Chapter 5

Differential Geometry III

5.1 The Lie derivative

5.1.1 The Pull-Back

Following (2.28), we considered one-parameter groups of diffeomorphisms

$$\gamma_t : \mathbb{R} \times M \to M \tag{5.1}$$

such that points $p \in M$ can be considered as being transported along curves

$$\gamma : \mathbb{R} \to M \tag{5.2}$$

with $\gamma(0) = p$. Similarly, the diffeomorphism γ_t can be taken at fixed $t \in \mathbb{R}$, defining a diffeomorphism

$$\gamma_t : M \to M \tag{5.3}$$

which maps the manifold onto itself and satisfies $\gamma_t \circ \gamma_s = \gamma_{s+t}$.

We have seen the relationship between vector fields and one-parameter groups of diffeomorphisms before. Let now v be a vector field on M and γ from (5.2) be chosen such that the tangent vector $\dot{\gamma}(t)$ defined by

$$(\dot{\gamma}(t))(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t) \tag{5.4}$$

is identical with v, $\dot{\gamma} = v$. Then γ is called an *integral curve* of v.

If this is true for *all* curves γ obtained from γ_t by specifying initial points $\gamma(0)$, the result is called the *flow* of v.

The domain of definition $\mathcal{D}$ of γ_t can be a subset of $\mathbb{R}\times M$. If $\mathcal{D} = \mathbb{R}\times M$, the vector field is said to be *complete* and γ_t is called the *global flow* of v.

If $\mathcal{D}$ is restricted to open intervals $I \subset \mathbb{R}$ and open neighbourhoods $U \subset M$, thus $\mathcal{D} = I \times U \subset \mathbb{R} \times M$, the flow is called *local*.

Pull-back

Let now M and N be two manifolds and $\phi : M \to N$ a map from M onto N. A function f defined at a point $q \in N$ can be defined at a point $p \in M$ with $q = \phi(p)$ by

$$\phi^* f : M \to \mathbb{R} \,, \quad (\phi^* f)(p) := (f \circ \phi)(p) = f[\phi(p)] \,. \tag{5.5}$$

The map ϕ^* "pulls" functions f on N "back" to M and is thus called the *pull-back*.

Similarly, the pull-back allows to map vectors v from the tangent space T_pM of M in p onto vectors from the tangent space T_qN of N in q. We can first pull-back the function f defined in $q \in N$ to $p \in M$ and then apply v on it, and identify the result as a vector $\phi_* v$ applied to f,

$$\phi_* : T_pM \to T_qN \,, \quad v \mapsto \phi_* v = v \circ \phi^* \,, \tag{5.6}$$

such that $(\phi_* v)(f) = v(\phi^* f) = v(f \circ \phi)$. This defines a vector from the tangent space of N in $q = \phi(p)$.

Push-forward

The map ϕ_* "pushes" vectors from the tangent space of M in p to the tangent space of N in q and is thus called the *push-forward*.

In a natural generalisation to dual vectors, we define their pull-back ϕ^* by

$$\phi^* : T_q^*N \to T_p^*M \,, \quad w \mapsto \phi^* w = w \circ \phi_* \,, \tag{5.7}$$

such that $(\phi^* w)(v) = w(\phi_* v) = w(v \circ \phi^*)$, where $w \in T_q^*N$ is an element of the dual space of N in q. This operation "pulls back" the dual vector w from the dual space in $q = \phi(p) \in N$ to $p \in M$.

The pull-back ϕ^* and the push-forward ϕ_* can now be extended to tensors. Let T be a tensor field of rank $(0, r)$ on N, then its pull-back is defined by

$$\phi^* : \mathcal{T}_r^0(N) \to \mathcal{T}_r^0(M) \,, \quad T \mapsto \phi^* T = T \circ \phi_* \,, \tag{5.8}$$

such that $(\phi^* T)(v_1 \, \ldots, v_r) = T(\phi_* v_1, \ldots, \phi_* v_r)$. Similarly, we can define the pull-back of a tensor field of rank $(r, 0)$ on N by

$$\phi^* : \mathcal{T}_0^r(N) \to \mathcal{T}_0^r(M) \,, \quad T \mapsto \phi^* T \tag{5.9}$$

such that $(\phi^* T)(\phi^* w_1, \ldots, \phi^* w_r) = T(w_1, \ldots, w_r)$.

