Chapter 12

Homogeneous, Isotropic Cosmology

12.1 Spherically-symmetric spacetimes

Physical cosmology aims at studying the structure and evolution of the universe as a whole. Of the four fundamental interactions of physics, only gravity is relevant on the largest scales because the strong and the weak interactions are confined to sub-atomic length scales, and the electromagnetic force is shielded on large scales by opposite charges. We thus expect that the spacetime of the Universe can be idealised as a solution of Einstein's field equations, satisfying certain simplicity requirements expressed by symmetries imposed on the form of the solution. In this chapter, we shall therefore first discuss spherically-symmetric spacetimes in general and then specialise them to cosmological solutions in particular.

12.1.1 Form of the metric

Generally, a spacetime (M, g) is called *spherically symmetric* if it admits the group $SO(3)$ as an isometry such that the group's orbits are two-dimensional, *space-like* surfaces.

For any point $p \in M$, we can then select the orbit $\Omega(p)$ of $SO(3)$ through p. In other words, we construct the spatial two-sphere containing p which is compatible with the spherical symmetry.

Next, we construct the set of all geodesics $N(p)$ through p which are orthogonal to $\Omega(p)$. Locally, $N(p)$ forms a two-dimensional surface which we also call $N(p)$. Repeating this construction for all $p \in M$ yields the surfaces N.

We can now introduce coordinates (r, t) on N and (ϑ, φ) on Ω, i.e. such that the group orbits Ω of $SO(3)$ are given by $(r, t) = \text{const.}$ and the surfaces N by $(\vartheta, \varphi) = \text{const.}$ This allows the following intermediate conclusion.

Metric of a spherically-symmetric spacetime

The line element of the metric of a spherically-symmetric spacetime M can be written in the form

$$\mathrm{d}s^2 = \mathrm{d}\tilde{s}^2 + R^2(t, r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) , \tag{12.1}$$

where $\mathrm{d}\tilde{s}^2$ is the line element of a yet unspecified metric $\tilde{g}$ in the coordinates (t, r) on the surfaces N.
Without loss of generality, we can now choose t and r such that the metric $\tilde{g}$ is diagonal, which allows us to write its line element as

$$\mathrm{d}\tilde{s}^2 = -\mathrm{e}^{2a(t,r)}c^2\mathrm{d}t^2 + \mathrm{e}^{2b(t,r)}\mathrm{d}r^2 , \tag{12.2}$$

with functions $a(t, r)$ and $b(t, r)$ to be determined.

?

Following similar arguments as presented here, how would you construct the metric for a cylindrically-symmetric spacetime?

As suggested by the line elements (12.1) and (12.2), we introduce the dual basis

$$\theta^0 = \mathrm{e}^a c\mathrm{d}t , \quad \theta^1 = \mathrm{e}^b\mathrm{d}r , \quad \theta^2 = R\mathrm{d}\vartheta , \quad \theta^3 = R\sin\vartheta\mathrm{d}\varphi \tag{12.3}$$

and find its exterior derivatives

$$\begin{aligned}
\mathrm{d}\theta^0 &= -a'\mathrm{e}^{-b}\,\theta^0\wedge\theta^1 , \\
\mathrm{d}\theta^1 &= \dot{b}\mathrm{e}^{-a}\,\theta^0\wedge\theta^1 , \\
\mathrm{d}\theta^2 &= \frac{\dot{R}}{R}\mathrm{e}^{-a}\,\theta^0\wedge\theta^2 + \frac{R'}{R}\mathrm{e}^{-b}\,\theta^1\wedge\theta^2 , \\
\mathrm{d}\theta^3 &= \frac{\dot{R}}{R}\mathrm{e}^{-a}\,\theta^0\wedge\theta^3 + \frac{R'}{R}\mathrm{e}^{-b}\,\theta^1\wedge\theta^3 + \frac{\cot\vartheta}{R}\,\theta^2\wedge\theta^3 ,
\end{aligned} \tag{12.4}$$

where the overdots and primes denote derivatives with respect to ct and r, respectively.

?

Carry out the calculations leading to (12.4) yourself. Can you confirm the results?

12.1.2 Connection and curvature forms

In the dual basis (12.3), the metric is Minkowskian, $g = \mathrm{diag}(-1, 1, 1, 1)$, thus $\mathrm{d}g = 0$, and Cartan's first structure equation (8.13) implies

$$\omega^\mu_\nu\wedge\theta^\nu = -\mathrm{d}\theta^\mu \tag{12.5}$$

for the connection 1-forms ω^μ_ν. From (12.5) and the results (12.4), we can read off

$$
\begin{aligned}
\omega^0_1 &= \omega^1_0 = a' \mathrm{e}^{-b}\,\theta^0 + \dot{b}\mathrm{e}^{-a}\,\theta^1 \,, &\qquad \omega^1_2 &= -\omega^2_1 = -\frac{R'}{R}\mathrm{e}^{-b}\,\theta^2 \,, \\
\omega^0_2 &= \omega^2_0 = \frac{\dot{R}}{R}\mathrm{e}^{-a}\,\theta^2 \,, & \omega^1_3 &= -\omega^3_1 = -\frac{R'}{R}\mathrm{e}^{-b}\,\theta^3 \,, \\
\omega^0_3 &= \omega^3_0 = \frac{\dot{R}}{R}\mathrm{e}^{-a}\,\theta^3 \,, & \omega^2_3 &= -\omega^3_2 = -\frac{\cot\vartheta}{R}\,\theta^3 \,.
\end{aligned}
\tag{12.6}
$$

