Chapter 5

Stellar Dynamics

5.1 The Jeans equations and Jeans' theorem

We begin this section by a derivation of the relaxation time scale (5.20), showing how long it takes a star orbiting through a system of stars to substantially change its velocity. In a way reminiscent of the derivation of the hydrodynamical equations, we then derive Jeans' equations (5.41) for the evolution of the number density and the mean velocity of stars in a stellar system. We then transform the Jeans equations to spherical polar coordinates and specialise them to stationary spherical systems in (5.56), showing that the main difference to hydrostatics is the anisotropy in velocity space. Next, we derive the tensor virial theorem (5.82) for stellar-dynamical systems and introduce Jeans' theorem.

5.1.1 Collision-less motion in a gravitational field

Particles in a gas or a fluid move almost unaccelerated until they meet another particle, which forces them to change their state of motion abruptly. As we have discussed before, hydrodynamics is based on the central assumption that the collisions occur on much smaller length scales λ than those macroscopic scales L that characterise the extent of the entire hydrodynamical system. In plasma physics, we had seen that the shielding of charges on the scale of the Debye length λ_D allows a hydrodynamical treatment despite the formally inifite range of electrostatic interactions, provided there are sufficiently many particles in the Debye volume $\approx \lambda_\mathrm{D}^3$. In all these cases, the interactions are effectively extremely short-ranged. Likewise, we had assumed in our treatment of local thermodynamical equilibrium in radiation transport that the mean free path of the photons be much smaller than the characteristic dimensions of the system under consideration.

Studying the motion of many point masses such as stars in a gravitational field, we encounter a fundamentally changed situation. The forces between the particles are now long-ranged and cannot be shielded. A single star in a galaxy, for instance, thus experiences not only the attraction of its nearest neighbours, but essentially the gravitational force exerted by all stars in the entire galaxy.

To give an illustrative example, let us consider a two-dimensional system, such as a galactic disk, which we shall assume to be infinitely extended for now and in whose centre we assume a star. The disk be randomly covered by stars in such a way that their mean number density is spatially constant (Figure 5.1).

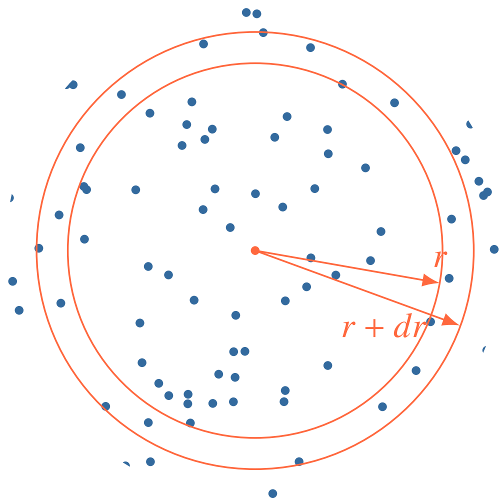

Figure 5.1 Illustration of a small section of a random star field, centered on an arbitrarily chosen star.

In a circular ring around the central star of radius r and width $\mathrm{d}r$, we find

$$\mathrm{d}N = 2\pi r \mathrm{d}r\, n \tag{5.1}$$

stars whose combined gravitational force on the central star is

$$\mathrm{d}F = 2\pi r \mathrm{d}r\, n \frac{Gm^2}{r^2} \tag{5.2}$$

if the mass m is assumed to be the same for all stars for simplicity. Of course, the directions of all forces cancel in the mean, but the contribution of arbitrarily distant rings diverges logarithmically,

$$\int \mathrm{d}F = 2\pi Gnm^2 \int \frac{\mathrm{d}r}{r} = 2\pi Gnm^2 \ln r\ . \tag{5.3}$$

Thus, the structure of the entire stellar system is important for the dynamics of the stars in the gravitational field.

In the spirit of the distinction between microscopic and macroscopic forces that we had made when introducing hydrodynamics, the forces in a system which is dominated by self-gravity are also macroscopic. Therefore, the collision terms, which describe the interaction on a microscopic scale, can be neglected here at least to first order of approximation. Thus, we begin our treatment of self-gravitating systems with the collision-less Boltzmann equation,

$$\mathrm{d}_t f\,(\vec{x}, \vec{v}, t) = \partial_t f + \dot{\vec{x}} \cdot \vec{\nabla} f + \dot{\vec{v}} \cdot \vec{\nabla}_{\vec{v}} f = 0 \tag{5.4}$$

5.1.2 The relaxation time scale

Before we turn to a detailed study of Eq. (5.4) in a gravitational field, we investigate approximately how the trajectory of a star through a galaxy which is composed of individual stars deviates from the trajectory through a hypothetical, "smooth" galaxy. We consider the passage of a star past another star employing Born's approximation, i.e. we integrate the deflection along a straight trajectory passing the deflecting star at an impact parameter b. The perpendicular force at the location x along the hypothetical, straight trajectory is

$$F_\perp = \left|-\vec{\nabla}_\perp \phi\right| = \left|-\frac{\partial}{\partial b}\frac{Gm^2}{\sqrt{b^2+x^2}}\right| = \frac{Gm^2 b}{(b^2+x^2)^{3/2}}\,, \tag{5.5}$$

where ϕ is the Newtonian gravitational potential. With $x \approx vt$, we have

$$F_\perp \approx \frac{Gm^2}{b^2}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-3/2}\,, \tag{5.6}$$

and Newton's second law $m\dot{v}_\perp = F_\perp$ thus implies

$$\delta v_\perp \approx \frac{Gm}{b^2}\int_{-\infty}^{\infty}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-3/2}\mathrm{d}t = \frac{2Gm}{bv}\int_0^\infty \left(1+\tau^2\right)^{-3/2}\mathrm{d}\tau = \frac{2Gm}{bv}\,. \tag{5.7}$$

?

How do you solve an integral like that in (5.7)?

Let N be the number of stars in the galaxy and R be its radius, then the fiducial test star experiences

$$\delta N = 2\pi b\delta b\, n = 2\pi b\delta b\frac{N}{\pi R^2} = \frac{2N}{R^2}b\delta b \tag{5.8}$$

such encounters with other stars at an impact parameter between b and $b+\delta b$. The mean quadratic velocity change is thus

$$\delta v_\perp^2 \approx \frac{2Nb\delta b}{R^2}\left(\frac{2Gm}{bv}\right)^2 = \frac{8NG^2m^2}{R^2v^2}\frac{\delta b}{b}\,. \tag{5.9}$$

Integrating this expression, we need to take into account that the assumption of Born's approximation requires that

$$\delta v_\perp \lesssim v \quad\Rightarrow\quad \frac{2Gm}{bv}\lesssim v \quad\Rightarrow\quad b \gtrsim b_{\mathrm{min}} = \frac{Gm}{v^2}\,, \tag{5.10}$$

and thus we obtain

$$\Delta v_\perp^2 = \int_{b_{\mathrm{min}}}^{\infty}\delta v_\perp^2 \approx 2N\left(\frac{2Gm}{Rv}\right)^2 \ln b\Big|_{b_{\mathrm{min}}}^{R} \equiv 2N\left(\frac{2Gm}{Rv}\right)^2\ln\Lambda\,, \tag{5.11}$$

where

$$\ln\Lambda \equiv \ln\frac{R}{b_{\mathrm{min}}} = \ln\frac{Rv^2}{Gm}\,; \tag{5.12}$$

is the so-called *Coulomb logarithm*. A typical velocity for the stars in a galaxy of mass $M = Nm$ is, according to the virial theorem,

$$v^2 \approx \frac{GMm}{R} \quad\Rightarrow\quad R \approx \frac{GNm}{v^2}\,. \tag{5.13}$$

Using this, we obtain

$$\frac{\Delta v_\perp^2}{v^2} \approx \frac{8 \ln \Lambda}{N} \,. \tag{5.14}$$

This shows by which relative amount the star's velocity is changed during one passage through the galaxy. The Coulomb logarithm $\ln \Lambda$ follows from

$$\ln \Lambda = \ln \frac{R}{b_{\min}} = \ln \frac{Rv^2}{Gm} \approx \ln N \,, \tag{5.15}$$

i.e. the relative velocity change is approximated by

$$\frac{\Delta v_\perp^2}{v^2} \approx \frac{8 \ln N}{N} \,. \tag{5.16}$$

After n_{cross} passages through the galaxy, the total relative velocity change will approximately be

$$n_{\text{cross}} \frac{8 \ln N}{N} \,. \tag{5.17}$$

For this expression to be of order unity, the number of passages needs to be

$$n_{\text{cross}} \approx \frac{N}{8 \ln N} \,. \tag{5.18}$$

Since one passage takes approximately the time

$$t_{\text{cross}} \approx \frac{R}{v} \,, \tag{5.19}$$

a substantial velocity change needs the *relaxation time*

$$t_{\text{relax}} \approx \frac{R}{v} \frac{N}{8 \ln N} \,. \tag{5.20}$$

Example: Relaxation of a galaxy

In a galaxy, we typically have a crossing time scale of

$$t_{\text{cross}} \approx \frac{10\,\text{kpc}}{200\,\text{km}\,\text{s}^{-1}} \approx 5 \cdot 10^7\,\text{yr} \tag{5.21}$$

and perhaps $N \approx 10^{11}$ stars. The relaxation time thus turns out to be

$$t_{\text{relax}} \approx 3 \cdot 10^{16}\,\text{yr} \,, \tag{5.22}$$

which is much more than the age of the Universe. This illustrates that in many, if not most astrophysically relevant systems, the collision-less Boltzmann equation can safely be used. ◂

5.1.3 The Jeans equations

The derivation of the Jeans equation, which will now follow, is formally similar to the derivation of the hydrodynamical equations. Yet, there are several important conceptual differences which justify going through the derivation again. First of all, we now have to do with a collection of individual, indentifiable

Example: Relaxation of a globular cluster

A counter-example is given by globular clusters. There, the number of stars is much smaller, $N \approx 10^5$, and crossing times are of order $t_{\rm cross} \approx 10^5$ yr. Their relaxation time scale is therefore

$$t_{\rm relax} \approx 10^8\ {\rm yr}\ , \tag{5.23}$$

which is short compared to the life time of the globular cluster. In such cases, therefore, collisions do play an essential role. ◄

"particles", namely the stars in a stellar system, orbiting under their mutual gravitational interaction and possibly in an external, more or less smooth gravitational potential ϕ. Second, when we integrated over the momentum subspace during the derivation of hydrodynamics, we introduced an integral measure to ensure that the integral was relativistically invariant. There is no need to do so here since we can treat the stars in a stellar system as non-relativistically moving objects. Third, on the way to hydrodynamics, we introduced the four-vector J^μ for the particle-current density and the energy-momentum tensor $T^{\mu\nu}$ of the fluid and showed that the hydrodynamical equations followed from the vanishing four-divergences of J^μ and $T^{\mu\nu}$. Since we now have a collection of individual point masses, the introduction of continuous quantities such as the current densities of stars, momentum and energy is not necessarily justified. What we shall introduce, though, is the mean spatial number density $n(t, \vec{x})$ of the stars at the position $\vec{x}$ and at the time t.

We thus begin again with Boltzmann's collision-less equation, in which the right-hand side is set to zero, $\mathrm{d}_t f = 0$. Consider $f(\vec{x}, \vec{v}, t)$ as a function of position, velocity and time, and replace the time derivative of the velocity according to Newton's second law,

$$\dot{\vec{v}} = \frac{\vec{F}}{m} = -\vec{\nabla}\phi \tag{5.24}$$

to obtain

$$\partial_t f + \vec{v}\cdot\vec{\nabla} f - \vec{\nabla}\phi\cdot\vec{\nabla}_{\vec{v}} f = 0\ . \tag{5.25}$$

Similar to the derivation of the hydrodynamical equations, we now form velocity moments of equation (5.25) by multiplying the collision-less Boltzmann equation with powers of the velocity and integrating over velocity space,

$$\partial_t \int \mathrm{d}^3 v\, f + \int \mathrm{d}^3 v\, \vec{v}\cdot\vec{\nabla} f - \vec{\nabla}\phi\cdot\int \mathrm{d}^3 v\,\vec{\nabla}_{\vec{v}} f = 0\ . \tag{5.26}$$

The last term here simply gives boundary terms at infinity which vanish under the assumption that there are no infinitely fast point masses,

$$f(\vec{x}, \vec{v}, t) \to 0 \quad \text{for} \quad \left|\vec{v}\right| \to \infty\ . \tag{5.27}$$

In the second term, the gradient can be pulled out of the integral since it operates on the spatial coordinates $\vec{x}$, while the integration is carried out over the velocity. Equation (5.26) then gives

$$\partial_t n + \vec{\nabla}\cdot\int \mathrm{d}^3 v\, f\vec{v} = 0\ . \tag{5.28}$$

Since the mean velocity is defined as

$$\langle\vec{v}\rangle = \frac{1}{n}\int \mathrm{d}^3v\, f\vec{v}\,, \tag{5.29}$$

we find the continuity equation for our point masses,

$$\partial_t n + \vec{\nabla}\cdot(n\langle\vec{v}\rangle) = 0\,, \tag{5.30}$$

as we might have expected. Notice in particular that we have introduced the mean spatial number density

$$n(\vec{x},t) = \int \mathrm{d}^3v\, f(\vec{x},\vec{v},t) \tag{5.31}$$

of the stars here. As an integral over the one-particle phase-space distribution f, this is a well-defined quantity, which should however not be confused with the smooth matter density of a fluid. Given the discrete nature of the stars in a stellar system, their spatial number density may fluctuate considerably. Moreover, it is not easily possible to move from the number density n to the matter density ρ by multiplying with a particle mass since the stars will typically have a wide mass distribution.