If the pull-back ϕ^* is a diffeomorphism, which implies in particular that the dimensions of M and N are equal, the pull-back and the push-forward are each other's inverses,

$$\phi_* = (\phi^*)^{-1} \,. \tag{5.10}$$

Irrespective of the rank of a tensor, we now denote by ϕ^* the pull-back of the tensor and by ϕ_* its inverse, i.e.

$$\begin{aligned} \phi^* &: \mathcal{T}^r_s(N) \to \mathcal{T}^r_s(M) \;, \\ \phi_* &: \mathcal{T}^r_s(M) \to \mathcal{T}^r_s(N) \;. \end{aligned} \tag{5.11}$$

The important point is that if $\phi^* : M \to M$ is a diffeomorphism and T is a tensor field on M, then $\phi^* T$ can be compared to T.

Symmetry transformations

If $\phi^* T = T$, ϕ^* is a *symmetry transformation* of T because T stays the same even though it was "moved" by ϕ^*. If the tensor field is the metric g, such a symmetry transformation of g is called an *isometry*.

5.1.2 The Lie Derivative

Lie derivative

Let now v be a vector field on M and γ_t be the flow of v. Then, for an arbitrary tensor $T \in \mathcal{T}^r_s$, the expression

$$\mathcal{L}_v T := \lim_{t \to 0} \frac{\gamma_t^* T - T}{t} \tag{5.12}$$

is called the *Lie derivative* of the tensor T with respect to v.

Caution While the covariant derivative determines how vectors and tensors change when moved across a given manifold, the Lie derivative determines how these objects change upon transformations of the manifold itself. ◀

Note that this definition naturally generalises the ordinary derivative with respect to "time" t. The manifold M is infinitesimally transformed by one element γ_t of a one-parameter group of diffeomorphisms. This could, for instance, represent an infinitesimal rotation of the two-sphere S^2. The tensor T on the manifold *after* the transformation is pulled back to the manifold before the transformation, where it can be compared to the original tensor T before the transformation.

Obviously, the Lie derivative of a rank-(r, s) tensor is itself a rank-(r, s) tensor. It is linear,

$$\mathcal{L}_v(t_1 + t_2) = \mathcal{L}_v(t_1) + \mathcal{L}_v(t_2) \;, \tag{5.13}$$

satisfies the Leibniz rule

$$\mathcal{L}_v(t_1 \otimes t_2) = \mathcal{L}_v(t_1) \otimes t_2 + t_1 \otimes \mathcal{L}_v(t_2) \;, \tag{5.14}$$

and it commutes with contractions. So far, these properties are easy to verify in particular after choosing local coordinates.

The application of the Lie derivative to a function f follows directly from the definition (5.4) of the tangent vector $\dot{\gamma}$,

$$\begin{aligned}\mathcal{L}_v f &= \lim_{t\to 0}\frac{\gamma_t^* f - f}{t} = \lim_{t\to 0}\frac{(f\circ\gamma_t)-(f\circ\gamma_0)}{t}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(f\circ\gamma) = \dot{\gamma}f = vf = \mathrm{d}f(v)\ .\end{aligned} \tag{5.15}$$

The additional convenient property

$$\mathcal{L}_x y = [x,y] \tag{5.16}$$

for vector fields y is non-trivial to prove.

Given two vector fields x and y, the Lie derivative further satisfies the linearity relations

$$\mathcal{L}_{x+y} = \mathcal{L}_x + \mathcal{L}_y\ , \quad \mathcal{L}_{\lambda x} = \lambda\mathcal{L}_x\ , \tag{5.17}$$

with $\lambda\in\mathbb{R}$, and the commutation relation

$$\mathcal{L}_{[x,y]} = [\mathcal{L}_x,\mathcal{L}_y] = \mathcal{L}_x\circ\mathcal{L}_y - \mathcal{L}_y\circ\mathcal{L}_x\ . \tag{5.18}$$

If and only if two vector fields x and y commute, so do the respective Lie derivatives,

$$[x,y]=0 \quad\Leftrightarrow\quad \mathcal{L}_x\circ\mathcal{L}_y = \mathcal{L}_y\circ\mathcal{L}_x\ . \tag{5.19}$$

If ϕ and ψ are the flows of x and y, the following commutation relation is equivalent to (5.19),

$$\phi_s\circ\psi_t = \psi_t\circ\phi_s\ . \tag{5.20}$$

Let $t\in\mathcal{T}_r^0$ be a rank-$(0,r)$ tensor field and $v_1,\dots,v_r$ be vector fields, then

$$\begin{aligned}(\mathcal{L}_x t)(v_1,\dots,v_r) &= x(t(v_1,\dots,v_r))\\ &\quad - \sum_{i=1}^r t(v_1,\dots,[x,v_i],\dots,v_r)\ .\end{aligned} \tag{5.21}$$

?