Cartan's second structure equation (8.13) then yields the curvature 2-forms Ω^i_j,

$$
\begin{aligned}
\Omega^0_1 &= \mathrm{d}\omega^0_1 && \equiv E\,\theta^0 \wedge \theta^1 \,, \\
\Omega^0_2 &= \mathrm{d}\omega^0_2 + \omega^0_1 \wedge \omega^1_2 && \equiv \tilde{E}\,\theta^0 \wedge \theta^2 + H\,\theta^1 \wedge \theta^2 \,, \\
\Omega^0_3 &= \mathrm{d}\omega^0_3 + \omega^0_1 \wedge \omega^1_3 + \omega^0_2 \wedge \omega^2_3 && = \tilde{E}\,\theta^0 \wedge \theta^3 + H\,\theta^1 \wedge \theta^3 \,, \\
\Omega^1_2 &= \mathrm{d}\omega^1_2 + \omega^1_0 \wedge \omega^0_2 && \equiv -H\,\theta^0 \wedge \theta^2 + \tilde{F}\,\theta^1 \wedge \theta^2 \,, \\
\Omega^1_3 &= \mathrm{d}\omega^1_3 + \omega^1_0 \wedge \omega^0_3 + \omega^1_2 \wedge \omega^2_3 && = -H\,\theta^0 \wedge \theta^3 + \tilde{F}\,\theta^1 \wedge \theta^3 \,, \\
\Omega^2_3 &= \mathrm{d}\omega^2_3 + \omega^2_0 \wedge \omega^0_3 + \omega^2_1 \wedge \omega^1_3 && \equiv F\,\theta^2 \wedge \theta^3 \,,
\end{aligned}
\tag{12.7}
$$

where the functions

$$
\begin{aligned}
E &= \mathrm{e}^{-2a}\left(\ddot{b} - \dot{a}\dot{b} + \dot{b}^2\right) - \mathrm{e}^{-2b}\left(a'' - a'b' + a'^2\right) \,, \\
\tilde{E} &= \frac{\mathrm{e}^{-2a}}{R}\left(\ddot{R} - \dot{a}\dot{R}\right) - \frac{\mathrm{e}^{-2b}}{R}a'R' \,, \\
H &= \frac{\mathrm{e}^{-a-b}}{R}\left(\dot{R}' - a'\dot{R} - \dot{b}R'\right) \,, \\
F &= \frac{1}{R^2}\left(1 - R'^2\mathrm{e}^{-2b} + \dot{R}^2\mathrm{e}^{-2a}\right) \,, \\
\tilde{F} &= \frac{\mathrm{e}^{-2a}}{R}\dot{b}\dot{R} + \frac{\mathrm{e}^{-2b}}{R}\left(b'R' - R''\right)
\end{aligned}
\tag{12.8}
$$

were defined for brevity.

? Repeat the calculations leading to (12.8) yourself, beginning reading off the connection forms (12.6).

According to (8.20), the curvature forms imply the components

$$
R_{\alpha\beta} = \Omega^\lambda_\alpha(e_\lambda, e_\beta) \tag{12.9}
$$

of the Ricci tensor, for which we obtain

$$
\begin{aligned}
R_{00} &= -E - 2\tilde{E} \,, &\qquad R_{01} &= -2H \,, &\qquad R_{02} &= 0 = R_{03} \,, \\
R_{11} &= E + 2\tilde{F} \,, & R_{12} &= 0 = R_{13} \\
R_{22} &= \tilde{E} + \tilde{F} + F = R_{33} \,, & R_{23} &= 0 \,,
\end{aligned}
\tag{12.10}
$$

the Ricci scalar

$$
\begin{aligned}
\mathcal{R} &= (E + 2\tilde{E}) + (E + 2\tilde{F}) + 2(\tilde{E} + \tilde{F} + F) \\
&= 2(E + F) + 4(\tilde{E} + \tilde{F}) \,,
\end{aligned}
\tag{12.11}
$$

and finally the components

$$G_{\alpha\beta} = \begin{pmatrix} F + 2\tilde{F} & -2H & 0 & 0 \\ -2H & -2\tilde{E} - F & 0 & 0 \\ 0 & 0 & -E - \tilde{E} - \tilde{F} & 0 \\ 0 & 0 & 0 & -E - \tilde{E} - \tilde{F} \end{pmatrix} \tag{12.12}$$

of the Einstein tensor

$$G = R - \frac{\mathcal{R}}{2} g \, . \tag{12.13}$$

12.1.3 Generalised Birkhoff's theorem

We can now state and prove Birkhoff's theorem in its general form:

Birkhoff's generalised theorem

Every C^2 solution of Einstein's vacuum equations which is spherically symmetric in an open subset $U \subset M$ is locally isometric to a domain of the Schwarzschild-Kruskal solution.

