0 Introduction

Subject of this course are numerical algorithms for solving problems in Linear Algebra, such as linear algebraic systems and corresponding matrix eigenvalue problems. The emphasis is on iterative methods suitable for large-scale problems arising, e. g., in the discretization of partial differential equations and in network problems.

0.1 Basic notation of Linear Algebra and Analysis

At first, we introduce some standard notation in the context of (finite dimensional) vector spaces of functions and their derivatives. Let $\mathbb{K}$ denote the field of real or complex numbers $\mathbb{R}$ or $\mathbb{C}$, respectively. Accordingly, for $n \in \mathbb{N}$, let $\mathbb{K}^n$ denote the n-dimensional vector space of n-tuples $x = (x_1, \ldots, x_n)$ with components $x_i \in \mathbb{K}$, $i = 1, \ldots, n$. For these addition and scalar multiplication are defined by:

$$x + y := (x_1 + y_1, \ldots, x_n + y_n), \qquad \alpha x := (\alpha x_1, \ldots, \alpha x_n), \quad \alpha \in \mathbb{K}.$$

The elements $x \in \mathbb{K}^n$ are, depending on the suitable interpretation, addressed as "points" or "vectors" . Here, one may imagine x as the end point of a vector attached at the origin of the chosen Cartesian[1] coordinate system and the components x_i as its "coordinates" with respect to this coordinate system. In general, we consider vectors as "column vectors". Within the "vector calculus" its row version is written as $(x_1, \ldots, x_n)^T$. The null (or zero) vector $(0, \ldots, 0)$ may also be briefly written as 0. Usually, we prefer this coordinate-oriented notation over a coordinate-free notation because of its greater clearness. A set of vectors $\{a^1, \ldots, a^k\}$ in $\mathbb{K}^n$ is called "linearly independent" if

$$\sum_{i=1}^{k} c_i a^i = 0, \;\; c_i \in \mathbb{K} \quad \Rightarrow \quad c_i = 0, \;\; i = 1, \ldots, k.$$

Such a set of $k = n$ linearly independent vectors is called a "basis" of $\mathbb{K}^n$, which spans all of $\mathbb{K}^n$, i. e., each element $x \in \mathbb{K}^n$ can be (uniquely) written as a linear combination of the form

$$x = \sum_{i=1}^{n} c_i a^i, \;\; c_i \in \mathbb{K}.$$

Each (finite dimensional) vector space, such as $\mathbb{K}^n$, possesses a basis. The special "Cartesian basis" $\{e^1, \ldots, e^n\}$ is formed by the "Cartesian unit vectors" $e^i := (\delta_{1i}, \ldots, \delta_{ni})$, $\delta_{ii} = 1$ and $\delta_{ij} = 0$, for $i \neq j$, being the usual Kronecker symbol. The elements of this basis are mutually orthonormal, i. e., with respect to the Euclidian scalar product, there holds $(e^i, e^j)_2 := \sum_{k=1}^{n} e^i_k e^j_k = \delta_{ij}$. "Matrices" $A \in \mathbb{K}^{n \times n}$ are two-dimensional square arrays of numbers from $\mathbb{K}$ written in the form $A = (a_{ij})_{i,j=1}^{n}$, where the first index, i,

[1] René Descartes (1596–1650): French mathematician and philosopher ("(ego) cogito ergo sum"); worked in the Netherlands and later in Stockholm; first to recognize the close relation between geometry and arithmetic and founded analytic geometry.

refers to the row and the second one, j, to the column (counted from the left upper corner of the array) at which the element a_{ij} is positioned. Usually, matrices are *square* arrays, but in some situations also rectangular matrices may occur. The set of (square) matrices forms a vector space with addition and scalar multiplication defined in the natural elementwise sense,

$$A = (a_{ij})_{i,j=1}^n,\ B = (b_{ij})_{i,j=1}^n,\ c \in \mathbb{K} \quad \Rightarrow \quad cA + B = (ca_{ij} + b_{ij})_{i,j=1}^n.$$

For matrices and vectors natural multiplications are defined by

$$Ax = \Big(\sum_{k=1}^d a_{ik} x_k \Big)_{i=1}^n \in \mathbb{K}^n, \quad AB = \Big(\sum_{k=1}^d a_{ik} b_{kj} \Big)_{i,j=1}^n \in \mathbb{K}^{n \times n}.$$