Compute the Lie derivative of a rank-$(1,0)$ tensor field.

To demonstrate this, we apply the Lie derivative to the tensor product of t and all v_i and use the Leibniz rule (5.14),

$$\begin{aligned}\mathcal{L}_x(t\otimes v_1\otimes\ldots\otimes v_r) &= \mathcal{L}_x t\otimes v_1\otimes\ldots\otimes v_r\\ &\quad + t\otimes\mathcal{L}_x v_1\otimes\ldots\otimes v_r + \ldots\\ &\quad + t\otimes v_1\otimes\ldots\otimes\mathcal{L}_x v_r\ .\end{aligned} \tag{5.22}$$

Then, we take the complete contraction and use the fact that the Lie derivative commutes with contractions, which yields

$$\begin{aligned}\mathcal{L}_x(t(v_1,\dots,v_r)) &= (\mathcal{L}_x t)(v_1,\dots,v_r)\\ &\quad + t(\mathcal{L}_x v_1,\dots,v_r) + \ldots + t(v_1,\dots,\mathcal{L}_x v_r)\ .\end{aligned} \tag{5.23}$$

Inserting (5.16), we now obtain (5.21).

As an example, we apply (5.21) to a tensor of rank $(0, 1)$, i.e. a dual vector w:

$$(\mathcal{L}_x w)(y) = x w(y) - w([x, y]) \ . \tag{5.24}$$

One particular dual vector is the differential of a function f, defined in (2.35). Inserting $\mathrm{d}f$ for w in (5.24) yields the useful relation

$$\begin{aligned}(\mathcal{L}_x \mathrm{d}f)(y) &= x\mathrm{d}f(y) - \mathrm{d}f([x, y]) \\ &= xy(f) - [x, y](f) = yx(f) \\ &= y\mathcal{L}_x f = \mathrm{d}\mathcal{L}_x f(y) \ ,\end{aligned} \tag{5.25}$$

and since this holds for any vector field y, we find

$$\mathcal{L}_x \mathrm{d}f = \mathrm{d}\mathcal{L}_x f \ . \tag{5.26}$$

Using the latter expression, we can derive coordinate expressions for the Lie derivative. We introduce the coordinate basis $\{\partial_i\}$ and its dual basis $\{\mathrm{d}x^i\}$ and apply (5.26) to $\mathrm{d}x^i$,

$$\mathcal{L}_v \mathrm{d}x^i = \mathrm{d}\mathcal{L}_v x^i = \mathrm{d}v(x^i) = \mathrm{d}v^j \partial_j x^i = \mathrm{d}v^i = \partial_j v^i \, \mathrm{d}x^j \ . \tag{5.27}$$

The Lie derivative of the basis vectors ∂_i is

$$\mathcal{L}_v \partial_i = [v, \partial_i] = -(\partial_i v^j)\partial_j \ , \tag{5.28}$$

where (2.32) was used in the second step.

Example: Lie derivative of a rank-$(1, 1)$ tensor field

To illustrate the components of the Lie derivative of a tensor, we take a tensor t of rank $(1, 1)$ and apply the Lie derivative to the tensor product $t \otimes \mathrm{d}x^i \otimes \partial x_j$,

$$\begin{aligned}\mathcal{L}_v(t \otimes \mathrm{d}x^i \otimes \partial_j) &= (\mathcal{L}_v t) \otimes \mathrm{d}x^i \otimes \partial_j \\ &\quad + t \otimes \mathcal{L}_v \mathrm{d}x^i \otimes \partial_j + t \otimes \mathrm{d}x^i \otimes \mathcal{L}_v \partial_j \ ,\end{aligned} \tag{5.29}$$

and now contract completely. This yields

$$\begin{aligned}\mathcal{L}_v t^i_j &= (\mathcal{L}_v t)^i_j + t(\partial_k v^i \mathrm{d}x^k, \partial_j) - t(\mathrm{d}x^i, \partial_j v^k \partial_k) \\ &= (\mathcal{L}_v t)^i_j + t^k_j \partial_k v^i - t^i_k \partial_j v^k \ .\end{aligned} \tag{5.30}$$

Solving for the components of the Lie derivative of t, we thus obtain

$$(\mathcal{L}_v t)^i_j = v^k \partial_k t^i_j - t^k_j \partial_k v^i + t^i_k \partial_j v^k \ , \tag{5.31}$$

and similarly for tensors of higher ranks. ◀

In particular, for a tensor of rank $(0, 1)$, i.e. a dual vector w,

$$(\mathcal{L}_v w)_i = v^k \partial_k w_i + w_k \partial_i v^k \ . \tag{5.32}$$

5.2 Killing vector fields

Killing vector fields

A Killing vector field K is a vector field along which the Lie derivative of the metric vanishes,

$$\mathcal{L}_K g = 0\ . \tag{5.33}$$

This implies that the flow of a Killing vector field defines a symmetry transformation of the metric, i.e. an *isometry*.