The proof proceeds in four steps:

1. If the surfaces $\{R(t, r) = \text{const.}\}$ are *time-like* in U and $\mathrm{d}R \neq 0$, we can choose $R(t, r) = r$, thus $\dot{R} = 0$ and $R' = 1$. Since $H = -\dot{b}\mathrm{e}^{-a-b}/R$ then, the requirement $G_{01} = 0$ implies $\dot{b} = 0$. The sum $G_{00} + G_{11} = 2(\tilde{F} - \tilde{E})$ must also vanish, thus
$$\frac{\mathrm{e}^{-2b}}{R}\,(b' + a') = 0 \, , \tag{12.14}$$
which means $a(t, r) = -b(r) + f(t)$. By a suitable choice of a new time coordinate, a can therefore be made time-independent as well. Moreover, we see that
$$0 = G_{00} = F + 2\tilde{F} = \frac{1 - \mathrm{e}^{-2b}}{R^2} + \frac{2b'\mathrm{e}^{-2b}}{R} \tag{12.15}$$
is identical to the condition (8.62) for the function b in the Schwarzschild spacetime. Thus, we have $\mathrm{e}^{-2b} = 1 - 2m/r$ as there, further $a(r) = -b(r)$, and the metric turns into the Schwarzschild metric.

?

Construct yourself the new time coordinate implied by the condition $a(t, r) = -b(r) + f(t)$.

2. If the surfaces $\{R(t, r) = \text{const.}\}$ are *space-like* in U and $\mathrm{d}R \neq 0$, we can choose $R(t, r) = t$ and proceed in an analogous way. Then, $\dot{R} = 1$ and $R' = 0$, thus $H = -a'\mathrm{e}^{-a-b}/R$, hence $G_{01} = 0$ implies $a' = 0$ and, again through $G_{00} + G_{11} = 0$, the condition $\dot{a} + \dot{b} = 0$ or $b(t, r) = -a(t) + f(r)$. This allows us to change the radial coordinate appropriately so that $b(t, r)$ also becomes independent of r. Then, $G_{00} = 0$ implies
$$0 = G_{00} = \frac{1}{R^2}\left(1 + \mathrm{e}^{-2a}\right) - 2\dot{a}\frac{\mathrm{e}^{-2a}}{R} \, , \tag{12.16}$$

where $\dot{b} = -\dot{a}$ was used. Since $R = t$, this is equivalent to

$$\partial_t(t\mathrm{e}^{-2a}) = -1 \quad \Rightarrow \quad \mathrm{e}^{-2a} = \mathrm{e}^{2b} = \frac{2m}{t} - 1\;, \tag{12.17}$$

with $t < 2m$. This is the Schwarzschild solution for $r < 2m$ because r and t change roles inside the Schwarzschild horizon.

3. If $\{R(t,r) = \text{const.}\}$ are space-like in some part of U and time-like in another, we obtain the respective different domains of the Schwarzschild spacetime.

4. Assume finally $\langle \mathrm{d}R, \mathrm{d}R\rangle = 0$ on U. If R is constant in U, $G_{00} = R^{-2} = 0$ implies $R = \infty$. Therefore, suppose $\mathrm{d}R$ is not zero, but light-like. Then, r and t can be chosen such that $R = t - r$ and $\mathrm{d}R = \mathrm{d}t - \mathrm{d}r$. For $\mathrm{d}R$ to be light-like,

$$\langle \mathrm{d}R, \mathrm{d}R\rangle = \tilde{g}(\mathrm{d}R, \mathrm{d}R) = -\mathrm{e}^{2a} + \mathrm{e}^{2b} = 0\;, \tag{12.18}$$

we require $a = b$. Then, $G_{00} + G_{11} = 0$ or

$$-\frac{e^{-2a}}{R}\left(\dot{a} + \dot{b} - a' - b'\right) = 0\;, \tag{12.19}$$

implies $\dot{a} = a'$, which again leads to $R = \infty$ through $G_{00} = 0$.

This shows that the metric reduces to the Schwarzschild metric in all relevant cases.

Cavity in spherically-symmetric spacetime

It is a corollary to Birkhoff's theorem that a spherical cavity in a spherically-symmetric spacetime has the Minkowski metric. Indeed, Birkhoff's theorem says that the cavity must have a Schwarzschild metric with mass zero, which is the Minkowski metric.

?

Compare Birkhoff's to Newton's theorem.

12.2 Homogeneous and isotropic spacetimes

12.2.1 Homogeneity and isotropy

There are good reasons to believe that the Universe at large is *isotropic* around our position. The most convincing observational data are provided by the cosmic microwave background, which is a sea of blackbody radiation at a temperature of $(2.725 \pm 0.001)\,\mathrm{K}$ whose intensity is almost exactly independent of the direction into which it is observed.

There is furthermore no good reason to believe that our position in the Universe is in any sense prefered compared to others. We must therefore conclude that any observer sees the cosmic microwave background as an isotropic source such as we do. Then, the Universe must also be *homogeneous*.

Caution While isotropy about our position in spacetime can be tested and is confirmed by observations, homogeneity is essentially impossible to test. ◂

We are thus led to the expectation that our Universe at large may be described by a homogeneous and isotropic spacetime. Let us now give these terms a precise mathematical meaning.

