B GRAVITY AND CONCEPTS OF RELATIVITY

B.1 *Metric structure of spacetime*

Spacetime is first of all a topological space, where the points are given coordinates by a continuous coordinate mapping (the system of open sets allows specifically to define continuity of a mapping), where the coordinates are arranged in a coordinate tuple, for instance $x^\mu = (ct, x^i)^t$. Unlike in vector spaces, differences between coordinates as distances have no meaning, but one needs a metric tensor to compute the line element $\mathrm{d}s^2$ from an infinitesimal coordinate difference $\mathrm{d}x^\mu$: As the metric tensor can change across the manifold, all definitions are only made in a *local* way.

Starting with a Euclidean manifold with a metric γ^{ij} one would write down for the line element

$$\mathrm{d}s^2 = \mathrm{d}x_i \mathrm{d}x^i = \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{B.30}$$

where the last equality is true for Cartesian coordinates as a particular coordinate choice, where $\gamma_{ij} = \delta_{ij}$. Euclidian space is a flat space with no curvature, and there is invariance of $\mathrm{d}s^2$ under rotations. Generalising to Minkowskian space, we get the line-element

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{B.31}$$

again with the last equality being applicable if Cartesian coordinates have been chosen. Minkowskian space is flat, too, there is no curvature and it is invariant under Lorentz-transformations. In opposite to the Euclidian line-element, the line-element is no longer positive definite, which means that there can be negative distances. In practice, this is never an actual issue, as only events with positive distances $\mathrm{d}s^2 > 0$ are causally related to each other. At the same time, $\mathrm{d}s^2 = 0$ defines a light cone structure for the manifold. Both examples are (pseudo-)Riemannian manifolds where the line element is given by a quadratic form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{B.32}$$

with a general metric tensor $g_{\mu\nu}$. In 4 dimensions there are 10 independent entries of $g_{\mu\nu}$ due to the symmetry $g_{\mu\nu} = g_{\nu\mu}$: Any anti-symmetric part would not be able to influence the value of $\mathrm{d}s^2$ as $\mathrm{d}x^\mu\mathrm{d}x^\nu$ is fully symmetric. On a manifold we will establish invariance of line elements as general scalars under *arbitrary* coordinate transforms, generalising the idea of the invariance of the Euclidean line element under rotations and the invariance of the Minkowski-line element under Lorentz-transforms. To make this specific, we have for an *invertible* and *differentiable* coordinate change (a so-called diffeomorphism):

$$x'^\rho = x'^\rho(x^\mu) \quad \rightarrow \quad \mathrm{d}x'^\rho = \frac{\partial x'^\rho}{\partial x^\mu}\mathrm{d}x^\mu \tag{B.33}$$

as well as the inverse

$$x^\mu = x^\mu(x'^\rho) \quad \rightarrow \quad \mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial x'^\rho}\mathrm{d}x'^\rho \tag{B.34}$$

If the line element is to be invariant as a scalar, the metric $g_{\mu\nu}$ needs to transform inversely to $\mathrm{d}x^\mu$:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = g_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}\mathrm{d}x'^\rho\mathrm{d}x'^\sigma = g'_{\rho\sigma}\mathrm{d}x'^\rho\mathrm{d}x'^\sigma \tag{B.35}$$

i.e.

$$g_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} = g'_{\rho\sigma} \tag{B.36}$$

and the general picture emerges that contravariant (superscript) indices transform with Jacobians $\frac{\partial x'^\rho}{\partial x^\mu}$ whereas covariant (subscript) indices transform with inverse Jacobians $\frac{\partial x^\mu}{\partial x'^\rho}$.

B.2 *Metric and inner products*

Picking up this idea lets us write for a vector υ^μ with contravariant indices

$$\upsilon^\mu \to \upsilon'^\rho = \frac{\partial x'^\rho}{\partial x^\mu}\upsilon^\mu \tag{B.37}$$

with a Jacobian $\frac{\partial x'^\rho}{\partial x^\mu}$ and for a linear form w_μ with covariant indices

$$w_\mu \to w'_\rho = \frac{\partial x^\mu}{\partial x'^\rho}w_\mu \tag{B.38}$$

with an inverse Jacobian $\frac{\partial x^\mu}{\partial x'^\rho}$, such that inner products stay invariant:

$$w_\mu\upsilon^\mu = g_{\mu\nu}w^\mu\upsilon^\nu \quad \to \quad w'_\mu\upsilon'^\mu = g'_{\mu\nu}w'^\mu\upsilon'^\nu = \underbrace{\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\sigma}}_{\delta^\rho_\mu} w_\rho\upsilon^\sigma = w_\sigma\upsilon^\sigma \tag{B.39}$$

as Jacobian cancels with the inverse Jacobian and simply a renaming of the indices is taking place.

The index shift carried out by the metric $\upsilon_\mu = g_{\mu\rho}\upsilon^\rho$ is undone by the inverse metric $\upsilon^\sigma = g^{\sigma\mu}\upsilon_\mu = g^{\sigma\mu}g_{\mu\rho}\upsilon^\rho = \delta^\sigma_\rho\upsilon^\rho$, such that the inverse metric fulfils

$$g^{\sigma\mu}g_{\mu\rho} = \delta^\sigma_\rho \tag{B.40}$$

Please keep in mind that

$$g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = 4 \tag{B.41}$$

in 4 dimensions, and *not equal* to 2, as on might (naively) think.

B.3 *Vectors and covariant derivatives*

Considering a curve $x^\mu(\lambda)$ with parameter λ cutting through a field $\varphi(x^\mu)$: How would φ change along the curve as λ changes? The chain rule suggests that

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi(x^\mu(\lambda)) = \underbrace{\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}}_{\text{tangent}}\frac{\partial\varphi}{\partial x^\mu} = u^\mu\frac{\partial\varphi}{\partial x^\mu} \tag{B.42}$$

with the tangent $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$, such that the rate of change of φ along the curve $x^\mu(\lambda)$ is given as a projection of the gradient field $\partial\varphi/\partial x^\mu = \partial_\mu\varphi$ onto the tangent u^μ. From this we recognise that u^μ as well as $\mathrm{d}x^\mu$ are vectors, and $\partial_\mu\varphi$ is a linear form. It is possible to run curves through a point A in all possible directions and construct vectors $\mathrm{d}x^\mu$ tangent to them, and the minimal collection of $\mathrm{d}x^\mu$ would constitute the basis of a tangent space $\mathrm{T_A M}$ of the manifold M at A, relative to which all tensor of vector fields can be expressed in components. Most sensibly, one would run these curves through A by changing a single coordinate at a time: But this implies that the construction of the basis for $\mathrm{T_A M}$ would depend on the coordinate choice and could be different at another point B! That has in fact profound implications when considering changes to a vector or tensor field across the manifold: The components of the field can become different because the basis has changed going from $\mathrm{T_A M}$ to $\mathrm{T_B M}$, or there could be a *genuine* change in the field, and the two cases would need to be distinguished.

But before we investigate that in detail, we should try out a remapping of the coordinates in equation B.42. Writing $x^\mu(x'^\alpha)$ we can introduce a "one" $\delta^\nu_\mu = \partial x^\nu/\partial x^\mu$ in the form of two mutually annihilating Jacobians,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\partial\varphi}{\partial x^\mu} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\delta^\nu_\mu\frac{\partial\varphi}{\partial x^\nu} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\underbrace{\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\alpha}}_{\delta^\nu_\mu}\frac{\partial\varphi}{\partial x^\nu} = \frac{\mathrm{d}x'^\alpha}{\mathrm{d}\lambda}\frac{\partial\varphi}{\partial x'^\alpha} \tag{B.43}$$

suggesting that vectors such as u^μ transform with the Jacobian $\frac{\partial x'^\alpha}{\partial x^\mu}$ while linear forms like $\partial_\mu\varphi$ transform with the inverse Jacobian $\frac{\partial x^\nu}{\partial x'^\alpha}$.