Matrices are used to represent linear mappings in $\mathbb{K}^d$ with respect to a given basis, mostly a Cartesian basis, $\varphi(x) = Ax$. By $\bar{A}^T = (a_{ij}^T)_{i,j=1}^n$, we denote the conjugate "transpose" of a matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{K}^{n \times n}$ with the elements $a_{ij}^T = \bar{a}_{ji}$. For matrices $A, B \in \mathbb{K}^{n \times n}$ there holds $(AB)^T = B^T A^T$. Matrices for which $A = \bar{A}^T$ are called "symmetric" in the case $\mathbb{K} = \mathbb{R}$ and "Hermitian" in the case $\mathbb{K} = \mathbb{C}$.

0.2 Linear algebraic systems and eigenvalue problems

Let A be an $m \times n$-matrix and b an m-vector,

$$A = (a_{jk})_{j,k=1}^{m,n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \qquad b = (b_j)_{j=1}^m = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

We seek an n-vector $x = (x_k)_{k=1,\ldots,n}$ such that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{0.2.1}$$

or written in short as $Ax = b$. This is called a "linear system" (of equations). It is called "underdetermined" for $m < n$, "quadratic" for $m = n$, and "overdetermined" for $m > n$. The linear system is solvable if and only if $\operatorname{rank}(A) = \operatorname{rank}([A, b])$ ($\operatorname{rank}(A)$ = number of linearly independent columns of A) with the composed matrix

$$[A, b] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right].$$

In the "quadratic" case the solvability of the system (0.2.1) is equivalent to any one of the following properties of the coefficient matrix $A \in \mathbb{K}^{n\times n}$:

- $Ax = 0$ implies $x = 0$.
- $\operatorname{rank}(A) = n$.
- $\det(A) \neq 0$.
- All eigenvalues of A are nonzero.

A number $\lambda \in \mathbb{C}$ is called "eigenvalue" of the (quadratic) matrix $A \in \mathbb{K}^{n\times n}$ if there exists a corresponding vector $w \in \mathbb{K}^n \setminus \{0\}$, called "eigenvector", such that

$$Aw = \lambda w. \tag{0.2.2}$$

Eigenvalues are just the zeros of the characteristic polynomial $\chi_A(z) := \det(A - zI)$ of A, so that by the fundamental theorem of Algebra each $n \times n$-matrix has exactly n eigenvalues counted accordingly to their (algebraic) multiplicities. The corresponding eigenvectors span linear subspaces of $\mathbb{K}^n$ called "eigenspaces".

Eigenvalue problems play an important role in many problems from science and engineering, e. g., they represent energy levels in physical models (e. g., Schrödinger equation in Quantum Mechanics) or determine the stability or instability of solutions of dynamical systems (e. g., Navier-Stokes equations in hydrodynamics).

0.3 Numerical approaches

We will mainly consider numerical methods for solving *quadratic* linear systems and associated eigenvalue problems. The emphasis will be on medium- and large-scale problems, i. e., problems of dimension $n \approx 10^4 - 10^9$, which at the upper end impose particularly strong requirements on the algorithms with respect to storage and work efficiency. Problems of that size usually involve matrices with special structure such as "band structure" and/or extreme "sparsity", i. e., only very few matrix elements in each row are non-zero. Most of the classical methods, which have originally been designed for "full" but smaller matrices, cannot be realistically applied to such large problems. Therefore, modern methods extensively exploit the particular sparsity structure of the matrices. These methods split into two classes, "direct methods" and "iterative methods".

Definition 0.1: *A "direct" method for the solution of a linear system $Ax = b$ is an algorithm, which (neglecting round-off errors) delivers the exact solution x in finitely many arithmetic steps. "Gaussian elimination" is a typical example of such a "direct method". In contrast to that an "iterative method" constructs a sequence of approximate solutions $\{x^t\}_{t\in\mathbb{N}}$, which only in the limit $t \to \infty$ converge to the exact solution, i. e., $\lim_{t\to\infty} x^t = x$. "Richardson iteration" or more general fixed-point methods of similar kind are typical example of such "iterative methods". In analyzing a direct method, we are mainly interested in the work count, i. e., the asymptotic number of arithmetic operations needed for achieving the final result depending on the problem size, e. g., $\mathcal{O}(n^3)$, while*

in an iterative method, we look at the work count needed for one iteration step and the number of iteration steps for reducing the initial error by a certain fixed factor, e. g., 10^{-1}*, or the asymptotic speed of convergence ("linear", "quadratic, etc.).*

However, there is no sharp separation between the two classes of "direct" or "iterative" methods as many theoretically "direct" methods are actually used in "iterative" form in practice. A typical method of this type is the classical "conjugate gradient (CG) method", which in principle is a direct method (after n iteration steps) but is usually terminated like an iterative methods already after $m \ll n$ steps.

