Chapter 1 Principles of Dynamics

1-1 Preamble

Classical mechanics is a truly vast subject. That being so, the material that appears in any text of this size and purpose has of necessity been through a severe selection process. The choice of topics to be presented here has been strongly influenced by a perceived relevance to modern physics. This might sound unfair to our subject: The builders of classical mechanics certainly never did intend their work to be seen only as stepping stones on the way to quantum mechanics and field theory. Yet much of it is.

We have also attempted to keep the discussion of mechanics as uniform as possible by using a Lagrangian formulation throughout. In many instances this restricts one to considering only conservative systems, or at least systems for which a Lagrange function can be constructed. This is hardly a disadvantage. Most of the systems of interest in physics are of this type; furthermore the generalizations to cover particular non-conservative systems are usually straightforward.

1-2 The Ingredients of Mechanics

We start by considering the motion of a single particle under the action of given forces, thereby introducing the two main ingredients of mechanics, the concepts of *particle* and *force*. A *qualitative* idea of a force is quite natural and instinctive, based on everyday experience. Gravitational forces and their effects are certainly familiar to anyone who has ever tried to defy them. But other forces also exist in nature that are not so easily experienced; electromagnetic forces that are responsible for the structure of matter in bulk, and forces which govern the dynamics inside the atomic nucleus and its components itself. What we will have to say in the rest of this book is generally applicable for any force system. In practice, of course, the natural forces that occur in classical mechanics are basically of two types: gravitational and electrical (even the action of a spring, or the friction on a rough surface can in the last analysis be traced back to the properties of the material involved, i.e. to the electrical forces between atoms). Examples of man-made forces in engineering applications like steam pressure, hydraulic systems, etc. abound of course.

Experience tells us that forces have both magnitude and direction: they are to be represented as *vector quantities*. But how does one measure a force, or compare the action of different forces? This is the *quantitative* aspect of the problem. The answer is also obvious: we must do an experiment. For this purpose we introduce the other ingredient mentioned at the beginning of this section: the idea of a *particle*, or *mass point* [1], whose internal structure is characterized by a single parameter m called its *mass*. The extent to which actual material bodies in the laboratory approximate particles is determined by to what extent properties other than their mass play a role in determining their motion.

[1] Sometimes also described as a *material point*.

To investigate the motion of a particle, we must measure its velocity $\mathbf{v}$ and changes in its velocity as a function of time, $\dot{\mathbf{v}}$. In order to perform such measurements the particle must be located in some *frame of reference* by a position vector $\mathbf{r}$. As the particle moves, $\mathbf{r}$ changes. The rates of change of $\mathbf{r}(t)$ and $\mathbf{v}(t)$, given by

$$\mathbf{v} = \frac{d}{dt}\mathbf{r}(t), \qquad \dot{\mathbf{v}} = \frac{d^2}{dt^2}\mathbf{r}(t),$$

represent the *velocity* and *acceleration* of the particle at time t. However, it is not the velocity $\mathbf{v}$, but rather the product $\mathbf{p} = m\mathbf{v}$, called the *linear momentum*, that plays a central role in determining the motion of the particle. For experience shows that $\mathbf{p}$ stays constant if no forces act on the particle, i.e. the motion is governed by the condition

$$\mathbf{p} = \text{constant}. \tag{1.1}$$

This is *Newton's first law of motion* or law of inertia. The effect of an applied force $\mathbf{F}$ acting on the particle is to change the momentum $\mathbf{p}$. The rate of change, $\dot{\mathbf{p}} = d\mathbf{p}/dt$, is dictated by *Newton's second law of motion*,

$$\dot{\mathbf{p}} = \mathbf{F} \tag{1.2}$$

This equation is the basic law governing the motion or *dynamics* of material systems. It is often called the *principle of linear momentum*.

We saw that it was necessary to specify a frame of reference in which to measure $\mathbf{p}$ and hence $\dot{\mathbf{p}}$. Are the laws of motion (1.1) and (1.2) valid irrespective of what reference frame is used? Experience shows that this is not the case. Newton's first two laws of motion only assume the simple forms given in special frames of reference, called *inertial* frames. In fact, (1.1) can be turned around to provide a definition of an inertial frame: a free particle moves with constant momentum in an inertial frame. In particular, a free particle initially at rest remains at rest in such a frame [2].

[2] A better definition of an inertial frame of reference can be given in General Relativity: The gravitational field vanishes in the immediate neighborhood of an inertial frame, whether or not this field is caused by gravitating masses or the non-uniform motion of the frame of reference.

The next question is very important. How many inertial frames are there? Suppose the point O is the origin of a set of rectangular axes O_{xyz} that form an inertial frame Σ. Now introduce a second frame made up of rectangular axes $O'_{x'y'z'}$ moving in a straight line with constant velocity $\mathbf{V}$ with respect to Σ. A particle at P which has position coordinate $\mathbf{r} = OP$ in Σ has position coordinate $\mathbf{r}' = O'P$ in Σ', see Fig. 1.1, where

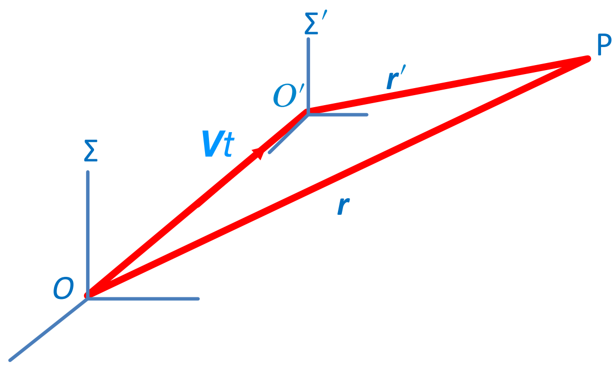

Figure 1.1: The particle at P moves with velocity $\mathbf{v}$ with respect to Σ and velocity $\mathbf{v}'$ with respect to Σ'.

$$\mathbf{r} = \mathbf{r}' + \mathbf{V}t \tag{1.3}$$

if Σ and Σ' coincide at $t = 0$. Hence,

$$\mathbf{v} = \mathbf{v}' + \mathbf{V} \tag{1.4}$$

where $\mathbf{v}'$ is the velocity of the particle as measured in Σ'. Since $\mathbf{V}$ is constant in time, $\dot{\mathbf{v}} = \dot{\mathbf{v}}'$ or $\dot{\mathbf{p}} = \dot{\mathbf{p}}'$. Thus, Newton's laws of motion hold in Σ' if they hold in Σ, if we assume that the force is the same in Σ and Σ', i.e. $\mathbf{F} = \mathbf{F}'$. All frames Σ' moving with constant velocity in a straight line are inertial frames, i.e. there are infinitely many inertial frames if there is one such frame, and Newton's laws are valid in all these frames. This is *Galileo's principle of relativity*. Stated another way, Galileo's principle asserts that the laws of motion are invariant in form under the transformation of coordinates (1.3). Equation (1.3) is called a *Galilean transformation*. Notice the tacit assumption

$$t = t' \tag{1.5}$$

in (1.4): the passage of time in the two frames of reference is the same. This is the classical point of view of the absolute nature of time in Galilean relativity. It will undergo a profound modification when we discuss the theory of relativity.

In the laboratory, the question of an inertial reference frame is easily settled: for most practical purposes a reference frame attached rigidly to the earth's surface will suffice. The effects coming from the earth's rotation are small and can be corrected for when necessary.

Laws 1 and 2 must be supplemented by two further statements. These are contained in
Law 3: *Action and reaction are equal*, or the forces exerted by two bodies in mutual interaction are equal and opposite, and
Law 4: The superposition principle for forces asserts that *forces add like vectors*. The content of this statement is simply that when several forces act simultaneously on a particle, each force causes changes in motion as if it alone were present; the other forces do not interfere with its action. It is a remarkable fact that these four laws of motion summarize the whole body of experience of experimental mechanics.

1-3 Systems of Particles

So far, we have spoken of a single particle and its motion under the action of an applied force. Now, consider what happens if the four laws of motion are applied to a system of interacting particles. We fix our attention on particle a of this system, moving with momentum $\mathbf{p}_a$. The forces on a arise from
(i) the sum $\Sigma_{b\neq a}\mathbf{F}_{ab}$ of all internal forces acting on a due to its interaction with all the other particles in the system. $\mathbf{F}_{ab}$ is the force exerted by b on a. By law 3, it is equal and opposite to the force exerted by a on b, $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$.
(ii) the sum of all external forces $\mathbf{F}_a^{(e)}$ that are applied from outside the system. The motion of particle a is thus determined by

$$\dot{\mathbf{p}}_a = \mathbf{F}_a^{(e)} + \sum_b \mathbf{F}_{ab} \tag{1.6}$$

so that

$$\sum_a \dot{\mathbf{p}}_a = \sum_a \mathbf{F}_a^{(e)} + \sum_a \sum_{b\neq a} \mathbf{F}_{ab}$$

summing over all particles. Now, $\Sigma_a \mathbf{p}_a = \mathbf{P}$ is the total linear momentum of the system, $\Sigma_a \mathbf{F}_a^{(e)} = \mathbf{F}$ the total *external* force on the system, and $\Sigma_a\Sigma_{b\neq a}\mathbf{F}_{ab} = 0$ because the internal forces cancel in pairs. Hence, the principle of linear momentum for a composite system of particles reads

$$\dot{\mathbf{P}} = \mathbf{F}. \tag{1.7}$$

Notice that the internal forces cannot effect changes in the total momentum; they simply drop out. In fact this equation has a simple meaning. If we locate particle a with the position vector $\mathbf{r}_a(t)$ in an inertial frame, the center of mass of the system is located at $\mathbf{R}$:

$$M\mathbf{R} = \sum_a m_a \mathbf{r}_a(t), \qquad M = \sum_a m_a \tag{1.8}$$

so that

$$\sum_a m_a \dot{\mathbf{r}}_a = \sum_a \mathbf{p}_a = M\dot{\mathbf{R}} = \mathbf{P}. \tag{1.9}$$

Thus, $\mathbf{P}$ is the total momentum of the system. If the external forces are removed, the momentum $\mathbf{P}$ remains constant no matter what motions the individual particles perform under their mutual interactions. Notice in passing that the definition of total momentum in (1.9) holds whether or not the particles are interacting.

Next, we derive the *principle of angular momentum*. The *angular momentum* of particle a about the origin O of the coordinate system for $\mathbf{r}_a$ is defined by the vector product [3].

$$\mathbf{r}_a \times \mathbf{p}_a = \mathbf{L}_a. \tag{1.10}$$

[3] The reader unfamiliar with standard vector operations may consult D.E. Rutherford, *Vector Methods*, Eighth Edition, Oliver and Boyd, Ltd., Edinburgh and London, 1954.

The time rate of change of $\mathbf{L}_a$ is

$$\dot{\mathbf{L}}_a = (\dot{\mathbf{r}}_a \times \mathbf{p}_a) + (\mathbf{r}_a \times \dot{\mathbf{p}}_a). \tag{1.11}$$

The first vector product vanishes since $\dot{\mathbf{r}}_a$ and $\mathbf{p}_a$ are parallel. For the second, we insert the dynamical information on $\dot{\mathbf{p}}_a$ from (1.6) and get

$$\dot{\mathbf{L}}_a = \mathbf{N}_a + \mathbf{N}'_a \tag{1.12}$$

where

$$\mathbf{N}_a = (\mathbf{r}_a \times \mathbf{F}_a^{(e)}) \quad \text{and} \quad \mathbf{N}'_a = (\mathbf{r}_a \times \Sigma_{b\neq a}\mathbf{F}_{ab}) \tag{1.13}$$

are the force moments or *torques* about 0 of the external and internal forces acting on the ath particle. As in the case of linear momentum, we can define the total angular momentum, $\sum_a \mathbf{L}_a = \mathbf{L}$, the total external torque $\sum_a \mathbf{N}_a = \mathbf{N}$ and find from (1.12) that

$$\dot{\mathbf{L}} = \mathbf{N} \tag{1.14}$$

since $\sum_a \mathbf{N}'_a = 0$ if $\mathbf{F}_{ab}$ is directed along ab. This equation expresses the principle of angular momentum for the composite system.