Spatially homogeneous spacetime

A spacetime (M, g) is called *spatially homogeneous* if there exists a one-parameter family of space-like hypersurfaces Σ_t that foliate the spacetime such that for each t and any two points $p, q \in \Sigma_t$, there exists an isometry ϕ of g which takes p into q.

Before we can define isotropy, we have to note that isotropy requires that the state of motion of the observer needs to be specified first because two observers moving with different velocities through a given point in spacetime will generally observe different redshifts in different directions.

Spatially isotropic spacetime

Therefore, we define a spacetime (M, g) as *spatially isotropic* about a point p if there exists a congruence of time-like geodesics through p with tangents u such that for any two vectors $v_1, v_2 \in T_pM$ orthogonal to u, there exists an isometry of g taking v_1 into v_2 but leaving u and p invariant. In other words, if the spacetime is spatially isotropic, no prefered spatial direction orthogonal to u can be identified.

Isotropy thus identifies a special class of observers, with four-velocities u, who cannot identify a prefered spatial direction. The spatial hypersurfaces Σ_t must then be orthogonal to u because otherwise a prefered direction could be identified through the misalignment of the normal direction to Σ_t and u, breaking isotropy.

We thus arrive at the following conclusions: a homogeneous and isotropic spacetime (M, g) is foliated into space-like hypersurfaces Σ_t on which g induces a metric h. There must be isometries of h carrying any point $p \in \Sigma_t$ into any other point $q \in \Sigma_t$. Because of isotropy, it must furthermore be impossible to identify prefered spatial directions on Σ_t. These are very restrictive requirements which we shall now exploit.

12.2.2 Spaces of constant curvature

Consider now the curvature tensor ${}^{(3)}\bar{R}$ induced on Σ_t (i.e. the curvature tensor belonging to the metric h induced on Σ_t). We shall write it in components with its first two indices lowered and the following two indices raised,

$$ {}^{(3)}\bar{R} = {}^{(3)}\bar{R}_{ij}{}^{kl} \; . \tag{12.20} $$

In this way, ${}^{(3)}\bar{R}$ represents a linear map from the vector space of 2-forms $\bigwedge^2$ into $\bigwedge^2$, because of the antisymmetry of ${}^{(3)}\bar{R}$ with respect to permutations of the first and the second pairs of indices. Thus, it defines an *endomorphism*

$$L : \bigwedge^2 \to \bigwedge^2 , \quad (L\omega)_{ij} = {}^{(3)}\bar{R}_{ij}{}^{kl}\omega_{kl} . \tag{12.21}$$

Caution Recall that an endomorphism is a linear map of a vector space into itself. ◂

Due to the symmetry (3.81) of ${}^{(3)}\bar{R}$ upon swapping the first with the second pair of indices, the endomorphism L is self-adjoint. In fact, for any pair of 2-forms $\alpha, \beta \in \bigwedge^2$,

$$\begin{aligned}\langle\alpha, L\beta\rangle &= {}^{(3)}\bar{R}_{ij}{}^{kl}\alpha^{ij}\beta_{kl} = {}^{(3)}\bar{R}_{ijkl}\alpha^{ij}\beta^{kl} = {}^{(3)}\bar{R}_{klij}\alpha^{ij}\beta^{kl} \\ &= {}^{(3)}\bar{R}_{kl}{}^{ij}\alpha_{ij}\beta^{kl} = \langle\beta, L\alpha\rangle ,\end{aligned} \tag{12.22}$$

which defines a self-adjoint endomorphism.

We can now use the theorem stating that the eigenvectors of a self-adjoint endomorphism provide an orthonormal basis for the vector space it is operating on. *Isotropy* now requires us to conclude that the eigenvalues of these eigenvectors need to be equal because we could otherwise define a prefered direction (e.g. by the eigenvector belonging to the largest eigenvalue). Then, however, the endomorphism L must be proportional to the identical map

$$L = 2k\,\mathrm{id} , \tag{12.23}$$

with some $k \in \mathbb{R}$.

By the definition (12.21) of L, this implies for the coefficients of the curvature tensor

$${}^{(3)}\bar{R}_{ij}{}^{kl} = k\left(\delta^k_i\delta^l_j - \delta^k_j\delta^l_i\right) \tag{12.24}$$

because ${}^{(3)}\bar{R}$ must be antisymmetrised. Lowering the indices by means of the induced metric h yields

$${}^{(3)}\bar{R}_{ijkl} = k\left(h_{ik}h_{jl} - h_{jk}h_{il}\right) . \tag{12.25}$$

The Ricci tensor is

$$\begin{aligned}{}^{(3)}R_{jl} &= {}^{(3)}\bar{R}^{i}{}_{jil} = kh^{is}\left(h_{si}h_{jl} - h_{ji}h_{sl}\right) = k\left(3h_{jl} - h_{jl}\right) \\ &= 2kh_{jl} ,\end{aligned} \tag{12.26}$$

and the Ricci scalar becomes

$${}^{(3)}\mathcal{R} = {}^{(3)}R^j_j = 6k . \tag{12.27}$$

?

Summarise the arguments leading to the Ricci tensor (12.26) and the Ricci scalar (12.27) in your own words.