The second moment of Boltzmann's equation is taken by multiplying equation (5.25) with the velocity $\vec{v}$ prior to the integration over velocity space. In this way, further using that

$$\begin{aligned}\left(\vec{\nabla}f\cdot\vec{v}\right)\vec{v} &= \vec{\nabla}f^{\top}(\vec{v}\otimes\vec{v}) = \vec{\nabla}\cdot(f\vec{v}\otimes\vec{v}) \quad \text{and}\\ \left(\vec{\nabla}\phi\cdot\vec{\nabla}_{\vec{v}}f\right)\vec{v} &= \vec{\nabla}\phi^{\top}\left(\frac{\partial f}{\partial\vec{v}}\otimes\vec{v}\right)\end{aligned} \tag{5.32}$$

we obtain

$$\partial_t\int \mathrm{d}^3v\, f\vec{v} + \vec{\nabla}\cdot\int \mathrm{d}^3v\, f\vec{v}\otimes\vec{v} - \vec{\nabla}\phi^{\top}\int \mathrm{d}^3v\left(\frac{\partial f}{\partial\vec{v}}\otimes\vec{v}\right) = 0\,. \tag{5.33}$$

?

Verify the expressions (5.32) by your own calculation.

We continue by considering the third term, which can be integrated by parts to yield

$$\int \mathrm{d}^3v\left(\frac{\partial f}{\partial\vec{v}}\otimes\vec{v}\right) = -\int \mathrm{d}^3v\, f\,\frac{\partial\vec{v}}{\partial\vec{v}} = -n\mathbb{1}_3\,, \tag{5.34}$$

if we can ignore boundary terms at infinity as before. This expression enables us to re-write (5.33) as

$$\partial_t(n\langle\vec{v}\rangle) + \vec{\nabla}\cdot(n\langle\vec{v}\otimes\vec{v}\rangle) + n\vec{\nabla}\phi = 0\,, \tag{5.35}$$

where

$$\langle\vec{v}\otimes\vec{v}\rangle \equiv \frac{1}{n}\int \mathrm{d}^3v\, f\vec{v}\otimes\vec{v} \tag{5.36}$$

is the velocity-dispersion tensor. This tensor can be re-written in terms of the velocity-correlation tensor and the average velocity components,

$$\langle\vec{v}\otimes\vec{v}\rangle = \langle(\vec{v}-\langle\vec{v}\rangle)\otimes(\vec{v}-\langle\vec{v}\rangle)\rangle + \langle\vec{v}\rangle\otimes\langle\vec{v}\rangle \equiv \sigma^2 + \langle\vec{v}\rangle\otimes\langle\vec{v}\rangle\,. \tag{5.37}$$

For convenience, we now substitute

$$\langle\vec{v}\rangle \to \vec{v} \tag{5.38}$$

since only averaged velocities and no velocities of individual particles remain. This allows us to write (5.35) as

$$\partial_t (n\vec{v}) + \vec{\nabla} \cdot \left(n\sigma^2\right) + \vec{\nabla} \cdot (n\vec{v} \otimes \vec{v}) + n\vec{\nabla}\phi = 0 \ . \tag{5.39}$$

Applying the product rule to the first and third terms and grouping terms conveniently, we can continue to write

$$\vec{v}\left[\partial_t n + \vec{\nabla} \cdot (n\vec{v})\right] + n\partial_t\vec{v} + \left(n\vec{v} \cdot \vec{\nabla}\right)\vec{v} + \vec{\nabla} \cdot \left(n\sigma^2\right) + n\vec{\nabla}\phi = 0 \ . \tag{5.40}$$

Noticing that the term in square brackets vanishes due to the continuity equation, we thus obtain the two equations

$$\partial_t n + \vec{\nabla} \cdot (n\vec{v}) = 0 \ ,$$

$$\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} = -\vec{\nabla}\phi - \frac{1}{n}\vec{\nabla} \cdot \left(n\sigma^2\right) \ . \tag{5.41}$$

These are the *Jeans equations* which were derived for the first time by Maxwell, but first applied to stellar-dynamical problems by Sir James Jeans. As an equation of motion for the mean velocity components, the second equation corresponds to Euler's equation in ideal hydrodynamics, where the divergence of the tensor $n\sigma^2$ takes the role of the pressure gradient,

$$\frac{\vec{\nabla}P}{\rho} = \rho^{-1}\vec{\nabla} \cdot (P\mathbb{1}_3) \to \vec{\nabla} \cdot \left(n\sigma^2\right) \ . \tag{5.42}$$

5.1.4 Jeans equations in cylindrical and spherical coordinates

It is useful for many applications to write the distribution function f as a function not of Cartesian but of such coordinates that are adapted to the symmetry of a specific stellar-dynamical system under investigation. The Jeans equation then needs to be transformed from the Cartesian basis vectors $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ to those of the new, curvi-linear coordinates system. Let us carry out this transformation for the two frequent cases of cylindrical and spherical coordinates. Doing so, the transformations (3.209) and (3.214) derived earlier for the components of the acceleration need to be taken into account.

Since the gradient in cylindrical coordinates is

$$\vec{\nabla} = \hat{e}_r\partial_r + \frac{\hat{e}_\varphi}{r}\partial_\varphi + \hat{e}_z\partial_z \ , \tag{5.43}$$

we first obtain the components

$$a_r = -\partial_r\phi \ , \quad a_\varphi = -\frac{\partial_\varphi\phi}{r} \ , \quad a_z = -\partial_z\phi \tag{5.44}$$

of the equation of motion. With (3.209), we find

$$\dot{v}_r - \frac{v_\varphi^2}{r} = -\partial_r\phi \ , \quad \dot{v}_\varphi + \frac{v_r v_\varphi}{r} = -\frac{\partial_\varphi\phi}{r} \ . \tag{5.45}$$

This implies that the collision-less Boltzmann equation in cylindrical coordinates reads

$$\partial_t f + v_r \partial_r f + \frac{v_\varphi}{r}\partial_\varphi f + v_z \partial_z f + \left(\frac{v_\varphi^2}{r} - \partial_r \phi\right)\partial_{v_r} f - \left(\frac{v_r v_\varphi}{r} + \frac{\partial_\varphi \phi}{r}\right)\partial_{v_\varphi} f - \partial_z \phi \partial_{v_z} f = 0 \; . \tag{5.46}$$

In spherical polar coordinates, we use the representation

$$\vec{\nabla} = \hat{e}_r \partial_r + \frac{\hat{e}_\theta}{r}\partial_\theta + \frac{\hat{e}_\varphi}{r\sin\theta}\partial_\varphi \tag{5.47}$$

of the gradient operator and transformation (3.214) of the acceleration components to find the fairly lengthy form

$$\partial_t f + v_r \partial_r f + \frac{v_\theta}{r}\partial_\theta f + \frac{v_\varphi}{r\sin\theta}\partial_\varphi f + \left[\frac{v_\theta^2 + v_\varphi^2}{r} - \partial_r \phi\right]\partial_{v_r} f - \left[\frac{v_r v_\theta}{r} - \frac{v_\varphi^2}{r}\cot\theta + \frac{\partial_\theta \phi}{r}\right]\partial_{v_\theta} f - \left[\frac{v_\varphi}{r}(v_r + v_\theta \cot\theta) + \frac{\partial_\varphi \phi}{r\sin\theta}\right]\partial_{v_\varphi} f = 0 \tag{5.48}$$

for the collision-less Boltzmann equation, whose physical meaning remains of course unchanged.

Whatever coordinates we choose, the zeroth moment of the collision-less Boltzmann equation must reproduce the continuity equation (5.30), with the appropriate representation of the divergence operator in the coordinate system chosen. Let us multiply the Boltzmann equation in spherical coordinates, (5.48), with the radial velocity component v_r and then integrate it over the complete velocity subspace of phase space. The result is the still lengthy expression

$$\partial_t \left(n\langle v_r\rangle\right) + \partial_r \left(n\left\langle v_r^2\right\rangle\right) + \frac{\partial_\theta}{r}\left(n\langle v_r v_\theta\rangle\right) + \frac{\partial_\varphi}{r\sin\theta}\left(n\left\langle v_r v_\varphi\right\rangle\right) - \frac{n}{r}\left\langle v_\theta^2 + v_\varphi^2\right\rangle + n\partial_r \phi + \frac{2n}{r}\left\langle v_r^2\right\rangle + \frac{n}{r}\langle v_r v_\theta\rangle\cot\theta = 0 \; , \tag{5.49}$$

where we have kept the order and the arrangement of the terms like those from which they originate in (5.48). The decisive step in deriving (5.49) are partial integrations in velocity space.

?

Derive (5.49) yourself, following the steps described in the text.

5.1.5 Application to spherical systems

Equation (5.49) can be considerably simplified under the following natural assumptions. First, let us assume that the average velocities in the polar and azimuthal directions vanish,

$$\left\langle v_\varphi \right\rangle = 0 = \langle v_\theta \rangle \; . \tag{5.50}$$

Then, let us further assume that the velocity components are statistically independent of each other,

$$\langle v_r v_\theta \rangle = 0 = \left\langle v_r v_\varphi \right\rangle \; , \tag{5.51}$$

and that the situation is static, allowing us to ignore the partial time derivative. If we further introduce the velocity dispersions as the averages

$$\sigma^2_{r,\theta,\varphi} = \frac{1}{n}\int \mathrm{d}^3v\, v^2_{r,\theta,\varphi} f\ , \tag{5.52}$$

we arrive at the much simpler equation

$$\partial_r\left(n\sigma_r^2\right) + \frac{n}{r}\left[2\sigma_r^2 - \left(\sigma_\theta^2 + \sigma_\varphi^2\right)\right] = -n\partial_r\phi\ . \tag{5.53}$$

Notice that we have neither used the continuity equation nor the explicit assumption of spherical symmetry here, but exclusively the first, radial moment of the collision-less Boltzmann equation together with an assumed isotropy in velocity space, expressed by the conditions (5.50) and (5.51).

Given this isotropy in velocity space, it is natural to assume that the polar and azimuthal velocity dispersions be equal,

$$\sigma_\theta^2 = \sigma_\varphi^2\ . \tag{5.54}$$

We relate them to the radial velocity dispersion σ_r^2 by an anisotropy parameter β such that

$$\sigma_\theta^2 = \sigma_r^2(1-\beta) = \sigma_\varphi^2\ . \tag{5.55}$$

Typically, the anisotropy parameter is non-negative, $\beta \geq 0$. If $\beta > 0$, radial motion dominates, while tangential motion dominates if $\beta < 0$. The anisotropy parameter itself cannot generally be assumed to be constant, but should be taken as depending on the radius r.

We are finally left with the radial Jeans equation

$$\partial_r\left(n\sigma_r^2\right) + \frac{2\beta(r)}{r}n\sigma_r^2 = -n\partial_r\phi\ , \tag{5.56}$$

which is a first-order, linear, ordinary and inhomogeneous differential equation for the quantity $n\sigma_r^2$. It is easily solved by variation of constants. The general homogeneous solution is quickly found to be

$$n\sigma_r^2 = C\exp\left(-2\int_0^r \frac{\beta(x)}{x}\mathrm{d}x\right)\ , \tag{5.57}$$

where the constant C is chosen such that $n\sigma_r^2 = C$ at the centre, $r = 0$, and x was introduced merely as a radial integration variable. For solving the inhomogeneous equation, we now allow C to vary with radius, $C = C(r)$. Then, (5.56) gives the differential equation

$$C'(r) = -n\partial_r\phi\exp\left(2\int_0^r \frac{\beta(x)}{x}\mathrm{d}x\right) \tag{5.58}$$

for $C(r)$ since the exponential from (5.57) was constructed to solve the homogeneous equation (5.56) in the first place. This equation can formally be integrated to give

$$C(r) = \int_r^\infty \mathrm{d}y\left[n(y)(\partial_r\phi)(y)\exp\left(2\int_0^y \frac{\beta(x)}{x}\mathrm{d}x\right)\right]\ , \tag{5.59}$$

where y was introduced as another radial integration variable and the boundary condition was chosen such that $C \to 0$ for $r \to \infty$ irrespective of what $\beta(r)$ may be. Returning with this result to (5.57), we obtain the solution

$$n\sigma_r^2 = \int_r^\infty \mathrm{d}y \left[n(y)(\partial_r\phi)(y) \exp\left(2 \int_r^y \frac{\beta(x)}{x} \mathrm{d}x \right) \right] \tag{5.60}$$

for the radial velocity dispersion σ_r^2 times the stellar density n. For a spherically-symmetric system, we can further write the radial derivative of the gravitational potential as

$$\partial_r\phi = \frac{GM(r)}{r^2} , \tag{5.61}$$

which enables us to write

$$n\sigma_r^2 = G \int_r^\infty \mathrm{d}y \left[\frac{M(y)n(y)}{y^2} \exp\left(2 \int_r^y \frac{\beta(x)}{x} \mathrm{d}x \right) \right] . \tag{5.62}$$

?

Carry out all steps yourself that lead from the radial Jeans equation (5.56) to the solution (5.62).

Many studies of stellar dynamics begin here. In principle, the radial stellar density $n(r)$ is observable through the surface brightness of an observed stellar system. By spectroscopy, the radial velocity dispersion σ_r^2 is accessible. If an anisotropy parameter $\beta(r)$ can now be reasonably guessed, (5.62) allows determining the mass (Figure 5.2).