The angular momentum and torque vectors $\mathbf{L}$ and $\mathbf{N}$ refer to the same fixed point in an inertial frame of reference Σ, and change in value if this point is changed. However, according to Galileo's principle of relativity, the principle of angular momentum should hold in any inertial frame. We express this analytically by saying that (1.14) is invariant in form under the Galilean transformation. To verify this, note that, by (1.3) and (1.4), the angular momentum $\mathbf{L} = \Sigma_a \mathbf{r}_a \times \mathbf{p}_a$ in Σ is related to the angular momentum $\mathbf{L}'$ in the moving frame Σ' through

$$\mathbf{L} = \mathbf{L}' + (\sum_a m_a \mathbf{r}'_a) \times \mathbf{V} + \mathbf{R} \times (\sum_a \mathbf{p}'_a) + \mathbf{R} \times m_a \mathbf{V}, \tag{1.15}$$

where $\mathbf{L}' = \sum_a \mathbf{r}' \times \mathbf{p}'_a$, and $\mathbf{R} = \mathbf{V}t$ is the distance between the origins of Σ and Σ' after a time t. Since $\mathbf{R}$ and $\mathbf{V}$ are parallel, the last term in (1.15) drops away. The time derivative of $\mathbf{L}'$ is simple to calculate:

$$\dot{\mathbf{L}}' = \dot{\mathbf{L}} - \mathbf{R} \times \sum_a \dot{\mathbf{p}}'_a = \sum_a (\mathbf{r}_a - \mathbf{R}) \times \mathbf{F}_a = \mathbf{N}', \tag{1.16}$$

where $\mathbf{N}' = \sum_a (\mathbf{r}_a - \mathbf{R}) \times \mathbf{F}_a = \sum_a \mathbf{r}'_a \times \mathbf{F}_a$ is the torque measured about the origin in Σ'. Thus, the *form* $\dot{\mathbf{L}} = \mathbf{N}$ is valid in any inertial frame, even though the *values* of $\mathbf{L}$ and $\mathbf{N}$ depend on the frame.

The principle of angular momentum is also valid relative to the *center of mass* of the moving system (this is *not* necessarily an inertial frame: the center of mass can accelerate). To prove this statement, simply reinterpret the coordinate $\mathbf{R}$ in (1.15) as the center of mass coordinate in Σ, thereby attaching the frame Σ' to the center of mass of the system. The primed quantities are then measured relative to the center of mass which moves with velocity $\mathbf{V}$. Now both $\sum_a m_a \mathbf{r}'_a$ and $\sum_a \mathbf{p}_a$ are zero. However, the last term in (1.15) survives since $\mathbf{R}$ and $\mathbf{V}$ are not necessarily parallel anymore. The time derivative of $\mathbf{L}'$ in the center of mass system becomes

$$\begin{aligned} \dot{\mathbf{L}}' &= \dot{\mathbf{L}} - \mathbf{R} \times \sum_a m_a \dot{\mathbf{V}} = \sum_a \mathbf{r} \times \mathbf{F}_a^{(e)} - \mathbf{R} \times \sum_a \dot{\mathbf{p}}_a \\ &= \sum_a \mathbf{r}'_a \times \mathbf{F}_a^{(e)} = \mathbf{N}'_a, \end{aligned} \tag{1.17}$$

using (1.7) in reverse for $\dot{\mathbf{P}} = \sum_a \dot{\mathbf{p}}_a$. The torque, or moment, $\mathbf{N}'$ of the external forces now refers to the center of mass. In applying the principle of angular momentum, we may take moments about a fixed point in space (in an inertial frame of course), or about the center of mass. The principle does not hold in a frame of reference performing an arbitrary motion in space.

If the external torque is removed, the total angular momentum $\mathbf{L}$ stays constant. While superficially similar to the statement of constant linear momentum $\mathbf{P}$ if $\mathbf{F} = 0$, the conservation of angular momentum has quite different consequences: If a system initially has zero total (linear) momentum, it stays zero throughout the subsequent motion. Internal forces cannot move the center of mass. If the angular momentum is initially zero, this is maintained throughout the subsequent motion. However, this does not say that the orientation of the system in space is maintained; the internal components can vary their relative orientation at will with the help of mutual interactions. A cat released from an inverted position makes use of internal muscular couplings between its posterior and anterior regions to drop feet first onto the floor! (See Prob. 1-2).

The principles of linear and angular momentum contain all the necessary information to study the motion of particle systems under given forces. We would only have to apply these principles to an endless variety of special situations in order to build up a complete scene of mechanics. To be quite specific, let us look at the motion of a single particle under a prescribed force $\mathbf{F}$. The equation of motion is given by (1.2). This single vector equation is short-hand for the three component equations

$$\dot{p}_x = F_x, \quad \dot{p}_y = F_y, \quad \dot{p}_z = F_z, \tag{1.18}$$

if we choose a rectangular reference frame O_{xyz} and identify the components of $\dot{\mathbf{p}}$ and $\mathbf{F}$ along the axes by subscripts. For m a constant, $\dot{p}_x = m\ddot{x}$, etc., where (x, y, z) are the coordinates of the particle at any time, and

$$m\ddot{x} = F_x, \quad m\ddot{y} = F_y, \quad m\ddot{z} = F_z. \tag{1.19}$$

These three second order differential equations in time may be integrated (numerically if need be) subject to known initial conditions of position and velocity to find the position $x(t)$, $y(t)$, $z(t)$ of the particle at all times. This process is called *solving for the motion*. Such *brute force* methods have their uses of course. However, we will show only passing interest in them since it turns out that often a great deal can be learned about the properties of a mechanical system *without* integrating the equations of motion directly. The latter point of view concerns itself in part with various invariance properties under coordinate transformations the system may possess and how these properties affect the dynamics. A formulation of mechanics that makes such invariance properties particularly transparent and moreover allows one to exploit them to advantage is to be found in the *principle of least action* [4] or *Hamilton's principle*.

We base the developments in these lectures on such a point of view. One of the many advantages is a formulation of dynamics that discusses with equal ease, and in a uniform fashion, the dynamics of such widely distinct systems as the motion of a planet, the oscillations of electrons in a solid, or the pattern of ripples around an insect swimming on the surface of a pond. Furthermore, the tendency in modern physics has been to assign the action principle a fundamental role in the dynamics of systems that have no classical counterpart. Consequently, the methods that have evolved by treating classical systems from this point of view are useful in many unrelated fields of research in theoretical physics.

[4] The name, principle of least action, is often restricted to refer to a principle enunciated by Maupertius in 1747, see Sec 1-8. However, we will use the phrase *generically* to refer to any of the several extremum principles that are available in mechanics.

1-4 The Action Principle

Newton's laws of motion are a differential formulation of mechanics. They determine the change in momentum produced by forces acting on a system for a time dt. Other formulations of mechanics are available that concern themselves with the motion of a system throughout a finite time interval from t_1 to t_2. Such formulations are called *integral principles* of mechanics. A particularly important principle of this type is the *principle of least action*, or Hamilton's principle. It simply states that a mechanical system moves between t_1 and t_2 in such a way that the integral

$$S = \int_{t_1}^{t_2} L dt \tag{1.20}$$

assumes a minimum [5] value. The function L in this integral is called the *Lagrange function* of the system, and depends on the coordinates

[5] More precisely, a stationary value S has a minimum value only if t_1 and t_2 are close together (see, for example, E.T. Whittaker, *A Treatise on the Analytical Dynamics of particles and rigid bodies*, p. 250, fourth edition, Cambridge University Press, London, digital printing 1999).

that describe the system as well as their time derivatives and on t. We will see later how L can be found when we derive this principle from Newton's laws.

The way the action principle has been stated above does not make clear how the value of S is to be judged or on what it depends. A simple example illustrates the basic idea. Consider the motion of a particle in one dimension. Its position $x = f(t)$ can be found at any time t by integrating Newton's second law of motion twice with respect to time. The value of the Lagrange function may thus be calculated as a function of t too, giving $L(x, \dot{x}) = L[f(t), \dot{f}(t)]$ from which we may calculate S. Obviously, S depends on the functional relation between x and t, given by $x = f(t)$ in this case. We say S is a *functional* of f and write

$$S = S[f] = \int_{t_1}^{t_2} L[f(t), \dot{f}(t)]dt.$$

Therefore S changes if the functional relation between x and t changes. If we replace the function $f(t)$ by *another function* $\bar{f}(t)$ that evolves differently with time, and set $x = \bar{f}(t)$, we get a different value for $L = L[\bar{f}(t), \dot{\bar{f}}(t)]$ and consequently another value for $S = S[\bar{f}]$. The principle of least action then asserts that [6] $S[f] < S[\bar{f}]$ if $x = f(t)$ describes the actual motion. The relation $x = \bar{f}(t)$, that could hold between the particle position and elapsed time if additional forces were present, but does not under the actual forces present, is termed a *virtual motion*. We may go from the actual to the virtual motion by making a *virtual displacement* in the coordinates. This means adding an *arbitrary function* of time, $\delta x(t)$ to the actual position $x = f(t)$, so that $\bar{f}(t) = f(t) + \delta x(t)$ or

$$\delta x(t) = \bar{f}(t) - f(t). \qquad (1.21)$$

[6] Recall that strictly speaking, it may be an extremum.

The functions $\delta x(t)$ are usefully regarded as *small*, i.e. $\bar{f}(t)$ and $f(t)$ are not to differ much in form [7]. In comparing the values of S in actual and virtual motions, we must be careful to let the virtual displacements $\delta x(t)$ vanish at both ends of the time interval under consideration,

$$\delta x(t_1) = \delta x(t_2) = 0, \qquad (1.22)$$

[7] More precisely, $\delta x(t)$ must be uniformly small, have a uniformly small first derivative, and satisfy (1.22).

that is, the actual and virtual motions must coincide at these two times. The necessity of such boundary conditions in time on the virtual displacements will be proven presently.

To get some feel for the action principle let us test it out on the following simple problem, before starting into a general proof. A particle falls from rest under gravity g. Its position is $x = f(t) = \frac{1}{2}gt^2$ after a time t has elapsed. This is the actual motion. We compare it to a virtual motion prescribed by

$$x = \bar{f}(t) = \frac{1}{2}gt_2^2(\frac{t}{t_2})^\beta, \quad \beta > 0,$$

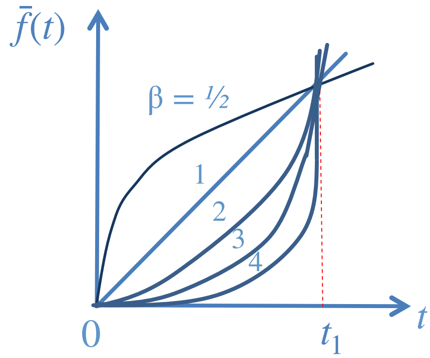

Figure 1.2: The family of curves $\bar{f}(t) = \frac{1}{2}gt_2^2(\frac{t}{t_2})^\beta$ for $\beta = \frac{1}{2}, 1, 2, 3, 4$.

as shown in Fig. 1.2.

The condition $\beta > 0$ ensures that $f(t)$ and $\bar{f}(t)$ agree at $t = t_1 = 0$ and $t = t_2$, so that $\delta x(t)$ vanishes at the endpoints of the motion. Since the Lagrange function for free fall under gravity is [see (1.34) and Prob. 1-3]

$$\begin{aligned} L &= \frac{1}{2}m\dot{x}^2 + mgx \\ &= m(\frac{1}{2}gt_2)^2[\frac{1}{2}\beta^2(\frac{t}{t_2})^{2\beta-2} + 2(\frac{t}{t_2})^\beta], \end{aligned}$$

we have

$$S[\bar{f}] = \int_0^{t_2} Ldt = \frac{1}{8}mt_2(gt_2)^2[\frac{\beta^2}{2\beta-1} + \frac{4}{\beta+1}].$$

For various values of β, the functions $\bar{f}(t)$ form a family of curves that intersect the path of the actual motion at $t = t_1 = 0$ and $t = t_2$. We try various forms of $\bar{f}$ by varying β. The resulting values of S are shown in Fig. 1.3. The minimum value of S at $\beta = 2$ confirms that $\bar{f} = \frac{1}{2}gt^2$ is indeed the actual motion. The other stationary point below $\beta = -1$ is excluded by the condition $\beta > 0$, i.e by the boundary conditions on $\delta x(t)$.

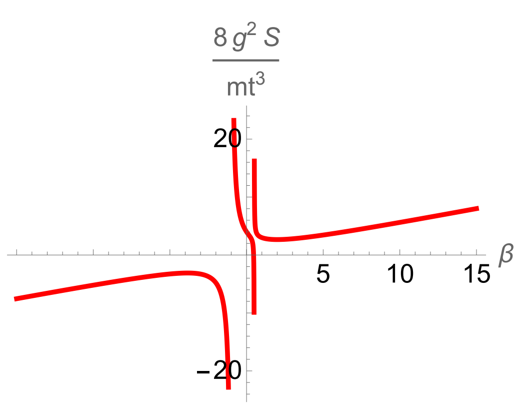

Figure 1.3: The function S (multiplied by the constant $8/mg^2t_2^3$) is shown as a function of the parameter β.