To find a coordinate expression, we use (5.31) to write

$$\begin{aligned}(\mathcal{L}_K g)_{ij} &= K^k\partial_k g_{ij} + g_{kj}\partial_i K^k + g_{ik}\partial_j K^k \\ &= K^k(\partial_k g_{ij} - \partial_i g_{kj} - \partial_j g_{ik}) + \partial_i(g_{kj}K^k) + \partial_j(g_{ik}K^k) \\ &= \nabla_i K_j + \nabla_j K_i = 0\ ,\end{aligned} \tag{5.34}$$

?

Derive the Killing equation (5.34) yourself.

where we have identified the Christoffel symbols (3.74) in the last step. This is the *Killing equation*.

Let γ be a geodesic, i.e. a curve satisfying

$$\nabla_{\dot\gamma}\dot\gamma = 0\ , \tag{5.35}$$

then the projection of a Killing vector K on the tangent to the geodesic $\dot\gamma$ is constant along the geodesic,

$$\nabla_{\dot\gamma}\langle\dot\gamma, K\rangle = 0\ . \tag{5.36}$$

This is easily seen as follows. First,

$$\nabla_{\dot\gamma}\langle\dot\gamma, K\rangle = \langle\nabla_{\dot\gamma}\dot\gamma, K\rangle + \langle\dot\gamma, \nabla_{\dot\gamma}K\rangle = \langle\dot\gamma, \nabla_{\dot\gamma}K\rangle \tag{5.37}$$

because of the geodesic equation (5.35).

Writing the last expression explicitly in components yields

$$\langle\dot\gamma, \nabla_{\dot\gamma}K\rangle = g_{ik}\dot\gamma^i\dot\gamma^j\nabla_j K^k = \dot\gamma^i\dot\gamma^j\nabla_j K_i\ , \tag{5.38}$$

changing indices and using the symmetry of the metric, we can also write it as

$$\langle\dot\gamma, \nabla_{\dot\gamma}K\rangle = g_{jk}\dot\gamma^j\dot\gamma^i\nabla_i K^k = \dot\gamma^j\dot\gamma^i\nabla_i K_j\ . \tag{5.39}$$

Adding the latter two equations and using the Killing equation (5.34) shows

$$2\langle\dot\gamma, \nabla_{\dot\gamma}K\rangle = \dot\gamma^i\dot\gamma^j\left(\nabla_i K_j + \nabla_j K_i\right) = 0\ , \tag{5.40}$$

which proves (5.36). More elegantly, we have contracted the symmetric tensor $\dot\gamma^i\dot\gamma^j$ with the tensor $\nabla_i K_j$ which is antisymmetric because of the Killing equation, thus the result must vanish.

Equation (5.36) has a profound meaning:

Conservation laws from Killing vector fields

Freely-falling particles and light rays both follow geodesics. The constancy of $\langle \dot{\gamma}, K \rangle$ along geodesics means that each Killing vector field gives rise to a conserved quantity for freely-falling particles and light rays. Since a Killing vector field generates an isometry, this shows that symmetry transformations of the metric give rise to conservation laws.

5.3 Differential forms

5.3.1 Definition

Differential p-forms are totally antisymmetric tensors of rank $(0, p)$. The most simple example are dual vectors $w \in T_p^*M$ since they are tensors of rank $(0, 1)$. A general tensor t of rank $(0, 2)$ is not antisymmetric, but can be antisymmetrised defining the two-form

$$\tau(v_1, v_2) \equiv \frac{1}{2} \left[t(v_1, v_2) - t(v_2, v_1) \right] \ , \tag{5.41}$$

with two vectors $v_1, v_2 \in V$.

To generalise this operation for tensors of arbitrary ranks $(0, r)$, we first define the *alternation operator* by

$$(\mathcal{A}t)(v_1, \ldots, v_r) := \frac{1}{r!} \sum_\pi \operatorname{sgn}(\pi) t(v_{\pi(1)}, \ldots, v_{\pi(r)}) \ , \tag{5.42}$$

where the sum extends over all permutations π of the integer numbers from 1 to r. The sign of a permutation, $\operatorname{sgn}(\pi)$, is negative if the permutation is odd and positive otherwise.

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As an exercise, explicitly apply the alternation operator to a tensor field of rank $(0, 3)$.