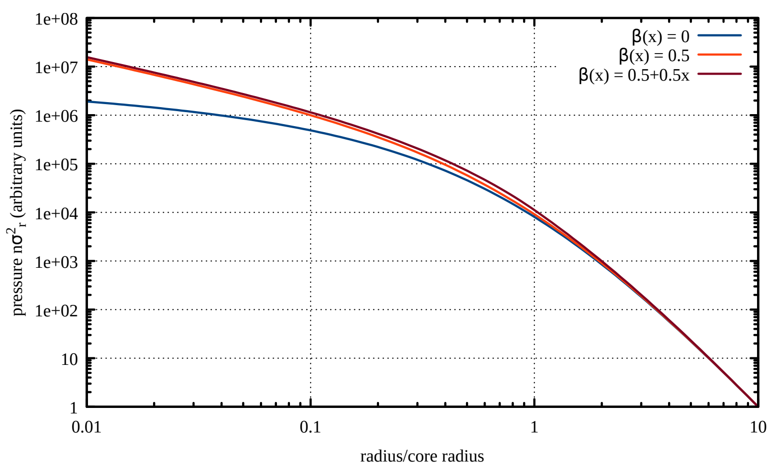

Figure 5.2 The effective kinematic pressure in a spherically-symmetric system according to Jeans' equation is shown as a function of the three-dimensional radius for different anisotropy parameters β.

The radial velocity dispersion multiplied with the stellar number density is of course not quite an observable quantity. Only the line-of-sight component of stellar velocities can typically be measured by the red- or blueshift of spectral lines. Since the red- and blueshifts of many stars generally appear superposed, lines appear broadened by the motion of the stars within the gravitational potential and shifted by the systemic velocity of the potential well as a whole relative to us as observers. The Doppler-broadened width of the spectral lines is the observable quantity to be measured, and it is directly related to the line-of-sight averaged velocity dispersion. Since the observed spectral line is a superposition of lines in the spectra of many stars, the observed line is dominated by those velocities that are represented by the most intense stellar light. Assuming that the stellar light is related to the stellar density by some constant factor, what we see is thus the density-weighted component of the stellar velocity along the line-of-sight.

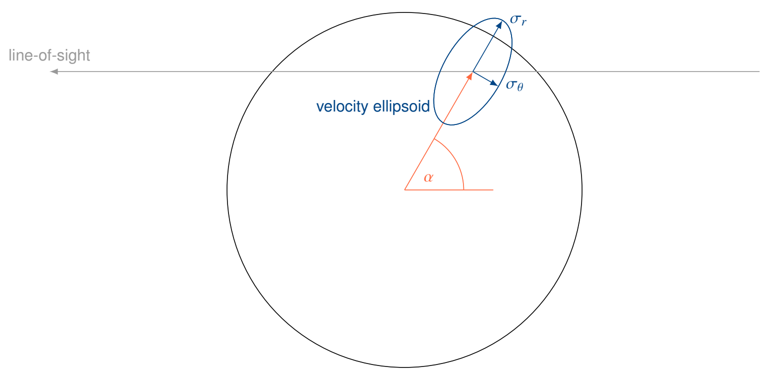

Figure 5.3 Illustration of the projection of the radial and tangential velocity dispersions along the line-of-sight.

Imagine a line-of-sight that passes through the spherical galaxy at a projected distance s from the line-of-sight through its centre at an azimuthal angle that we can without loss of generality assume to be zero, $\varphi = 0$. Further, let α be the angle between this line-of-sight and the radial direction. Then, the projected velocity component parallel to the line-of-sight is (Figure 5.3)

$$v_\| = v_r \cos\alpha + v_\theta \sin\alpha \, , \tag{5.63}$$

whose density-weighted, averaged square we see,

$$\sigma_\|^2 = \frac{\int_{-\infty}^{\infty} \mathrm{d}z \, n(s, z) \left\langle v_\|^2 \right\rangle}{\int_{-\infty}^{\infty} \mathrm{d}z \, n(s, z)} \, . \tag{5.64}$$

The denominator normalises the line-of-sight weighting with the stellar density. Since it is proportional to the surface brightness of the stellar light, we abbreviate it by $I(s)$. The integral along the z direction is conveniently converted into an integral in radial direction by noting that $r^2 = s^2 + z^2$ such that, at constant projected distance s, we have $r\mathrm{d}r = z\mathrm{d}z$. We can thus write the normalisation integral as

$$I(s) = \int_{-\infty}^{\infty} \mathrm{d}z \, n(s, z) = 2 \int_s^{\infty} \frac{r\mathrm{d}r}{z} n(s, z) = 2 \int_s^{\infty} \frac{r\mathrm{d}r \, n(r)}{\sqrt{r^2 - s^2}} \tag{5.65}$$

and the projected velocity dispersion as

$$\sigma_\|^2 = \frac{2}{I(s)} \int_s^{\infty} \frac{r\mathrm{d}r \, n(r)}{\sqrt{r^2 - s^2}} \left\langle (v_r \cos\alpha + v_\theta \sin\alpha)^2 \right\rangle \, . \tag{5.66}$$

The average is easily carried out. The mixed average $\langle v_r v_\theta \rangle$ vanishes due to our isotropy assumption (5.51) such that

$$\begin{aligned}\left\langle (v_r \cos\alpha + v_\theta \sin\alpha)^2 \right\rangle &= \sigma_r^2 \cos^2\alpha + \sigma_\theta^2 \sin^2\alpha \\ &= \sigma_r^2 \left[\cos^2\alpha + (1-\beta)\sin^2\alpha\right] \\ &= \sigma_r^2 \left(1 - \beta \sin^2\alpha\right) \end{aligned} \tag{5.67}$$

remains, where the anisotropy parameter β from (5.55) was inserted. By definition of the angle α, we can further substitute

$$\sin^2\alpha = \frac{s^2}{r^2} \tag{5.68}$$

in (5.67) and (5.66). This gives the relation

$$\sigma_\parallel^2 = \frac{2}{I(s)} \int_s^\infty \frac{r\mathrm{d}r\, n(r)\sigma_r^2}{\sqrt{r^2 - s^2}} \left(1 - \frac{\beta(r)s^2}{r^2}\right) \tag{5.69}$$

between the observable, density-weighted line-of-sight velocity dispersion $\sigma_\parallel^2$ and the radial velocity dispersion σ_r^2 (Figure 5.4).

?

Confirm the expression (5.69) for the line-of-sight velocity dispersion.

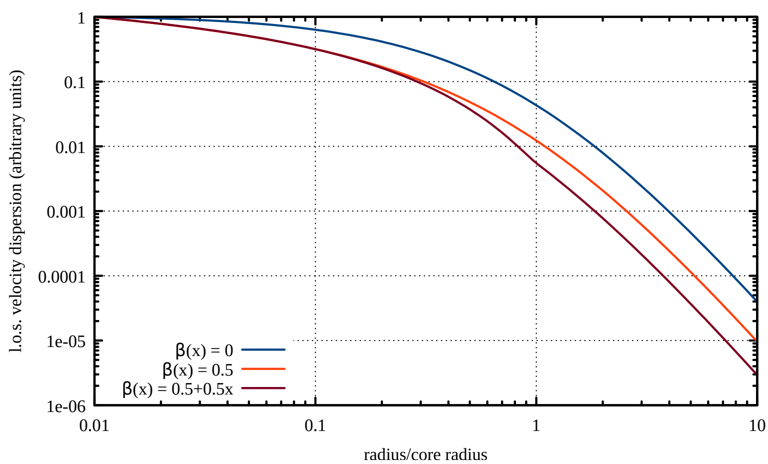

Figure 5.4 The observable line-of-sight velocity dispersion in a spherical system is shown as a function of projected radius for different anisotropy parameters β.

The anisotropy parameter can finally be eliminated by means of (5.56), since

$$\frac{\beta(r)}{r} n\sigma_r^2 = -\frac{1}{2}\left[\partial_r \left(n\sigma_r^2\right) + n\partial_r\phi\right] . \tag{5.70}$$

Inserting this expression into (5.69) and rearranging leads us to

$$I(s)\sigma_\parallel^2 - Gs^2 \int_s^\infty \frac{\mathrm{d}r\, n(r)M(r)}{r^2\sqrt{r^2 - s^2}} = \int_s^\infty \frac{r\mathrm{d}r}{\sqrt{r^2 - s^2}} \left[2n(r)\sigma_r^2 + \frac{s^2\partial_r(n\sigma_r^2)}{r}\right] . \tag{5.71}$$

This is an integro-differential equation for the density-weighted, radial velocity-dispersion profile $n\sigma_r^2$, determined by two observables, the surface-brightness profile $I(s)$ and the line-of-sight velocity dispersion $\sigma_\parallel^2$, together with a mass model $M(r)$. With (5.70), the solution can be used to constrain the anisotropy parameter $\beta(r)$.

5.1.6 The tensor virial theorem in stellar dynamics

We return to the Jeans equation in the form (5.35),

$$\partial_t \left(n \langle \vec{v} \rangle\right) + \vec{\nabla} \cdot \left(n \langle \vec{v} \otimes \vec{v} \rangle\right) + n\vec{\nabla}\phi = 0 \; . \tag{5.72}$$

Multiplication with the particle mass m and the spatial position vector $\vec{x}$, followed by integration over d^3x yields

$$\int d^3x \, \vec{x} \otimes \partial_t \left(\rho \langle \vec{v} \rangle\right) = -\int d^3x \, \vec{x} \otimes \vec{\nabla} \cdot \left(\rho \langle \vec{v} \otimes \vec{v} \rangle\right) - \int d^3x \, \vec{x} \otimes \rho \vec{\nabla}\phi \; . \tag{5.73}$$

We had seen already in (3.178) that the second term on the right-hand side is Chandrasekhar's tensor of the potential energy,

$$U = -\int d^3x \, \vec{x} \otimes \rho \vec{\nabla}\phi \; , \tag{5.74}$$

whose trace is the system's potential energy, as was shown in (3.180),

$$\mathrm{Tr}\, U = \frac{1}{2} \int d^3x \, \rho\phi \; . \tag{5.75}$$

Now we return to the first term on the right-hand side of the spatial integral (5.73), which we write as an integral over a complete divergence and a correction term,

$$\begin{aligned} \int d^3x \, \vec{x} \otimes \vec{\nabla} \cdot \left(\rho \langle \vec{v} \otimes \vec{v} \rangle\right) = & \int d^3x \, \vec{\nabla} \cdot \left(\rho \langle \vec{v} \otimes \vec{v} \rangle \otimes \vec{x}\right) \\ & - \int d^3x \, \rho \langle \vec{v} \otimes \vec{v} \rangle \left(\vec{\nabla} \otimes \vec{x}\right) \; . \end{aligned} \tag{5.76}$$

By Gauss' law, the integral over the divergence is the surface integral over $\rho\langle \vec{v} \otimes \vec{v} \rangle \otimes \vec{x}$, which vanishes if the surface completely encloses the system such that the density vanishes there. The remaining term is related to the tensor K of the kinetic energy,

$$\int d^3x \, \rho \langle \vec{v} \otimes \vec{v} \rangle \left(\vec{\nabla} \otimes \vec{x}\right) = \int d^3x \, \rho \langle \vec{v} \otimes \vec{v} \rangle = 2K \; , \tag{5.77}$$

whose trace is the total kinetic energy of the system. By means of the velocity-correlation tensor σ^2 defined in (5.37), we can split up the tensor of kinetic energy into a part T due to the bulk motion of the system, and another part Π due to the random motion of the stars about the mean motion. Specifically, we define

$$K = \frac{1}{2}T + \frac{1}{2}\Pi \; , \tag{5.78}$$

where T and Π are defined by

$$T \equiv \int d^3x \, \rho \langle \vec{v} \rangle \langle \vec{v} \rangle \; , \quad \Pi \equiv \int d^3x \, \rho\sigma^2 \; . \tag{5.79}$$

Quite evidently, the tensor T corresponds to the stress-energy tensor in ideal hydrodynamics up to the pressure term, while the tensor Π describes the momentum transport by unordered motion and is thus represents the pressure.

On the left-hand side of the spatial integral (5.73), the term

$$\int \mathrm{d}^3x\, \vec{x} \otimes \partial_t \left(\rho \langle \vec{v} \rangle\right) \tag{5.80}$$

remains. We symmetrise it by bringing it into the form

$$\frac{1}{2} \int \mathrm{d}^3x\, \left[\vec{x} \otimes \partial_t \left(\rho \langle \vec{v} \rangle\right) + \partial_t \left(\rho \langle \vec{v} \rangle\right) \otimes \vec{x}\right] \tag{5.81}$$

which, by comparison with (3.184), equals the second absolute time derivative of the inertial tensor I. We thus obtain the tensor virial theorem for collision-less systems,

$$\frac{1}{2} \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = T + \Pi + U \ . \tag{5.82}$$

?

Why is it appropriate to symmetrise the tensor given by (5.80)?

Taking the trace of this equation leads us back to the ordinary (scalar) virial theorem, if the mass distribution is static,

$$\frac{\mathrm{d}^2\, \mathrm{Tr}\, I}{\mathrm{d}t^2} = 0 \quad \Rightarrow \quad \mathrm{Tr}\, T + \mathrm{Tr}\, \Pi + \mathrm{Tr}\, U = 0 \ . \tag{5.83}$$

Now, the sum of the traces of T and Π is twice the trace of the total kinetic-energy tensor,

$$\mathrm{Tr}\, T + \mathrm{Tr}\, \Pi = 2\, \mathrm{Tr}\, K = \int \mathrm{d}^3x\, \rho v^2 \ , \tag{5.84}$$

and thus twice the total kinetic energy K, while $\mathrm{Tr}\, U$ is the total potential energy, as we have seen before. Thus,

$$2\, \mathrm{Tr}\, K = -\, \mathrm{Tr}\, U \ , \tag{5.85}$$

which is the ordinary scalar virial theorem.