Now consider an arbitrary, interacting system of n particles, possibly also subject to external forces. We wish to derive the principle of least action for this system. To that end, introduce cartesian coordinates (x_1, x_2, x_3) for particle 1, (x_4, x_5, x_6) for particle 2, etc. and the force components (F_1, F_2, F_3), acting on particle 1, (F_4, F_5, F_6) acting on particle 2, etc. This notation is not as clumsy as it looks, since we may write Newton's law of motion as

$$F_k - m_k \ddot{x}_k = 0 \tag{1.23}$$

for each particle in the system by simply letting $k = 1, 2, \ldots, 3n$. The masses m_k are equal in triplets, $m_1 = m_2 = m_3$, $m_4 = m_5 = m_6$, etc., giving the masses of particles $1, 2, \ldots, 3n$ and so forth.

Now subject each coordinate $x_k(t)$ to a virtual displacement, $x_k(t) \to x_k(t) + \delta x_k(t)$. By doing so, we force the system to perform a virtual motion described by the functions

$$x_k(t) + \delta x_k(t), \quad k = 1, 2, \cdots, 3n. \tag{1.24}$$

Notice that the virtual displacements $\delta x_k(t)$ are *instantaneous*. They have nothing to do with the displacement $dx_k = \dot{x}_k dt$ occurring in time dt during the actual motion. Corresponding positions during the actual and virtual motions are tagged by the same value of t. However, the virtual displacements are certainly *functions* of time. Therefore, not only positions but also velocities change in a virtual displacement. The change in velocity $\delta \dot{x}$ is found from the difference

$$\delta \dot{x}_k = \frac{d}{dt}(x_k(t) + \delta x_k(t)) - \frac{d}{dt} x_k(t) = \frac{d}{dt}(\delta x_k). \tag{1.25}$$

Since $\delta \dot{x}_k = \delta(dx_k/dt)$, we see that the operations δ and d/dt actually *commute* for the virtual displacements we are considering. This fact will be important in what follows.

We now form the expression

$$\sum_k (F_k - m_k \ddot{x}_k) \delta x_k, \tag{1.26}$$

where the sum over k runs over all coordinates. This sum is actually *zero* by Newton's second law. Before using this fact let us perform a mathematical transformation on the acceleration terms by writing down the identity

$$\sum m_k \ddot{x}_k \delta x_k = \frac{d}{dt}\Big(\sum_k m_k \dot{x}_k \delta x_k\Big) - \sum_k m_k \dot{x}_k \frac{d}{dt}(\delta x_k).$$

We interchange d/dt and δ in the last term in accordance with our findings in (1.25) and get

$$\sum m_k \ddot{x}_k \delta x_k = \frac{d}{dt}\Big(\sum_k m_k \dot{x}_k \delta x_k\Big) - \delta\Big(\sum_k \frac{1}{2} m_k \dot{x}_k^2\Big).$$

The sum

$$T = \sum_k \frac{1}{2} m_k \dot{x}_k^2 \tag{1.27}$$

is the total *kinetic energy* of the system and $\delta(\sum_k \frac{1}{2} m_k \dot{x}_k^2) = \sum_k m_k \dot{x}_k \delta \dot{x}_k$ its change, or *variation* in a virtual displacement. Our expression (1.26) thus reads

$$\sum_k F_k \delta x_k + \delta T - \frac{d}{dt}(\sum_k m_k \dot{x}_k \delta x_k) = 0 \tag{1.28}$$

if we use Newton's laws. We may also identify the first sum in this equation. The expression

$$\sum_k F_k \delta x_k = \delta W \tag{1.29}$$

is just the work done by the forces F_k in the virtual displacement. It is called the *virtual work*. Notice that any forces which do no work drop out at this stage. Such workless forces are present when a system is made to move in a definite way by *constraints* on its motion. We postpone discussion of such forces of constraint, except to note that they do not contribute to the virtual work δW. Forces which do contribute to δW are termed *applied forces*.

Returning to (1.28) we now integrate over time from t_1 to t_2. Then

$$\int_{t_1}^{t_2} \delta W \, dt + \int_{t_1}^{t_2} \delta T \, dt - \left[\sum_k m_k \dot{x}_k \delta x_k \right]_{t_1}^{t_2} = 0,$$

and if we only employ virtual displacements that vanish at both ends of the time interval,

$$\delta x_k(t_1) = \delta x_k(t_2) = 0, \quad \text{for all} \quad k, \tag{1.30}$$

then

$$\int_{t_1}^{t_2} \delta W \, dt + \int_{t_1}^{t_2} \delta T \, dt = 0 \tag{1.31}$$

is seen to be a consequence of Newton's laws of motion plus the boundary condition on the $\delta x_k(t)$.

We can transform this result somewhat further by observing that, since time is not varied in calculating δT, we can move δ outside the integral sign and write

$$\int_{t_1}^{t_2} \delta T \, dt = \delta \int_{t_1}^{t_2} T \, dt.$$

Similarly, we might be tempted to write

$$\int_{t_1}^{t_2} \delta W \, dt = \delta \int_{t_1}^{t_2} W \, dt.$$

However, this procedure is *ambiguous* since only δW is well-defined and not the total work W. In general W depends on the path followed by the system between t_1 and t_2. Only for systems in which the work is independent of the path can we use this transformation. Such systems are called *conservative*, and possess *potential energy* V equal to the negative of the work done in moving from t_1 to t_2. They occur sufficiently often in physics to merit rewriting (1.31) specially for them as

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0, \tag{1.32}$$

since

$$\int_{t_1}^{t_2} \delta W\, dt = -\int_{t_1}^{t_2} \delta V\, dt = -\delta \int_{t_1}^{t_2} V\, dt.$$

The structure of (1.32) suggests that the function

$$L = T - V \tag{1.33}$$

plays a fundamental role in dynamics. It does. This is the Lagrange function introduced at the beginning of this section. The condition (1.32)

$$\delta \int_{t_1}^{t_2} L\, dt = 0, \quad \text{where} \quad L = T - V \tag{1.34}$$

asserts that the integral

$$S = \int_{t_1}^{t_2} L\, dt$$

is stationary for the actual motion of the system, subject to the boundary conditions (1.30). The integral S itself is called the *action*. The conditions (1.34), or $\delta S = 0$ for short, emphasize that not S itself but its *stationary character* is important, i.e. the value of S is unchanged to first order if the coordinates $x_k(t)$ describing the actual motion are replaced by coordinates $x_k(t) + \delta x_k(t)$.

The derivation we have presented gives the impression that the variational principle (1.34) only holds when $L = T - V$. This is not so. Equation (1.34) is true for the motion of any system for which a Lagrange function can be constructed, even if this is not equal to $T - V$. For completeness, we also make the obvious remark that the more general principle (1.31) is available whether a Lagrange function can be constructed or not.

The condition $\delta S = 0$ is called a *variational principle* for determining the functions $x_k(t)$. This principle is a compact formulation of Newtonian dynamics that emphasizes a remarkable fact: a *single* scalar function L completely determines the dynamics of a system. In fact, (1.34) embodies an entirely new point of view of mechanics. A. Sommerfeld summarizes its essence on p. 209 of his beautiful treatise on Mechanics[8] *find the Lagrange function!* We here re-emphasize Sommerfeld's admonition by pointing out that (1.34) indicates that L is indeed the key that unlocks the door to the dynamics of classical (and presumably also

[8] A. Sommerfeld, *Lectures on Theoretical Physics, Vol. I, Mechanics*, Academic Press Inc., New York, 1952.

quantum) systems. How unique is L? The variational principle answers this question immediately. For example, the condition $\delta S = 0$ is unaffected by (i) adding a constant to L, (ii) multiplying L by a constant, (iii) adding the total time derivative of any function $f(x_1, x_2, x_3, \ldots, t)$ of the coordinates and time to L. Changes (i) and (ii) merely affect the zero of energy and units used in a particular problem. To appreciate (iii) we write $L' = L + df/dt$ and find

$$\delta \int_{t_1}^{t_2} L'\, dt = \delta \int_{t_1}^{t_2} L\, dt + [\delta f]_{t_1}^{t_2},$$

where δf is the variation of f in a virtual displacement. This must vanish at t_1 and t_2 by the boundary conditions on $\delta x_k(t)$, so L' and L satisfy the same variational principle.

1-5 Lagrange's Equations

In the previous section, we showed that the variational principle $\delta S = 0$ is a consequence of Newton's equations of motion. Now we turn the question around and ask what equations of motion are implied by this principle. At first sight, this seems a silly thing to do because we should surely just get back to the equations of motion we started out from! In a sense this is true and in another sense it is not. The point is that, while we derived the condition $\delta S = 0$ from Newton's laws of motion expressed in cartesian coordinates, the result must be true in *any* set of coordinates, since S only depends on the scalar function L and hence is a scalar itself.

The advantage of starting out from the variational principle is then that we may pick any system of coordinates that we fancy. Let $\{q_1, q_2, \ldots, q_n\}$ or q_k for short, be such a set of n *generalized* coordinates. We may pass to them via *point transformations* of the form

$$x_k = f_k(q_1, q_2, \ldots, q_n, t) \tag{1.35}$$

for each x_k. The time derivatives of the x_k transform like $\dot{x}_k = \dot{f}_k$, or

$$\dot{x}_k = \sum_l \frac{\partial f_k}{\partial q_l} \dot{q}_l + \frac{\partial f_k}{\partial t}. \tag{1.36}$$

The last factor, $\partial f_k / \partial t$, is only present if the transformation is time dependent.

How does the Lagrange function respond to such a transformation? Since L is a scalar, its value is unchanged of course. However, its *functional* dependence on the coordinates and their time derivatives changes. For example, the Lagrange function given in (1.33) reads

$$L = \sum_k \frac{1}{2} m_k \dot{x}_k^2 - V(x_1, x_2, \ldots) \tag{1.37}$$

in cartesian coordinates, and changes into

$$L = \sum_k \frac{1}{2} m_k \dot{f}_k^2 - V(f_1, f_2, \dots)$$

under the tranformations (1.35) and (1.36). Thus, the potential energy becomes a function of the q_k, the kinetic energy (through $\dot{f}_k$) a function of the $\dot{q}_k$ *and* the q_k. If the transformation is time-dependent, both T and V become a function of time also. We write

$$L = L(q_1, q_2, \dots; \dot{q}_1, \dot{q}_2, \dots, t), \quad \text{or} \quad L(q, \dot{q}, t)$$

for short, thereby completing our definition of the variational principle in (1.34): L is to be considered a function of q, $\dot{q}$ and t. The time derivatives $\dot{q}_k$ are called the *generalized velocities*.

The actual motion of the system is given by knowing the coordinates q_k as functions of the time: $q_k = q_k(t)$. It is the functions $q_k(t)$ that we now *seek*. The stationary character of S tells us how to find them. For if we subject the $q_k(t)$ to virtual displacements the value of S should not change to first order in these displacements. Changing $q_k(t)$ to $q_k(t) + \delta q_k(t)$ induces a corresponding change in $\dot{q}_k(t)$ to $\dot{q}_k(t) + \delta \dot{q}_k(t)$, so that L varies by an amount

$$\begin{aligned} \delta L &= L(q + \delta q, \dot{q} + \delta \dot{q}, t) - L(q, \dot{q}, t) \\ &= \sum_k \frac{\partial L}{\partial q_k} \delta q_l + \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k. \end{aligned}$$

Consequently S changes by

$$\int_{t_1}^{t_2} \delta L \, dt = \int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right] dt.$$

This the final form for the condition $\delta S = 0$. We can meet this condition *separately* for each q_k, *provided that these coordinates are independent.* For if this is the case, we can choose all virtual displacements to be zero barring the kth one, $\delta q_k(t)$. For it, we choose a function of time that not only vanishes at t_1 and t_2 but everywhere inbetween, except in a small region Δt around some time t. The sum drops away in the above equation, which now specializes to

$$\int_t^{t+\Delta t} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k(t) \, dt = 0. \tag{1.38}$$

Since $\delta q_k(t)$ can be an arbitrary function of time in t, we must conclude that

$$\left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] = 0. \tag{1.39}$$

By choosing each δq_k to be non-vanishing in turn, we find this equation holds for each independent coordinate. Thus, the action S is rendered stationary if each $q_k(t)$ satisfies (1.39). These are the celebrated

Lagrange equations of motion (1788) for systems possessing a Lagrange function.