In components, we briefly write

$$(\mathcal{A}t)_{i_1 \ldots i_r} = t_{[i_1 \ldots i_r]} \tag{5.43}$$

so that p-forms ω are defined by the relation

$$\omega_{i_1 \ldots i_p} = \omega_{[i_1 \ldots i_p]} \tag{5.44}$$

between their components. For example, for a 2-form ω we have

$$\omega_{ij} = \omega_{[ij]} = \frac{1}{2} \left(\omega_{ij} - \omega_{ji} \right) \ . \tag{5.45}$$

The vector space of p-forms is denoted by $\bigwedge^p$. Taking the product of two differential forms $\omega \in \bigwedge^p$ and $\eta \in \bigwedge^q$ yields a tensor of rank $(0, p + q)$

which is not antisymmetric, but can be antisymmetrised by means of the alternation operator. The result

$$\omega \wedge \eta \equiv \frac{(p+q)!}{p!q!}\mathcal{A}(\omega \otimes \eta) \tag{5.46}$$

is called the *exterior product* . Evidently, it turns the tensor $\omega \otimes \eta \in \mathcal{T}^0_{p+q}$ into a $(p+q)$-form.

The definition of the exterior product implies that it is bilinear, associative, and satisfies

$$\omega \wedge \eta = (-1)^{pq}\, \eta \wedge \omega \; . \tag{5.47}$$

A basis for the vector space $\bigwedge^p$ can be constructed from the basis $\{\mathrm{d}x^i\}$, $1 \leq i \leq n$, of the dual space V^* by taking

$$\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p} \quad \text{with} \quad 1 \leq i_1 < \ldots < i_p \leq n \; , \tag{5.48}$$

which shows that the dimension of $\bigwedge^p$ is

$$\begin{pmatrix} n \\ p \end{pmatrix} \equiv \frac{n!}{p!(n-p)!} \tag{5.49}$$

for $p \leq n$ and zero otherwise. The skewed commutation relation (5.47) implies

$$\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i \; . \tag{5.50}$$

Given two vector spaces V and W above the same field F, the Cartesian product $V \times W$ of the two spaces can be turned into a vector space by defining the vector-space operations component-wise. Let $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$, then the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \; , \quad \lambda(v, w) = (\lambda v, \lambda w) \tag{5.51}$$

with $\lambda \in F$ give $V \times W$ the structure of a vector space $V \oplus W$ which is called the *direct sum* of V and W.

Vector space of differential forms

Similarly, we define the vector space of differential forms

$$\bigwedge \equiv \bigoplus_{p=0}^{n} \bigwedge^p \tag{5.52}$$

as the direct sum of the vector spaces of p-forms with arbitrary $p \leq n$.

Recalling that a vector space V attains the structure of an *algebra* by defining a vector-valued product between two vectors,

$$\times : V \times V \to V \; , \quad (v, w) \mapsto v \times w \; , \tag{5.53}$$

we see that the exterior product $\wedge$ gives the vector space $\bigwedge$ of differential forms the structure of a *Grassmann algebra*,

$$\wedge : \bigwedge \times \bigwedge \to \bigwedge , \quad (\omega, \eta) \mapsto \omega \wedge \eta . \tag{5.54}$$

The *interior product* of a p-form ω with a vector $v \in V$ is a mapping

$$V \times \bigwedge^p \to \bigwedge^{p-1} , \quad (v, \omega) \mapsto i_v \omega \tag{5.55}$$

defined by

$$(i_v\omega)(v_1, \ldots, v_{p-1}) \equiv \omega(v, v_1, \ldots, v_{p-1}) \tag{5.56}$$

and $i_v\omega = 0$ if ω is 0-form (a number or a function).

Caution A Grassmann algebra (named after Hermann Graßmann, 1809–1877) is an associative, skew-symmetric, graduated algebra with an identity element. ◂

5.3.2 The Exterior Derivative

For p-forms ω, we now define the *exterior derivative* as a map d,

$$\mathrm{d} : \bigwedge^p \to \bigwedge^{p+1} , \quad \omega \mapsto \mathrm{d}\omega , \tag{5.57}$$

with the following three properties:

(i) d is an *antiderivation* of degree 1 on $\bigwedge$, i.e. it satisfies

$$\mathrm{d}\,(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta \tag{5.58}$$

for $\omega \in \bigwedge^p$ and $\eta \in \bigwedge$.

(ii) $\mathrm{d} \circ \mathrm{d} = 0$.

(iii) For every function $f \in \mathcal{F}$, $\mathrm{d}f$ is the differential of f, i.e. $\mathrm{d}f(v) = v(f)$ for $v \in TM$.