We need the concept of a parallel transport to quantify changes in a vector field υ^μ across the manifold M. The components of the vector are given in terms of a local coordinate frame which is the basis of $\mathrm{T_A M}$, and which might differ from the frame at $\mathrm{T_B M}$, implying that the same abstract vector υ could have different components at A and B: We need to disentangle changes of the tangent space from genuine changes in the vector field! For this purpose, one introduces parallel transport, which moves a vector perfectly from A to B and tracks only the change in tangent space. If the two points are separated by δx, the parallel-transported, perfect copy $\upsilon^\mu_\parallel$ at the point B with coordinates $x + \delta x$ of the original vector υ^μ at point A with coordinates x is given at linear order

$$\upsilon^\mu_\parallel(x+\delta x) = \upsilon^\mu(x) - \Gamma^\mu{}_{\alpha\beta}\,\upsilon^\alpha(x)\delta x^\beta + \ldots \tag{B.44}$$

where the minus-sign is chosen by convention. The coefficients $\Gamma^\mu{}_{\alpha\beta}$ form the Christoffel-symbol. A vector field would now change genuinely if it differs at position $x + \delta x$ from the parallel-transported vector field. We are now only comparing two vector fields $\upsilon^\mu(x+\delta x)$ and $\upsilon^\mu_\parallel(x+\delta x)$ at the same point within the same tangent space

T_BM, as opposed to a direct comparison of $\upsilon^\mu(x+\delta x)$ with $\upsilon^\mu(x)$ which is senseless as the tangent spaces T_AM and T_BM are in general different and the component expansion of υ exists in two different bases:

$$\lim_{\delta x^\beta \to 0} \frac{\upsilon^\mu(x+\delta x) - \upsilon^\mu_\parallel(x+\delta x)}{\delta x^\beta} = \lim_{\delta x^\beta \to 0} \frac{\upsilon^\mu(x+\delta x) - \upsilon^\mu(x)}{\delta x^\beta} + \Gamma^\mu{}_{\alpha\beta}\,\upsilon^\alpha(x)\frac{\delta x^\beta}{\delta x^\beta} = \partial_\beta \upsilon^\mu + \Gamma^\mu{}_{\alpha\beta}\,\upsilon^\alpha \equiv \nabla_\beta \upsilon^\alpha \quad \text{(B.45)}$$

Here, we have identified a straightforward index-by-index change of the vector field over the shift δx as the partial differentiation $\partial_\beta \upsilon^\mu$, which gets corrected by the Christoffel-symbol tracking the change of the tangent spaces.

It is important to realise that the covariant differentiation becomes only relevant for fields that have internal degrees of freedom, whose decomposition in components depend on the change in tangent space moving from T_AM to T_BM. Scalar fields are oblivious to these changes, and therefore the covariant differentiation falls back on the conventional partial differentiation:

$$\nabla_\beta \varphi = \partial_\beta \varphi \quad \text{(B.46)}$$

For higher-order tensorial fields one needs a Christoffel-symbol for each index: You can imagine that the basis for such an object is the Cartesian product, and that the differentiation fulfils a Leibnitz-rule, such that we get

$$\nabla_\beta T^{\mu\nu} = \partial_\beta T^{\mu\nu} + \Gamma^\mu{}_{\beta\alpha}\,T^{\alpha\nu} + \Gamma^\nu{}_{\beta\alpha}\,T^{\mu\alpha} \quad \text{(B.47)}$$

Let's now have a look at the differentiation of a covariant vector or, equivalently, a linear form. A contraction between the vector υ^μ and the linear form w_μ is scalar, so the covariant differentiation falls back onto the partial one:

$$\nabla_\beta(\upsilon^\mu w_\mu) = \partial_\beta(\upsilon^\mu w_\mu) = \partial_\beta \upsilon^\mu \cdot w_\mu + \upsilon^\mu \partial w_\mu \quad \text{(B.48)}$$

If, on the other side, the covariant differentiation comes with a Leibnitz-rule for dealing with products we would write

$$\nabla_\beta(\upsilon^\mu w_\mu) = \nabla_\beta \upsilon^\mu \cdot w_\mu + \upsilon^\mu \nabla w_\mu = \underbrace{(\partial_\beta \upsilon^\mu + \Gamma^\mu{}_{\alpha\beta}\,\upsilon^\alpha)}_{\partial_\beta \upsilon^\mu \text{ from above}} w_\mu + \upsilon^\mu \underbrace{\nabla_\beta w_\mu}_{\text{isolate this term}} \quad \text{(B.49)}$$

Isolating the covariant derivative $\nabla_\beta w_\mu$ of the linear form w_μ we get:

$$\upsilon^\mu \nabla_\beta w_\mu = \upsilon^\mu \partial_\beta w^\mu - \Gamma^\mu{}_{\alpha\beta}\,\upsilon^\alpha w_\mu = \upsilon^\mu \partial_\beta w^\mu - \Gamma^\alpha{}_{\mu\beta}\,\upsilon^\mu w_\alpha \quad \text{(B.50)}$$

after renaming indices and finally

$$\nabla_\beta w_\mu = \partial_\beta w_\mu - \Gamma^\alpha{}_{\mu\beta}\,w_\alpha \quad \text{(B.51)}$$

implying that a linear form needs a negative Christoffel-symbol, as opposed to a vector with a positive Christoffel-term.

With this definition the covariant derivative depends completely on the choice of the connection coefficients $\Gamma^{\alpha}{}_{\mu\nu}$, but we should be guided by the idea that the two structures that exist on the manifold, (i) the metric structure which allows the measurements of angles between vectors and determinations of their lengths, and (ii) the differential structure which quantifies rates of change of vectors, should be compatible with each other. Specifically, if two vectors are parallel-transported, their length and relative orientation should not change, and as a consequence their scalar product should be unaffected. With the covariant derivative

$$\nabla_\beta \upsilon^\mu = \lim_{\delta x^\beta \to 0} \frac{\upsilon^\mu(x+\delta x) - \upsilon^\mu_{\|}(x+\delta x)}{\delta x^\beta} \tag{B.52}$$

based on the parallel transport

$$\upsilon^\mu_{\|}(x+\delta x) = \upsilon^\mu(x) + \Gamma^{\mu}{}_{\alpha\beta}\, \upsilon^\alpha \delta x^\beta \tag{B.53}$$

we can reformulate parallel transport in an operator notation: The vector $\upsilon^\mu(x+\delta x)$ must be equal to $\upsilon^\mu_{\|}(x+\delta x) + \delta x^\beta \nabla_\beta \upsilon^\mu$. Perfect parallel transport means that the vector $\upsilon^\mu(x+\delta x)$ at $T_B M$ and $\upsilon^\mu_{\|}(x+\delta x)$ transported from $T_A M$ to $T_B M$ by the shift δx are now identical, and in this case $\delta x^\beta \nabla_\beta \upsilon^\mu$ must be zero. This shifting operator $\delta x^\beta \nabla_\beta$ can be applied to scalar quantities as well, such as in particular the scalar product $g_{\mu\nu}\upsilon^\mu w^\nu$:

$$\delta x^\beta \nabla_\beta (g_{\mu\nu}\upsilon^\mu w^\nu) = \delta x^\beta (\nabla_\beta g_{\mu\nu} \cdot \upsilon^\mu w^\nu + g_{\mu\nu} \underbrace{\nabla_\beta \upsilon^\mu}_{=0} \cdot w^\nu + g_{\mu\nu}\upsilon^\mu \underbrace{\nabla_\beta w^\nu}_{=0}) = \delta x^\beta \nabla_\beta g_{\mu\nu} \cdot \upsilon^\mu w^\nu = 0 \tag{B.54}$$

as a consequence of the Leibnitz rule, with a single term remaining:

$$\nabla_\beta g_{\mu\nu} = 0 \tag{B.55}$$

which is referred to as the metric compatibility condition: If it is true, the scalar product over perfectly parallel transported vectors does not change across the manifold. As the metric itself is a tensor with covariant indices, the covariant derivative is computed as

$$\nabla_\beta g_{\mu\nu} = \partial_\beta g_{\mu\nu} - \Gamma^{\alpha}{}_{\beta\mu}\, g_{\alpha\nu} - \Gamma^{\alpha}{}_{\beta\nu}\, g_{\mu\alpha} \tag{B.56}$$