The extension to systems not possessing a Lagrange function is straightforward. We merely adopt the general form of the variational principle given in (1.31). We have already noted that the kinetic energy T becomes a function of q and $\dot{q}$ in generalized coordinates. Therefore, the contribution from δT to its time integral is

$$\int_{t_1}^{t_2} \delta T \, dt = \int_{t_1}^{t_2} \sum_k \left[\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\right] \delta q_k(t) \, dt, \tag{1.40}$$

using the properties and boundary conditions of the $\delta q_k(t)$ once more. We have no information on δW in terms of the virtual displacements δq_k, so we simply write

$$\delta W = \sum_k Q_k \delta q_k, \tag{1.41}$$

thereby *defining* new force components Q_k. The Q_k are called *generalized forces*. We add the time integral of δW to (1.40) and set

$$\int_{t_1}^{t_2} \sum_k \left[Q_k + \frac{\partial T}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\right] \delta q_k(t) dt = 0 \tag{1.42}$$

to find the actual motion.

As before, we can meet this condition separately for each q_k, provided they are independent, by putting

$$Q_k + \frac{\partial T}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = 0 \tag{1.43}$$

for all k. These are often called *generalized Lagrange equations*.

The forces Q_k introduced above can be determined by inspection. We write δW in its alternative forms

$$\sum_k F_k \, \delta x_k \quad \text{and} \quad \sum_k Q_k \delta q_k$$

and eliminate the virtual displacements δx_k in favor of the δq_k. The transformation is

$$\delta x_k = \sum_l \frac{\partial f_k}{\partial q_l} \delta q_l$$

from (1.44) (notice there is no term $(\partial f_k / \partial t)\delta t$; the time does *not* vary!). The two expressions for δW are identical if

$$Q_k = \sum_l F_l \frac{\partial f_l}{\partial q_k}. \tag{1.44}$$

Finally, if some of the forces contributing to δW are derivable from a potential V,

$$\delta W = \sum_k Q_k \delta q_k - \delta V. \tag{1.45}$$

The variational principle (1.42) is modified accordingly to read

$$\int_{t_1}^{t_2} \sum_k [Q_k + \frac{\partial L}{\partial q_k} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k})] \delta q_k(t) dt = 0,$$

implying that

$$Q_k + \frac{\partial L}{\partial q_k} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k}) = 0 \tag{1.46}$$

for all k. Equation (1.46) is a *hybrid* Lagrange equation, where some of the forces have been included through V in a Lagrange function $L = T - V$. On comparing their contribution to δW in (1.42), we have

$$\sum_k Q'_k \delta q_k = -\delta V = -\sum_k \frac{\partial V}{\partial q_k} \delta q_k$$

or that

$$Q'_k = -\frac{\partial V}{\partial q_k}. \tag{1.47}$$

A word on the dimensions of the q_k and the Q_k that have been introduced: The q_k need not have dimensions of length, but may individually assume any convenient dimension. Hence Q_k does not always have the dimensions of force. In fact, its dimension is determined by the q_k it associates with, since each term $Q_k \delta q_k$ in the sum (1.41) must have the dimensions of work, that is, energy.

The generality of the results found in this section cannot be overemphasized. The Lagrange equations we derived are true for *any* set of generalized coordinates. There is one very important proviso: the q_k must all be independent of each other. Otherwise we cannot make the essential step from the variational principle to the Lagrange equations it implies for each coordinate. How then should the q_k be chosen; how many do we need? These questions are closely related to the number of coordinates a system requires and the degrees of freedom that it has. This relationship is discussed in the next section.

1-6 Constraints

A system has as many *degrees of freedom* as there are independent coordinates required to specify its instantaneous state completely. For example, a system of N particles moving arbitrarily in space has $3N$ degrees of freedom, since three cartesian coordinates are required to position each particle in space. However, if a system moves subject to constraints, the number of degrees of freedom is reduced. A *constraint* is simply some additional condition that is imposed on the way a system shall move. A wheel rolling without slipping, a ride on a roller coaster, or simply that N atoms move together as a rigid body, are examples of motions performed under constraints. No slipping means the point on the rim of the wheel in contact with the surface is momentarily at rest;

the roller coaster forces the rider to follow a particular path in space; the relative separation between atoms must stay constant during the motion of a rigid body. Forces are needed to enforce these constraints. Friction causes the wheel not to slip, guide rails (exerting reactions) define the path of the roller coaster, while electrical forces between atoms give the rigid body its rigidity. Such forces are called *forces of constraint*. As the above examples show, we may always replace them by their effects on the motion of a system in the form of constraints. One should realize, however, that considering the constraints instead of the forces that enforce them is an idealization of Nature. No rigid body is perfectly rigid. The electrical forces between atoms can never exactly "freeze" them into permanent positions in a solid. When a solid is subjected to stresses, the relative positions of its constituent atoms *do* change (as when a sound wave passes through it). If such changes are unimportant for our purposes, we idealize the situation by considering the solid to be *rigid*. In this way we circumvent the enormous, but for us irrelevant problem of the internal motion of the constituent atoms making up a rigid body.

Forces of constraint have one very important property in common: they do no work in a virtual displacement that is consistent with the constraints on the system. A general proof of this assertion is difficult[9], but our three examples of constrained motion illustrate the workless nature of these forces: the wheel does not "scratch" the surface it is rolling on, the guide rail reaction is perpendicular to the direction of motion of the roller coaster, the internal forces do no work when a rigid body is displaced as a whole. In this connection the role of frictional forces should be noted with care. *Static* friction, as for the rolling wheel, is a force of constraint; it makes rolling possible. However, *sliding* or *kinetic* friction comes into play if the wheel slips on the surface. The "rolling" constraint falls away and friction becomes an applied force. It does work during a virtual displacement of the system and thus contributes to δW.

[9] Lagrange attempts one in his *Mechanique Analytique, Tome premier, Quatrieme edition, Librairie Scientifique et Technique, A. Blanchard, Paris, 1965.*

Quite generally, then, the distinguishing feature about forces of constraint is that they have to drop out of the variational principle because they do no work. We do not need to know them to calculate δW. However, their effects must show up somewhere, and they do, in that the number of independent coordinates are reduced through the constraints imposed by these forces. Thus, the $3N$ cartesian coordinates x_k for an N-particle system are not all independent if the system has constraints on its motion: r constraint conditions leave it with only $3N - r$ degrees of freedom and as many independent coordinates. The Lagrange equations of the previous section *will therefore not hold for all the* x_k unless the constraining forces are explicitly included among all the forces. There is a simple way around this difficulty: express the x_k in terms of a smaller number of coordinates that *are* independent by introducing $3N$ transformations

$$x_k = f_k(\underbrace{q_1, q_2, \ldots}_{3N-r \text{ coordinates}}, t) \tag{1.48}$$

to new coordinates q_k. We might well ask: can these transformations to a smaller number of independent coordinates always be constructed? The answer depends on the type of constraint involved. If the constraints are conditions among the coordinates themselves that do not involve their time derivatives, such transformations are always possible. Constraints of this type are termed *holonomic* (from the Greek word *holos*, or whole) and can be displayed in the general form[10]

$$F_\alpha(x_1, x_2, \cdots, t) = 0 \tag{1.49}$$

where the label $\alpha = 1, 2, \ldots, r$ distinguishes between r different functions F. We are thus provided with r relations between $3N$ coordinates x_k, so only $3N - r$ of them are independent. If we now choose the $3N$ functions $f_k(q_1, q_2, \ldots, t)$ in (1.48) such that the r relations (1.49) hold as *identities*, i.e. $F(f_1, f_2, \ldots, t) = 0$, the reduction of the number of coordinates in the problem is complete. The Lagrange function can be expressed in terms of the q_k and the $\dot{q}_k$, and satisfies

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

in these coordinates. We have succeeded in eliminating the constraints entirely by means of a point transformation.

[10] In the literature these constraints are further classified as *rheonomous* (fluid) or *scleronomous* (fixed) depending on whether they contain the time or not.

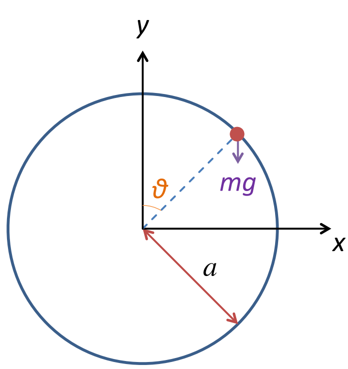

Figure 1.4: A particle of mass m constrained to move in a vertical circular path in the gravitational field of the earth.

A simple example illustrates these ideas, see Fig. 1.4. A particle of mass m moves in a circle in a vertical plane. Find its equations of motion if gravity is the only applied force. The motion has one degree of freedom, since the cartesian coordinates (x, y) of the particle must satisfy

$$F(x, y) = x^2 + y^2 - a^2 = 0$$

choosing the center of the circle of radius a as the origin. Taking the y axis vertical, the potential energy of the particle in any position is gy per unit mass, so the Lagrange function is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy.$$

The Lagrange equations

$$\frac{\partial L}{\partial x} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = -m\ddot{x} = 0;$$
$$\frac{\partial L}{\partial y} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) = -mg - m\ddot{y} = 0$$

are *incorrect*; x and y are not independent. However, if we first eliminate x and $\dot{x}$ by the relations

$$x = \sqrt{a^2 - y^2}; \quad \dot{x} = -\frac{y\dot{y}}{\sqrt{a^2 - y^2}} \tag{1.50}$$

following from the constraint condition, then

$$L = \frac{1}{2}m\frac{a^2}{a^2 - y^2}\dot{y}^2 - mgy$$

and

$$\frac{\partial L}{\partial y} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) = 0,$$

that is

$$-g(1 - \frac{y^2}{a^2}) - \frac{y\dot{y}^2}{a^2 - y^2} - \ddot{y} = 0 \tag{1.51}$$

gives the correct equation of motion for y. The elimination of x and $\dot{x}$ amounts to choosing a new variable $q = y$ in (1.48), where

$$x = f_1(y) = \sqrt{a^2 - y^2}; \quad y = f_2(y) = y$$

so that f_1 and f_2 automatically satisfy the constraint condition $F(x, y) = 0$.

We have purposely stuck to the rather clumsy choice of $q = y$ as our independent variable to emphasize that it is not necessary to introduce the "obvious" angular variable θ giving the angular position of the particle from the vertical. But it is much more convenient to do so. The choice $q = \theta$, where

$$x = f_1(\theta) = a\sin\theta; \quad y = f_2(\theta) = a\cos\theta$$

gives

$$L = \frac{1}{2}m(a\dot{\theta})^2 - mga\cos\theta,$$

from which the much simpler equation of motion in θ,

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = ma(g\sin\theta - a\ddot{\theta}) = 0$$

follows.

Constraints that do not fall into the above category are termed *non-holonomic*. Their elimination depends specifically on the form they take. Non-holonomic constraints of special interest in mechanics are of the differential type

$$\sum_k F_{\alpha k} dx_k + G_\alpha dt = 0, \quad \alpha = 1, 2, \ldots, r, \tag{1.52}$$

where the $3n \times r$ functions $F_{\alpha k}$ and the r functions G_α can only depend on the x_k and the time. The trouble with such constraints is that we cannot eliminate them from the problem as we could holonomic constraints. Introducing new coordinates q_k by point transformations like (1.35) or (1.48) merely turns (1.52) into similar differential constraints on the q_k:

$$\sum_k F_{\alpha k} dx_k + G_\alpha dt = \sum_k F'_{\alpha k} dx_k + G'_\alpha dt, \tag{1.53}$$

where the $F_{\alpha k}$ and G_α in the second expression are new functions

$$F'_{\alpha k} = \sum_l F_{\alpha l} \frac{\partial f_l}{\partial q_k}; \quad G'_\alpha = G_\alpha + \sum_k F_{\alpha k} \frac{\partial f_k}{\partial t}$$

that depend on the q_k and the time. To see what constraints among the coordinates themselves are implied by such differential relations, we would first have to integrate the latter with respect to time. But because differentials of the coordinates as well as the coordinates themselves appear, we cannot do this until the coordinates are known functions of time, i.e. until the dynamical problem has been solved.