0.4 Applications and origin of problems

we present some applications, from which large linear algebra problems originate. This illustrates how the various possible structures of matrices may look like. Thereby, we have to deal with scalar or vector-valued functions $u = u(x) \in \mathbb{K}^n$ for arguments $x \in \mathbb{K}^n$. For derivatives of differentiable functions, we use the notation

$$\partial_x u := \frac{\partial u}{\partial x}, \quad \partial_x^2 u := \frac{\partial^2 u}{\partial^2 x}, \ \dots \ , \quad \partial_i u := \frac{\partial u}{\partial x_i}, \quad \partial_{ij}^2 u := \frac{\partial^2 u}{\partial x_i \partial x_j}, \ \dots \ ,$$

and analogously also for higher-order derivatives. With the nabla operator ∇ the "gradient" of a scalar function and the "divergence" of a vector function are written as $\operatorname{grad} u = \nabla u := (\partial_1 u, ..., \partial_d u)^T$ and $\operatorname{div} u = \nabla \cdot u := \partial_1 u_1 + ... + \partial_d u_d$, respectively. For a vector $\beta \in \mathbb{R}^d$ the derivative in direction β is written as $\partial_\beta u := \beta \cdot \nabla u$. Combination of gradient and divergence yields the so-called "Laplacian operator"

$$\nabla \cdot \nabla u = \Delta u = \partial_1^2 u + ... + \partial_d^2 u.$$

The symbol $\nabla^m u$ denotes the "tensor" of all partial derivatives of order m of u, i. e., in two dimensions $u = u(x_1, x_2)$, $\nabla^2 u = (\partial_1^i \partial_2^j u)_{i+j=2}$.

0.4.1 Gaussian equalization calculus

A classical application in Astronomy is the Gaussian equalization calculus (method of least error-squares): For given functions $u^1, \dots, u^n$ and points $(x_j, y_j) \in \mathbb{R}^2$, $j = 1, \dots, m$, $m > n$, a linear combination

$$u(x) = \sum_{k=1}^{n} c_k u^k(x)$$

is to be determined such that the "mean deviation"

$$\Delta_2 := \Big(\sum_{j=1}^{m} |u(x_j) - y_j|^2 \Big)^{1/2}$$

becomes minimal. (The "Chebyshev[2] equalization problem" in which the "maximal deviation" $\Delta_\infty := \max_{j=1,\dots,m} |u(x_j) - y_j|$ is minimized poses much more severe difficulties and is therefore used only for smaller n.) For the solution of the Gaussian equalization problem, we set $y := (y_1, \dots, y_m)$, $c := (c_1, \dots, c_n)$ and

$$a^k := (u^k(x_1), \dots, u^k(x_m)) \,,\ k = 1, \dots, n \,, \qquad A \equiv [a^1, \dots, a^n].$$

Using this notation, now the quadratic functional

$$F(c) = \Big(\sum_{j=1}^{m} |(Ac - y)_j|^2 \Big)^{1/2}$$

is to be minimized with respect to $c \in \mathbb{R}^n$. This is equivalent to solving the overdetermined linear system $Ac = y$ in the sense of finding a vector c with minimal mean error-squares, i. e., with minimal "defect". In case that $\operatorname{rank}(A) = n$ this "minimal-defect solution" c is determined by the so-called "normal equation"

$$A^T A c = A^T y, \tag{0.4.3}$$

a linear $n \times n$-system with a positive definite (and hence regular) coefficient matrix $A^T A$. In the particular case of polynomial fitting, i. e., $u^k(x) = x^{k-1}$, the "optimal" solution

$$u(x) = \sum_{k=1}^{n} c_k x^{k-1}$$

called "Gaussian equalization parabola" for the points (x_j, y_j), $j = 1, \dots, m$. Because of the regularity of the "Vandermondian[3] determinant"

$$\det \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix} = \prod_{j,k=1, j<k}^{n} (x_k - x_j) \neq 0,$$

for mutually distinct points x_j there holds $\operatorname{rank}(A) = n$, i. e., the equalization parabola is uniquely determined.