If in addition we assume that the parallel transport is torsion-free the Christoffel-symbol is symmetric in the lower two indices,

$$\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}\,. \tag{B.57}$$

This implies that we write out the combination $\nabla_\mu g_{\beta\nu} + \nabla_\nu g_{\mu\beta} - \nabla_\beta g_{\mu\nu} = 0$ (metric compatibility ensures that the terms vanish already individually!) and solve for the Christoffel-symbol $\Gamma^{\alpha}{}_{\mu\nu}$, which comes out as

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}) \tag{B.58}$$

A connection $\Gamma^{\alpha}{}_{\mu\nu}$ which is metric-compatible ($\nabla_\alpha g_{\mu\nu} = 0$) and torsion free ($\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}$) is called a Levi-Civita connection; it is uniquely compatible with the metric structure on the manifold, as the connection can be computed from the metric and its derivatives alone. A metric manifold with a Levi-Civita connection and the corresponding covariant derivative defines Riemannian geometry.

At this point, a beautiful conceptual picture emerges: Spacetime is a manifold with, first of all, a topological structure, which allows a continuous mapping of coordinates onto spacetime. Then, there is in addition a metric structure, which allows measurements of lengths and angles in vector fields on the manifold: As other fields, the metric tensor may vary across the manifold. We've introduced a differentiable structure on the manifold, in addition, by defining parallel transport and the covariant derivative. This differentiable structure has to be compatible with the metric structure, which is made sure by metric compatibility. Later in this course, we'll see that there is a second notion of derivation, called a Lie-derivative, which is needed to describe symmetries: Those are made compatible with covariant derivative by the requirement of torsion-free connections, giving further support to Levi-Civita connections. A physical motivation for choosing torsion-free connections is the compatibility of covariant derivatives with Lie-derivatives which are used for characterising symmetries of spacetimes.

B.4 *Geodesics and autoparallelity*

A particle drifting through spacetime follows a trajectory $x^\mu(\lambda)$ in a given coordinate choice, parameterised by the affine parameter λ. Then, the rate of change of the coordinates with λ would be the velocity u^μ,

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \dot{x}^\mu \tag{B.59}$$

or equivalently the tangent to the trajectory $x^\mu(\lambda)$. The velocity u^μ and the coordinate differential $\mathrm{d}x^\mu$ are vectors, in contrast to the coordinate tuple x^μ itself. With the idea the operator for parallel transport we might construct a curve whose tangent $u^\mu = \dot{x}^\mu$ stays parallel to itself, exactly through the autoparallelity condition

$$\dot{x}^\mu \nabla_\mu \dot{x}^\nu = 0 \tag{B.60}$$

i.e. $u^\mu = \dot{x}^\mu$ is always a parallel-transported version of itself. It is suggestive to imagine that these curves describe inertial motion through spacetime, as no accelerations are felt, because the velocity $\boldsymbol{u}$ as an abstract vector does not change, only its components u^μ can be different as there can be different tangent spaces along the curve. Taking this thought a little further leads us to the realisation that there is actually no difference between inertial motion and freely falling motion, as both cases are characterised by the absence of physical accelerations.

Surely, $\mathrm{d}u^\mu/\mathrm{d}\lambda$ can be nonzero, but the abstract vector $\boldsymbol{u}$ is conserved.

$$\dot{x}^\mu \nabla_\mu \dot{x}^\nu = \dot{x}^\mu \partial_\mu \dot{x}^\nu + \Gamma^\nu{}_{\mu\alpha}\, \dot{x}^\mu \dot{x}^\alpha = \ddot{x}^\nu + \Gamma^\nu{}_{\alpha\mu}\, \dot{x}^\mu \dot{x}^\alpha = 0 \tag{B.61}$$

using

$$\ddot{x}^\nu = \frac{\mathrm{d}\dot{x}^\nu}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\partial \dot{x}^\nu}{\partial x^\mu} = \dot{x}^\mu \partial_\mu \dot{x}^\nu \tag{B.62}$$

to obtain the second derivative $\ddot{x}^\nu$. The result is the geodesic equation, reading

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\lambda} + \Gamma^\alpha{}_{\mu\nu}\, u^\mu u^\nu = 0, \quad \text{or} \quad \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha{}_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0. \tag{B.63}$$

if formulated in terms of the tangent vector u^μ. It is a fun realisation that the tangent vector of the Earth's orbit *in 4 dimensions* is autoparallel, in a spacetime which is non-Minkowskian with a very slight curvature introduced by the Sun.

It is possible to tease out the geodesic equation from Newton's equation of motion. In fact,

$$\ddot{x}^i + \partial^i \Phi = 0 \tag{B.64}$$

describes the freely falling motion of a test particle in the gravitational potential Φ. It follows a force-free trajectory, which is straight according to the inertial law formulated by Newton. Surely, we don't make a mistake by writing

$$\ddot{x}^i + \partial^i \frac{\Phi}{c^2} \cdot c \cdot c = 0 \tag{B.65}$$

where now c^2 provides a scale for the potential Φ: Because c has no particular relevance for Galilean physics one would think that the division by c^2 just makes the potential dimensionless. In the slow-motion limit of relativity particles follow trajectories with $\dot{x}^t = c$, so the formula becomes

$$\ddot{x}^i + \partial^i \frac{\Phi}{c^2} \dot{x}^t \dot{x}^t = 0 \tag{B.66}$$

But the terms $\dot{x}^t$ are just the t-components of the velocities, which in the slow-motion limit $\dot{x}^\mu = (c, \upsilon^i)^t$, where proper time and coordinate time are identical, $t = \tau$ identical and consequently $\gamma = 1$. Then,

$$\ddot{x}^\alpha + \partial^\alpha \frac{\Phi}{c^2} \dot{x}^t \dot{x}^t = 0, \quad \text{suggesting that} \quad \Gamma^\alpha{}_{tt} \sim \partial^\alpha \frac{\Phi}{c^2} \tag{B.67}$$

by identifying the derivative of the potential with the Christoffel-symbol, consolidating the idea that Newton's equation of motion is the weak-field and slow-motion limit of the geodesic equation,

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu = 0 \tag{B.68}$$

If we try out an extremal principle for the trajectory as in classical mechanics and impose Hamilton's principle $\delta S = 0$ on an action integral

$$S = \int \mathrm{d}t\, \mathcal{L} \quad \text{with} \quad \mathcal{L} = \frac{1}{2}\dot{x}_i\dot{x}^i - \Phi \tag{B.69}$$

we end up with the Euler-Lagrange equation

$$\ddot{x}^i + \partial^i\Phi = 0 \tag{B.70}$$

from classical mechanics. In a similar calculation $\delta s = 0$ of the line-element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\mathrm{d}\lambda^2 \to \mathrm{d}s = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\mathrm{d}\lambda \quad \text{and} \quad s = \int \mathrm{d}s \tag{B.71}$$

provides the geodesic equation: Straight lines in the sense of autoparallelity are at the same time extremal in their arc length.