Lagrange introduced a method for dealing with non-holonomic constraints like (1.52) or (1.53). The method depends on the stipulation that in using the variational principle we must only allow virtual displacements that are consistent with the constraints on the system. Suppose that r conditions of the type (1.53) constrain the motion of a system that is described by n generalized coordinates q_k. Since the time is not varied during a virtual displacement, this means that the displacements in the q_k must obey (1.53) in the form (we drop the primes on the $F'_{\alpha k}$ and G'_α from now on)

$$\sum_k F_{\alpha k} \delta q_k = 0; \quad \delta t = 0. \tag{1.54}$$

The δq_k are not independent any more; equations (1.48) provide r subsidiary conditions they must satisfy. This is where the effects of constraints come in. We can no longer regard the δq_k as independent, and the form of Lagrange equations derived in Sec. 1-5 will not follow from the action principle. The method of *Lagrange multipliers* circumvents this difficulty: multiply the r relations (1.54) by arbitrary functions λ_α (one for each relation) and add. The result is still zero:

$$\sum_\alpha \lambda_a F_{\alpha k} \delta q_k = 0. \tag{1.55}$$

We add this sum to the integrand of (1.47), which then reads

$$\int_{t_1}^{t_2} \sum_k \left[\sum_\alpha \lambda_\alpha F_{\alpha k} + \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right] \delta q_k \, dt = 0. \qquad (1.56)$$

The r relations in (1.54) leave us only $n - r$ independent virtual displacements to choose at will. Call these δq_k with $k = 1, 2, \ldots, n - r$. The remaining virtual displacements δq_k for $k = n - r + 1, n - r + 2, \ldots, n$ are dependent on the values of the first $n - r$ displacements. We eliminate the dependent displacements from consideration in (1.56) by forcing their cofactors to be zero, that is, by choosing

$$\sum_\alpha \lambda_\alpha F_{\alpha k} + \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

for $k = n - r + 1, n - r + 2, \ldots, n$. These r equations are to be regarded as conditions on the functions λ_α. If these conditions are met, the sum in (1.56) will only contain terms involving the independent virtual displacements δq_k for $k \leq n - r$, and may be written as

$$\int_{t_1}^{t_2} \sum_{k=1}^{n-r} \left[\sum_\alpha \lambda_\alpha F_{\alpha k} + \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right] \delta q_k(t) \, dt = 0. \qquad (1.57)$$

This condition is satisfied by letting each q_k satisfy

$$\sum_\alpha \lambda_\alpha F_{\alpha k} + \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \qquad (1.58)$$

for $k = 1, 2, \ldots, n - r$. Both sets of equations (1.56) and (1.58) are identical in form but very different in origin. Together they provide n equations for the n unknown coordinates q_k and r unknown functions λ_α. Therefore, we need further r equations to make the problem determinate. These are supplied by (1.53). Notice that we must employ the constraint conditions as they unfold with time, since we are interested in the actual motion. The virtual displacements have served their purpose; they are out of the picture now. Equations (1.58) with $k = 1, 2, \ldots, n$ are the Lagrange equations in the presence of non-holonomic constraints (1.53). The same procedure adds the sum $\sum_\alpha \lambda_\alpha F_{\alpha k}$ to the two other versions of the Lagrange equations that we derived in Sec. 1-5.

The functions λ_α are called undetermined or Lagrange multipliers. They are introduced as a device to take care of the constraints on the system while the $n - r$ virtual displacements in (1.57) are allowed to vary independently. What is their physical significance? Referring to (1.58), we see that the motion is unchanged if the constraints are removed, but additional forces

$$Q_k'' = \sum_\alpha \lambda_\alpha F_{\alpha k} \qquad (1.59)$$

are introduced. The forces Q_k'' are not included in the Lagrange function; they must therefore represent the forces of constraint that enforce the constraints imposed by (1.53). If so, the Q_k'' should do no work in a virtual displacement. They don't. Their contribution to the virtual work is

$$\sum_k Q_k''\delta q_k = \sum_{\alpha k} \lambda_\alpha F_{\alpha k}\delta q_k$$

which is zero by (1.55). However, *time-dependent* constraints do work on the system in time dt during the actual motion. A proof of this statement and its physical origin waits in the next section.

One remark remains to be made: holonomic constraints are a special kind of constraint. For if we calculate the change of the holonomic constraints (1.49) with time, there results

$$dF_\alpha = \sum_l \frac{\partial F_\alpha}{\partial x_k} dx_k + \frac{\partial F_\alpha}{\partial t} dt = 0 \tag{1.60}$$

which is a special case of (1.52) with $F_{\alpha k} = \frac{\partial F_\alpha}{\partial x_k}$ and $G = \frac{\partial F_\alpha}{\partial t}$. Thus, holonomic constraints can also be treated with Lagrange multipliers; we do not *have to* eliminate them. In fact, if the forces of constraint are sought we *must not* eliminate them. Once the functions λ_α are determined, these forces are given by

$$Q_k'' = \sum_\alpha \lambda_\alpha \frac{\partial F_\alpha}{\partial x_k}. \tag{1.61}$$

The example of a particle moving in a circle under gravity that we discussed is a case in point. If we want to calculate the forces of constraint necessary to maintain the circular path we must keep both coordinates x and y in the Lagrange function

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

but restrict the virtual displacements in x and y. For circular motion this restriction is

$$\lambda(x\delta x + y\delta y) = 0$$

after multiplying by the single Lagrange multiplier λ required to enforce it. The two modified Lagrange equations

$$\lambda x + \frac{\partial L}{\partial x} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = \lambda x - m\ddot{x} = 0 \tag{1.62}$$

$$\lambda y + \frac{\partial L}{\partial y} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) = \lambda y - mg - m\ddot{y} = 0 \tag{1.63}$$

plus the constraint condition

$$x\dot{x} + y\dot{y} = 0 \tag{1.64}$$

are three equations that determine x, y and λ according to the general theory of this section. Multiply the Lagrange equations by x and y respectively and add to determine λ. We get

$$\lambda a^2 = mgy + m(x\ddot{x} + y\ddot{y}) = mgy - m(\dot{x}^2 + \dot{y}^2) \tag{1.65}$$

after using the condition $x^2 + y^2 = a^2$ and its second time derivative to obtain the result on the right. Since $v^2 = \dot{x}^2 + \dot{y}^2$ is just the velocity squared of the particle, λ is finally given by

$$\lambda a = mg\frac{y}{a} - \frac{mv^2}{a} = R.$$

Now (1.59) identifies the components of the constraint force along x and y as

$$Q_x'' = \lambda x = (mg\frac{y}{a} - \frac{mv^2}{a})\frac{x}{a} = R\frac{x}{a} \tag{1.66}$$

$$Q_y'' = \lambda y = (mg\frac{y}{a} - \frac{mv^2}{a})\frac{y}{a} = R\frac{y}{a} \tag{1.67}$$

of magnitude $R = \lambda a$. Since Q_x'' and Q_y'' are in the same ratio as x to y the direction of R is along the radius vector to the particle from the center of the circle. These relations are also shown in Fig. 1.5.

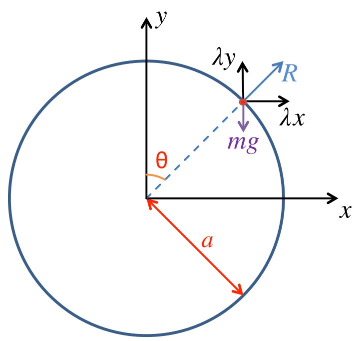

Figure 1.5: The particle of mass m constrained to a vertical circular path and acted on by gravity.

To find the motion of the particle, we either proceed as before and eliminate the holonomic constraint, or we can eliminate λ directly from the equation of motion in x and y: Taking the latter course of action, we multiply (1.62) by x, (1.63) by y, subtract the former equation from the latter and express the derivatives as a total derivative. The result

$$\frac{d}{dt}[m(x\dot{y} - \dot{x}y)] = -mgx$$

is equivalent to the principle of angular momentum (1.14) applied to the present problem. Now eliminating x and $\dot{x}$ by using the constraint equations (1.50) gets us back to the equation of motion

$$\ddot{y} + \frac{y\dot{y}^2}{a^2 - y^2} + g(1 - \frac{y^2}{a^2}) = 0$$

found previously for y and given in (1.51).

But working in x and y coordinates is just a difficult way to solve a simple problem. We can always go to any other coordinates that are more convenient. So, if the polar coordinates (r, θ) are introduced instead, where

$$x = f_1(r, \theta) = r \sin\theta, \quad y = f_2(r, \theta) = r \cos\theta$$

things become much simpler. The Lagrange function transforms into

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

and the constraint condition into

$$r - a = 0$$

by direct substitution. Again, we must not eliminate the holonomic constraint $r =$ from L if the force of constraint is desired. Instead, we introduce a multiplier λ' to take care of the constraint on the variation of r,

$$\lambda' \delta r = 0.$$

Then,

$$\begin{aligned}
\lambda' + \frac{\partial L}{\partial r} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) &= \lambda' - mg\cos\theta + mr\dot{\theta}^2 - m\ddot{r} = 0 && (1.68)\\
\frac{\partial L}{\partial \theta} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) &= mgr\sin\theta - \frac{d}{dt}(mr^2\dot{\theta}) = 0 && (1.69)\\
\dot{r} &= 0 && (1.70)
\end{aligned}$$

are the equations determining r, θ and λ'. But since $\dot{r} = 0$, we get $\ddot{r} = 0$ by differentiation and $r = a$ by integration. The equation in r, (1.68), gives

$$\lambda' = mg\cos\theta - ma\dot{\theta}^2$$

which, together with (1.59), identifies the constraint force for the rth coordinate (i.e. in the radial direction) as

$$Q''_r = R = mg\cos\theta - ma\dot{\theta}^2. \quad (1.71)$$

Since $a\dot{\theta} = v$ and $y = a\cos\theta$ this is the same result as obtained previously, cf. (1.66) and (1.67). The equation for θ does not depend on R (or λ) and is identical with the equation $ma\ddot{\theta} = mg\sin\theta$ found before.

Notice that the physical origin of R does not have to be specified in the problem we have just discussed, only its effect.

1-7 *Properties of L*

We return to systems that have a Lagrange function and introduce the quantity

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \quad (1.72)$$

into the Lagrange equations (1.39). These equations then become

$$\dot{p}_k = \frac{\partial L}{\partial q_k}. \tag{1.73}$$

The p_k are called *canonical momenta* and the pair (q_k, p_k) are referred to as canonical variables[11].

[11] The reason for this terminology will become clear in Chap. 7.

The version (1.73) of the Lagrange equations is reminiscent of Newton's form of the equations of motion, to which it reduces in cartesian coordinates. In that case

$$\frac{\partial L}{\partial x_k} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}_k}) = -\frac{\partial V}{\partial x_k} - m_k\ddot{x}_k = F_k - m_k\ddot{x}_k = 0,$$

using the Lagrange function given by (1.37) and the fact that $\sum F_k\delta x_k = -\delta V$ is the work done in a virtual displacement.

Equations (1.72) and (1.73) have further important consequences that relate to the question of conservation laws. We explore these next. As a system moves, the coordinates q_k and velocities $\dot{q}_k$ change with time. However, it often happens that one or more functions of these quantities stay constant in time during motion. Such functions are said to be conserved, or to obey conservation laws. It is of utmost importance to discover such conservation laws because they can often provide a complete picture of how a system moves. A powerful and systematic way of doing so involves studying the invariance properties of the Lagrange function under coordinate and time translations. For suppose L does not depend on a particular coordinate q_s. Then, changing q_s infinitesimally to $q_s + \Delta q_s$ cannot induce any change in L, so that

$$L(q_s + \Delta q_s, \dot{q}_s, t) - L(q_s, \dot{q}_s, t) = \frac{\partial L}{\partial q_s}\Delta q_s = 0,$$

indicating that

$$\frac{\partial L}{\partial q_s} = 0. \tag{1.74}$$

Therefore, the Lagrange equation for the coordinate q_s reads

$$\dot{p}_s = 0, \quad p_s = \frac{\partial L}{\partial \dot{q}_s} = \text{constant}. \tag{1.75}$$

The canonical momentum associated with a missing coordinate q_s is conserved. Observe carefully that this statement *is only true if q_s is an independent coordinate*, i.e. if all constraints have been eliminated from L. Otherwise, constraint forces enter (1.75) and spoil the conservation of p_s. The canonical momentum that is conserved may either be a linear or an angular momentum component, or some more complicated entity. The dimension of p_s depends on the dimension of q_s. Most commonly q_s is either a distance or an angle. Then, p_s is either a linear or an angular momentum. Notice in passing that the replacement $q_s \to q_s + \Delta q_s$ to probe L for its conserved momenta is *not* a virtual displacement; the velocity $\dot{q}_s$ is not allowed to change during this probe. Coordinates like q_s that do not appear in L are called *cyclic* or *ignorable*.