[2] Pafnuty Lvovich Chebyshev (1821–1894): Russian mathematician; prof. in St. Petersburg; contributions to number theory, probability theory and especially to approximation theory; developed the general theory of orthogonal polynomials.

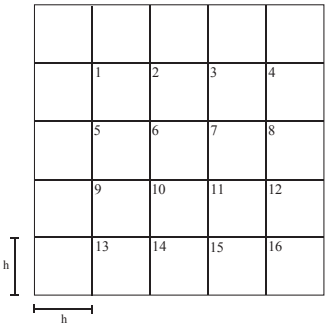
[3] Alexandre-Thophile Vandermonde (1735–1796): French mathematician; gifted musician, came late to mathematics and published here only four papers (nevertheless member of the Academy of Sciences in Paris); contributions to theory of determinants and combinatorial problem (curiously enough the determinant called after him does not appear explicitly in his papers).

0.4.2 Discretization of elliptic PDEs

The numerical solution of partial differential equations requires an appropriate "discretization" of the differential operator, e. g., by a "difference approximation" of the derivatives. Consider, for example, the "first boundary value problem of the Laplacian[4] operator",

$$Lu := -\Delta u = f \ \text{ in } \ \Omega, \quad u = g \ \text{ on } \ \partial\Omega, \tag{0.4.4}$$

posed on a domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$. Here, for a given (continuous) right-hand side function $f = f(x_1, x_2)$ and boundary function $g = g(x_1, x_2)$ a function $u = u(x_1, x_2)$ is to be determined, which is twice differentiable on Ω and continuous on $\bar{\Omega}$, such that (0.4.4) holds. The region $\overline{\Omega}$, e. g., the unit square, is covered by an equidistant Cartesian mesh Ω_h with "mesh boundary" $\partial\Omega_h$. The mesh points $P \in \Omega_h$ may be numbered row-wise.



$h = \frac{1}{m+1}$ mesh width

$n = m^2$ "interior" mesh points

Figure 1: *Finite difference mesh*

At "interior" mesh points $P \in \Omega_h$ the differential operators in x_1- and x_2-direction are approximated by second-order central difference quotients, which act on mesh functions $u_h(P)$. This results in "difference equations" of the form

$$L_h u_h(P) := \sum_{Q \in N(P)} \sigma(P,Q) u_h(Q) = f_h(P), \quad P \in \Omega_h, \tag{0.4.5}$$

$$u_h(P) = g_h(P), \quad P \in \partial\Omega_h, \tag{0.4.6}$$

with certain mesh neighborhoods $N(P) \subset \Omega \cup \partial\Omega_h$ of points $P \in \Omega_h$ and approximations $f_h(\cdot)$ to f and $g_h(\cdot)$ to g. We set the coefficients $\sigma(P,Q) := 0$ for points $Q \notin N(P)$. The considered difference operator based on second-order central difference quotients for approximating second derivatives is called "5-point difference operator" since it uses 5 points (Accordingly, its three-dimensional analogue is called "7-point difference opera-

[4]Pierre Simon Marquis de Laplace (1749–1827): French mathematician and astronomer; prof. in Paris; founded among other fields probability calculus.

tor"). Then, for $P \in \Omega_h$ there holds

$$\sum_{Q\in\Omega_h} \sigma(P,Q)u_h(Q) = f_h(P) - \sum_{Q\in\partial\Omega_h} \sigma(P,Q)g_h(Q). \tag{0.4.7}$$

For any numbering of the mesh points in Ω_h and $\partial\Omega_h$, $\Omega_h = \{P_i,\ i = 1, ..., n\}$, $\partial\Omega_h = \{P_i,\ i = n+1, ..., n+m\}$, we obtain a quadratic linear system for the vector of approximate mesh values $U = (U_i)_{i=1}^N$, $U_i := u_h(P_i)$.

$$AU = F, \tag{0.4.8}$$

with $A = (a_{ij})_{i,j=1}^n$, $F = (b_j)_{j=1}^n$, where

$$a_{ij} := \sigma(P_i, P_j), \quad b_j := f_h(P_j) - \sum_{i=n+1}^{n+m} \sigma(P_j, P_i)g_h(P_i).$$