5.1.7 Jeans' theorem

An integral of the motion in any field of force is any function $Q(\vec{x}, \vec{v})$ of the phase-space coordinates that satisfies

$$\frac{\mathrm{d}Q\left(\vec{x}, \vec{v}\right)}{\mathrm{d}t} = 0 \tag{5.86}$$

along all possible particle trajectories $[\vec{x}(t), \vec{v}(t)]$ through phase space. Integrals of the motion should not be confused with constants of the motion, which are less strongly defined as quantities that do not depend on time along one particular orbit. Any integral of the motion turns into a constant of the motion when evaluated along a particular orbit, but the reverse is not generally true.

An orbit of a classical particle in a Hamiltonian system always has six constants of the motion. Namely, let the orbit be specified by $\vec{x}(t)$ and $\vec{v}(t)$, then it can be uniquely traced back to an initial phase-space point $(\vec{x}_0, \vec{v}_0)$ by means of the equations of motion. These six numbers are constants of the motion, since they are independent of time along any trajectory.

For Hamiltonian systems, a potential $\phi(\vec{x})$ exists and the Hamiltonian equations of motion require $\dot{\vec{v}} = -\vec{\nabla}\phi(\vec{x})$. The condition (5.86) for Q to be an integral of the motion can then be cast into the form

$$\frac{\mathrm{d}Q\,(\vec{x},\vec{v})}{\mathrm{d}t} = \dot{\vec{x}}\cdot\vec{\nabla}Q + \dot{\vec{v}}\cdot\frac{\partial Q}{\partial\vec{v}} = \vec{v}\cdot\vec{\nabla}Q - \vec{\nabla}\phi\cdot\frac{\partial Q}{\partial\vec{v}} = 0\,. \tag{5.87}$$

By comparison with the collision-less Boltzmann equation, we see that Q is an integral of the motion if and only if it is a stationary solution of the collision-less Boltzmann's equation, i.e. a solution satisfying

$$\frac{\partial Q}{\partial t} = 0\,. \tag{5.88}$$

This leads us to *Jeans' theorem*:

Any stationary solution of the collision-less Boltzmann equation depends on the phase-space coordinates only through integrals of the motion, and conversely any function depending only on integrals of the motion is a stationary solution of the collision-less Boltzmann equation.

The *proof* of the first statement has already been given: If Q is a stationary solution of the collision-less Boltzmann equation it is by itself an integral of the motion. Regarding the second statement, let I_i, $1 \le i \le n$ be an arbitrary number n of integrals of the motion, and let $Q(I_1, I_2, \ldots, I_n)$ an arbitrary function exclusively depending on these integrals. Then,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial Q}{\partial I_i}\frac{\mathrm{d}I_i}{\mathrm{d}t} = 0\,, \tag{5.89}$$

and Q solves the collision-less Boltzmann equation.

Jeans' theorem is important because it guides the construction of physically meaningful phase-space densities. For example, in a static, spherically symmetric potential with isotropic orbits, the phase-space density can only be a function of the energy $E = \vec{v}^2/2 + \phi(r)$. If the potential remains spherically symmetric, but the orbits become anisotropic, the phase-space density will depend on E and the absolute value of the angular momentum L. In a static system with axial symmetry, the energy E and the component L_z of the angular momentum along the symmetry axis will be integrals of the motion, and the phase-space density will depend only on those.

One distinguishes isolating and non-isolating integrals of the motion. An isolating integral of the motion defines a subspace of phase space with dimension lowered by one to which an orbit is confined. Let I_1 be a first isolating integral of the motion, for example the energy. In the originally six-dimensional phase space, I_1 defines a five-dimensional subspace from which no orbit can escape. An further isolating integral I_2 will confine orbits to a four-dimensional subspace, and so forth. Isolating integrals such as the energy E or the angular-momentum vector $\vec{L}$ constrain the orbits. If n isolating integrals exist, orbits must be confined to a $(6-n)$-dimensional subspace of phase space. Isolating integrals are extraordinarily important while non-isolating integrals have no practical importance for stellar dynamics.

Orbits are called regular if they have as many isolating integrals as there are spatial dimensions; otherwise they are called irregular. Regular orbits in d

Example: Harmonic oscillator

The example of a harmonic oscillator in one spatial dimension may perhaps be instructive. Its phase space is two-dimensional, its energy is conserved. The constant energy confines the phase-space orbits of the oscillator to one-dimensional subspaces of phase space which are the ellipses defined by

$$E = \frac{m}{2}\left(\vec{v}^2 + \omega^2 x^2\right) = \text{const} . \tag{5.90}$$

Any one-dimensional harmonic oscillator must remain on the ellipse defined by its energy, and no harmonic oscillator with another energy will ever enter that subspace of phase space. This illustrates the isolating effect of the energy in phase space. ◂

spatial dimensions are thus confined to $2d - d = d$-dimensional subspaces of phase space. The one-dimensional harmonic oscillator is one example for a system with regular orbits.

Problems

1. A convenient model density profile for different kinds of astrophysical objects is the Hernquist profile

$$\rho(x) = \frac{\rho_0}{x(1+x)^3} , \tag{5.91}$$

proposed by L. Hernquist (1990). The dimension-less radius x is defined as $x = r/a$, with a scale or core radius a.

   (a) Write the density amplitude ρ_0 in terms of the total mass M contained in the profile (5.91).
   (b) Derive the Newtonian potential $\phi(x)$ of objects with the Hernquist density profile.
   (c) Assuming that the number-density $n(x)$ of the stars in a Hernquist-like object follows the matter-density profile, and assuming that the anisotropy parameter $\beta = 1/2$ is independent of the radius, solve (5.62) for the radial velocity dispersion.
   (d) Calculate the profile (5.69) of the observable, line-of-sight velocity dispersion profile.

2. For the Hernquist profile (5.91), calculate Chandrasekhar's tensor $U^i{}_j$ of the potential energy.

5.2 Equilibrium and Stability

This section discusses issues of equilibrium and stability of self-gravitating systems. The isothermal sphere (5.111) is introduced first as a simple example for a solution of the static Jeans equation in spherical symmetry.

Equilibrium considerations are briefly mentioned, emphasising that self-gravitating systems have no stable equilibrium state. A linear perturbation analysis reveals the close analogy (5.130) between perturbations of the gravitational potential in a stellar-dynamical system and the longitudinal dielectricity in a plasma. The Jeans wave number (5.136) is derived as the boundary between stable and unstable perturbations. A detour on two-dimensional, self-gravitating systems leads to the solution (5.159) of Poisson's equation for a disk with a given surface-mass density. Finally, the dispersion relation (5.185) for linear perturbations of disks is derived, leading to Toomre's stability criterion (5.193) for disks.

5.2.1 The Isothermal Sphere

By Noether's theorem, spherical systems which are independent of time have orbits with at least the four integrals of the motion, which are the energy E and the angular momentum $\vec{L}$. Jeans' theorem then tells us that any (non-negative) function $f(E, \vec{L})$ of these integrals of the motion is a stationary solution of the collision-less Boltzmann equation and may thus represent a stable, self-gravitating system. Generally, the gravitational potential generated by a system with a phase-space distribution function f is determined by Poisson's equation,

$$\vec{\nabla}^2\phi = 4\pi G\rho = 4\pi Gm \int \mathrm{d}^3v\, f \,, \tag{5.92}$$

where m is the particle mass, assumed to be the same for all particles. If the system is also isotropic in velocity space, the phase-space density cannot depend on the direction of the angular momentum either. Then, the phase-space density may be taken to be a function of E and the absolute value $L = m|\vec{x} \times \vec{v}|$ of the angular momentum only,

$$f\left(E, \vec{L}\right) = f(E, L) \,. \tag{5.93}$$

Writing the Laplacian operator in spherical symmetry,

$$\vec{\nabla}^2 = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) \,, \tag{5.94}$$

the equation for the gravitational potential

$$\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) = 4\pi Gm \int \mathrm{d}^3v\, f\left(\frac{mv^2}{2} + m\phi, m\left|\vec{x} \times \vec{v}\right|\right) \tag{5.95}$$

follows as the fundamental equation for self-gravitating spherical systems in equilibrium.

It is now convenient to re-scale the gravitational potential ϕ and the energy $E = mv^2/2 + m\phi$ by subtracting a constant potential ϕ_0 and defining the shifted potential ψ and the shifted specific energy $\mathcal{E}$,

$$\psi \equiv -\phi + \phi_0 \,, \quad \mathcal{E} \equiv -\frac{E}{m} + \phi_0 = \psi - \frac{v^2}{2} \,. \tag{5.96}$$

Let us consider a simple example specified by a phase-space distribution function f entirely independent of L and depending exponentially on the shifted specific energy $\mathcal{E}$,

$$f(\mathcal{E}) = \frac{\tilde{n}}{(2\pi\sigma^2)^{3/2}} \mathrm{e}^{\mathcal{E}/\sigma^2} = \frac{\tilde{n}}{(2\pi\sigma^2)^{3/2}} \exp\left(\frac{\psi - v^2/2}{\sigma^2}\right) , \tag{5.97}$$

where the constant $\tilde{n}$ appears for normalisation. Integration over all velocities yields the number density n of the particles,

$$\int \mathrm{d}^3v\, f(\mathcal{E}) = \frac{4\pi\tilde{n}\mathrm{e}^{\psi/\sigma^2}}{(2\pi\sigma^2)^{3/2}} \int_0^\infty \mathrm{d}v\, v^2 \mathrm{e}^{-v^2/(2\sigma^2)} = \tilde{n}\mathrm{e}^{\psi/\sigma^2} = n , \tag{5.98}$$

as must be the case for all phase-space densities. Poisson's equation for this system then reads

$$\frac{1}{r^2}\partial_r \left(r^2 \partial_r \psi\right) = -4\pi G n m = -4\pi G m \tilde{n} \mathrm{e}^{\psi/\sigma^2} . \tag{5.99}$$

Eliminating the re-scaled potential ψ and the exponential by means of (5.98),

$$\psi = \sigma^2 \ln \frac{n}{\tilde{n}} = \sigma^2 (\ln n - \ln \tilde{n}) , \quad \mathrm{e}^{\psi/\sigma^2} = \frac{n}{\tilde{n}} , \tag{5.100}$$

and substituting

$$\partial_r \psi = \sigma^2 \partial_r \ln n , \tag{5.101}$$

we can turn Poisson's equation (5.99) into an equation for the spatial number density n,

$$\frac{1}{r^2}\partial_r \left(r^2 \partial_r \ln n\right) = -\frac{4\pi G m}{\sigma^2} n , \tag{5.102}$$

which can of course also be considered as an equation for the mass density $\rho = nm$.

In ideal hydrodynamics, we had derived the equation (3.267),

$$M(r) = -\frac{r k_\mathrm{B} T}{m G} \left(\frac{\mathrm{d}\ln\rho_\mathrm{gas}}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}\right) \tag{5.103}$$

for a spherical gas mass in hydrostatic equilibrium with the gravitational-potential well given by its mass $M(r)$ enclosed by the radius r. If this gas is isothermal, $\mathrm{d}T/\mathrm{d}r = 0$, we can re-write (5.103) as

$$r^2 \partial_r \ln\rho_\mathrm{gas} = -\frac{mG}{k_\mathrm{B} T} \int_0^r \mathrm{d}r'\, r'^2 \rho_\mathrm{gas}(r') \tag{5.104}$$

if we consider the mass as being only contributed by the gas without any dark matter. Differentiating (5.104) with respect to r and dividing by r^2 yields

$$\frac{1}{r^2}\partial_r \left(r^2 \partial_r \ln\rho_\mathrm{gas}\right) = -\frac{4\pi G m}{k_\mathrm{B} T}\rho_\mathrm{gas} . \tag{5.105}$$

This equation is identical with our result (5.102) which we had previously derived from Jeans' theorem if we identify the velocity dispersion σ with the specific thermal energy,

$$\sigma^2 = \frac{k_\mathrm{B} T}{m} . \tag{5.106}$$

Thus, the corresponding self-gravitating, stellar-dynamical model with constant velocity dispersion σ^2 is called the *isothermal sphere*. The mean-squared velocity in the isothermal sphere is

$$\left\langle v^2 \right\rangle = \frac{1}{n} \int \mathrm{d}^3 v\, v^2 f = \frac{\int \mathrm{d}v\, v^4 \exp\left(\frac{-v^2}{2\sigma^2}\right)}{\int \mathrm{d}v\, v^2 \exp\left(\frac{-v^2}{2\sigma^2}\right)} = 3\sigma^2 \; . \tag{5.107}$$

Since no direction is preferred due to the spherical symmetry, the three individual velocity components thus have the same mean square

$$\left\langle v_x^2 \right\rangle = \left\langle v_y^2 \right\rangle = \left\langle v_z^2 \right\rangle = \sigma^2 \; . \tag{5.108}$$

One solution of the equation (5.102) for the density of the isothermal sphere can be obtained with the *ansatz*

$$n = C r^{-\alpha} \; . \tag{5.109}$$

The operations on the left-hand side of (5.102) yield

$$-\frac{\alpha}{r^2} \partial_r \left(r^2 \partial_r \ln r \right) = -\frac{\alpha}{r^2} \; . \tag{5.110}$$

Since the right-hand side scales with the radius as r^{-2}, the two sides can equal if and only if $\alpha = 2$. Therefore, the *ansatz* (5.109) is indeed a solution of (5.102) if $\alpha = 2$ and $C = \sigma^2/(2\pi G m)$, giving the matter density

$$\rho(r) = m n(r) = \frac{\sigma^2}{2\pi G r^2} \; . \tag{5.111}$$

This solution is called the *singular isothermal sphere*. It has the considerable advantage that the velocity of test particles on circular orbits around its centre is independent of radius,

$$v_{\mathrm{circ}}^2 = \frac{G M(r)}{r} = 4\pi G \int_0^r r^2 \mathrm{d}r\, \rho(r) = 2\sigma^2 \; , \tag{5.112}$$

which is observed in the vast majority of galaxies. Besides the central singularity, a substantial conceptual disadvantage is that its mass grows linearly with the radius and is thus formally infinite. Of course, this is an inevitable consequence of the assumption that the gas is isothermal: If this is so, the gas distribution must extend to infinity.