In the coordinate-free representation, the curvature is

$$\bar{R}(x, y)v = k\left(\langle x, v\rangle y - \langle y, v\rangle x\right) . \tag{12.28}$$

from (8.18) and (12.25), we find the curvature forms

$$\begin{aligned}\Omega^i_j &= \frac{1}{2}{}^{(3)}\bar{R}^i{}_{jkl}\,\theta^k\wedge\theta^l = \frac{k}{2}h^{is}\left(h_{sk}h_{jl}-h_{jk}h_{sl}\right)\theta^k\wedge\theta^l \\ &= k\,\theta^i\wedge\theta_j \end{aligned} \tag{12.29}$$

in a so far arbitrary dual basis θ^i.

The *curvature parameter* k must be (spatially) constant because of homogeneity. Space-times with constant curvature can be shown to be *conformally flat*, which means that coordinates can be introduced in which the line element $\mathrm{d}l^2$ of the metric h reads

$$\mathrm{d}l^2 = \frac{1}{\psi^2}\sum_{i=1}^{3}(\mathrm{d}x^i)^2\ , \tag{12.30}$$

with a yet unknown arbitrary function $\psi = \psi(x^j)$. This leads us to introduce the dual basis

$$\theta^i \equiv \frac{1}{\psi}\,\mathrm{d}x^i\ , \tag{12.31}$$

from which we find

$$\mathrm{d}\theta^i = -\frac{\partial_j\psi}{\psi^2}\,\mathrm{d}x^j\wedge\mathrm{d}x^i = \psi_j\,\theta^i\wedge\theta^j\ , \tag{12.32}$$

where $\psi_j = \partial_j\psi$ abbreviates the partial derivative of ψ with respect to x^j.

In this basis, the metric h is represented by $h = \mathrm{diag}(1,1,1)$. Therefore, we do not need to distinguish between raised and lowered indices, and $\mathrm{d}h = 0$. Hence Cartan's first structure equation (8.13) implies the connection forms

$$\omega_{ij} = \psi_i\,\theta_j - \psi_j\,\theta_i\ . \tag{12.33}$$

According to Cartan's second structure equation, the curvature forms are

$$\begin{aligned}\Omega_{ij} &= \mathrm{d}\omega_{ij} + \omega_{ik}\wedge\omega^k_j \\ &= \psi\left(\psi_{ik}\,\theta^k\wedge\theta_j - \psi_{jk}\,\theta^k\wedge\theta_i\right) - \psi_k\psi^k\,\theta_i\wedge\theta_j\ ,\end{aligned} \tag{12.34}$$

but at the same time we must satisfy (12.29). This immediately implies

$$\psi_{ik} = 0 \quad (i\neq k)\ , \tag{12.35}$$

thus ψ has to be of the form

$$\psi = \sum_{k=1}^{3} f_k(x^k) \tag{12.36}$$

because otherwise the mixed derivatives could not vanish.

Inserting this result into (12.34) shows

$$\Omega_{ij} = \psi\left(f_i'' + f_j'' - \frac{f_k' f'^k}{\psi}\right)\theta_i\wedge\theta_j\ . \tag{12.37}$$

In order to satisfy (12.29), we must have

$$f_i'' + f_j'' = \frac{k + f_k' f'^k}{\psi} \; . \tag{12.38}$$

Since the two sides of these equations (one for each combination of i and j) depend on different sets of variables, the second derivatives f_i'' and f_j'' must all be equal and constant, and thus the f_i must be quadratic in x^i with a coefficient of $x^{i\,2}$ which is independent of x^i. Therefore, we can write

$$\psi = 1 + \frac{k}{4} \sum_{i=1}^{3} x^{i\,2} \tag{12.39}$$

because, if the linear term is non-zero, it can be made zero by translating the coordinate origin, and a constant factor on ψ is irrelevant because it simply scales the coordinates.

12.3 Friedmann's equations

12.3.1 Connection and curvature forms

Robertson-Walker metric

According to the preceding discussion, the homogeneous and isotropic spatial hypersurfaces Σ_t must have a metric h with a line element of the form

$$\mathrm{d}l^2 = \frac{\sum_{i=1}^{3} \mathrm{d}x^{i\,2}}{(1 + kr^2/4)^2} \; , \quad r^2 \equiv \sum_{i=1}^{3} x^{i\,2} \; . \tag{12.40}$$

By a suitable choice of the time coordinate t, the line element of the metric of a spatially homogeneous and isotropic spacetime can then be written as

$$\mathrm{d}s^2 = -c^2 \mathrm{d}t^2 + a^2(t) \mathrm{d}l^2 \; , \tag{12.41}$$

because the scaling function $a(t)$ must not depend on the x^i in order to preserve isotropy and homogeneity. The metric (12.41) of a spatially homogeneous and isotropic spacetime is called *Robertson-Walker metric*.

Correspondingly, we choose the appropriate dual basis

$$\theta^0 = c\mathrm{d}t \; , \quad \theta^i = \frac{a(t)\,\mathrm{d}x^i}{1 + kr^2/4} \; , \tag{12.42}$$

in terms of which the metric coefficients are $g = \mathrm{diag}(-1, 1, 1, 1)$.