In the considered special case of the unit square and row-wise numbering of the interior mesh points Ω_h the 5-point difference approximation of the Laplacian yields the following sparse matrix of dimension $n = m^2$:

$$A = \frac{1}{h^2}\begin{bmatrix} B_m & -I_m & & \\ -I_m & B_m & -I_m & \\ & -I_m & B_m & \ddots \\ & & \ddots & \ddots \end{bmatrix}\Bigg\}n \qquad B_m = \begin{bmatrix} 4 & -1 & & \\ -1 & 4 & -1 & \\ & -1 & 4 & \ddots \\ & & \ddots & \ddots \end{bmatrix}\Bigg\}m,$$

where I_m is the $m\times m$-unit matrix. The matrix A is a very sparse band matrix with half-band width m, symmetric and (irreducibly) diagonally dominant. This implies that it is regular and positive definite. In three dimensions the corresponding matrix has dimension $n = m^3$ and half-band width m^2 and shares all the other mentioned properties of its two-dimensional analogue. In practice, $n \gg 10^4$ up to $n \approx 10^7$ in three dimensions. If problem (0.4.4) is only part of a larger mathematical model involving complex domains and several physical quantities such as (in chemically reacting flow models) velocity, pressure, density, temperature and chemical species, the dimension of the complete system may reach up to $n \approx 10^7 - 10^9$.

To estimate a realistic size of the algebraic problem oriented by the needs of a practical application, we consider the above model problem (Poisson equation on the unit square) with an adjusted right-hand side and boundary function such that the exact solution is given by $u(x, y) = \sin(\pi x)\sin(\pi y)$,

$$-\Delta u = 2\pi^2 u =: f \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega. \tag{0.4.9}$$

For this setting the error analysis of the difference approximation yields the estimate

$$\max_{\overline{\Omega}_h} |u - u_h| \approx \tfrac{1}{24} d_\Omega^2 M_4(u) h^2 \approx 8h^2, \tag{0.4.10}$$

where $M_4(u) = \max_{\bar{\Omega}} |\nabla^4 u| \approx \pi^4$ (see the lecture notes Rannacher [3]). In order to guarantee a relative error below $\mathrm{TOL} = 10^{-3}$, we have to choose $h \approx 10^{-2}$ corresponding to $n \approx 10^4$ in two and $n \approx 10^6$ in three dimension. The concrete structure of the matrix A depends on the numbering of mesh points used:

i) Row-wise numbering: The lexicographical ordering of mesh points leads to a band matrix with band width $2m+1$. The sparsity within the band would be largely reduced by Gaussian elimination (so-called "fill-in").

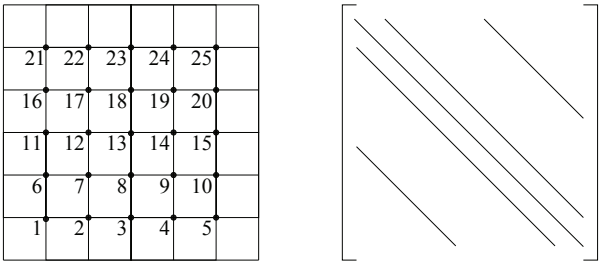


Figure 2: *Lexicographical ordering of mesh points*

ii) Diagonal numbering: The successive numbering diagonally to the Cartesian coordinate directions leads to a band matrix with less band volume. This results in less fill-in within Gaussian elimination.

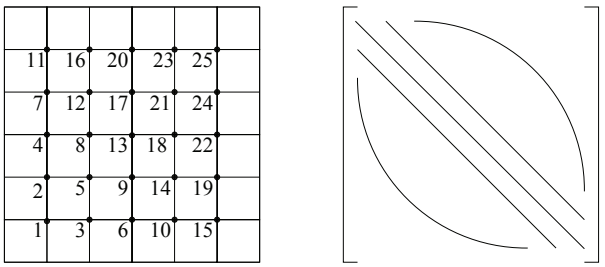


Figure 3: *Diagonal mesh-point numbering*

iii) Checker-board numbering: The staggered row-wise and column-wise numbering leads to a 2×2-block matrix with diagonal main blocks and band width $2m+1 \approx h^{-1}$.