Another solution of (5.102) which avoids the central singularity can be found numerically. For doing so, we conveniently introduce the dimension-less quantities

$$x \equiv \frac{r}{r_0} \; , \quad y \equiv \frac{\rho}{\rho_0} \; , \tag{5.113}$$

where ρ_0 is meant to be the finite central density. Then, the equation for the scaled density y is

$$\partial_x \left(x^2 \partial_x \ln y \right) = -\frac{4\pi G}{\sigma^2} \rho_0 r_0^2\, y x^2 \; . \tag{5.114}$$

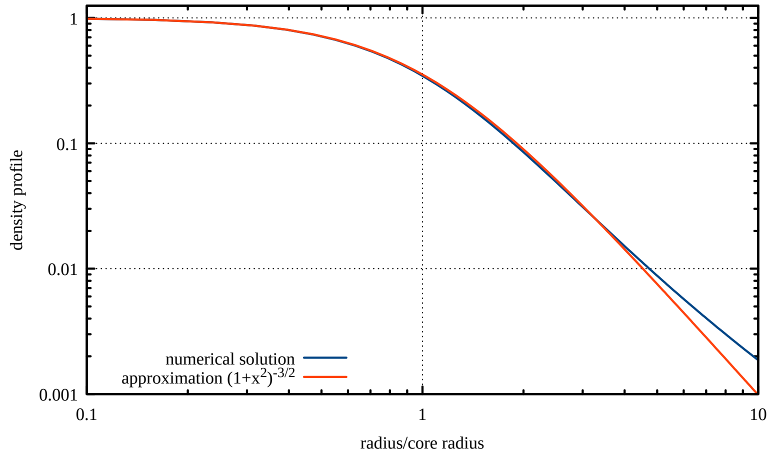

Figure 5.5 The density profile of the non-singular isothermal solution is shown together with its approximation.

If we define the scale radius r_0 to be related to the central density and the velocity dispersion by

$$r_0 \equiv \sqrt{\frac{9\sigma^2}{4\pi G\rho_0}} , \tag{5.115}$$

equation (5.114) simplifies to

$$\partial_x \left(x^2 \partial_x \ln y\right) = -9yx^2 , \tag{5.116}$$

which can numerically be integrated with the appropriate boundary conditions

$$y(0) = 1 , \quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_0 = 0 . \tag{5.117}$$

?

Compare (5.116) with the Lane-Emden equation (3.259) and the scale radius r_0 from (5.115) with r_0 from (3.258). Should they be related, and if so, for which n?

These boundary conditions mean that the central density is indeed ρ_0 and that the density profile is flat at the centre. For sufficiently small radii, the numerical result is very well approximated by (Figure 5.5)

$$y(x) \approx \left(1 + x^2\right)^{-3/2} . \tag{5.118}$$

For $x \lesssim 4.5$ or $r \lesssim 4.5r_0$, the relative deviation between the true numerical solution of (5.116) and the approximate solution (5.118) is $\lesssim 10\%$. As expected from isothermality, the total mass of the non-singular isothermal sphere still diverges. Since the density falls off asymptotically like r^{-3} for $r \gg r_0$, the mass must diverge logarithmically for $r \to \infty$.

5.2.2 Equilibrium and Relaxation

Is there an equilibrium state of a self-gravitating system, which corresponds to an entropy maximum? The entropy

$$S \propto -\int_{\text{phase space}} \mathrm{d}^3x\mathrm{d}^3p\, p \ln p \tag{5.119}$$

is maximised if and only if p is the distribution function of the isothermal sphere. However, the isothermal sphere has infinite mass and energy and can thus not be an exact description of a thermodynamical equilibrium state. This implies that there is no thermodynamical equilibrium of a self-gravitating system, and that self-gravitating systems cannot have stable final configurations, but at best long-lived transient states!

If we populate a narrow region in phase space with N stars, their orbits will have slightly different initial conditions. As time proceeds, they will progressively evolve away from each other and thus occupy a growing part of phase space. This *phase mixing* causes the averaged phase-space distribution $\bar{f}$ to decrease, because the averaged phase-space density is progressively diluted. Thus, the *macroscopic entropy*

$$\bar{S} \propto -\int \mathrm{d}^3x\mathrm{d}^3v\, \bar{f} \ln \bar{f} \tag{5.120}$$

does indeed increase.

This process of phase mixing is in fact hardly different from the thermodynamical trend to equilibrium. There, too, the increase of entropy is caused by macroscopically averaging over processes which are otherwise reversible. If the potential is changed while the particles are moving through it, energy can be transported from particles to others. If, for example, the system contracts while a star approaches its centre, the potential deepens and the star looses energy. Other stars can gain considerable amounts of energy; this process is called *violent relaxation* (Lynden-Bell).

5.2.3 Linear analysis and the Jeans swindle

In a way very similar to the derivation of the dielectricity tensor in plasma physics, we now consider an equilibrium solution f_0, ϕ_0 of the coupled system of the collision-less Boltzmann equation and the Poisson equation,

$$\begin{aligned} \frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f - \vec{\nabla}\phi\cdot\frac{\partial f}{\partial \vec{v}} = 0\;, \\ \vec{\nabla}^2\phi = 4\pi G m \int \mathrm{d}^3v\, f\;. \end{aligned} \tag{5.121}$$

In a stationary equilibrium state, $\partial f_0/\partial t = 0$. As usual for a linear stability analysis, we perturb f_0 and ϕ_0 by small amounts δf and $\delta\phi$ and linearise the equations in these perturbations. The result is

$$\begin{aligned} \frac{\partial \delta f}{\partial t} + \vec{v}\cdot\vec{\nabla}\delta f - \vec{\nabla}\phi_0\cdot\frac{\partial \delta f}{\partial \vec{v}} - \vec{\nabla}\delta\phi\cdot\frac{\partial f_0}{\partial \vec{v}} = 0\;, \\ \vec{\nabla}^2\delta\phi = 4\pi G m \int \mathrm{d}^3v\, \delta f\;. \end{aligned} \tag{5.122}$$

Poisson's equation implies one peculiarity here which needs to be emphasised. Suppose we adopt as the unperturbed equilibrium state an infinitely extended, homogeneous phase-space distribution f_0, which implies a constant density ρ_0 and a potential ϕ_0 given by

$$\vec{\nabla}^2\phi_0 = 4\pi G\rho_0\;. \tag{5.123}$$

Due to the infinite extent of this matter distribution and its homogeneity, we must have

$$\vec{\nabla}\phi_0 = 0 \tag{5.124}$$

because there cannot be any gravitational force on a test particle in a surrounding homogeneous matter density. This condition complies with the Poisson equation if and only if $\rho_0 = 0$. An infinitely extended, homogeneous matter distribution is possible in Newtonian gravity only if it has no matter; it is therefore generally inconsistent with Newtonian gravity. The reason is profound: In Newtonian gravity, the boundary conditions for the potential are of decisive importance for the solution of the Poisson equation, but an infinitely extended mass distribution has no boundary. Only in General Relativity, this problem is satisfactorily solved.

We rather invoke the "Jeans swindle" and set $\phi_0 = 0$. This is practically permissible if we study perturbations whose spatial scales are small compared to possible scales in the smooth background density ρ_0. By the "Jeans swindle", we simply ignore the potential ϕ_0 and obtain the linearised equations

$$\begin{aligned}\frac{\partial\delta f}{\partial t} + \vec{v}\cdot\vec{\nabla}\delta f - \vec{\nabla}\delta\phi\cdot\frac{\partial f_0}{\partial\vec{v}} &= 0\ ,\\ \vec{\nabla}^2\delta\phi &= 4\pi Gm\int \mathrm{d}^3v\,\delta f\ .\end{aligned} \tag{5.125}$$

Now, we decompose the spatial and temporal dependence of the perturbations into plane waves,

$$\delta f = \delta\hat{f}\,\mathrm{e}^{\mathrm{i}\left(\vec{k}\cdot\vec{x}-\omega t\right)}\ ,\quad \delta\phi = \delta\hat{\phi}\,\mathrm{e}^{\mathrm{i}\left(\vec{k}\cdot\vec{x}-\omega t\right)}\ , \tag{5.126}$$

where the amplitude $\delta\hat{f}$ of the phase-space distribution function will generally still depend on the velocity coordinates of phase space. The perturbation amplitudes must then satisfy the equations

$$\begin{aligned}-\mathrm{i}\omega\delta\hat{f} + \mathrm{i}\vec{v}\cdot\vec{k}\delta\hat{f} - \mathrm{i}\delta\hat{\phi}\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}} &= 0\ ,\\ -k^2\delta\hat{\phi} &= 4\pi Gm\int \mathrm{d}^3v\,\delta\hat{f}\ .\end{aligned} \tag{5.127}$$

The perturbed Boltzmann equation can be solved to relate the perturbation of the phase-space distribution $\delta\hat{f}$ to the potential perturbation $\delta\hat{\phi}$,

$$\delta\hat{f} = \delta\hat{\phi}\,\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega}\ , \tag{5.128}$$

which can in turn be inserted into the perturbed Poisson equation to find

$$-k^2\delta\hat{\phi} = 4\pi Gm\int \mathrm{d}^3v\,\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}\,\frac{\delta\hat{\phi}}{\vec{k}\cdot\vec{v}-\omega}\ . \tag{5.129}$$

Since the potential perturbation $\delta\hat{\phi}$ does not depend on $\vec{v}$, (5.129) shows that non-vanishing perturbations $\delta\hat{\phi} \neq 0$ are possible only if

$$1 + \frac{4\pi Gm}{k^2}\int \mathrm{d}^3v\,\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega} = 0\ . \tag{5.130}$$

This correponds exactly to the longitudinal dielectricity $\hat{\varepsilon}_{\|}$ (4.69) from plasma physics,

$$\hat{\varepsilon}_{\|} = 1 - \frac{4\pi e^2}{k^2} \int \mathrm{d}^3 p\, \vec{k} \cdot \frac{\partial f_0}{\partial \vec{p}} \frac{1}{\vec{k} \cdot \vec{v} - \omega} \tag{5.131}$$

which has to vanish for longitudinal electromagnetic waves to propagate. Just as longitudinal electromagnetic waves undergo Landau damping in a plasma, so do potential fluctuations in a stellar-dynamical system.

5.2.4 Jeans length and Jeans mass

Analysing the stability of a self-gravitating system, we need to distinguish potential fluctuations which oscillate from others. Accordingly, we seek the boundary between oscillating and unstable solutions requiring that the frequency should vanish there, $\omega = 0$. If we assume that the unperturbed equilibrium state has a Maxwellian velocity distribution with a velocity dispersion σ^2 and a homogeneous spatial number density n_0,

$$f_0(\vec{v}) = \frac{n_0}{(2\pi\sigma^2)^{3/2}} \mathrm{e}^{-v^2/(2\sigma^2)} , \tag{5.132}$$

the velocity gradient of f_0 required in (5.130) is

$$\frac{\partial f_0}{\partial \vec{v}} = -f_0(\vec{v}) \frac{\vec{v}}{\sigma^2} . \tag{5.133}$$

Without loss of generality, we rotate the coordinate frame such that the positive x axis points into the direction of the wave vector $\vec{k}$, the condition (5.130) simplifies to

$$1 - \frac{4\pi G m n_0}{k^2\sigma^2(2\pi\sigma^2)^{3/2}} \int \mathrm{d}^3 v \frac{k v_x \mathrm{e}^{-v^2/(2\sigma^2)}}{k v_x - \omega} = 0 . \tag{5.134}$$

For $\omega = 0$, the remaining integral is easily solved since

$$\int \mathrm{d}^3 v\, \mathrm{e}^{-v_x^2/(2\sigma^2)} = \left(2\pi\sigma^2\right)^{3/2} , \tag{5.135}$$

and we find from (5.134) the condition

$$k^2\Big|_{\omega=0} \equiv k_\mathrm{J}^2 = \frac{4\pi G\rho_0}{\sigma^2} . \tag{5.136}$$

The wave number k_J satisfying this equation is called the *Jeans wave number*. Gravitational instability sets in for larger perturbations, that is, for wave numbers $k < k_\mathrm{J}$ or wave lengths exceeding the so-called Jeans wave length

$$\lambda_\mathrm{J} \equiv \frac{2\pi}{k_\mathrm{J}} = \frac{2\pi\sigma}{\sqrt{4\pi G\rho_0}} = \frac{\sqrt{\pi}\sigma}{\sqrt{G\rho_0}} . \tag{5.137}$$

?

Why is $\omega = 0$ relevant for separating between gravitational stability and instability?

The Jeans wave length or Jeans length defines the Jeans volume λ_J^3 and thereby the *Jeans mass* $M_\mathrm{J} = \rho_0\lambda_\mathrm{J}^3$. An interesting insight follows if we compare the

Jeans mass to the actual mass M of an object and its velocity dispersion. Solving the equation (5.137) for the Jeans length for the density,

$$\rho_0 = \frac{\pi\sigma^2}{G\lambda_\mathrm{J}^2} , \tag{5.138}$$

and multiplying with the actual radius R of an object, we find

$$M \approx \rho_0 R^3 = \frac{\pi\sigma^2 R^3}{G\lambda_\mathrm{J}^2} . \tag{5.139}$$

However, due to the virial theorem, we must further obey the relation

$$\sigma^2 \approx \frac{GM}{R} . \tag{5.140}$$

Eliminating the velocity dispersion between (5.140) and (5.139) shows that

$$R \approx \frac{\lambda_\mathrm{J}}{\sqrt{\pi}} . \tag{5.141}$$

The radius of the system is thus necessarily comparable to the Jeans length. This means that the assumption of homogeneity on the scale of the Jeans length cannot be satisfied and that the nature of the instability needs to be studied for each system in detail once its geometry is specified. Nonetheless, the Jeans length defines an order-of-magnitude estimate for the boundary between stability and instability.