Another important conservation law is connected with the time-dependence of L. During the motion of a system from time t to $t + dt$ the value of L varies on two accounts: the change wrought by changes in q_k and $\dot{q}_k$, and the change due to an explicit dependence on time. In symbols,

$$dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt,$$

where the last term contributes only if L depends explicitly on time. This equation is a mathematical statement for dL. We now put in the dynamics by inserting the value of $\partial L/\partial q_k$ from the Lagrange equation (1.73), and find the relation

$$d(\sum_k p_k \dot{q}_k) = dL - dt \frac{\partial L}{\partial t} \tag{1.76}$$

for the total change dL in time dt. Equivalently,

$$dH = -dt \frac{\partial L}{\partial t}, \tag{1.77}$$

where H is the function

$$H = \sum_k p_k \dot{q}_k - L. \tag{1.78}$$

If $\partial L/\partial t = 0$, i.e. if L does not contain the time explicitly, (1.77) shows that H is conserved. In that case H is called Jacobi's first integral of motion (later on H, whether conserved or not, will have another, more illustrious name).

The two relations, (1.77) and (1.78), have a familiar physical content if: (i) L is given by $T - V$ and does not depend explicitly on time, (ii) the kinetic energy T is a *homogeneous* quadratic function of the $\dot{q}$, and (iii) the potential V is only a function of the q. Then

$$\sum_k p_k \dot{q}_k = \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k = \sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T \tag{1.79}$$

by Euler's theorem on homogeneous functions[12], so that (1.77) reduces to

$$dT + dV = 0, \quad \text{or} \quad dT = -dV = dW. \tag{1.80}$$

[12] See for example R.P. Gillespie, *Partial Differentiation*, Oliver and Boyd, Edinburgh and London, 1951, p. 28.

This is just the equation of differential energy balance in time dt: the change in kinetic energy equals the work done $-dV = dW$ by the applied forces. Equation (1.78) is the time-integrated version of this statement, or the conservation of the total energy E,

$$H = 2T - (T - V) = T + V = E = \text{constant}. \tag{1.81}$$

We can thus establish whether the laws of momentum and energy conservation hold for any given system by simply examining the structure of its Lagrange function! This fundamental feature of L will be used repeatedly in our future discussions.

Two additional points deserve mention. If L is independent of time, V must also be. Otherwise, an additional time derivative shows up on the right hand side of (1.77) and upsets the conservation of H:

$$dH = dt\frac{\partial V}{\partial t}.$$

The other point concerns V being independent of time, but depending on the $\dot{q}_k$ in addition to the q_k. Such a dependence replaces $\sum_k p_k\dot{q}_k$ in (1.79) by

$$\sum_k p_k\dot{q}_k = 2T - \sum_k \frac{\partial V}{\partial \dot{q}_k}\dot{q}_k$$

so that, while

$$H = (T+V) - \sum_k \frac{\partial V}{\partial \dot{q}_k}\dot{q}_k \tag{1.82}$$

is still conserved, it does not equal the total energy $T+V$ anymore (which is *not* conserved!). The conservation of H and conservation of energy are two separate issues. Physically, the important quantity is H, since like L it plays a fundamental role in dynamics, too. We explore this facet of H in Chap. 7. The results we have just found also hold true, with one important qualification, when the system is subject to constraints. The qualification relates to *time-dependent* constraints. The effect of constraints is to inject the additional forces of constraint $Q_k'' = \sum_\alpha \lambda_\alpha F_{\alpha k}$ into the relation (1.76) for dL. That relation is changed to read

$$d(\sum_k p_k\dot{q}_k) - \sum_{\alpha,k}\lambda_\alpha F_{\alpha k}dq_k = dL - dt\frac{\partial L}{\partial t}.$$

The second term on the left would be zero in a virtual displacement. However, dq_k is the actual displacement of q_k in time dt. Therefore, $\sum F_{\alpha k}dq_k = -G_\alpha dt$ from the constraint equations (1.53), and

$$dH + dt\sum_\alpha \lambda_\alpha G_\alpha = -dt\frac{\partial L}{\partial t} \tag{1.83}$$

replaces (1.77). If now $H = T+V$ and L is independent of time, we get

$$dT = dW + dW', \quad dW' = -dt\sum_\alpha \lambda_\alpha G_\alpha \tag{1.84}$$

in place of the usual law $dT = dW$ or differential energy balance. Time-dependent constraints do work on the system! The work dW of applied forces only accounts for part of the change in kinetic energy. The rest is supplied by the forces of constraint working at a rate $dW'/dt = -\sum_\alpha \lambda_\alpha G_\alpha$.

We can visualize the physical origin of this additional energy by means of our particle moving in a circle under gravity again. Imagine the particle is a bead sliding on a hoop and rotate the hoop with a pre-assigned angular velocity ω about the vertical y axis. The motion then takes place in three dimensions and we must supply the azimuthal angle ϕ to locate the bead in spherical polar coordinates (r, θ, ϕ), see Fig. 1.6.

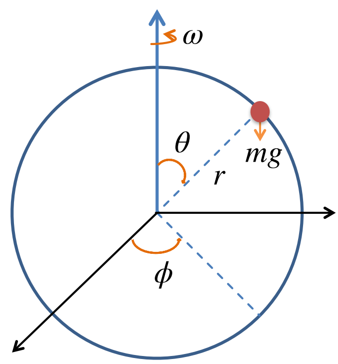

Figure 1.6: Bead on a rotating hoop.

The bead now has an additional velocity component $r \sin\theta\dot{\phi}$ perpendicular to the plane of the hoop, so the Lagrange function reads

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2) - mgr\cos\theta$$

subject to the two constraints

$$r - a = 0, \quad \phi - \omega t = 0,$$

ensuring that the bead stays on the hoop, and turns with it. To illustrate the work done by the second constraint that is time-dependent, we must not eliminate it. The first one, $r = a$, may go however, so

$$L = \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) - mga\cos\theta \tag{1.85}$$

is the Lagrange function for a system with two coordinates θ and ϕ, subject to the one time-dependent constraint

$$d\phi - \omega dt = 0,$$

in order to identify with (1.53). The constraint

$$\lambda\delta\phi = 0$$

on virtual displacements of ϕ is taken care of by a single multiplier λ in the equation of motion for ϕ, which then determines λ. The work done by the time-dependent constraint is

$$-dt\sum_\alpha \lambda_\alpha G_\alpha = \omega\lambda\, dt$$

according to (1.84). But we also know the hoop forces the bead to rotate with it by pushing the bead with a reaction $Q''_\phi = \lambda$. However, since ϕ is an angle, Q''_ϕ will be a *moment* of the reaction R, so

$$Q''_\phi = \lambda = Ra \sin\theta$$

and the work done in time dt is

$$dt\, \omega Ra \sin\theta = R(a \sin\theta d\phi),$$

which is just the work done by R in pushing the bead an angle $d\phi = dt\omega$ in the horizontal plane. The machine (or person!) cranking the hoop around has to supply this energy. Therefore, the energy balance for the motion of the bead is

$$dT = mga \sin\theta d\theta + Ra \sin\theta d\phi,$$

where $dW = mga \sin\theta d\theta$ is the work done by gravity.

But suppose we also eliminated the second constraint $\phi - \omega t$ from L so that it only contains the independent variable θ:

$$L \to L' = \frac{1}{2} ma^2(\dot\theta^2 + \omega^2 \sin^2\theta) - mga \cos\theta.$$

How does this Lagrange function *know* that it describes a system which does not convert work into kinetic energy according to the equation $dT = dW$? We notice that condition (ii) from (1.79) to hold is violated: the kinetic energy in L' is not a homogeneous function in $\dot\theta$ and $\dot\phi$ as was L. Therefore,

$$H' = \frac{\partial L}{\partial \theta}\theta - L' = \frac{1}{2} ma^2(\dot\theta^2 - \omega^2 \sin^2\theta) + mga \cos\theta$$

is *not* the total energy; the minus sign on the $\sin^2\theta$ prevents this from happening. But H' is conserved. The equation $dH' = 0$ still holds since L' is independent of time and shows that the kinetic energy $T' = \frac{1}{2} ma^2(\dot\theta^2 + \omega^2 \sin^2\theta)$ obeys the relation

$$dT = mga \sin\theta d\theta + dt \frac{d}{dt}(ma^2\omega^2 \sin^2\theta).$$

But, from the equation of motion for ϕ as given by L, we have

$$\lambda + \frac{\partial L}{\partial \phi} - \frac{d}{dt}(\frac{\partial L}{\partial \dot\phi}) = \lambda - \frac{d}{dt}(ma^2\omega \sin^2\theta) = 0,$$

which determines λ. Thus dT is given by the same expression as before, since the second term indeed equals $dt\omega\lambda$, the work done by the time-dependent constraint during the actual motion.

We have also remarked that the presence of constraints (time-dependent or not) can upset the conservation of momentum laws. For instance, since L in (1.85) is independent of ϕ, we might be tempted to conclude that

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ma^2 \sin^2 \theta \dot{\phi}$$

is a constant of motion. This would be wrong: p_ϕ actually changes at a rate $\dot{p}_\phi = \lambda$ according to the equation we found above for λ. The forces of constraint regulate how p_ϕ varies with time. The error in our argument is again the fact that ϕ is not an independent variable for the problem. The structure of the Lagrange function L' from which we have eliminated all constraints shows that the only constant of motion for the problem is H'.

Our example illustrates a general property. If a system with a time-independent Lagrange function L is subjected to time-dependent, holonomic constraints $F_\alpha = 0$, then, from (1.83)

$$dH + dt \sum \lambda_\alpha \frac{\partial F_\alpha}{\partial t} = 0 \tag{1.86}$$

holds if these constraints are not eliminated, and from (1.87)

$$dH' + dt \frac{\partial L'}{\partial t} = 0 \tag{1.87}$$

holds if they are (we use primes as a reminder that H and L are now functions of independent coordinates). The effect of the constraints in the latter equation is made up for partly in the functional dependence of H' on the new coordinates and partly by the time-dependence induced in L'. The difference between energy conservation and the conservation of H is also brought out by (1.86) and (1.87) if L' happens to stay independent of time when the constraints are eliminated, i.e. if $\partial L'/\partial t = 0$. If H is the total energy of the system, this changes at a rate $-\sum_\alpha \lambda_\alpha \partial F_\alpha / \partial t$ according to (1.86). But $dH' = 0$ if $\partial L'/\partial t = 0$ or H' stays constant during motion.

1-8 Other Variational Principles

The principle of least action that is discussed in Sec. 1-4. is but one of a whole host of variational formulations that are available in mechanics, that are grouped under the general title of *least action principles*. The emphasis here on *least* is historical rather than factual. All such principles only require that the action in question attains a *stationary* value for the actual motion. No statement about the value of the action itself is involved. We have seen one example of this in Sec. 1-4. We will see another one in the principle of Maupertuis that we are about to derive.

Maupertuis' principle only concerns itself with systems that conserve energy. The Lagrange function for such systems can be written as

$$L = \sum p_k \dot{q}_k - H, \tag{1.88}$$

where $H = E$ is the total energy, and the action S then becomes

$$S = \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt - E(t_2 - t_1). \tag{1.89}$$

We can take advantage of this breakup of S into two parts by designing another type of variation. As before, we induce variations in S by replacing $q_k(t)$ by new functions $s_k(t) + \delta q_k(t)$, but now also allow the *transit time* to vary during the virtual motion. *However, we maintain the same total energy for all virtual motions.* Then,

$$\delta' S = \delta' \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt - E(\delta t_2 - \delta t_1) \tag{1.90}$$

follows from (1.89) where δt_1 and δt_2 are the variations in time at the endpoints of the motion. This new type of variation, symbolized by δ', is clearly different from the δ-variation visualized in Sec. 1-4. There, we required that the transit time for all motions be the same $(\delta t_1 = \delta t_2 = 0)$. Now, we demand that the *energy* be the same $(\delta E = 0)$.

