is discussed in Chapter 7.)

2-10 Show that the radius vector r to the particle P in Fig. 2.4 sweeps out equal areas in equal times in central motion.

2-11 Calculate the differential cross-section for the scattering of particles by a repulsive square well potential [replace $-V_0$ by $+V_0$ in (2.58)].

2-12 Obtain the relation (2.59) of the text by making use of conservation laws only.

2-13 Calculate the equation for the orbit and the differential scattering cross-section for particle motion in the central field

$$V(r) = \frac{\alpha}{r} + \frac{\beta}{r^2},$$

where α and β are constants. Do the relative and/or absolute signs of α and β matter? Comment.

2-14 Verify the statements made below (2.66) in the text by calculating explicitly the orbits *in space* for two particles m' and m'' interacting via an attractive inverse square force.

2-15 Construct the Lagrange function for a free particle moving relative to a frame of reference that performs an arbitrary motion.