5.2.5 Disk potentials

Disk-like structures are of particular importance for stellar-dynamical systems. Imagine a disk in the x-y plane, centred on the coordinate origin, with a spatial matter density

$$\rho(x, y, z) = \Sigma(x, y)\delta_\mathrm{D}(z) . \tag{5.142}$$

The disk is thus infinitely thin and has a surface-mass density $\Sigma(x, y)$. Assume further that the disk is axially symmetric about the $\hat{e}_z$ axis. The surface-mass density can then only depend on the radius s in the x-y plane, $\Sigma = \Sigma(s)$. In the appropriate cylindrical coordinates, Poisson's equation then reads

$$\frac{1}{s}\partial_s (s\partial_s\phi) + \partial_z^2\phi = 0 \tag{5.143}$$

everywhere outside the x-y plane. The structure of this equation suggests a separation ansatz for ϕ,

$$\phi(s, z) = \psi(s)\chi(z) . \tag{5.144}$$

Inserting this into (5.143), we obtain

$$\frac{1}{s\psi(s)}\partial_s (s\partial_s\psi) = \frac{\partial_z^2\chi(z)}{\chi(z)} . \tag{5.145}$$

Since the left- and right-hand sides of this equation depend on different variables, s and z, respectively, they must individually equal the same constant, which we call $-k^2$. From the oscillator equation

$$\frac{\partial_z^2\chi(z)}{\chi(z)} = -k^2 \tag{5.146}$$

with negative squared frequency $-k^2$, we immediately infer that $\chi(z)$ must be an exponential function,

$$\chi(z) = \chi_0 e^{\pm kz} . \tag{5.147}$$

Since the potential should tend to zero far away from the disk, the positive sign applies to negative z and vice versa. The potential therefore decreases exponentially in the direction perpendicular to the disk. Turning to the s dependence, $\psi(s)$ must satisfy the equation

$$\partial_s (s\partial_s\psi) + k^2 s\psi = 0 . \tag{5.148}$$

Substituting $x = ks$ turns this equation into Bessel's differential equation of order zero,

$$x\psi''(x) + \psi'(x) + x\psi(x) = 0 , \tag{5.149}$$

where the prime denotes the derivative with respect to x. Its solution with the appropriate boundary condition that $\psi(s)$ remains regular at the centre $s = 0$ is the ordinary, zeroth-order Bessel function $J_0(x)$. Our solution of Poisson's equation is thus

$$\phi_k(s, z) = e^{-k|z|} J_0(ks) . \tag{5.150}$$

Any linear superposition of such potential modes $\phi_k(s, z)$ will also be solutions of Poisson's equation.

For taking the disk into account, we enclose an arbitrary point $(s, 0)$ on the disk by a small cylinder of height h and cross section A such that the disk plane is perpendicular to the symmetry axis of the cylinder and cuts through its centre. We then apply Gauss' law to $\vec{\nabla}^2\phi$ in the cylinder and let the height h become arbitrarily small to find

$$\int dV\vec{\nabla}^2\phi = \int_{\partial V} \vec{\nabla}\phi \cdot d\vec{A} \to A \left(\partial_z\phi\big|_{z\to0+} - \partial_z\phi\big|_{z\to0-} \right) . \tag{5.151}$$

By Poisson's equation, this integral must equal $4\pi G$ times the mass contained in the cylinder, which is $\Sigma(s)A$. Hence, the surface density causes the discontinuity

$$4\pi G\Sigma(s) = \partial_z\phi\big|_{z\to0+} - \partial_z\phi\big|_{z\to0-} \tag{5.152}$$

in the potential gradient $\partial_z\phi$ perpendicular to the disk, Inserting (5.150) here, we see that

$$kJ_0(ks) = 2\pi G\Sigma_k(s) \tag{5.153}$$

must hold for Poisson's equation to be satisfied. However, we generally wish to determine the gravitational potential of disks with arbitrary surface densities $\Sigma(s)$. We can do so if we can find a function $S(k)$ such that

$$\Sigma(s) = \int_0^\infty dk\, S(k)\Sigma_k(s) = \frac{1}{G}\int_0^\infty \frac{k dk}{2\pi} S(k) J_0(ks) . \tag{5.154}$$

In this way, the arbitrary surface density $\Sigma(s)$ would then be assembled by linear superposition of modes $\Sigma_k(s)$ that individually satisfy Poisson's equation, and the complete potential would be given by

$$\phi(s, z) = \int_0^\infty dk\, S(k)\phi_k(s, z) = \int_0^\infty dk\, e^{-k|z|} S(k) J_0(ks) . \tag{5.155}$$

?

Strictly speaking, we should write the general solution of (5.146) as $\chi(z) = \chi_0 e^{kz} + \chi_1 e^{-kz}$ with two constants $\chi_{0,1}$. Why is it appropriate to proceed as described in the text?

To see how such a function $S(k)$ could be found, consider the inverse Fourier transform of an arbitrary, two-dimensional and axi-symmetric function $\hat{f}(k)$,

$$f(s) = \int \frac{k\mathrm{d}k\mathrm{d}\varphi}{(2\pi)^2}\,\hat{f}(k)\,\mathrm{e}^{\mathrm{i}ks\cos\varphi}\;. \tag{5.156}$$

The azimuthal integral can be carried out, giving

$$\int_0^{2\pi}\mathrm{d}\varphi\,\mathrm{e}^{\mathrm{i}ks\cos\varphi} = \int_0^{2\pi}\mathrm{d}\varphi\,\cos(ks\cos\varphi) = 2\pi\,J_0(ks)\;. \tag{5.157}$$

?

Why does the solution (5.157) have no imaginary part?

Comparing with (5.154), we see that the sought function $S(k)$ is simply G times the Fourier transform of the surface-density $\Sigma(s)$,

$$S(k) = G\int s\mathrm{d}s\mathrm{d}\varphi\,\Sigma(s)\,\mathrm{e}^{\mathrm{i}ks\cos\varphi} = 2\pi G\int_0^\infty s\mathrm{d}s\,\Sigma(s)J_0(ks)\;. \tag{5.158}$$

Inserting this back into (5.155) shows that the potential is related to the surface mass density by

$$\phi(s,z) = 2\pi G\int_0^\infty \mathrm{d}k\,\mathrm{e}^{-k|z|}J_0(ks)\int_0^\infty s'\mathrm{d}s'\Sigma(s')J_0(ks')\;. \tag{5.159}$$

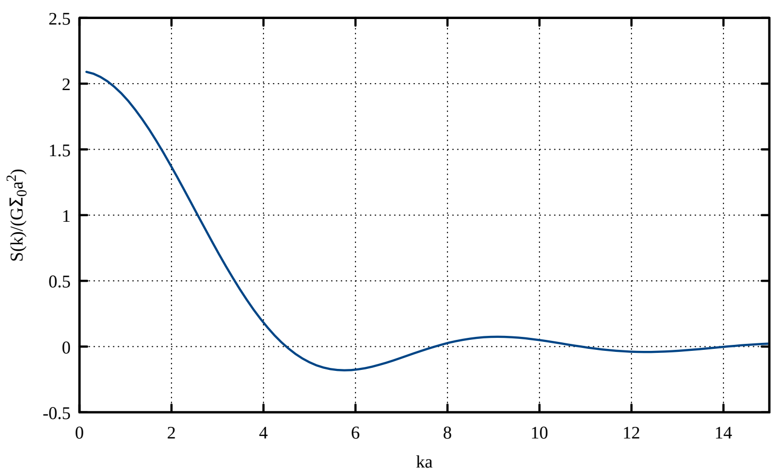

Figure 5.6 The function $S(k)$ is shown here for the Maclaurin disk model, normalised by $G\Sigma_0 a^2$.

5.2.6 Fluid equations for two-dimensional systems

As the simplest example for a rotating disk, we now consider an infinitely thin disk which is rigidly rotating about the z axis with the constant angular velocity $\vec{\Omega} = \Omega\hat{e}_z$. The disk thus fills the x-y plane and has a constant surface-mass density Σ.

We consider perturbations in the plane of the disk and neglect warps or twists. Furthermore, we transform into a co-rotating coordinate frame and study the

Example: The Maclaurin disk

The integrals in (5.159) can be solved analytically only for a surprisingly small class of surface densities $\Sigma(s)$. One example is the so-called *Maclaurin disk*, for which

$$\Sigma(s) = \Sigma_0 \left(1 - \frac{s^2}{a^2}\right)^{1/2} \tag{5.160}$$

for $s \leq a$ and zero otherwise. For this disk model, the function $S(k)$ is

$$S(k) = G\Sigma_0 \sqrt{\frac{2\pi^3 a}{k^3}} J_{3/2}(ka) \;, \tag{5.161}$$

where $J_{3/2}(x)$ is the cylindrical Bessel function of the first kind of fractional order $3/2$. This function is shown in Fig. 5.6. For $s \leq a$, the potential in the plane of the disk is

$$\phi(s, 0) =: \phi_0(s) = \frac{\pi^2 G\Sigma_0}{4a}\left(2a^2 - s^2\right) = -\frac{\pi^2 G\Sigma_0}{4a} s^2 + \text{const} \;. \tag{5.162}$$

Except for an irrelevant constant, the potential within the disk is therefore quadratic in the two-dimensional radius s,

$$\phi_0(s) = -\frac{1}{2}\Omega_0^2 s^2 \;, \quad \Omega_0^2 = \frac{\pi^2 G\Sigma_0}{2a} \;. \tag{5.163}$$

Deriving these results, we have used the integrals

$$\int_0^1 \mathrm{d}x\, x\sqrt{1 - x^2}\, J_0(kx) = \sqrt{\frac{\pi}{2k^3}} J_{3/2}(k) \tag{5.164}$$

and

$$\int_0^\infty \mathrm{d}k\, J_0(ks) J_{3/2}(ka) k^{-3/2} = \frac{1}{4}\sqrt{\frac{\pi}{2a^3}}\left(2a^2 - s^2\right) \;, \tag{5.165}$$

which may not be obvious. ◂

disk in the substantially simpler fluid approximation. We begin with the continuity equation, insert the spatial density

$$\rho(\vec{x}) = \Sigma(t)\delta_{\rm D}(z) \tag{5.166}$$

there and integrate over dz. The result is

$$\partial_t \Sigma + \vec{\nabla} \cdot (\Sigma \vec{v}) = 0 \ , \tag{5.167}$$

where the divergence is now two-dimensional and operates in the x-y plane only. Next, we take the x and y components of the Euler equation of ideal hydrodynamics in the form

$$\Sigma \delta_{\rm D}(z) \frac{{\rm d}\vec{v}}{{\rm d}t} = -\vec{\nabla} P - \Sigma \delta_{\rm D}(z) \vec{\nabla}\phi \ , \tag{5.168}$$

integrate again over dz and obtain

$$\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\frac{\vec{\nabla} P}{\Sigma} - \vec{\nabla}\phi \ , \tag{5.169}$$

where $\vec{\nabla}$ is again reduced to two dimensions. Poisson's equation reads

$$\vec{\nabla}^2 \phi = 4\pi G \Sigma \delta_{\rm D}(z) \tag{5.170}$$

with the three-dimensional Laplacian. We now move into a coordinate system co-rotating with the disk. Since we are then in a non-inertial frame, we must augment Euler's equation with the specific Coriolis and centrifugal force terms, $-2\vec{\Omega} \times \vec{v}$ and $\vec{\Omega}^2 \vec{r}$, respectively. With these terms, Euler's equation becomes

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\frac{\vec{\nabla} P}{\Sigma} - \vec{\nabla}\phi - 2\vec{\Omega} \times \vec{v} + \vec{\Omega}^2 \vec{r} \ . \tag{5.171}$$

The physical quantities occuring here have two spatial dimensions, $\vec{v}(x, y, t)$, $\Sigma(x, y, t)$ and so on. For the pressure, we assume a barotropic equation-of-state, $P = P(\Sigma)$.

The unperturbed quantities are obviously a vanishing velocity $\vec{v} = 0$ in the co-rotating frame, a constant surface-mass density $\Sigma = \Sigma_0$, and the pressure $P_0 = P(\Sigma_0)$ according to the equation-of-state. This solution satisfies the continuity equation trivially. Since the pressure gradient vanishes because Σ_0 is constant, Euler's equation reads

$$\vec{\nabla}\phi_0 = \Omega^2 \vec{r} \ , \tag{5.172}$$

while Poisson's equation is

$$\vec{\nabla}^2 \phi_0 = 4\pi G \Sigma_0 \delta_{\rm D}(z) \ . \tag{5.173}$$

Since no direction can be preferred in a homogeneous disk, $\vec{\nabla}\phi_0$ must point along the z axis, which contradicts Euler's equation. Thus, there is no gravitational force yet to balance the centrifugal force. Therefore, we have to assume that the disk can only exist if it is embedded into a surrounding gravitational field which compensates the centrifugal force, such as the halo of a galaxy.