The affine parameter λ can be chosen arbitrarily as the geodesic equation is invariant under affine transforms of λ, $\lambda \to a\lambda + b$, but there are two practical choices: In the case of a massive particle which follows a time-like geodesic with $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu > 0$ one can choose proper time $\lambda = \tau$, such that the normalisation is given by $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = c^2$. Photons, on the other hand, follow null-geodesics with $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$, which is incompatible with proper time as an affine parameter. As parallel transport is with Levi-Civita connection is constructed to conserve norms, we can conclude that in both cases the normalisation of the tangent $u^\mu = \dot{x}^\mu$ for both τ or λ is conserved.

B.5 *Spacetime curvature*

The geodesic equation

$$\ddot{x}^\alpha + \Gamma^\alpha_{\ \mu\nu}\,\dot{x}^\mu\dot{x}^\nu = 0 \tag{B.72}$$

is unable to differentiate between inertial motion in the absence of gravity or freely-falling motion in a gravitational field: This is absolutely sensible because in both cases one would not *feel* or *measure* any acceleration, so the two situations are physically equivalent. This has profound implications which we should clarify: The Christoffel-symbol has 40 entries (For every choice of α, 4 in total, there are because of the symmetry 10 different choices in the index pair μ, ν), which are all measurable through the acceleration $\ddot{x}^\alpha$ for a given choice of $\dot{x}^\mu\dot{x}^\nu$, for both cases of massive and massless particles. Acceleration in this context is a non-uniform passage of the coordinates along the path of the particle, which should not be interpreted as a physical acceleration. In this sense, the geodesic equation only takes care of the non-uniformity of the coordinate choice and does *not* contain information about gravity or curvature! A good example might be inertial motion through Euclidean space in polar coordinates r, φ, and a situation where the particle moves off-centre relative to origin of the coordinate frame. There, the velocities $\dot{r}$ and $\dot{\varphi}$ are not constant and show accelerations $\ddot{r} \neq 0 \neq \ddot{\varphi}$, but clearly, there are no physical accelerations present.

To summarise this important point: Neither the metric, nor the geodesic equation, nor the covariant derivative and nor the Christoffel-symbols contain information about gravity, $\Gamma^{\alpha}{}_{\mu\nu} = 0$ does not imply the absence of gravity, and neither does $\nabla_\mu = \partial_\mu$. All these things are consequences of the coordinate choice. That is in fact sensible, as there is always a coordinate choice that sets *locally* $g_{\mu\nu}$ to $\eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu} = 0$, i.e. the metric becomes Minkowskian and the Christoffel-symbol vanishes.

Information about the gravitational field is contained in curvature, which is in Riemannian-geometry ultimately computed from the second derivatives of the metric and which can not be set to zero by a suitable coordinate transform in the general case. Curvature is present if covariant derivatives ∇_μ into different directions do not commute, or equivalently, if shifts $\delta x^\mu \nabla_\mu$ into different directions carried out after each other, affect the internal degrees of a vector or tensor. The non-commutativity of covariant derivatives directly defines the Riemann-curvature,

$$\left[\nabla_\mu \nabla_\nu\right] \upsilon^\alpha = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\upsilon^\alpha = R^{\alpha}{}_{\beta\mu\nu}\upsilon^\beta \tag{B.73}$$

It can be shown that the effect of parallel-transport around a loop would be a rotated vector $R^{\alpha}{}_{\beta\mu\nu}\upsilon^\beta$ relative to υ^α, where parallel-transport conserves the norm of the vector υ^α due to metric compatibility. This is in fact the best way to visualise the effect of $R^{\alpha}{}_{\beta\mu\nu}$ as an operator and to memorise the index structure. By definition, $R^{\alpha}{}_{\beta\mu\nu}$ is antisymmetric for every choice of μ, ν, and in the index pair α, β is is an antisymmetric rotation matrix. In 4 dimensions, $R^{\alpha}{}_{\beta\mu\nu}$ has 20 entries, as opposed to the 40 entries of $\Gamma^{\alpha}{}_{\mu\nu}$.

The Riemann-curvature vanishes in flat spaces

$$R^{\alpha}{}_{\beta\mu\nu} = 0 \tag{B.74}$$

in every coordinate choice, even though the Christoffel-symbols $\Gamma^{\alpha}{}_{\mu\nu}$ only vanish in Cartesian coordinates. Following the formal definition of curvature as the non-commutativity of shifts in different coordinate directions leads us to

$$\upsilon^\mu(x + \delta x) = \upsilon^\mu(x) - \Gamma^{\mu}{}_{\alpha\beta}(x)\upsilon^\alpha \delta x^\beta \tag{B.75}$$

and in a second step to

$$\begin{aligned}\upsilon^\mu((x + \delta\bar{x}) + \delta x) &= \upsilon^\mu(x + \delta\bar{x}) - \Gamma^{\alpha}{}_{\mu\nu}(x + \delta\bar{x})\upsilon^\alpha(x + \delta\bar{x})\delta x^\beta = \\ &\upsilon^\mu(x) - \Gamma^{\alpha}{}_{\mu\nu}\, v(x)\upsilon^\alpha(x)\delta\bar{x}^\beta - (\Gamma^{\alpha}{}_{\mu\nu}(x) + \underbrace{\partial_\gamma \Gamma^{\mu}{}_{\alpha\beta}\,\delta\bar{x}^\gamma}_{\text{Taylor}})(\upsilon^\alpha(x) - \Gamma^{\alpha}{}_{\gamma\delta}\,\upsilon^\gamma \delta x^\gamma)\delta x^\beta\end{aligned} \tag{B.76}$$

and therefore the order of the shifts matters:

$$\upsilon^\mu((x + \delta x) + \delta\bar{x}) \neq \upsilon^\mu((x + \delta\bar{x}) + \delta x) \tag{B.77}$$

Computing the difference shows that

$$\upsilon^\mu((x + \delta\bar{x}) + \delta x) - \upsilon^\mu((x + \delta x) + \delta\bar{x}) = R^{\mu}{}_{\alpha\beta\nu}\upsilon^\alpha \delta x^\beta \delta\bar{x}^\nu \tag{B.78}$$

with the Riemann-curvature-tensor $R^{\mu}{}_{\alpha\beta\nu}$

$$R^{\mu}{}_{\alpha\beta\nu} = \partial_\beta \Gamma^{\mu}{}_{\alpha\nu} - \partial_\nu \Gamma^{\mu}{}_{\alpha\beta} + \Gamma^{\mu}{}_{\delta\beta}\, \Gamma^{\delta}{}_{\alpha\nu} - \Gamma^{\mu}{}_{\delta\nu}\, \Gamma^{\delta}{}_{\alpha\beta} \tag{B.79}$$

which depends, as expected on the derivatives of the Christoffel-symbols as well as their "squares". But ultimately, due to the choice of a (pseudo-)Riemannian geometry, the curvature tensor can be computed from the metric and its first and second derivatives.