The exterior derivative is *unique*. By properties (i) and (ii), we directly find

$$\mathrm{d}\omega = \sum_{i_1<\ldots<i_p} \mathrm{d}\omega_{i_1\ldots i_p} \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p} \tag{5.59}$$

for any p-form

$$\omega = \sum_{i_1<\ldots<i_p} \omega_{i_1\ldots i_p}\, \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p} . \tag{5.60}$$

According to (5.59), the components of the exterior derivative of a p-form ω can be written as

$$(\mathrm{d}\omega)_{i_1\ldots i_{p+1}} = (p+1)\, \partial_{[i_1}\omega_{i_2\ldots i_{p+1}]} . \tag{5.61}$$

Since $\omega_{i_2 \dots i_{p+1}}$ is itself antisymmetric, this last expression can be brought into the form

$$(\mathrm{d}\omega)_{i_1 \dots i_{p+1}} = \sum_{k=1}^{p+1} (-1)^{k+1} \partial_{i_k} \omega_{i_1, \dots, \hat{i}_k, \dots i_{p+1}} \,, \tag{5.62}$$

with $1 \le i_1 < \ldots < i_p < i_{p+1} \le n$. Indices marked with a hat are left out.

The Lie derivative, the interior product and the exterior derivative are related by *Cartan's equation*

$$\mathcal{L}_v = \mathrm{d} \circ i_v + i_v \circ \mathrm{d} \,. \tag{5.63}$$

?

Verify the expressions (5.61) and (5.62).

Cartan's equation implies the convenient formula for the exterior derivative of a p-form ω

$$\begin{aligned} \mathrm{d}\omega(v_1, \dots, v_{p+1}) = &\sum_{i=1}^{p+1} (-1)^{i+1} v_i \omega(v_1, \dots, \hat{v}_i, \dots, v_{p+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([v_i, v_j], v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{p+1}) \,, \end{aligned} \tag{5.64}$$

where the hat over a symbol means that this object is to be left out.

Example: Exterior derivative of a 1-form

For an example, let us apply these relations to a 1-form $\omega = \omega_i \mathrm{d}x^i$. For it, equation (5.59) implies

$$\mathrm{d}\omega = \mathrm{d}\omega_i \wedge \mathrm{d}x^i = \partial_j \omega_i \, \mathrm{d}x^j \wedge \mathrm{d}x^i \tag{5.65}$$

while (5.64) specialises to

$$\mathrm{d}\omega(v_1, v_2) = v_1 \omega(v_2) - v_2 \omega(v_1) - \omega([v_1, v_2]) \,. \tag{5.66}$$

With (5.61) or (5.62), we find the components

$$\mathrm{d}\omega_{ij} = \partial_i \omega_j - \partial_j \omega_i \tag{5.67}$$

of the exterior derivative of the 1-form. ◂

In $\mathbb{R}^3$, the expression (5.65) turns into

$$\begin{aligned} \mathrm{d}\omega = &(\partial_1 \omega_2 - \partial_2 \omega_1) \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 + (\partial_1 \omega_3 - \partial_3 \omega_1) \, \mathrm{d}x^1 \wedge \mathrm{d}x^3 \\ &+ (\partial_2 \omega_3 - \partial_3 \omega_2) \, \mathrm{d}x^2 \wedge \mathrm{d}x^3 \,. \end{aligned} \tag{5.68}$$

Closed and exact forms

A differential p-form α is called *exact* if a $(p-1)$-form β exists such that $\alpha = \mathrm{d}\beta$. If $\mathrm{d}\alpha = 0$, the p-form α is called *closed*. Obviously, an exact form is closed because of $\mathrm{d} \circ \mathrm{d} = 0$.

5.4 Integration

5.4.1 The Volume Form and the Codifferential

An atlas of a differentiable manifold is called *oriented* if for every pair of charts h_1 on $U_1 \subset M$ and h_2 on $U_2 \subset M$ with $U_1 \cap U_2 \neq 0$, the Jacobi determinant of the coordinate change $h_2 \circ h_1^{-1}$ is positive.

Volume form

An n-dimensional, paracompact manifold M is orientable if and only if a C^∞, n-form exists on M which vanishes nowhere. This is called a *volume form*.
The *canonical volume form* on a pseudo-Riemannian manifold (M, g) is defined by

$$\eta \equiv \sqrt{|g|}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \; . \tag{5.69}$$

This definition is independent of the coordinate system because it transforms proportional to the Jacobian determinant upon coordinate changes.