The exterior derivatives of the dual basis are

$$
\begin{aligned}
\mathrm{d}\theta^0 &= 0\ , \\
\mathrm{d}\theta^i &= \frac{\dot{a}\,\mathrm{d}t\wedge\mathrm{d}x^i}{1+kr^2/4} - \frac{a}{(1+kr^2/4)^2}\,\frac{k}{2}x_j\,\mathrm{d}x^j\wedge\mathrm{d}x^i \\
&= \frac{\dot{a}}{ca}\,\theta^0\wedge\theta^i + \frac{kx_j}{2a}\,\theta^i\wedge\theta^j\ .
\end{aligned} \tag{12.43}
$$

Since the exterior derivative of the metric is $\mathrm{d}g = 0$, Cartan's first structure equation (8.13) implies

$$
\omega^i_{\ j}\wedge\theta^j = -\mathrm{d}\theta^i\ , \tag{12.44}
$$

suggesting the curvature forms

$$
\begin{aligned}
\omega^0_{\ i} &= \omega^i_{\ 0} = \frac{\dot{a}}{ca}\,\theta^i\ , \\
\omega^i_{\ j} &= -\omega^j_{\ i} = \frac{k}{2a}\left(x_i\,\theta^j - x_j\,\theta^i\right)\ ,
\end{aligned} \tag{12.45}
$$

which evidently satisfy (12.44).

Their exterior derivatives are

$$
\begin{aligned}
\mathrm{d}\omega^0_{\ i} &= \frac{\ddot{a}a-\dot{a}^2}{c^2a^2}\,\theta^0\wedge\theta^i + \frac{\dot{a}}{ca}\,\mathrm{d}\theta^i \\
&= \frac{\ddot{a}}{c^2a}\,\theta^0\wedge\theta^i + \frac{k\dot{a}x_j}{2ca^2}\,\theta^i\wedge\theta^j
\end{aligned} \tag{12.46}
$$

and

$$
\begin{aligned}
\mathrm{d}\omega^i_{\ j} &= -\frac{k\dot{a}}{2a^2}\,\theta^0\wedge\left(x_i\theta^j - x_j\theta^i\right) \\
&\quad + \frac{k}{2a}\left(\mathrm{d}x_i\wedge\theta^j - \mathrm{d}x_j\wedge\theta^i + x_i\mathrm{d}\theta^j - x_j\mathrm{d}\theta^i\right) \\
&= \frac{k}{a^2}\left(1+\frac{k}{4}r^2\right)\theta^i\wedge\theta^j + \frac{k^2}{4a^2}\left(x_ix_k\,\theta^j\wedge\theta^k - x_jx_k\,\theta^i\wedge\theta^k\right)\ .
\end{aligned} \tag{12.47}
$$

Cartan's second structure equation (8.13) then gives the curvature forms

$$
\begin{aligned}
\Omega^0_{\ i} &= \mathrm{d}\omega^0_{\ i} + \omega^0_{\ k}\wedge\omega^k_{\ i} = \frac{\ddot{a}}{c^2a}\,\theta^0\wedge\theta^i\ , \\
\Omega^i_{\ j} &= \mathrm{d}\omega^i_{\ j} + \omega^i_{\ 0}\wedge\omega^0_{\ j} + \omega^i_{\ k}\wedge\omega^k_{\ j} = \frac{k+\dot{a}^2/c^2}{a^2}\,\theta^i\wedge\theta^j\ ,
\end{aligned} \tag{12.48}
$$

from which we obtain the components of the Ricci tensor

$$
R_{\mu\nu} = \bar{R}^{\alpha}{}_{\mu\alpha\nu} = \Omega^\alpha_{\ \mu}(e_\alpha, e_\nu) \tag{12.49}
$$

as

$$
R_{00} = -\frac{3\ddot{a}}{c^2a}\ , \quad R_{11} = R_{22} = R_{33} = \frac{\ddot{a}}{c^2a} + 2\,\frac{k+\dot{a}^2/c^2}{a^2}\ . \tag{12.50}
$$

The Ricci scalar is then

$$\mathcal{R} = R^\mu_\mu = 6\left(\frac{\ddot a}{c^2 a} + \frac{k + \dot a^2/c^2}{a^2}\right) . \tag{12.51}$$

Einstein tensor for a spatially homogeneous and isotropic spacetime

The Einstein tensor of a spatially homogeneous and isotropic spacetime has the components

$$G_{00} = 3\,\frac{k + \dot a^2/c^2}{a^2} , \quad G_{11} = G_{22} = G_{33} = -\frac{2\ddot a}{c^2 a} - \frac{k + \dot a^2/c^2}{a^2} . \tag{12.52}$$

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Beginning with the dual basis (12.43), carry out all calculations leading to (12.50) and (12.51) yourself.

12.3.2 From Einstein to Friedmann

For Einstein's field equations to be satisfied, the energy-momentum tensor must be diagonal, and its components must not depend on the spatial coordinates in order to preserve isotropy and homogeneity. We set $T_{00} = \rho c^2$, which is the total energy density, and $T_{ij} = p\delta_{ij}$, where p is the pressure.

This corresponds to the energy-momentum tensor of an ideal fluid,

$$T = \left(\rho + \frac{p}{c^2}\right) u^\flat \otimes u^\flat + pg \tag{12.53}$$

as seen by a fundamental observer (i.e. an observer for whom the spatial hypersurfaces are isotropic). For such an observer, $u = c\partial_t$, and since the metric is Minkowskian in the tetrad (12.42), the components of T are simply $T_{00} = \rho c^2$ and $T_{ii} = p$.