B.6 *Covariant divergence*

The idea of using the divergence for expressing conserved quantities like $g^{\alpha\mu}\nabla_\alpha J_\mu = 0$ for the electric charge or $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ for the energy-momentum tensor is very central to physics. Formulated in a covariant way, it behaves properly as a tensor under coordinate transforms. The covariant divergence needs a pecular index-combination in the Christoffel-symbol, where two of the indices become equal.

$$\nabla_\mu v^\mu = \partial_\mu v^\mu + \Gamma^{\mu}{}_{\mu\alpha}\, v^\alpha \tag{B.80}$$

In particular, a Levi-Civita connection would have

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{g^{\mu\beta}}{2} \cdot \left[\partial_\mu\, g_{\beta\alpha} + \partial_\alpha\, g_{\mu\beta} - \partial_\beta\, g_{\mu\alpha}\right] = \frac{1}{2}\left[g^{\mu\beta}\, \partial_\mu\, g_{\beta\alpha} + g^{\mu\beta}\, \partial_\alpha\, g_{\mu\beta} - g^{\mu\beta}\, \partial_\beta\, g_{\mu\alpha}\right] \tag{B.81}$$

i.e. essentially

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{2} g^{\mu\beta}\, \partial_\alpha\, g_{\mu\beta} \tag{B.82}$$

Curiously, there is a relation between the covariant divergence and the covolume $g = \det(g_{\mu\nu})$. My third most favourite formula in theoretical physics says that

$$g = \det(g_{\mu\nu}) = \exp \ln\, \det(g_{\mu\nu}) = \exp\, \mathrm{tr}\, \ln(g_{\mu\nu}) \tag{B.83}$$

relating the logarithm of the determinant with the trace of the matrix-valued logarithm, which is easily checked in the principal axis frame. Then,

$$\partial_\alpha\, g = g \cdot \partial_\alpha \mathrm{tr}\, \ln(g_{\mu\nu}) = g \cdot \mathrm{tr}\, \partial_\alpha\, \ln(g_{\mu\nu}) = g \cdot \mathrm{tr}\left(g^{-1} \cdot \partial_\alpha\, g_{\mu\nu}\right) = g \cdot g^{\mu\nu} \cdot \partial_\alpha\, g_{\mu\nu} \tag{B.84}$$

using the linearity of the derivative as well as the inverse metric. With the derivative of the square root one then obtains

$$g^{\mu\nu}\, \partial_\alpha\, g_{\mu\nu} = \frac{1}{g} \partial_\alpha\, g, \quad \text{and therefore} \quad \frac{1}{2} g^{\mu\nu}\, \partial_\alpha\, g_{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\alpha\, \sqrt{-g}. \tag{B.85}$$

With this result one can write for the contracted Christoffel-symbol

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{\sqrt{-g}}\, \partial_\alpha\, \sqrt{-g} \tag{B.86}$$

and finally for the covariant divergence

$$\nabla_\mu\, \upsilon^\mu = \partial_\mu\, \upsilon^\mu + \Gamma^\mu{}_{\mu\alpha}\, \upsilon^\alpha = \partial_\mu\, \upsilon^\mu + \frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g}\cdot\upsilon^\alpha$$

$$\stackrel{\mu\leftrightarrow\alpha}{=} \partial_\mu \upsilon^\mu + \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\cdot\upsilon^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,\upsilon^\mu\right) \quad \text{(B.87)}$$

using the Leibnitz-rule. An interesting application of the covariant divergence is the wave equation

$$g^{\mu\nu}\,\nabla_\mu\nabla_\nu\,\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,\partial^\mu\,\phi\right) = 0 \quad \text{(B.88)}$$

which is obviously not just $\partial_\mu\partial^\mu\phi = 0$; there is clearly an influence from the background onto wave propagation. For our particular case of FLRW-cosmologies, the covolume is quickly computed in comoving coordinates to be

$$\sqrt{-\det g} = ca^3 \quad \text{(B.89)}$$

with physical time t, and as

$$\sqrt{-\det g} = ca^4 \quad \text{(B.90)}$$

with conformal time η.

B.7 *Geodesic deviation: experiencing curvature*

A freely falling particle experiences perfect weightlessness and spacetime appears to be locally Minkowskian, $g_{\mu\nu} = \eta_{\mu\nu}$ with a vanishing first derivative $\partial_\alpha g_{\mu\nu} = 0$, which enables the *local* choice of Cartesian coordinates. But that does not imply that a second particle, likewise in a state of perfect free fall, moves at constant velocities relative to the first particle: This is exactly the idea of geodesic deviation. The relative distance δ^μ of two freely falling particles, each one following its geodesic, obeys

$$\frac{d^2\delta^\mu}{d\tau^2} = R^\mu{}_{\alpha\beta\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\delta^\nu \quad \text{(B.91)}$$

which follows from expanding $\Gamma(\bar{x})$ for the second particle in the geodesic equation in terms of $\Gamma(x)$ for the first geodesic. Only if the manifold is flat, the Riemann curvature $R^\alpha{}_{\beta\mu\nu} = 0$ vanishes, resulting in

$$\frac{d^2}{d\tau^2}\delta^\mu = 0 \quad \rightarrow \quad \frac{d}{d\tau}\delta^\mu = a^\mu \quad \rightarrow \quad \delta^\mu = a^\mu\tau + b^\mu \quad \text{(B.92)}$$

with two integration constants a^μ and b^μ: The particles would drift apart at a constant rate, and accelerations $\ddot{\delta}^\mu$ only appear if there is curvature. Please keep in mind, that this also applies for time component, as we use 4d coordinates. Classically we would get an analogous statement

$$\frac{d^2}{dt^2}\delta^i = -\underbrace{\partial^i\partial_j\Phi}_{\text{tidal tensor}}\,\delta^j = \partial^i g_j \delta^j \quad \text{(B.93)}$$

with no velocity dependence of the gravitational force and universal time instead of proper time. This underlines the idea that the tidal field tensor $\partial^i \partial_j \Phi$ is the Newtonian analogue of the Riemann curvature.

B.8 *Curvature invariants and curvature tensors*

The Riemann-curvature contains the complete information about curvature if the connection is chosen to be torsion-free and metric compatible, otherwise one would need the torsion tensor and the non-metricity scalar in addition. From the Riemann-curvature, one can compute further measures of curvature, which are physically relevant, such as the Ricci-tensor $R_{\beta\nu}$

$$R_{\beta\nu} = R^{\alpha}{}_{\beta\alpha\nu} = g^{\alpha\mu} R_{\alpha\beta\mu\nu} \tag{B.94}$$

and curvature scalars by complete contraction, for instance the Ricci-scalar R

$$R = R^{\alpha}{}_{\alpha} = g^{\beta\nu} R_{\beta\nu} \tag{B.95}$$

or the Kretschmann-scalar K

$$K = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma} R_{\alpha\beta\gamma\delta} R_{\mu\nu\rho\sigma} \tag{B.96}$$

which are both coordinate-invariant measures of curvature.