To see what consequences this new variation has for S we also compute $\delta' S$ directly from (1.20). Then,

$$\delta' S = \int_{t_1+\delta t_1}^{t_2+\delta t_2} L(q+\delta q, \dot{q}+\delta\dot{q}, t)\, dt - \int_{t_1+\delta t_1}^{t_2+\delta t_2} L(q, \dot{q}, t)\, dt \simeq \int_{t_1}^{t_2} \delta L\, dt + [L\delta t]_{t_1}^{t_2},$$

if all virtual displacements are considered small. Now, the time integral of δL has already been calculated in (1.38). We introduce that expression in the evaluation of $\delta' S$ and find that

$$\delta' S = \int_{t_1}^{t_2} \sum_k (\frac{\partial L}{\partial q_k} - \dot{p}_k)\delta q_k\, dt + [\sum_k p_k(\delta q_k + \dot{q}_k \delta t)]_{t_1}^{t_2} - [H\delta t]_{t_1}^{t_2} \tag{1.91}$$

after inserting p_k for $\partial L/\partial \dot{q}_k$ and $\sum_k p_k \dot{q}_k - H$ for L. This expression is true for any system, whether conservative or not. We now specialize it as follows: first only the actual motion of the system is considered. This drops out the first term on the right due to the validity of the Lagrange equations. The second term drops out if we replace our former boundary conditions, $\delta q_k(t_1) = \delta q_k(t_2) = 0$, by

$$\delta q_k(t) + \dot{q}_k(t)\delta t = 0, \quad \text{for} \quad t_1 \quad \text{and} \quad t_2. \tag{1.92}$$

This simply means that all virtual motions are made to pass through the same endpoints, even if they do so at different times. Finally we assume H *is* constant and equal to the total energy E. Then $\delta' S = E(\delta t_2 - \delta t_1)$. Inserting this result into (1.90) we learn

$$\delta' \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k\, dt = 0 \tag{1.93}$$

under the stated variations and boundary conditions. This is Maupertuis' principle. The action in this case is represented by

$$S_0 = \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k \, dt \tag{1.94}$$

and is often called the *reduced action* to distinguish it from S.

The contents of this principle can be visualized as follows: represent the system by plotting its coordinates as a point in the n-dimensional space they define. As the system moves this point moves too, tracing out a curve in n dimensions. To apply the variational principle we run the representative point along all paths joining A and B in Fig. 1.7 for which the energy of the system remains the same. Then, according to (1.93) and (1.94) the system moves from A to B in such a way that the integral S_0 is a minimum[13] *when integrated over whatever transit time that is required to go from A to B while keeping the energy constant.*

[13] Actually an extremum.

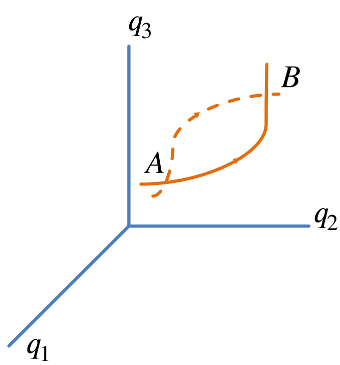

Figure 1.7: Illustration of two possible paths of the representative point for a system with three degrees of freedom.

We have thus shown that (1.93) is true if the Lagrange equations describe the motion. To complete the demonstration of Maupertius' principle we must also show that it leads to the correct equation of motion. Carrying out the variation of the action S_0 we get a sum of three terms:

$$\delta' S_0 = \int_{t_1}^{t_2} \sum_k \delta p_k \dot{q}_k \, dt + \int_{t_1}^{t_2} \sum_k p_k \delta \dot{q}_k \, dt + [\sum_k p_k \dot{q}_k \delta t]_{t_1}^{t_2}. \tag{1.95}$$

We know from previous work that $\delta \dot{q}_k = [(d/dt)(\delta q_k)]$. But what is δp_k? The condition that all virtual paths are to have the same energy enters at this point. We always have

$$\delta H = \delta(\sum_k p_k \dot{q}_k) - \delta L = \sum_k (p_k - \frac{\partial L}{\partial \dot{q}_k}) \delta \dot{q}_k + \sum (\delta p_k \dot{q}_k - \frac{\partial L}{\partial q_k} \delta q_k).$$

But $p_k = \partial L / \partial \dot{q}_k$ by definition and $\delta H = 0$ by design. Therefore,

$$\sum_k \delta p_k \dot{q}_k = \sum_k \frac{\partial L}{\partial q_k} \delta q_k$$

and one finds that

$$\delta' S_0 = \int_{t_1}^{t_2} (\frac{\partial L}{\partial q_k} - \dot{p}_k)\delta q_k \, dt + [\sum p_k(\delta q_k + \dot{q}_k \delta t)]_{t_1}^{t_2} \tag{1.96}$$

after interchanging δ and $\frac{d}{dt}$ as before in the second term on the right of (1.95) and integrating by parts. The boundary conditions at the end-points get rid of the last term and

$$\delta' S_0 = \int_{t_1}^{t_2} \sum_k (\frac{\partial L}{\partial q_k} - \dot{p}_k)\delta q_k \, dt = 0$$

results, a *condition* determining the equations of motion. It obviously leads to the Lagrange equations again! It also goes without saying that the question of independence of the δq_k must be raised again if there are constraints on the motion. The reader will have an opportunity to worry about such matters in Probs. 1-4. and 1-5.

We realize from the preceding discussion that the only important point about H is that it must be constant. It need not be the total energy. However, if H is the total energy, one can cast (1.93) into a number of equivalent forms, each having its own special significance. Thus, if $H = E = T + V$, then T must be a homogeneous quadratic function of the velocities and V independent of them. Therefore, $\sum_k p_k \dot{q}_k = 2T$ and

$$\delta' \int_{t_1}^{t_2} 2T \, dt = 0 \tag{1.97}$$

is another rendering of (1.91) in terms of the total kinetic energy of the system. In particular, if the motion takes place under no forces, then T is also constant, and

$$\delta' \int_{t_1}^{t_2} dt = \delta(t_2 - t_1) = 0.$$

This is the *principle of least time* for motion under no forces. It first gained recognition as a principle governing the propagation of light (Fermat). For such applications it is useful to introduce the optical path length $n\,ds$ ($n =$ the index of refraction) of the light ray and write

$$\delta' \int_{s_1}^{s_2} n \, ds = 0,$$

since $dt = ds/v$, where $v = c/n$ is the velocity of light in the medium, c its velocity *in vacuo*. This transformation then gives a variational principle for determining the *path* of a light ray from s_1 to s_2 in a refractive medium. The time variable has disappeared from the scene completely.

A similar transformation is possible in dynamics. We go back to (1.97) and apply it to the motion of a particle of mass m. Then

$$2T \, dt = 2T \frac{ds}{v} = mv \, ds$$

where ds is the arc length travelled in time dt with speed v. Consequently[14],

$$\delta' \int_{s_1}^{s_2} mv\, ds = 0 \tag{1.98}$$

determines how the particle moves from s_1 to s_2. However, we still have to express v in terms of the position of the particle in order to clear (1.98) of all time variables. This is simple to do using the energy equation: if the particle moves in a potential field V, then $mv = \sqrt{2m(E-V)}$, and

$$\delta' \int_{s_1}^{s_2} \sqrt{E-V}\, ds = 0. \tag{1.99}$$

This form is due to Jacobi (1842). It is in complete analogy with Fermat's principle. The potential field throughout which the particle moves acts on the particle like a heterogeneous medium with refractive index $\sim \sqrt{E-V}$, and (1.99) is a variational principle for determining the *path* of the particle. As a corollary, we see that a free particle moves along the shortest path from s_1 to s_2,

$$\delta' \int_{s_1}^{s_2} ds = \delta(s_2 - s_1) = 0$$

since mv is constant in this case and factors out of (1.98) or (1.99).

Let us apply Jacobi's form to the simple case where a free particle enters a region of space where the potential suddenly decreases from zero to a constant value $-V_0$ (the particle is suddenly accelerated at O as it crosses the boundary between the two regions, see Fig. 1.8). The

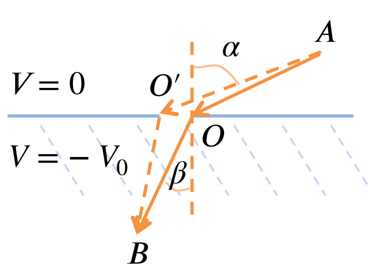

Figure 1.8: Application of Fermat's principle to the motion of a particle.

particle starts at A with energy E and has to reach B. We try the path AOB and calculate the reduced action:

$$S_0 = \sqrt{2mE}(AO) + \sqrt{2m(E+V)}(OB).$$

S_0 has to be stationary for the actual path. In order to minimize S_0 we try various paths by moving the point O where the particle strikes the boundary, and so determine the relation between the angles α and β that holds for the actual motion. Moving O to O', i.e. going along the dotted path, changes AO and OB by $OO' \sin\alpha$ and $-OO' \sin\beta$. Hence, S_0 changes by

$$\delta' S_0 \sim (\sqrt{E}\sin\alpha - \sqrt{E+V_0}\sin\beta)OO'.$$

This is zero for arbitrary displacements OO' if

$$\frac{\sin\alpha}{\sin\beta} = \sqrt{1+\frac{V_0}{E}}.$$

Thus, the particle is *refracted* at the interface between the two potential regions: the actual path is *not* simply the straight line joining A and B. If we call $\sqrt{1+\frac{V_0}{E}}$ the relative index of refraction of the region of negative potential to free space, then this result is analogous to Snell's law of light refraction. Physically, the origin of this effect is obvious: the particle receives an impulse normal to the interface that kinks its path.

It should be realized, however, that Jacobi's form holds for a much wider class of systems than that of a single particle moving in a potential field. The only requirement is that the Lagrange function be expressible in the form

$$L = \frac{1}{2}\sum_{kl} a_{kl}\dot{q}_k\dot{q}_l - V(q_1, q_2, \ldots).$$

The a_{kl} only depend on the coordinates and not their time derivatives. Then, the element of action $2T\,dt$ can be re-expressed as

$$2T\,dt = \sqrt{2T}\sqrt{\sum_{kl} a_{kl}\dot{q}_k\dot{q}_l}\,dt = \sqrt{2T}\,ds$$

to *define* the arc length

$$ds = \sqrt{\sum_{kl} a_{kl}dq_k dq_l}\,dt \qquad (1.100)$$

travelled by the system point in time dt with "speed" $\sqrt{2T}$. Using this extended meaning of ds, (1.99) is unaltered in form, but its meaning is different. It now gives the "path" of the representative point of the system in n-dimensional space. This only coincides with the path of the system in ordinary space if the degrees of freedom are 3 or less and refer to the position of the system in space.

Differential equations determining this path may readily be derived from (1.99). We carry out the derivation for the special, but most usual case of a particle moving in a potential V in two dimensions. Let (x, y) be the cartesian coordinates of the particle. Following the tradition for curvilinear coordinates in two dimensions, we introduce new coordinates $q_1 = u$ and $q_2 = v$, where the grid of curves

$$\begin{aligned} u(x,y) &= \text{constant} \\ v(x,y) &= \text{constant} \end{aligned}$$

defines these coordinates. The line element ds is written as

$$(ds)^2 = E_u(du)^2 + 2F_{uv}\,du\,dv + G_v(dv)^2, \tag{1.101}$$

where E_u, F_{uv} and G_v are the *first differential parameters* introduced by Gauss. They are functions of u and v in general.

Since the variation in (1.99) means considering different paths passing through the same endpoints, the line element ds must also be varied. We consider variations in u only and find

$$\delta' \int_{s_1}^{s_2} \sqrt{E-V}\,ds = \int_{s_1}^{s_2} (-)\frac{\partial V}{\partial u}\frac{\delta u}{2\sqrt{E-V}}\,ds + \int_{s_1}^{s_2} \sqrt{E-V}\,\delta(ds) = 0.$$

Now,

$$ds\,\delta(ds) = (E_u\,du + F_{uv}\,dv)\delta(du) + \frac{1}{2}\Big[\frac{\partial E_u}{\partial u}(du)^2 + 2\frac{\partial F_{uv}}{\partial u}\,du\,dv + \frac{\partial G_v}{\partial u}(dv)^2\Big]\delta u.$$

The term multiplying (du) in this expression gives the contribution

$$-\int_{s_1}^{s_2} \frac{d}{ds}\{\sqrt{E-V}(E_u\frac{du}{ds} + F_{uv}\frac{dv}{ds})\}\,\delta u\,ds,$$

after integrating by parts and enforcing the boundary conditions $\delta u(s_1) = \delta u(s_2) = 0$. Therefore (1.99) is satisfied in the present case if

$$\frac{1}{\sqrt{E-V}}\frac{d}{ds}\{\sqrt{E-V}(E_u\frac{du}{ds} + F_{uv}\frac{dv}{ds})\} = -\frac{1}{2}\frac{1}{E-V}\frac{\partial V}{\partial u} + \frac{1}{2}\{\frac{\partial E_u}{\partial u}(\frac{du}{ds})^2 + 2\frac{\partial F_{uv}}{\partial u}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial G_v}{\partial u}(\frac{dv}{ds})^2\}. \tag{1.102}$$

The equation for v follows by interchanging u and v, E_u and G_v. We have thus obtained differential equations for the path, or *orbit*, of a particle moving in two dimensions in an arbitrary potential field $V(u, v)$. These equations contain as a special case the differential equations for the orbit of a particle in a central potential $V(r)$, $r = \sqrt{x^2 + y^2}$, as will be shown in Chap. 2. Finally, if the particle is moving under no forces, (1.102) becomes

$$\frac{d}{ds}(E_u\frac{du}{ds} + F_{uv}\frac{dv}{ds}) = \frac{1}{2}\{\frac{\partial E_u}{\partial u}(\frac{du}{ds})^2 + 2\frac{\partial F_{uv}}{\partial u}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial G_v}{\partial u}(\frac{dv}{ds})^2\}.$$

This is one of the differential equations determining the shortest path between s_1 and s_2. Such a path is called a *geodesic*. A particle moving freely in the u, v space thus follows one of the geodesics in that space.