5.2.7 Dispersion relation

As usual in linear stability analyses, we perturb our infinitely extended, rigidly rotating disk by small amounts of surface density $\delta\Sigma$, velocity $\delta\vec{v}$ and gravitational potential $\delta\phi$ and linearise the equations in these perturbations. This implies

$$\begin{aligned} \partial_t\delta\Sigma + \Sigma_0\vec{\nabla}\cdot\delta\vec{v} &= 0 \;, \\ \partial_t\delta\vec{v} &= -\frac{c_\mathrm{s}^2}{\Sigma_0}\vec{\nabla}\delta\Sigma - \vec{\nabla}\delta\phi - 2\vec{\Omega}\times\delta\vec{v} \;, \\ \vec{\nabla}^2\delta\phi &= 4\pi G\delta\Sigma\delta_\mathrm{D}(z) \;, \end{aligned} \tag{5.174}$$

where the sound velocity was introduced as the derivative of the pressure with respect to the surface-mass density,

$$c_\mathrm{s}^2 = \left.\left(\frac{\mathrm{d}P(\Sigma)}{\mathrm{d}\Sigma}\right)\right|_{\Sigma_0} \;, \tag{5.175}$$

taken at the unperturbed surface-mass density Σ_0. Next, we decompose the perturbations into plane waves with amplitudes $\delta\Sigma_A$, $\delta\vec{v}_A$ and $\delta\phi_A$,

$$\begin{pmatrix} \delta\Sigma \\ \delta v_x \\ \delta v_y \\ \delta\phi \end{pmatrix} = \begin{pmatrix} \delta\Sigma_A \\ \delta v_{Ax} \\ \delta v_{Ay} \\ \delta\phi_A \end{pmatrix} \mathrm{e}^{\mathrm{i}\left(\vec{k}\cdot\vec{x}-\omega t\right)} \tag{5.176}$$

confined to the x-y plane and therefore valid at $z = 0$. Without loss of generality, we turn the x axis into the direction of the wave vector $\vec{k}$. The first two perturbation equations (5.174) turn into

$$\begin{aligned} \omega\delta\Sigma_A - k\Sigma_0\delta v_{Ax} &= 0 \;, \\ \omega\delta v_{Ax} - \frac{c_\mathrm{s}^2 k}{\Sigma_0}\delta\Sigma_A - k\delta\phi_A - 2\mathrm{i}\Omega\delta v_{Ay} &= 0 \;, \\ \omega\delta v_{Ay} + 2\mathrm{i}\Omega\delta v_{Ax} &= 0 \;, \end{aligned} \tag{5.177}$$

while Poisson's equation needs a more detailed treatment. Within the disk plane, the potential perturbation behaves like the plane wave given by the third equation in (5.176), while the Laplace equation

$$\vec{\nabla}^2\delta\phi = 0 \tag{5.178}$$

must hold otherwise. A separation ansatz demonstrates that this can only be achieved by the function

$$\delta\phi = \delta\phi_\mathrm{A}\mathrm{e}^{\mathrm{i}(kx-\omega t)-|kz|} \;, \tag{5.179}$$

where the modulus is taken of the product kz since $k = k_x$ can have either sign.

?

Verify that the function $\delta\phi$ from (5.179) is the only one satisfying all conditions required here.

The preceding discussion leading to (5.151) and (5.152), specified to our perturbed disk, shows that the potential derivatives in z direction above and below the disk need to obey

$$\left.\partial_z\delta\phi\right|_{z\to0+} - \left.\partial_z\delta\phi\right|_{z\to0-} = 4\pi G\delta\Sigma = 4\pi G\delta\Sigma_A\mathrm{e}^{\mathrm{i}(kx-\omega t)} \;. \tag{5.180}$$

However, we see at the same time from (5.179) that

$$\partial_z\phi\big|_{z\to0+} - \partial_z\phi\big|_{z\to0-} = -2|k|\delta\phi_{\mathrm{A}}\mathrm{e}^{\mathrm{i}(kx-\omega t)}\ , \tag{5.181}$$

which implies that the fluctuation amplitudes in the gravitational potential and in the surface-mass density must be related through

$$-2|k|\delta\phi_{\mathrm{A}} = 4\pi G\delta\Sigma_{\mathrm{A}}\ . \tag{5.182}$$

This enables us to eliminate the amplitude $\delta\phi_A$ of the potential fluctuations from (5.177), leaving us with the linear system of equations

$$\begin{pmatrix} \omega & -k\Sigma_0 & 0 \\ k\left(\frac{2\pi G}{|k|} - \frac{c_{\mathrm{s}}^2}{\Sigma_0}\right) & \omega & -2\mathrm{i}\Omega \\ 0 & 2\mathrm{i}\Omega & \omega \end{pmatrix} \begin{pmatrix} \delta\Sigma_A \\ \delta v_{Ax} \\ \delta v_{Ay} \end{pmatrix} = 0 \tag{5.183}$$

?

Confirm the dispersion relation (5.184) by setting the determinant of the coefficient matrix in (5.183) to zero.

for the remaining variables $\delta\Sigma_A$, δv_{Ax} and δv_{Ay}.

Non-trivial solutions exist if and only if the determinant of the coefficient matrix vanishes, which leads us to the dispersion relation

$$\omega\left(\omega^2 - 4\Omega^2\right) + \omega k^2\Sigma_0\left(\frac{2\pi G}{|k|} - \frac{c_{\mathrm{s}}^2}{\Sigma_0}\right) = 0 \tag{5.184}$$

for the perturbations of the disk. This shows that perturbations must either be stationary, $\omega = 0$, or obey

$$\omega^2 = 4\Omega^2 + k^2c_{\mathrm{s}}^2 - 2\pi G|k|\Sigma_0\ . \tag{5.185}$$

This dispersion relation describes the non-stationary, propagating modes of the perturbed, rigidly rotating, uniform disk. The modes are stable for $\omega^2 \geq 0$ and unstable for $\omega^2 < 0$.

5.2.8 Toomre's criterion

Let us now analyse the dispersion relation (5.185). If $\Omega = 0$, which is certainly not the most exciting case of a rotating disk, the disk is unstable if the wave number satisfies

$$|k| < k_{\mathrm{J}} \equiv \frac{2\pi G\Sigma_0}{c_{\mathrm{s}}^2}\ , \tag{5.186}$$

where k_{J} plays the rôle of the Jeans wave number for the disk. If the sound speed can be arbitrarily low, $c_{\mathrm{s}} \to 0$, perturbations are unstable for arbitrarily large k or arbitrarily small wave length. The rate of the exponential growth of unstable perturbations is given by the imaginary part of ω,

$$\gamma = \mathrm{Im}\omega = \left(2\pi G\Sigma_0|k| - k^2c_{\mathrm{s}}^2\right)^{1/2}\ . \tag{5.187}$$

For a cold disk, $c_{\mathrm{s}} \to 0$, small perturbations with $\lambda \to 0$ and $|k| \to \infty$ grow at a rate increasing linearly with $|k|$, i.e. cold, non-rotating disks fragment violently on small scales.

This violent fragmentation is not suppressed by rotation either. For $c_s \to 0$, the oscillation frequency ω of the linear perturbations becomes imaginary for wave numbers

$$|k| > \frac{2\Omega^2}{\pi G \Sigma_0} \,, \tag{5.188}$$

i.e. even then the instability sets in on the smallest scales.

Pressure and rotation are therefore not able to stabilise the disk individually. For $\Omega = 0$, (5.186) shows that warm disks are unstable on large scales,

$$|k| < \frac{2\pi G \Sigma_0}{c_s^2} \,, \tag{5.189}$$

while cold disks with vanishing pressure are unstable on small scales despite any rotation, as we have seen in (5.188). However, pressure and rotation can be stabilising if they act *together*, since then the dispersion relation (5.185) has a minimum where

$$0 = \frac{\partial \omega^2}{\partial k} = \frac{\partial}{\partial k}\left(4\Omega^2 + k^2 c_s^2 - 2\pi G|k|\Sigma_0\right) \,, \tag{5.190}$$

which yields

$$2|k|c_s^2 = 2\pi G \Sigma_0 \quad \text{or} \quad |k| = \frac{\pi G \Sigma_0}{c_s^2} = \frac{k_J}{2} \,. \tag{5.191}$$

The disk can be stable if and only if $\omega^2 \geq 0$ at this wave number because it is then positive for all wave numbers. Thus, the condition for global stability is $\omega^2(k_J/2) \geq 0$ or

$$4\Omega^2 - \left(\frac{\pi G \Sigma_0}{c_s^2}\right) \geq 0 \,, \tag{5.192}$$

which constrains the sound speed c_s and the angular velocity Ω for a globally stable disk with mean surface-mass density Σ_0 by *Toomre's criterion*

$$\frac{c_s \Omega}{G \Sigma_0} \geq \frac{\pi}{2} \approx 1.57 \,. \tag{5.193}$$

A similar criterion can also be derived for collision-less systems (recall that we had adopted the fluid approximation!). Then,

$$\frac{c_s \Omega}{G \Sigma_0} \gtrsim 1.68 \tag{5.194}$$

is the condition for global stability.

Problems

1. Show that the scalar virial theorem reads

$$2K = -pU \tag{5.195}$$

   if the potential energy scales like $U \propto r^{-p}$ with the separation r between two particles. Derive for which values of p the heat capacity of self-gravitating systems is negative.

5.3 Dynamical Friction

This final section discusses the friction experienced by a test mass moving through an infinite system of other masses due to gravity. We first calculate the velocity changes (5.213) and (5.215) of the test mass perpendicular and parallel to its direction of motion due to encounters with the surrounding masses and average over impact parameters to arrive at Chandrasekhar's formula (5.228) for the acceleration due to dynamical friction. A specialisation of this formula for a Maxwellian velocity distribution in the surrounding system of masses is given in (5.232).

5.3.1 Deflection of point masses

An interesting effect occurs if a mass M moves through a system of masses m which are homogeneously distributed around the mass M. Although the motion of the masses can be considered collision-less, a deceleration occurs which is called *dynamical friction*.

Let us begin analysing a single two-body encounter between the mass M and a mass m, with $\vec{v}_M$ and $\vec{v}_m$ being their respective velocities and $\vec{x}_M$ and $\vec{x}_m$ being their positions. We introduce the separation vector

$$\vec{r} \equiv \vec{x}_m - \vec{x}_M \tag{5.196}$$

from M to m and the relative velocity

$$\vec{v} \equiv \dot{\vec{r}} = \vec{v}_m - \vec{v}_M \tag{5.197}$$

of m with respect to M. The two-body system of two point masses obeys an effective equation of motion around a fixed force centre of a single body with the *reduced mass*,

$$\left(\frac{mM}{m+M}\right)\ddot{\vec{r}} = -\frac{GMm}{r^2}\hat{e}_r \equiv -\frac{\alpha}{r^2}\hat{e}_r \;, \tag{5.198}$$

where $\hat{e}_r$ is the unit vector in radial direction away from M. By definition of the centre-of-mass $\vec{X}$,

$$\vec{X} = \frac{m\vec{x}_m + M\vec{x}_M}{m+M} \;, \tag{5.199}$$

its velocity remains unchanged,

$$\dot{\vec{X}} = \frac{m\vec{v}_m + M\vec{v}_M}{m+M} = 0 \;. \tag{5.200}$$

Any changes $\Delta\vec{v}_m$ and $\Delta\vec{v}_M$ in the velocites of m and M are thus related by

$$M\Delta\vec{v}_M = -m\Delta\vec{v}_m \tag{5.201}$$

with each other and by

$$\Delta\vec{v}_M = -\frac{m}{M}\Delta\vec{v}_m = -\frac{m}{M}(\Delta\vec{v} + \Delta\vec{v}_M) \quad \text{or} \quad \Delta\vec{v}_M = -\frac{m}{M+m}\,\Delta\vec{v} \tag{5.202}$$

to any change

$$\Delta\vec{v} = \Delta\vec{v}_m - \Delta\vec{v}_M \tag{5.203}$$

in the relative velocity of m and M. We shall now determine the relative velocity change $\Delta\vec{v}$ in a two-body encounter.

The fictitious particle with the reduced mass, $Mm/(M+m)$, follows a hyperbolic orbit around the (resting) centre of force.