Let's apply these ideas to a flat FLRW-cosmology, where the line element has the form

$$ds^2 = c^2 dt^2 - a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2(\theta) d\varphi^2)) \tag{B.97}$$

in terms of comoving coordinates and physical time. The trivial derivatives of the metric are

$$\partial_t g_{tt} = 0, \quad g_{t\alpha} = g_{\alpha t} = 0 \tag{B.98}$$

and the non-trivial derivatives can be summarised in the Christoffel-symbols

$$\Gamma^t{}_{\alpha\beta} = \frac{1}{2} \partial_t g_{\alpha\beta} \rightarrow \Gamma^t{}_{ij} = \frac{1}{2} \frac{d}{dt} a^2 = \dot{a} a = a^2 H \tag{B.99}$$

with $H = \dot{a}/a$, and

$$\Gamma^i{}_{\alpha\beta} = \frac{1}{2a^2} \left(\partial_\beta g_{i\alpha} + \partial_\alpha g_{\beta i} \right) \rightarrow \Gamma^i{}_{it} = \Gamma^i{}_{ti} = \frac{1}{2a^2} 2 \partial_t g_{ii} = \frac{\dot{a}}{a} = H \tag{B.100}$$

For the Ricci-scalar of a flat FLRW-spacetime we get, by contracting $g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\mu\nu}$

$$R = 6a \left(\frac{H}{c} \right)^2 (1 - q) \quad \text{with} \quad q = -\frac{\ddot{a} a}{\dot{a}^2} \tag{B.101}$$

The metric usually has no units (since $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ and coordinates are usually chosen to have units of length, and consequently would ds have units of length, but please keep in mind that this is purely conventional), whereas curvature as composed of second derivatives (with respect to x^μ) have units of a inverse length squared, which results in the curvature scale

$$\frac{1}{\sqrt{R}} = \frac{c}{H} \tag{B.102}$$

ignoring pre-factors of order one: The curvature scale of a FLRW-cosmology $R^{-1/2}$ is the Hubble-scale c/H, implying that on scales larger than c/H one can see effects of strong gravity, whereas on scales smaller than c/H spacetime can be approximated to be Minkowksian. In fact, light propagation effects associated with horizons appear on this scale.

B.9 *Raychaudhuri-equation*

The Riemann-tensor as a complete characterisation of spacetime curvature decomposes into two distinct types of curvature: The Ricci-curvature contained in $R_{\mu\nu}$ and the Weyl-curvature $C_{\alpha\beta\mu\nu}$, both tensors having 10 entries in 4 dimensions. The Ricci-tensor $R_{\mu\nu}$ at one point in spacetime reflects the energy momentum tensor $T_{\mu\nu}$ at the same point, as a consequence of the field equation, and is necessarily only a function of time. As there are no spatial derivatives in a FLRW-geometry, we are not concerned with propagation effects of gravity, so the Weyl-tensor $C_{\alpha\beta\mu\nu}$ is zero, and we're dealing in FLRW-cosmologies with a system of pure Ricci-curvature.

The effects of Ricci- and Weyl-curvature on test particles can be understood in an extension to geodesic equation, which is known as the Raychaudhuri-equation. Here, one considers not a pair, but an entire cloud of freely falling test particles and monitors the change of volume or the change in shape of that cloud. Ricci-curvature, which FLRW-spacetimes carry exclusively, induce a pure change in volume while conserving shape, while Weyl-curvature does the opposite: It causes a cloud of test particles to change its shape while conserving the volume.

The idea of the Raychaudhuri-equation is to have a look at the time evolution of the area enclosed by a bundle of geodesics. The Riemann-curvature splits in two different parts:

- The Ricci-curvature changes the volume enclosed by a bundle of geodesics but keeps the shape (typical for FLRW-cosmologies)
- The Weyl-curvature changes shape but conserves the volume (typical for gravitational waves)

Let's try a classical approach: Two test particles at coordinates x'^i and x^i have a relative motion at velocity υ^i

$$x'^i = x^i + \upsilon^i \Delta t \tag{B.103}$$

The relative coordinate mapping is encapsulated in the Jacobian

$$\frac{\partial x'^i}{\partial x^j} = \delta^i_j + \frac{\partial \upsilon^i}{\partial x^j}\Delta t \tag{B.104}$$

which we could think of as the first-order Taylor expansion of of a matrix-valued exponential

$$\frac{\partial x'^i}{\partial x^j} = \exp\left(\frac{\partial \upsilon^i}{\partial x^j}\Delta t\right) \tag{B.105}$$

The coordinate mapping would introduce a change in volume elements given by the functional determinant

$$\mathrm{d}^3x' = \det\left(\frac{\partial x'}{\partial x}\right)\mathrm{d}^3x \tag{B.106}$$

where we can write for the logarithmic change

$$\ln \mathrm{d}^3x' = \ln\det\left(\frac{\partial x'}{\partial x}\right)+\ln \mathrm{d}^3x = \ln \mathrm{d}^3x' = \mathrm{tr}\ln\exp\left(\frac{\partial \upsilon^i}{\partial x^j}\Delta t\right)+\ln \mathrm{d}^3x = \Delta t\,\mathrm{tr}\frac{\partial \upsilon^i}{\partial x^j}+\ln \mathrm{d}^3x \tag{B.107}$$

In this relation, one can identify $\mathrm{tr}(\partial \upsilon^i/\partial x^j)$ with the divergence of the velocity field: Intuitively, if this divergence is nonzero, the volume should change.

The Newton equation of motion for small time differences Δt is

$$\upsilon^i = -\partial^i\Phi\Delta t \tag{B.108}$$

which suggests for the velocity divergence

$$\partial_i \upsilon^i = -\partial_i\partial^i\Phi\Delta t = -\Delta\Phi\Delta t \tag{B.109}$$

with the Laplace-operator $\Delta = \partial_i\partial^i$, and therefore

$$\ln \mathrm{d}^3x' - \ln \mathrm{d}^3x = -\Delta\Phi(\Delta t)^2 \approx 8\pi G\rho\frac{(\Delta t)^2}{2} \tag{B.110}$$

Therefore, the logarithmic volume change is proportional to the density inside the volume and $\Delta\Phi \propto \rho$ from Poisson's equation. The volume change measures effectively the enclosed mass, and suggests that Riemann curvature and tidal field are analogous quantities, as well as the Ricci-curvature (as the trace of the Riemann curvature) and the Laplacian of the potential, and that both are coupled to the matter density as the source of the gravitational field.

The proper relativistic Raychaudhuri-equation makes a statement about the velocity divergence Θ,

$$\Theta = \nabla_\mu u^\mu = g^{\mu\nu}\nabla_\mu u_\nu \tag{B.111}$$

and states that for the time evolution that

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = -\frac{\Theta^2}{3} - \underbrace{\sigma_{\mu\nu}\sigma^{\mu\nu}}_{=0\text{ in FLRW}} + \underbrace{\omega_{\mu\nu}\omega^{\mu\nu}}_{=0\text{ in FLRW}} - R_{\mu\nu}u^\mu u^\nu + \nabla_\mu \dot{u}^\mu. \tag{B.112}$$

The terms in the Raychaudhuri-equation are the shear,

$$\sigma_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{1}{3}\Theta p_{\mu\nu} - (\dot{u}_\mu u_\nu + u_\mu \dot{u}_\nu) \tag{B.113}$$

and the vorticity

$$\omega_{\mu\nu} = \nabla_\mu u_\nu - \nabla_\nu u_\mu - (\dot{u}_\mu u_\nu + u_\mu \dot{u}_\nu) \tag{B.114}$$

and finally the proper acceleration

$$\dot{u}^\mu = u^\nu \nabla_\nu u^\mu \tag{B.115}$$

which vanishes, if u^μ is tangent to the geodesic (from geodesic equation), i.e. with no non-gravitational accelerations.