Then, Einstein's field equations in the form (6.80) *with* the cosmological constant Λ reduce to

$$\begin{aligned} 3\frac{k + \dot a^2/c^2}{a^2} &= \frac{8\pi\mathcal{G}}{c^2}\rho + \Lambda , \\ -\frac{2\ddot a}{c^2 a} - \frac{k + \dot a^2/c^2}{a^2} &= \frac{8\pi\mathcal{G}}{c^4}p - \Lambda . \end{aligned} \tag{12.54}$$

Adding a third of the first equation to the second, and re-writing the first equation, we find *Friedmann's equations*.

Friedmann's equations

For a spatially homogeneous and isotropic spacetime with the Robertson-Walker metric (12.41), Einstein's field equations reduce to Friedmann's equations,

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\mathcal{G}}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi\mathcal{G}}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \ . \tag{12.55}$$

A Robertson-Walker metric whose scale factor satisfies Friedmann's equations is called *Friedmann-Lemaître-Robertson-Walker metric*.

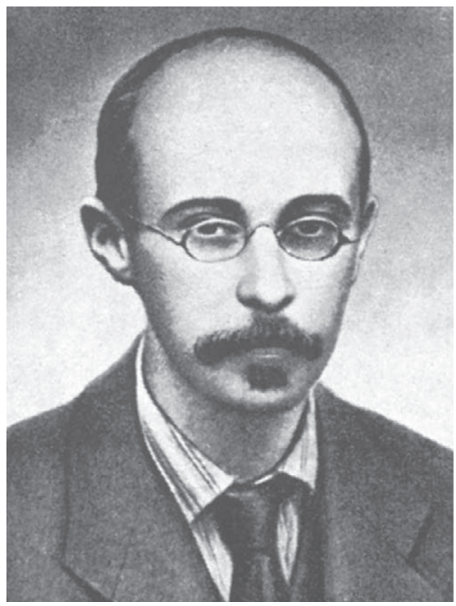

Figure 12.1 Alexander A. Friedmann (1888–1925), Russian physicist and mathematician. Source: Wikipedia

12.4 Density evolution and redshift

12.4.1 Density evolution

After multiplication with $3a^2$ and differentiation with respect to t, Friedmann's first equation gives

$$6\dot{a}\ddot{a} = 8\pi\mathcal{G}(\dot{\rho}a^2 + 2\rho a\dot{a}) + 2\Lambda c^2 a\dot{a} \ . \tag{12.56}$$

If we eliminate

$$6\dot{a}\ddot{a} = -8\pi G a\dot{a}\left(\rho + \frac{3p}{c^2}\right) + 2\Lambda c^2 a\dot{a} \tag{12.57}$$

by means of Friedmann's second equation, we find

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{12.58}$$

for the evolution of the density ρ with time.

This equation has a very intuitive meaning. To see it, let us consider the energy contained in a volume V_0, which changes over time in proportion to $V_0 a^3$, and employ the first law of thermodynamics,

$$\mathrm{d}(\rho c^2 V_0 a^3) + p\mathrm{d}(V_0 a^3) = 0 \quad \Rightarrow \quad \mathrm{d}(\rho c^2 a^3) + p\mathrm{d}(a^3) = 0\ . \tag{12.59}$$

We can use the first law of thermodynamics here because isotropy forbids any energy currents, thus no energy can flow into or out of the volume a^3.

Equation (12.59) yields

$$a^3\dot{\rho} + 3\rho a^2\dot{a} + \frac{3p}{c^2}a^2\dot{a} = 0\ , \tag{12.60}$$

which is identical to (12.58). This demonstrates that (12.58) simply expresses energy-momentum conservation. Consequently, one can show that it also follows from the contracted second Bianchi identity, $\nabla \cdot T = 0$.

?

Why are energy and momentum conserved here, but not in general?

Two limits are typically considered for (12.58). First, if matter moves non-relativistically, $p \ll \rho c^2$, and we can assume $p \approx 0$. Then,

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}\ , \tag{12.61}$$

which implies

$$\rho = \rho_0 a^{-3} \tag{12.62}$$

if ρ_0 is the density when $a = 1$.

Second, relativistic matter has $p = \rho c^2/3$, with which we obtain

$$\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a} \tag{12.63}$$

and thus

$$\rho = \rho_0 a^{-4}\ . \tag{12.64}$$

This shows that the density of non-relativistic matter drops as expected in proportion to the inverse volume, but the density of relativistic matter drops faster by one order of the scale factor. An explanation will be given below.

12.4.2 Cosmological redshift

We can write the line element (12.41) in the form

$$ds^2 = -c^2 dt^2 + a^2(t) dl^2 \,, \tag{12.65}$$

where dl^2 is the line element of a three-space with constant curvature k. Since light propagates on null geodesics, (12.65) implies

?