1-9 Properties of the Action Functions

We remarked at the beginning of this section that the adjective *least* attached to the various variational principles can be misleading; we could equally well have a *greatest* action as far as the equations of motion are concerned. We did not ever have to raise the issue of whether S is a maximum or a minimum in finding the equations of motion. Typically, physical systems in motion are only prepared to admit to the stationary character of the action. Just as typically, however, the various principles of least action have been made the basis of many a philosophical foray into the "deeper" meaning of the laws of dynamics. The trouble always seems to be the appearance of a finite time interval in the calculation of say the action S. This gives the impression that the system already "knows" at time t_1 where it is going to be at t_2, i.e. it endows mechanics with a *teleological*[15] character. Actually this impression is false. As we have seen, only the stationary character of the action and not its value determines the equations of motion.

[15] Teleology, *n*, (from Greek *telos* "end" and *logos* "reason"), explanation by reference to some purpose, end, goal, or function. Traditionally, it was also described as final causality, in contrast with explanation solely in terms of efficient causes (the origin of a change or a state of rest in something). *Encyclopaedia Britannica*, 2023.

This does not mean, however, that the action functions themselves are devoid of meaning. For consider (1.91) and (1.96); up to now we have always used these expressions to find the equations of motion by stipulating boundary conditions such that the contribution at the "boundaries" t_1 and t_2 vanishes. Now, we turn the procedure around and evaluate the variations in S and S_0 due to changes in the coordinates and the time at these boundaries *for the actual motion*. We set $\delta q_k(t_1)$ and δt_1 equal to zero and call it the time at the other endpoint of the action. Then,

$$\delta' S = \sum_k p_k(\delta q_k + \dot{q}_k \delta t) - H\delta t \tag{1.103}$$

expresses the variation in S as the coordinates and the time at the other endpoint are allowed to vary. The circumstance of this variation should be clearly distinguished from the variations employed so far. We are now assuming that Lagrange equations hold along all paths that are considered. Therefore, (1.103) shows how S depends on the coordinates of the motion at time t. In fact, by setting $\delta t = 0$, we obtain the *partial* derivatives of S with respect to the q_k:

$$\delta' S = \sum_k \frac{\partial S}{\partial q_k}\delta q_k = \sum_k p_k \delta q_k \quad \text{for} \quad \delta t = 0,$$

or

$$\frac{\partial S}{\partial q_k} = p_k. \tag{1.104}$$

On the other hand, setting all the δq_k equal to zero does not define the partial derivative with respect to time. This requires in addition that the position of the endpoint not vary when t is varied. Therefore, not δq_k but $\delta q_k + \dot{q}_k \delta t$ must be set equal to zero at time t. Then

$$\delta' S = -H\delta t$$

defines the partial derivative

$$\frac{\partial S}{\partial t} = -H. \tag{1.105}$$

Thus, the partial time derivative of S gives H; by contrast its total time derivative gives L:

$$\frac{dS}{dt} = L. \tag{1.106}$$

The latter statement follows from (1.20) with $t_2 = t$.

The corresponding partial derivatives of S_0 follow from (1.96) in a similar fashion. From

$$\delta' S_0 = \sum_k p_k(\delta q_k + \dot{q}_k \delta t) \tag{1.107}$$

we conclude that

$$\frac{\partial S_0}{\partial q_k} = p_k, \quad \text{and} \ \frac{\partial S_0}{\partial t} = 0, \tag{1.108}$$

provided of course that the energy is conserved (otherwise (1.107) is not valid!). Because of this requirement S_0 contains the energy E as a parameter. The derivative $\partial S/\partial E$ has a surprising significance. We go back to (1.90) and also vary the energy from path to path. Then,

$$\delta' S = \delta' S_0 - E\delta t - \delta E(t - t_1).$$

But, according to (1.106), $\delta' S$ equals $-E\delta t$ if $H = E$. Therefore,

$$\delta' S_0 - \delta E(t - t_1) = 0$$

defines the partial derivative we seek:

$$\frac{\partial S_0}{\partial E} = t - t_1. \tag{1.109}$$

Thus, the change of S_0 with energy gives the transit time from t_1 to t. This result allows one to express the coordinates as a function of time by inserting the value for S_0 and differentiating:

$$\frac{\partial}{\partial E} \int_{s_1}^{s_2} \sqrt{2(E - V)}\, ds = \int_{s_1}^{s_2} \frac{ds}{\sqrt{2(E - V)}} = t - t_1. \tag{1.110}$$

Here $s - s_1$ is the distance travelled in time $t - t_1$. Together with the equation of the path this relation completely determines the motion.

The relations (1.106) through (1.109) will appear again in Chap. 7 from a different point of view. However, it is useful and interesting to indicate one basic feature of (1.106) at this point. From its definition H on the right of that equation depends on q, $\dot{q}$ and t. On the other hand, $\dot{q}$ may be eliminated in favour of p by inverting the relations $p_k = \partial L/\partial \dot{q}_k$; then p may be re-expressed as in (1.104) and we get

$$\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0. \tag{1.111}$$

This is the famous *Hamilton-Jacobi partial differential equation* for determining S. It will emerge again in Chap. 7 - ostensibly from an entirely different point of view. If H is conserved and also happens to be the total energy, then the Hamilton-Jacobi equation turns into a partial differential equation for S_0,

$$H(q, \frac{\partial S_0}{\partial q}) - E = 0. \tag{1.112}$$

The meaning and practical significance of both these equations are discussed at length in Chap. 7.

1-10 D'Alembert's Principle

The action principles we have discussed so far are *integral* principles of mechanics in this respect as they concern themselves with properties of the motion over a finite time interval $t_2 - t_1$. *Differential* principles are also available which concern themselves with the motion at a given instant t and small variations about this motion. Perhaps the most important principle of the latter type is due to d'Alembert (1758)[16]. His principle simply states that the inertial forces are in equilibrium with the applied forces on any system. To appreciate this statement we go back to Newton I in (1.1) and notice that all bodies remain at rest or in uniform rectilinear motion if no forces act. A measure of their resistance to change, or *inertia*, therefore is $\dot{p}$. D'Alembert calls $\tilde{\mathbf{F}} = -\dot{\mathbf{p}}$ the *inertial force*. If k again labels a typical particle, then (1.23) gives the equation of motion for this particle in the form

[16] Jean-le-Rond d'Alembert (1717 - 1783), a famous French mathematician and philosopher.

$$F_k + \tilde{F}_k = 0.$$

Multiplying both sides by the virtual displacements δx_k and summing on all k, one has

$$\sum_k (F_k + \tilde{F}_k)\delta x_k = 0. \tag{1.113}$$

This all seems rather trivial until we realize that the forces of constraint drop out of the virtual work $\sum_k F_k \delta x_k$. Therefore, this equation only contains the applied forces. It states that the applied forces are in equilibrium with the inertial forces (the total virtual work is zero).

Equation (1.113) is the mathematical realization of d'Alembert's principle. It is of course identical with our former equation, (1.28),

$$\sum_k (F_k + \tilde{F}_k)\delta x_k = \sum_k F_k \delta x_k + \delta T - \frac{d}{dt}(\sum_k m_k \dot{x}_k \delta x_k) = 0 \tag{1.114}$$

if we restore the definition of $\tilde{F}_k$. We can also use this equation to determine the equations of motion. Take the case first where there are no constraints on the system. Then $F_k - \dot{p}_k = 0$ must follow if (1.114) is to hold for arbitrary displacements δx_k. But if constraints are present, this result does *not* follow. A transformation to independent coordinates is necessary before any conclusions can be drawn. But we can introduce such transformations since (1.114) is a *scalar* relation and therefore valid in any set of coordinates. Accordingly, we pass to new coordinates $x_k \to f_k(q_1, q_2, \dots)$ and observe the various parts of (1.114) transform:

$$\sum_k F_k \delta x_k = \sum_k Q_k \delta q_k$$

from the two forms (1.29) and (1.41) for the virtual work,

$$\delta T = \delta(\sum_k \frac{1}{2} m_k \dot{f}_k^2) = \sum_k \frac{\partial T}{\partial q_k}\delta q_k + \sum_k \frac{\partial T}{\partial \dot{q}_k}\delta \dot{q}_k$$

from the functional form of T in the new coordinates, and

$$\sum_k m_k \dot{x}_k \delta x_k = \sum_{k,l} m_k \dot{f}_k \frac{\partial f_k}{\partial q_l}\delta q_l$$

from the definition of the virtual displacements δx. But

$$\frac{\partial f_k}{\partial q_l} = \frac{\partial \dot{x}_k}{\partial \dot{q}_l} = \frac{\partial \dot{f}_k}{\partial \dot{q}_l}$$

from (1.36). Therefore,

$$\sum m_k \dot{x}_k \delta x_k = \sum_{k,l} m_k \dot{f}_k \frac{\partial \dot{f}_k}{\partial \dot{q}_k}\delta q_l = \sum_k \frac{\partial}{\partial \dot{q}_k}(\sum_l \frac{1}{2} m_l \dot{f}_l)\delta q_k = \sum_k \frac{\partial T}{\partial \dot{q}_k}\delta q_k. \tag{1.115}$$

This is the crucial relation. Knowing it, we can write (1.114) as

$$\sum_k [Q_k + \frac{\partial T}{\partial q_k} - \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_k})]\delta q_k = 0. \tag{1.116}$$

Now, assume that there are no constraints on the motion, or if there are, of the holonomic variety. Then, all the δq_k can be chosen independently so that

$$Q_k + \frac{\partial T}{\partial q_k} - \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_k}) = 0.$$

These are the, by now, familiar Lagrange equations for holonomic systems.

In the form (1.114) d'Alembert's principle has an interesting consequence. If a system is in *equilibrium* under applied forces, then $T = 0$ and all $\dot{x}_k = 0$ so that

$$\delta W = 0. \tag{1.117}$$

The virtual work of the applied forces must be zero for equilibrium. This succinct statement summarizes the entire subject of *statics* (systems in equilibrium).

Two further differential principles, Gauss' principle of least constraint, and Hertz's principle of least curvature (which is a special case of Gauss' principle) are also available, but do not introduce any new insight that is not already contained in d'Alembert's principle. In summary, we remark: the differential principles only require a knowledge of differentiation to derive the equations of motion. On the other hand, the integral principles (which require the calculus of variations) have perhaps a more dependable "feel", since their implementation does not depend on spotting relations like (1.116) and (1.113) ahead of time. Their dependence on a single scalar function L makes it manifest that the equations they lead to, *must* have the same form in all sets of coordinates. This is also true of course for the differential principles, but perhaps less obvious to the uninitiated (see Prob. 1-6).

These comments bring us to the end of our discussion of the general principles of mechanics. Our next task (it will occupy the remainder of the book!) is to apply these principles to the motion of particles and systems of particles that are of interest in physics.

Problems

1-1. Find out how Newton's law of motion, (1.2), is modified when referred to a frame of reference that moves with constant acceleration.

1-2. Describe the sequence of events that allow a cat to reorient while falling from an inverted position.

1-3. Verify the calculation of the action $S[\bar{f}]$ given in the text for a particle falling under gravity. Guess at some other forms $\bar{f}(t)$ for the virtual motion and verify that $S[\bar{f}]$ is stationary for the actual motion in each case.

1-4 .Discuss the validity of Maupertuis' principle in the presence of holonomic constraints. Pay particular attention to the case where such constraints may be time-dependent.

1-5. Discuss the validity of Maupertuis' principle in the presence of nonholonomic constraints of the type given in (1.52). Pay particular attention to the case where such constraints may be time-dependent.

1-6. Show by a *direct* transformation of coordinates $q_k = f_k(q'_1, q'_2, \dots)$ that the Lagrange equations are unaltered in form. This property is referred to as the *covariance* of these equations.

1-7. Find the geodesics joining any two points of a sphere. Can you always find the shortest distance between the points in question? Explain.