FLRW-cosmologies have no preferred axis due to the isotropy postulate, resulting in vanishing vorticity ω and vanishing shear $\sigma = 0$ and therefore

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = -\frac{\Theta^2}{3} - R_{\mu\nu} u^\mu u^\nu \tag{B.116}$$

for the evolution of the velocity divergence. By using comoving FLRW-coordinates we see no expansion as all, all galaxies stay in their spatial coordinate and only move in the time-direction. Consequently we choose $u^\mu = (c, 0)^t$ and can now compute the covariant divergence as

$$\Theta = \nabla_\mu u^\mu = \frac{1}{\sqrt{-\det g}} \partial_\mu \left(\sqrt{-\det g}\, u^\mu \right) \sim \partial_t \ln(\text{volume}) \tag{B.117}$$

Specifically, with the FLRW-metric in comoving coordinates

$$g_{\mu\nu} = \begin{pmatrix} c^2 & & & \\ & -a^2 & & \\ & & -a^2 & \\ & & & -a^2 \end{pmatrix} \tag{B.118}$$

suggesting for the velocity divergence, or equivalently, the volume evolution

$$\sqrt{-\det g} = ca^3 \tag{B.119}$$

and finally

$$\Theta = \frac{1}{a^3} \partial_t \left(a^3 \right) = 3\frac{\dot{a}a^2}{a^3} = 3\frac{\dot{a}}{a} = 3H(t) \tag{B.120}$$

such when replacing proper time by coordinate or cosmic time ($t = \tau$) as specifically allowed by FLRW-cosmologies one obtains:

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = -\frac{\Theta^2}{3} - R_{\mu\nu} u^\mu u^\nu \tag{B.121}$$

with

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{a}}{a} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 \tag{B.122}$$

In summary, the volume evolution of a FLRW universe is expressed in terms of the scale factor a and its derivatives, and depends on the term $R_{\mu\nu} u^\mu u^\nu$, which we can provide through the field equation of gravity and which depends on the energy-momentum content of spacetime: It acts as the source of the gravitational field and introduces curvature, the Ricci-part of which affects the evolution of Θ. It is amazing to see how volume evolution of the FLRW-spacetime works in Newtonian gravity and relativity alike, and that the Raychaudhuri-equation makes a sensible statement about the evolution of the velocity divergence of comoving velocity (which one would naively visualise as a perfectly parallel vector field in comoving coordinates). Progress beyond this result is only possible if we assume a specific form of the field equation, in order to relate the Ricci-tensor to the energy momentum-tensor. Surprising as it may seem, the Raychaudhuri-equation is a purely geometric statement about the divergence of a vector field, and does not assume anything specific about the gravitational theory.

B.10 *Energy-momentum tensor*

The energy and momentum content of spacetime sources the gravitational field: Very similar to the case of Maxwell-electrodynamics which is the theory of electric and magnetic fields for charge-conserving systems, general relativity is the theory of gravity for energy and momentum conserving systems (although that can be only formulated locally in the form of $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ with the energy-momentum tensor $T_{\mu\nu}$). In a fluid picture, energy and momentum conservation would characterise the dynamics of an ideal (i.e. inviscid) fluid, which can only have three properties: velocity u^μ, density ρ and pressure p, assembled into the energy momentum-tensor $T^{\mu\nu}$.

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu} \tag{B.123}$$

In a frame where the fluid is at rest, $u^\mu = (c, 0)^t$ and adopting locally flat, Cartesian coordinates one falls back on a diagonal form,

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{B.124}$$

Ideal fluids are characterised by the conservation law

$$g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0 \tag{B.125}$$

which is not straightforward to interpret, as it is a vectorial statement (in the index ν): This is in contrast to e.g. the law of charge conservation $g^{\alpha\mu}\nabla_\alpha J_\mu = 0$ in electrodynamics, which is scalar.

It might be intuitive to project the vector $\nabla_\mu T^{\mu\nu}$ onto the velocity u_ν and to a plane perpendicular to it, as u^μ is defining naturally a direction. Keeping in mind that the covariant derivative fulfils a Leibnitz-rule one gets:

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + \left(\rho + \frac{p}{c^2}\right) \nabla_\mu (u^\mu u^\nu) - \nabla_\mu p g^{\mu\nu} \tag{B.126}$$

with

$$\nabla_\mu (u^\mu u^\nu) = u^\nu \nabla_\mu u^\mu + \underbrace{u^\mu \nabla_\mu u^\nu}_{=0 \text{ if } u^\mu \text{ tangent to a geodesic}} \tag{B.127}$$

while any covariant derivative of the metric (and its inverse) vanishes due to metric compatibility. Carrying out the projection

$$u_\nu \nabla_\mu T^{\mu\nu} = 0 \tag{B.128}$$

we end up at

$$u_\nu \nabla_\mu T^{\mu\nu} = \nabla_\mu \left(\rho + \frac{p}{c^2}\right) u^\mu \underbrace{u_\nu u^\nu}_{c^2} + \left(\rho + \frac{p}{c^2}\right) \underbrace{u_\nu u^\nu}_{c^2} \nabla_\mu u^\mu - \nabla_\mu p \underbrace{g^{\mu\nu} u_\nu}_{u^\mu} = 0 \tag{B.129}$$

which can be further simplified to

$$u_\nu \nabla_\mu T^{\mu\nu} = \nabla_\mu \rho u^\mu c^2 + \rho \nabla_\mu u^\mu c^2 + p \nabla_\mu u^\mu = \nabla_\mu (\rho u^\mu) c^2 + p \nabla_\mu u^\mu = 0. \tag{B.130}$$

The projection of $u_\nu T^{\mu\nu} = 0$ onto a plane perpendicular to u_ν yields the Euler-equation

$$\left(\rho + \frac{p}{c^2}\right) \nabla_\mu u^\nu u^\mu = \left(g^{\mu\nu} - \frac{u^\mu u^\nu}{c^2}\right) \nabla_\mu p \tag{B.131}$$

B.11 *Equation of state for ideal fluids*

Ideal fluids are characterised by just two quantities (apart from their velocity field u^μ): density ρ and pressure p, and often it is the case that the two are related by an equation of state which reflects internal properties of the substance. Like in the case of an ideal classical or relativistic gas there could be a proportionality

$$p = w \rho c^2 \tag{B.132}$$

with the equation of state parameter w, which in relativity is often assumed to be constant (Please remember that energy density and pressure have identical units!). In a frame where the fluid is at rest we would write $u^\mu = (c, 0)^t$ and covariant energy momentum conservation $\nabla_\mu T^{\mu\nu} = 0$ becomes in the choice of comoving coordinates,

$$\nabla_\mu (\rho u^\mu) + \frac{p}{c^2} \nabla_\mu u^\mu = 0 \tag{B.133}$$

The first term $\nabla_\mu (\rho u^\mu)$ can be taken apart with the Leibnitz-rule

$$\nabla_\mu (\rho u^\mu) = \nabla_\mu \rho \cdot u^\mu + \rho \nabla_\mu u^\mu = \partial_\mu \rho \cdot u^\mu + \rho \nabla_\mu u^\mu \tag{B.134}$$

resulting in

$$\partial_\mu \rho + \left(\rho + \frac{p}{c^2}\right) \nabla_\mu u^\mu = 0 \tag{B.135}$$

by substituting the equation of state $p = w \rho c^2$. The covariant divergence of the velocity u^μ is

$$\nabla_\mu u^\mu = \underbrace{\partial_\mu u^\mu}_{=0} + \Gamma^\mu{}_{\alpha\mu} u^\alpha = \underbrace{\Gamma^i{}_{it}}_{=H/c} u^t = 3\frac{\dot{a}}{a} = 3H(t) \tag{B.136}$$

and using the fact that ρ is homogeneous and only a function of time, there is just a single derivative left, $\partial_\mu\rho = \partial_t\rho = \dot{\rho}$. With the argument we arrive at the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1+w)\rho = 0 \tag{B.137}$$

which reflects covariant energy momentum-conservation $\nabla_\mu T^{\mu\nu} = 0$ in a FLRW-spacetime.

Separation of the variables and assuming a constant equation of state parameter w results in

$$\frac{d\rho}{\rho} = d\ln\rho = -3(1+w)\frac{da}{a} = -3(1+w)d\ln a \tag{B.138}$$

which can be solved by

$$\rho \propto a^{-3(1+w)} \tag{B.139}$$

In fact, non-relativistic matter with $w = 0$ would dilute $\rho \propto a^{-3}$ as it is simply dispersed over a larger volume a^3, but for relativistic matter with $w = 1/3$ there would be an additional redshifting effect leading to $\rho \propto a^{-4}$.