Equation (5.69) implies that the components of the canonical volume form in n dimensions are proportional to the n-dimensional Levi-Civita symbol,

$$\eta_{i_1 \ldots i_n} = \sqrt{|g|}\, \varepsilon_{i_1 \ldots i_n} \; , \tag{5.70}$$

which is defined such that it is +1 for even permutations of the $i_1, \ldots, i_n$, −1 for odd permutations, and vanishes if any two of its indices are equal. A very useful relation is

$$\varepsilon^{j_1 \ldots j_q k_1 \ldots k_p} \varepsilon_{j_1 \ldots j_q i_1 \ldots i_p} = p!q!\, \delta^{k_1}_{[i_1} \delta^{k_2}_{i_2} \ldots \delta^{k_p}_{i_p]} \; , \tag{5.71}$$

where the square brackets again denote the complete antisymmetrisation. In three dimensions, one specific example for (5.71) is the familiar formula

$$\varepsilon^{ijk} \varepsilon_{klm} = \varepsilon^{kij} \varepsilon_{klm} = \delta^i_l \delta^j_m - \delta^i_m \delta^j_l \; . \tag{5.72}$$

Note that $p = 1$ and $q = 2$ here, but the factor $2! = 2$ is cancelled by the antisymmetrisation.

Hodge star operator

The *Hodge star operator* ($*$-operation) turns a p form ω into an $(n - p)$-form $(*\omega)$,

$$* : \bigwedge^p \to \bigwedge^{n-p} \; , \quad \omega \mapsto *\omega \; , \tag{5.73}$$

which is uniquely defined by its application to the dual basis.

For the basis $\{\mathrm{d}x^i\}$ of the dual space $T_p^* M$,

$$*(\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p}) := \frac{\sqrt{|g|}}{(n-p)!} \varepsilon^{i_1 \ldots i_p}{}_{i_{p+1} \ldots i_n}\, \mathrm{d}x^{i_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{i_n} \; . \tag{5.74}$$

If the dual basis $\{e^i\}$ is orthonormal, this simplifies to

$$*(e^{i_1} \wedge \ldots \wedge e^{i_p}) = e^{i_{p+1}} \wedge \ldots \wedge e^{i_n} \ . \qquad (5.75)$$

In components, we can write

$$(*\omega)_{i_{p+1}\ldots i_n} = \frac{1}{p!}\,\eta_{i_1\ldots i_n}\omega^{i_1\ldots i_p} \ , \qquad (5.76)$$

i.e. $(*\omega)$ is the volume form η contracted with the p-form ω. A straightforward calculation shows that

$$*(*\omega) = \mathrm{sgn}(g)(-1)^{p(n-p)}\omega \ . \qquad (5.77)$$

?

Verify the statement (5.77).

Example: Hodge dual in three dimensions

For a 1-form $\omega = \omega_i \mathrm{d}x^i$ in $\mathbb{R}^3$, we can use

$$*\mathrm{d}x^1 = \mathrm{d}x^2 \wedge \mathrm{d}x^3 \ , \quad *\mathrm{d}x^2 = \mathrm{d}x^3 \wedge \mathrm{d}x^1 \ , \quad *\mathrm{d}x^3 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \qquad (5.78)$$

to find the Hodge-dual 2-form

$$*\omega = \omega_1 \mathrm{d}x^2 \wedge \mathrm{d}x^3 - \omega_2 \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \omega_3 \mathrm{d}x^1 \wedge \mathrm{d}x^2 \ , \qquad (5.79)$$

while the 2-form $\mathrm{d}\omega$ (5.68) has the Hodge dual 1-form

$$\begin{aligned} *\mathrm{d}\omega &= (\partial_2\omega_3 - \partial_3\omega_2)\mathrm{d}x^1 - (\partial_1\omega_3 - \partial_3\omega_1)\mathrm{d}x^2 \\ &\quad + (\partial_1\omega_2 - \partial_2\omega_1)\mathrm{d}x^3 = \varepsilon_i^{\ jk}\partial_j\omega_k \mathrm{d}x^i \ . \end{aligned} \qquad (5.80)$$

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Codifferential

The *codifferential* is a map

$$\delta : \bigwedge^p \to \bigwedge^{p-1} \ , \quad \omega \mapsto \delta\omega \qquad (5.81)$$

defined by

$$\delta\omega \equiv \mathrm{sgn}(g)(-1)^{n(p+1)}(*\mathrm{d}*)\omega \ . \qquad (5.82)$$

$\mathrm{d} \circ \mathrm{d} = 0$ immediately implies $\delta \circ \delta = 0$.