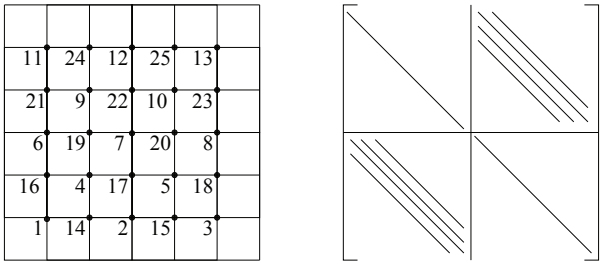


Figure 4: *Checkerboard mesh-point numbering*

For large linear systems of dimension $n > 10^5$ direct methods such as Gaussian elimination are difficult to realize since they are generally very storage and work demanding. For a matrix of dimension $n = 10^6$ and band width $m = 10^2$ Gaussian elimination requires already about 10^8 storage places. This is particularly undesirable if also the band is sparse as in the above example with at most 5 non-zero elements per row. In this case those *iterative* methods are more attractive, in which essentially only matrix-vector multiplications occur with matrices of similar sparsity pattern as that of A.

As illustrative examples, we consider simple fixed-point iterations for solving a linear system $Ax = b$ with a regular $n \times n$-coefficient matrix. The system is rewritten as

$$a_{jj}x_j + \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{jk}x_k = b_j, \quad j = 1, \ldots, n.$$

If $a_{jj} \neq 0$, this is equivalent to

$$x_j = \frac{1}{a_{jj}} \Big\{ b_j - \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{jk}x_k \Big\}, \quad j = 1, \ldots, n.$$

Then, the so-called "Jacobi method" generates iterates $x^t \in \mathbb{R}^n$, $t = 1, 2, \ldots$, by successively solving

$$x_j^t = \frac{1}{a_{jj}} \Big\{ b_j - \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{jk}x_k^{t-1} \Big\}, \quad j = 1, \ldots, n. \tag{0.4.11}$$

When computing x_j^t the preceding components $x_r^t, r < j$, are already known. Hence, in order to accelerate the convergence of the method, one may use this new information in the computation of x_j^t. This idea leads to the "Gauß-Seidel[5] method":

$$x_j^t = \frac{1}{a_{jj}} \Big\{ b_j - \sum_{k<j} a_{jk}x_k^t - \sum_{k>j} a_{jk}x_k^{t-1} \Big\}, \quad j = 1, \ldots, n. \tag{0.4.12}$$

[5]Philipp Ludwig von Seidel (1821–1896): German mathematician; Prof. in Munich; contributions to analysis (method of least error-squares) and celestial mechanics and astronomy.

The Gauß-Seidel method has the same arithmetic complexity as the Jacobi method but under certain conditions (satisfied in the above model situation) it converges twice as fast. However, though very simple and maximal storage economical, both methods, Jacobi as well as Gauß-Seidel, are by far too slow in practical applications. Much more efficient iterative methods are the Krylov-space methods. The best known examples are the classical "conjugate gradient method" ("CG method") of Hestenes and Stiefel for solving linear systems with positive definite matrices and the "Arnoldi method" for solving corresponding eigenvalue problems. Iterative methods with minimal complexity can be constructed using multi-scale concepts (e. g., geometric or algebraic "multigrid methods"). The latter type of methods will be discussed below.

0.4.3 Hydrodynamic stability analysis

Another origin of large-scale eigenvalue problems is hydrodynamic stability analysis. Let $\{\hat{v}, \hat{p}\}$ be a solution (the "base flow") of the stationary Navier-Stokes equation

$$\begin{aligned} &-\nu\Delta\hat{v} + \hat{v}\cdot\nabla\hat{v} + \nabla\hat{p} = 0, \quad \nabla\cdot\hat{v} = 0, \quad \text{in } \Omega, \\ &\hat{v}|_{\Gamma_{\text{rigid}}} = 0, \;\; \hat{v}|_{\Gamma_{\text{in}}} = v^{\text{in}}, \;\; \nu\partial_n\hat{v} - \hat{p}n|_{\Gamma_{\text{out}}} = P, \;\; \nu\partial_n\hat{v} - \hat{p}n|_{\Gamma_Q} = q, \end{aligned} \tag{0.4.13}$$

where $\hat{v}$ is the velocity vector field of the flow, $\hat{p}$ its hydrostatic pressure, ν the kinematic viscosity (for normalized density $\rho \equiv 1$), and q the control pressure. The flow is driven by a prescribed flow velocity v^{in} at the Dirichlet (inflow) boundary (at the left end), a prescribed mean pressure P at the Neumann (outflow) boundary (at the right end) and the mean pressure q at the control boundary Γ_Q. The (artificial) "free outflow" (also called "do nothing") boundary condition in (0.4.13) has proven successful especially in modeling pipe flow since it is satisfied by Poiseuille flow (see Heywood et al. [42]).