Sometimes you find a reformulation of continuity in this way: Multiplying eqn. B.137 with the volume a^3 yields

$$a^3\frac{d\rho}{da} + 3a^2\left(\rho + \frac{p}{c^2}\right) = 0 \quad \rightarrow \quad \frac{d}{da}(\rho c^2 a^3) = -p\frac{d}{da}(a^3) \tag{B.140}$$

with the interpretation that the energy density ρc^2 of the fluid changes if the volume changes, performing work against pressure p, reminiscent of the first law of thermodynamics, $dU = -p dV$: This is why the relation is sometimes called the adiabatic law.

General relativity is prepared to provide gravity for both fields and fluids, as both fields and fluids obey covariant conservation of energy and momentum, $\nabla_\mu T^{\mu\nu} = 0$ or $g^{\mu\alpha}\nabla_\mu T_{\alpha\nu} = 0$. But comparing the two cases fluids and fields, in the first case one would speak of the Poynting-law, and in the second case about the fluid mechanical equations: It is actually amazing that general relativity can provide gravity for such different concepts of matter.

B.12 *Relativistic fluid mechanics*

Let's start again at the expression for the energy-momentum tensor of an ideal fluid,

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu - g^{\mu\nu}p \tag{B.141}$$

of which one can directly compute the trace,

$$g_{\mu\nu}T^{\mu\nu} = \rho c^2 - 3p \tag{B.142}$$

keeping in mind that $g_{\mu\nu}u^\mu u^\nu = c^2$ for material particles, which approaches $p/(\rho c^2) = 1/3$ for relativistic, massless particles, for which $g_{\mu\nu}T^{\mu\nu} = 0$.

Computing the covariant divergence of $T^{\mu\nu}$ one gets

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho c^2 + p) \cdot u^\mu u^\nu + (\rho c^2 + p)\nabla_\mu(u^\mu u^\nu) - \nabla_\mu p \cdot g^{\mu\nu} \tag{B.143}$$

keeping in mind that the covariant derivative of the metric vanishes for a metric-compatible connection, in particular $\nabla_\mu g^{\mu\nu} = 0$ as well for the inverse metric.

Let's pursue this divergence and let's make a deliberate mistake by assuming that the fluid elements follow geodesics, i.e. that an autoparallelity condition applies to the velocities u^μ:

$$\nabla_\mu(u^\mu u^\nu) = \nabla_\mu u^\mu \cdot u^\nu + \underbrace{u^\mu \nabla_\mu u^\nu}_{=0} \tag{B.144}$$

such that the divergence of the velocity field is the only contributing term, which we've already encountered in the discussion of the Raychaudhuri-equation. Then,

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho c^2 + p) \cdot u^\mu u^\nu + (\rho c^2 + p)\nabla_\mu u^\mu \cdot u^\nu - \nabla_\mu p \cdot g_{\mu\nu} \tag{B.145}$$

Now we are projecting (as mentioned above) on the observer's world line with $u_\nu = \mathrm{d}x_\nu/\mathrm{d}\tau$, and by contraction

$$u_\nu \nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho c^2 + p) \cdot u^\mu \underbrace{u_\nu u^\nu}_{=1} + (\rho c^2 + p)\nabla_\mu \underbrace{u_\nu u^\nu}_{=1} - \nabla_\mu p \cdot \underbrace{g^{\mu\nu} u_\nu}_{=u^\mu} \tag{B.146}$$

and arrive the relativistic continuity equation

$$u_\nu \nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho c^2 u^\mu) + p\nabla_\mu u^\mu = 0 \tag{B.147}$$

in which we used the covariant conservation in the last step. In the non-relativistic limit with $p \ll \rho c^2$ this leaves us with only the first summand $\nabla_\mu(\rho c^2 u^\mu) = 0$ left. Furthermore, u^μ is given by $u^\mu = (1, \beta^i)^T$ with $\beta^i = \frac{v^i}{c}$ and of course $|\beta| \ll 1$ in the slow motion limit. Lastly, ∇_μ becomes ∂_μ by adopting locally Cartesian coordinates, so:

$$u_\nu \nabla_\mu T^{\mu\nu} = 0 = \nabla_\mu(\rho c^2 u^\mu) = \partial_{ct}(\rho c^2) + \partial_i(\rho c^2 \beta^i) \tag{B.148}$$

i.e. the classical continuity equation,

$$\partial_t \rho + \partial_i(\rho v^i) = 0. \tag{B.149}$$

In the real world it turns out that there are incompressible fluids which are characterised by $\partial_i v^i = 0$ (Please watch out: Incompressibility is a statement about the velocity field and has little to do with pressure!) and therefore the continuity equation becomes

$$\partial_t \rho + \partial_i \rho \cdot v^i = 0 \tag{B.150}$$

for incompressible fluids.

We can substitute the relativistic continuity equation back into the conservation law $\nabla_\mu T^{\mu\nu} = 0$ to arrive at

$$\nabla_\mu T^{\mu\nu} = \underbrace{(\nabla_\mu(\rho c^2 u^\mu) + p\nabla_\mu u^\mu)}_{=0 \text{ continuity}} u^\nu + \nabla_\mu p u^\mu u^\nu - \nabla_\mu p g^{\mu\nu} = \nabla_\mu p(u^\mu u^\nu - g^{\mu\nu}) = 0 \quad \text{(B.151)}$$

which states that there are no pressure gradients perpendicular to u^μ as u^μ is tangent to a geodesic (or otherwise the fluid doesn't follow a geodesic in equivalence to the previous expression). This is particularly relevant for FLRW-cosmologies, as it implies that there can not be any spatial gradients in pressure, $\nabla_i p = \partial_i p = 0$, in accordance to the Copernican-principle. If those gradient would exist, the motion of fluid elements can not be inertial.

Let's restart by imposing no condition on geodesic motion of fluid elements. Then,

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho c^2 + p) \cdot u^\mu u^\nu + (\rho c^2 + p)\nabla_\mu(u^\mu u^\nu) - \nabla_\mu p \cdot g^{\mu\nu} = 0 \quad \text{(B.152)}$$

A useful auxiliary statement can be obtained from the normalisation of u_ν, where

$$0 = \nabla_\mu(\underbrace{u^\nu u_\nu}_{=1}) = u^\nu \nabla_\mu u_\nu + \nabla_\mu u^\nu \cdot u_\nu = 2u_\nu \nabla_\mu u^\nu \quad \text{(B.153)}$$

If one contracts the conservation law with u_ν, the second term $u^\mu \nabla_\mu u^\nu$ becomes $u^\mu u_\nu \nabla_\mu u^\nu = 0$. We therefore end up at

$$u_\nu T^{\mu\nu} = \nabla_\mu(\rho c^2 + p) \cdot u^\mu + (\rho c^2 + p)\nabla_\mu u^\mu - \nabla_\mu p u^\mu = \nabla_\mu(\rho c^2 u^\mu) + p\nabla_\mu u^\mu = 0 \quad \text{(B.154)}$$

even if the fluid follows non-geodesic motion. If we resubstitute back into the full, non-geodesic conservation equation we obtain

$$\nabla_\mu T^{\mu\nu} = (u^\mu u^\nu - g^{\mu\nu})\nabla_\mu p + (\rho c^2 + p)u^\mu \nabla_\mu u^\nu = 0 \quad \text{(B.155)}$$

with an additional term being present for non-geodesic motion caused by gradients in pressure.