By successive application of (5.71) and (5.62), it can be shown that the coordinate expression for the codifferential is

$$(\delta\omega)^{i_1\ldots i_{p-1}} = \frac{1}{\sqrt{|g|}}\partial_k\left(\sqrt{|g|}\omega^{k i_1\ldots i_{p-1}}\right) \ . \qquad (5.83)$$

Comparing this with (4.59), we see that this generalises the divergence of ω. To see this more explicitly, let us work out the codifferential of a 1-form in $\mathbb{R}^3$ by first taking the exterior derivative of $*\omega$ from (5.79),

$$\mathrm{d}*\omega = (\partial_1\omega_1 + \partial_2\omega_2 + \partial_3\omega_3)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \ , \qquad (5.84)$$

whose Hodge dual is

$$\delta\omega = \partial_1\omega_1 + \partial_2\omega_2 + \partial_3\omega_3 \ . \tag{5.85}$$

Example: Maxwell's equations

The Faraday 2-form is defined by

$$F \equiv \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ . \tag{5.86}$$

Application of (5.62) shows that

$$\begin{aligned}(\mathrm{d}F)_{\lambda\mu\nu} &= \partial_\lambda F_{\mu\nu} - \partial_\mu F_{\lambda\nu} + \partial_\nu F_{\lambda\mu} \\ &= \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \ ,\end{aligned} \tag{5.87}$$

i.e. the homogeneous Maxwell equations can simply be expressed by

$$\mathrm{d}F = 0 \ . \tag{5.88}$$

Similarly, the components of the codifferential of the Faraday form are, according to (5.83) and (4.62)

$$(\delta F)^\mu = \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} F^{\nu\mu} \right) = \nabla_\nu F^{\nu\mu} = -\frac{4\pi}{c}\, j^\mu \ . \tag{5.89}$$

Introducing further the current 1-form by $j = j_\mu \mathrm{d}x^\mu$, we can thus write the inhomogeneous Maxwell equations as

$$\delta F = -\frac{4\pi}{c}\, j \ . \tag{5.90}$$

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5.4.2 Integrals and Integral Theorems

The integral over an n-form ω,

$$\int_M \omega \ , \tag{5.91}$$

is defined in the following way: Suppose first that the support $U \subset M$ of ω is contained in a single chart which defines positive coordinates $(x^1, \ldots, x^n)$ on U. Then, if $\omega = f\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ with a function $f \in \mathcal{F}(U)$,

$$\int_M \omega = \int_U f(x^1, \ldots, x^n) \mathrm{d}x^1 \ldots \mathrm{d}x^n \ . \tag{5.92}$$

Note that this definition is *independent* of the coordinate system because upon changes of the coordinate system, both f and the volume element $\mathrm{d}x^1 \dots \mathrm{d}x^n$ change in proportion to the Jacobian determinant of the coordinate change.

If the domain of the n-form ω is contained in multiple maps, the integral (5.92) needs to be defined piece-wise, but the principle remains the same.

The integration of functions $f \in \mathcal{F}(M)$ is achieved using the canonical volume form η,

$$\int_M f \equiv \int_M f\eta \,. \tag{5.93}$$

Caution Like $\flat$ lowers a note by a semitone in music, the $\flat$ operator lowers the index of vector components and thus turns them into dual-vector components. Analogously, $\sharp$ raises notes by semitones in music, and indices of dual-vector components. ◂

Integral theorems

Stokes' theorem can now be formulated as follows: let M be an n-dimensional manifold and the region $D \subset M$ have a smooth boundary ∂D such that $\bar{D} \equiv D \cup \partial D$ is compact. Then, for every $n-1$-form ω, we have

$$\int_D \mathrm{d}\omega = \int_{\partial D} \omega \,. \tag{5.94}$$

Likewise, *Gauss' theorem* can be brought into the form

$$\int_D \delta x^\flat \eta = \int_{\partial D} * x^\flat \,, \tag{5.95}$$

where $x \in TM$ is a vector field on M and $x^\flat$ is the 1-form belonging to this vector field.

Musical operators

Generally, the *musical operators* $\flat$ and $\sharp$ are isomorphisms between the tangent spaces of a manifold and their dual spaces given by the metric,

$$\flat : TM \to T^*M \,, \quad v \mapsto v^\flat \,, \quad v^\flat_i = g_{ij}v^j \tag{5.96}$$

and similarly by the inverse of the metric,

$$\sharp : T^*M \to TM \,, \quad w \mapsto w^\sharp \,, \quad (w^\sharp)^i = g^{ij}w_j \,. \tag{5.97}$$

The essence of the differential-geometric concepts introduced here are summarised in Appendix B.