Figure 5: *Configuration of the flow control problem.*

Fig. 5 shows the configuration of a channel flow around an obstacle controlled by pressure prescription at Γ_Q, and Figure 6 the computational mesh and streamline plots of two flows for different Reynolds numbers and control values, one stable and one unstable.

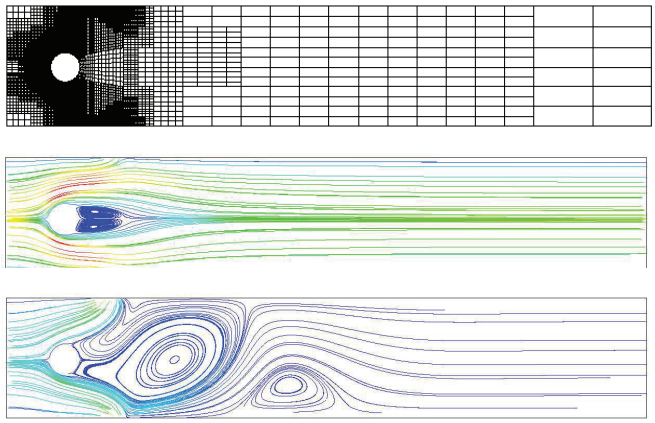


Figure 6: *Computational mesh (top), uncontrolled stable (middle) and controlled unstable (bottom) stationary channel flow around an obstacle.*

For deciding whether these base flows are stable or unstable, within the usual linearized stability analysis, one investigates the following eigenvalue problem corresponding to the Navier-Stokes operator linearized about the considered base flow:

$$\begin{aligned} &-\nu\Delta v + \hat{v}\cdot\nabla v + v\cdot\nabla\hat{v} + \nabla q = \lambda v, \quad \nabla\cdot v = 0, \quad \text{in } \Omega, \\ &v|_{\Gamma_{\text{rigid}}\cup\Gamma_{\text{in}}} = 0, \quad \nu\partial_n v - qn|_{\Gamma_{\text{out}}\cup\Gamma_Q} = 0. \end{aligned} \tag{0.4.14}$$

From the location of the eigenvalues in the complex plane, one can draw the following conclusion: If an eigenvalue $\lambda \in \mathbb{C}$ of (0.4.14) has $\operatorname{Re}\lambda < 0$, the base flow is unstable, otherwise it is said to be "linearly stable". This means that the solution of the linearized nonstationary perturbation problem

$$\begin{aligned} &\partial_t w - \nu\Delta w + \hat{v}\cdot\nabla w + w\cdot\nabla\hat{v} + \nabla q = 0, \quad \nabla\cdot w = 0, \quad \text{in } \Omega, \\ &w|_{\Gamma_{\text{rigid}}\cup\Gamma_{\text{in}}} = 0, \quad \nu\partial_n w - qn|_{\Gamma_{\text{out}}\cup\Gamma_Q} = 0 \end{aligned} \tag{0.4.15}$$

corresponding to an initial perturbation $w_{|t=0} = w_0$ satisfies a bound

$$\sup_{t\geq 0} \|w(t)\| \leq A\|w_0\|, \tag{0.4.16}$$

with some constant $A \geq 1$. After discretization the eigenvalue problem (0.4.14) in function space is translated into an nonsymmetric algebraic eigenvalue problem, which is usually of high dimension $n \approx 10^5 - 10^6$. Therefore its solution can be achieved only by iterative methods.

However, "linear stability" does not guarantee full "nonlinear stability" due to effects caused by the "non-normality" of the operator governing problem (0.4.14), which may cause the constant A to become large. This is related to the possible "deficiency" (discrepancy of geometric and algebraic multiplicity) or a large "pseudo-spectrum" (range

of large resolvent norm) of the critical eigenvalue. This effect is commonly accepted as explanation of the discrepancy in the stability properties of simple base flows such as Couette flow and Poiseuille flow predicted by linear eigenvalue-based stability analysis and experimental observation (see, e. g., Trefethen & Embree [22] and the literature cited therein).