What do the two different signs in (12.66) mean or imply?

$$c dt = \pm a(t) dl \,. \tag{12.66}$$

Suppose a light signal leaves the source at the coordinate time t_0 and reaches the observer at t_1, then (12.66) shows that the coordinate time satisfies the equation

$$\int_{t_0}^{t_1} \frac{c dt}{a(t)} = \int_{\text{source}}^{\text{observer}} dl \,, \tag{12.67}$$

whose right-hand side is time-independent. Thus, for another light signal leaving the source at $t_0 + dt_0$ and reaching the observer at $t_1 + dt_1$, we have

$$\int_{t_0}^{t_1} \frac{c dt}{a(t)} = \int_{t_0+dt_0}^{t_1+dt_1} \frac{c dt}{a(t)} \,. \tag{12.68}$$

Since this implies

$$\int_{t_0}^{t_0+dt_0} \frac{dt}{a(t)} = \int_{t_1}^{t_1+dt_1} \frac{dt}{a(t)} \,, \tag{12.69}$$

we find for sufficiently small $dt_{0,1}$ that

$$\frac{dt_0}{a(t_0)} = \frac{dt_1}{a(t_1)} \,. \tag{12.70}$$

We can now identify the time intervals $dt_{0,1}$ with the inverse frequencies of the emitted and observed light, $dt_i = \nu_i^{-1}$ for $i = 0, 1$. This shows that the emitted and observed frequencies are related by

$$\frac{\nu_0}{\nu_1} = \frac{a(t_1)}{a(t_0)} \,. \tag{12.71}$$

Since the redshift z is defined in terms of the wavelengths as

$$z = \frac{\lambda_1 - \lambda_0}{\lambda_0} = \frac{\nu_0 - \nu_1}{\nu_1} \,, \tag{12.72}$$

we find that light emitted at t_0 and observed at t_1 is redshifted by

$$1 + z = \frac{\lambda_1}{\lambda_0} = \frac{a(t_1)}{a(t_0)} \,. \tag{12.73}$$

Cosmological redshift

The expansion or contraction of spacetime according to Friedmann's equations causes the wavelength of light to be increased or decreased in the same proportion as the universe itself expands or contracts.

We can now interpret the result (12.64) that the density of relativistic matter drops by one power of a more than expected by mere dilution: as the universe expands, relativistic particles are redshifted by another factor a and thus loose energy in addition to their dilution.

12.4.3 Alternative forms of the metric

Before we proceed, we bring the spatial line element $\mathrm{d}l$ from (12.40) into a different form. We first write it in terms of spherical polar coordinates as

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)}{(1 + kr^2/4)^2} \tag{12.74}$$

and introduce a new radial coordinate u defined by

$$u = \frac{r}{1 + kr^2/4} \ . \tag{12.75}$$

Requiring that $r \approx u$ for small r and u, we can uniquely solve (12.75) to find

$$r = \frac{2}{ku}\left(1 - \sqrt{1 - ku^2}\right) \ , \tag{12.76}$$

which implies the differential

$$\mathrm{d}(ru) = \frac{2u\mathrm{d}u}{\sqrt{1 - ku^2}} \ . \tag{12.77}$$

At the same time, (12.75) requires

$$\mathrm{d}(ru) = \mathrm{d}\left(\frac{r^2}{1 + kr^2/4}\right) = \frac{2r\mathrm{d}r}{(1 + kr^2/4)^2} = \frac{2u\mathrm{d}r}{1 + kr^2/4} \tag{12.78}$$

and thus

$$\frac{\mathrm{d}r}{1 + kr^2/4} = \frac{\mathrm{d}u}{\sqrt{1 - ku^2}} \ . \tag{12.79}$$

In terms of the new radial coordinate u, we can thus write the spatial line element of the metric in the frequently used form

$$\mathrm{d}l^2 = \frac{\mathrm{d}u^2}{1 - ku^2} + u^2\mathrm{d}\Omega^2 \ , \tag{12.80}$$

where $\mathrm{d}\Omega$ abbreviates the solid-angle element. The constant k can be positive, negative or zero, but its absolute value does not matter since it merely scales the coordinates. Therefore, we can normalise the coordinates such that $k = 0, \pm 1$.

Yet another form of the metric is found by introducing a radial coordinate w such that

$$\mathrm{d}w = \frac{\mathrm{d}u}{\sqrt{1-ku^2}} \,. \tag{12.81}$$

Integrating both sides, we find that this is satisfied if

$$u = f_k(w) \equiv \begin{cases} k^{-1/2}\sin\left(k^{1/2}w\right) & (k>0) \\ w & (k=0) \\ |k|^{-1/2}\sinh\left(|k|^{1/2}w\right) & (k<0) \end{cases} \,. \tag{12.82}$$

?

What advantages or disadvantages could alternative forms of the Friedmann-Lemaître-Robertson-Walker (or any other) metric have?

Equivalent forms of the Robertson-Walker metric

We thus find that the homogeneous and isotropic class of cosmological models based on Einstein's field equations are characterised by the line element

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}w^2 + f_k^2(w)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)\right] \tag{12.83}$$

which is equivalent to

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}u^2}{1-ku^2} + u^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)\right] \tag{12.84}$$

with u related to w by (12.82), and the scale factor $a(t)$ satisfies the Friedmann equations (12.55).

Metrics with line elements of the form (12.83) or (12.84) are called *Robertson-Walker metrics*, and *Friedmann-Lemaître-Robertson-Walker metrics* if their scale factor satisfies Friedmann's equations.