Earlier in this book, we came across another situation where one particle moves on a hyperbolic orbit around another one, namely when we studied the emission of a plasma electron scattering off an ion. We can thus refer to the treatment there. We recall the relation (2.99) between the distance r of the orbiting particle from the force centre and the polar angle φ, from which we read off that the particle reaches infinity if and when

$$\cos\varphi = -\frac{1}{\varepsilon}\,, \tag{5.204}$$

where ε is the orbit's eccentricity. From (2.100), we infer that the squared eccentricity is

$$\varepsilon^2 = 1 + \frac{2El^2}{\alpha^2\mu}\,, \tag{5.205}$$

where $\alpha = GMm$ replaces the product Ze^2 in (2.100) since now the coupling is gravitational rather than electromagnetic, and the reduced mass μ replaces the electron mass m. Let now v be the velocity at infinity of our fictitious particle with the reduced mass μ, and b its impact parameter. Then, its conserved angular momentum is $l = \mu v b$, and its equally conserved energy is $E = \mu v^2/2$. We can then re-write the squared eccentricity as

$$\varepsilon^2 = 1 + \frac{\mu^2 b^2 v^4}{\alpha^2}\,. \tag{5.206}$$

Since the total scattering angle is $\theta = 2\varphi - \pi$, we have

$$\sin\frac{\theta}{2} = \sin\left(\varphi - \frac{\pi}{2}\right) = -\cos\varphi = \frac{1}{\varepsilon}\,. \tag{5.207}$$

Since the cosine of $\theta/2$ is

$$\cos\frac{\theta}{2} = \sqrt{1-\sin^2\frac{\theta}{2}} = \frac{\sqrt{\varepsilon^2-1}}{\varepsilon}\,, \tag{5.208}$$

we find for the scattering angle itself

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{\sqrt{\varepsilon^2-1}}{\varepsilon^2} = \frac{2\mu b v^2\alpha}{\mu^2 b^2 v^4 + \alpha^2} \tag{5.209}$$

and

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{\varepsilon^2-2}{\varepsilon^2} = \frac{\mu^2 b^2 v^4 - \alpha^2}{\mu^2 b^2 v^4 + \alpha^2}\,. \tag{5.210}$$

5.3.2 Velocity changes

The fictitious particle is approaching from infinity with velocity v. When it is leaving towards infinity, it will have been scattered by the angle θ. Relative to its incoming direction, its outgoing velocity will have the perpendicular and parallel components

$$v_\perp = -v\sin\theta = -\frac{2\mu b v^3\alpha}{\mu^2 b^2 v^4 + \alpha^2} \tag{5.211}$$

and

$$v_\parallel = v\cos\theta = v\,\frac{\mu^2 b^2 v^4 - \alpha^2}{\mu^2 b^2 v^4 + \alpha^2}\ , \tag{5.212}$$

where (5.209) and (5.210) were used. Since the velocity change perpendicular to the incoming direction is $\Delta v_\perp = v_\perp$, we find from (5.202) that the velocity change of the mass M perpendicular to its initial direction of motion is

$$\Delta v_{M\perp} = -\frac{m}{M+m}v_\perp = -\frac{\mu}{M}v_\perp = \frac{2\mu^2 b v^3\alpha}{M\,(\mu^2 b^2 v^4 + \alpha^2)}\ . \tag{5.213}$$

Parallel to the incoming direction, we have

$$\Delta v_\parallel = v - v_\parallel = -\frac{2v\alpha^2}{\mu^2 b^2 v^4 + \alpha^2}\ , \tag{5.214}$$

hence the velocity change of the mass M in the direction parallel to its initial motion is

$$\Delta v_{M\parallel} = -\frac{\mu}{M}\Delta v_\parallel = \frac{2v\mu\alpha^2}{M\,(\mu^2 b^2 v^4 + \alpha^2)}\ . \tag{5.215}$$

5.3.3 Chandrasekhar's formula

Having studied the effect of a single encounter on the velocity of the mass M, we shall proceed to determine the combined effect of many encounters. If the mass M is moving through a homogeneous "sea" of masses m, all velocity changes (5.213) perpendicular to the direction of motion must cancel, while the parallel velocity changes (5.215) must add up. Therefore, the mass M will experience a steady deceleration parallel to its direction of motion from the combined effect of many encounters with the masses m. Let $f(\vec{v}_m)$ be the phase-space density of the stars with mass m which constitute that background "sea" of point masses. Then, the rate at which the mass M encounters collisions with stars with an impact parameter between b and $b + \mathrm{d}b$ is

$$(2\pi b\mathrm{d}b)\cdot v\cdot(f(\vec{v}_m)\mathrm{d}^3v_m)\ , \tag{5.216}$$

where the first factor is the area of the ring with radius b and width $\mathrm{d}b$ and the third is the spatial number density of masses m with velocity v_m. The velocity v appearing in between is the relative velocity of M and m. According to (5.215), these collisions change the velocity of M by

$$\frac{\mathrm{d}\vec{v}_M}{\mathrm{d}t} = \vec{v}f(\vec{v}_m)\mathrm{d}^3v_m\int_0^{b_{\max}}2\pi b\mathrm{d}b\,\frac{2v\mu\alpha^2}{M\,(\mu^2 b^2 v^4 + \alpha^2)}\ , \tag{5.217}$$

The integral is easily carried out and returns a logarithm,

$$\begin{aligned}\int_0^{b_{\max}}2\pi b\mathrm{d}b\,\frac{2v\mu\alpha^2}{M\,(\mu^2 b^2 v^4 + \alpha^2)} &= \frac{2\pi v\mu\alpha^2}{M}\int_0^{b^2_{\max}}\frac{\mathrm{d}(b^2)}{\mu^2 b^2 v^4 + \alpha^2}\\ &= \frac{2\pi\alpha^2}{M\mu v^3}\ln\left(1+\Lambda^2\right)\ ,\end{aligned} \tag{5.218}$$

where the abbreviation

$$\Lambda := \frac{b_{\max}\mu v^2}{\alpha} = \frac{b_{\max} v^2}{G(M+m)} \tag{5.219}$$

was introduced. Since $\vec{v}$ in (5.217) is the relative velocity between the masses m and M, we need to set $\vec{v} = \vec{v}_m - \vec{v}_M$. Inserting this expression and (5.218) into (5.217), we find

$$\frac{\mathrm{d}\vec{v}_M}{\mathrm{d}t} = \frac{2\pi\alpha^2}{M\mu} \ln\left(1+\Lambda^2\right) \frac{\vec{v}_m - \vec{v}_M}{\left|\vec{v}_m - \vec{v}_M\right|^3} f(\vec{v}_m)\,\mathrm{d}^3 v_m \;. \tag{5.220}$$

The quantity Λ is typically $\Lambda \gg 1$, allowing us to approximate

$$\ln(1+\Lambda^2) \approx \ln\Lambda^2 = 2\ln\Lambda \;. \tag{5.221}$$

Typical values for this so-called *Coulomb logarithm* are

$$5 \lesssim \ln\Lambda \lesssim 20 \;. \tag{5.222}$$

The expression (5.220) is the deceleration of the mass M by those of the surrounding stars with mass m whose velocity falls within the volume element $\mathrm{d}^3 v_m$ around $\vec{v}_m$ in velocity space. We obtain the total deceleration only after a further integration over all velocities $\vec{v}_m$. We proceed to doing so in two steps. First, we specialise to isotropic velocity distributions, $f(\vec{v}_m) = f(v_m)$. We further abbreviate $\vec{x} = \vec{v}_m$ and $\vec{y} = \vec{v}_M$ and begin by simplifying the velocity integral as

$$\int \mathrm{d}^3 x \, \frac{f(x)\,(\vec{x}-\vec{y})}{\left|\vec{x}-\vec{y}\right|^3} = -\int \mathrm{d}^3 x \, f(x) \vec{\nabla} \frac{1}{\left|\vec{x}-\vec{y}\right|} = \vec{\nabla}_{\vec{y}} \int \mathrm{d}^3 x \, \frac{f(x)}{\left|\vec{x}-\vec{y}\right|} \;. \tag{5.223}$$

For solving the remaining integral, we turn the coordinate system such that $\vec{y}$ points into the $\hat{e}_z$ direction, introduce spherical polar coordinates and continue writing

$$\int \mathrm{d}^3 x \, \frac{f(x)}{\left|\vec{x}-\vec{y}\right|} = 2\pi \int_0^\infty x^2 \mathrm{d}x \, f(x) \int_{-1}^1 \frac{\mathrm{d}\mu}{\sqrt{x^2+y^2-2xy\mu}} \;, \tag{5.224}$$

where the direction cosine $\mu = \cos\theta$ was introduced. The factor of 2π out front is the result of the azimuthal integration. The μ integral is now easily carried out, giving

$$\int_{-1}^1 \frac{\mathrm{d}\mu}{\sqrt{x^2+y^2-2xy\mu}} = \frac{|x+y|-|x-y|}{xy} = \begin{cases} \frac{2}{x} & (x>y) \\ \frac{2}{y} & (\text{else}) \end{cases} \;. \tag{5.225}$$

?

Carry out the integral in (5.225) yourself.

Returning to (5.224), we write

$$\int \mathrm{d}^3 x \, \frac{f(x)}{\left|\vec{x}-\vec{y}\right|} = 4\pi \left(\frac{1}{y} \int_0^y x^2 \mathrm{d}x \, f(x) + \int_y^\infty x \mathrm{d}x \, f(x) \right) \;. \tag{5.226}$$

The gradient with respect to $\vec{y}$, required by (5.223), finally leaves us with

$$\int \mathrm{d}^3 x \, \frac{f(x)\,(\vec{x}-\vec{y})}{\left|\vec{x}-\vec{y}\right|^3} = -4\pi \frac{\vec{y}}{y^3} \int_0^y x^2 \mathrm{d}x \, f(x) \;. \tag{5.227}$$

Restoring the velocities $\vec{v}_M = \vec{y}$ and $\vec{v}_m = \vec{x}$ in this expression and returning with it to the deceleration (5.220), we obtain *Chandrasekhar's equation* for dynamical friction (Figure 5.7),

$$\frac{d\vec{v}_M}{dt} = -16\pi^2 G^2 m(M+m) \ln\Lambda \frac{\vec{v}_M}{v_M^3} \int_0^{v_M} v_m^2 dv_m f(v_m) \;, \tag{5.228}$$

where we have also expanded $\alpha = GMm$ and $\mu = Mm/(M+m)$.

Limiting cases are instructive. If v_M is small compared to the typical velocity of the stars m, the remaining integral can be approximated by

$$\int_0^{v_M} dv_m\, v_m^2 f(v_m) \approx \frac{v_M^3}{3} f(0) \;. \tag{5.229}$$

In this case, the dynamical friction becomes proportional to the velocity $\vec{v}_M$,

$$\frac{d\vec{v}_M}{dt} = -\frac{16\pi^2}{3} G^2 m(M+m) \ln\Lambda\, f(0)\, \vec{v}_M \;, \tag{5.230}$$

which is characteristic for Stokes-type friction. In the opposite limiting case of sufficiently large v_M, the integral covers most or all of the velocity distribution of the masses m and thus converges to a constant. Then, the friction force becomes proportional to v_M^{-2}.

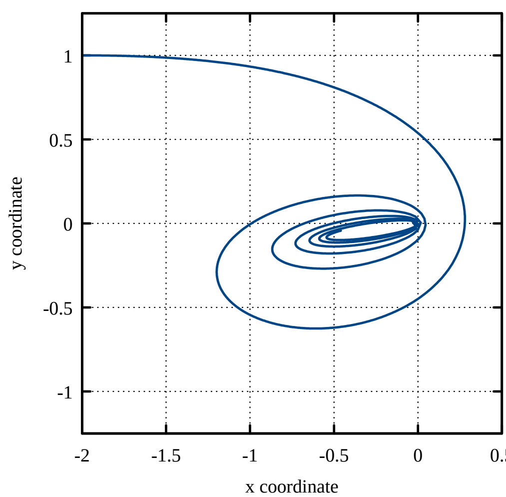

Figure 5.7 An example is shown for the trajectory of a test particle under the combined effect of gravity and dynamical friction.

For a Maxwellian velocity distribution with velocity dispersion σ,

$$f(v_m) = \frac{n_0}{(2\pi\sigma^2)^{3/2}} e^{-v^2/(2\sigma^2)} \;, \tag{5.231}$$

the friction force becomes

$$\frac{d\vec{v}_M}{dt} = -4\pi G^2 \ln\Lambda\, (M+m)\rho_0 \frac{\vec{v}_M}{v_M^3} \left[\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}} e^{-X^2} \right] \;, \tag{5.232}$$

where $X \equiv v_M/(\sqrt{2}\sigma)$ is the velocity v_M appropriately scaled by the velocity dispersion. The mass density ρ_0 of the masses m is $\rho_0 = mn_0$. If $M \gg m$, $(M+m) \approx M$, and the friction only depends on the density ρ_0 of the scatteres stars, but not on their mass any more,

$$\frac{\mathrm{d}\vec{v}_M}{\mathrm{d}t} = -4\pi G^2 \ln\Lambda\,\rho_0 M \frac{\vec{v}_M}{v_M^3}\left[\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}}\mathrm{e}^{-X^2}\right] . \tag{5.233}$$

In this case, the friction force is proportional to M^2 because the deceleration is proportional to M. An example for an orbit decaying via the dynamical friction in a system with Maxwellian velocity distribution is shown in Fig. 5.7.

Problems

1. Approximate the velocity distribution of a hypothetical stellar-dynamical system by the step function

$$f(v_m) = \Theta(v_0 - v_m) \tag{5.234}$$

with some maximal velocity $v_0 > 0$.

(a) Solve Chandrasekhar's equation for dynamical friction (5.228) for a star moving through this system with an initial velocity $V_{M0} > v_0$.

(b) How long will it take for the star to reach v_0, and when will it stop completely?

Suggested further reading: [19, 4]

Brief summary and concluding remarks

Our introductory tour through theoretical astrophysics is coming to an end. Its main goal should be seen as achieved if it could reveal the roots of four main branches of the field: the largely classical physics of electromagnetic radiation processes, ideal and viscous hydrodynamics, the physics of plasmas with and without magnetic fields, and stellar dynamics.

Larmor's equation, underlying the strictly classical electromagnetic radiation processes, follows directly from the Liénard-Wiechert potentials and thus from the retarded Greens function of electrodynamics. The back-reaction of the radiation on the radiating charges needs to be added by hand to classical electrodynamics because it is a linear theory. For Compton scattering, the photon picture needs to be introduced, and quantum mechanics, in particular Fermi's Golden Rule, is required to describe the internal degrees of freedom in systems interacting with radiation.

Hydrodynamics, including viscous, relativistic and magnetised fluids, follows from the conservation law for the energy-momentum tensor, which can be derived from moments of the Boltzmann equation. The phenomenology of the resulting equations is very rich, but global statements can be derived by integration. Powerful examples are Bernoulli's law, which is an integral of Euler's equation, and the Rankine-Hugoniot jump conditions at shocks. Linear stability analysis, following a well-defined procedure, reveals a rich variety of dispersion relations and instabilities, some of which were discussed. Turbulence, at the boundary between ordered, macroscopic and unordered, microscopic motion, could only be touched briefly.

Plasma physics adds electromagnetic properties to a fluid which are encapsulated in the dielectric tensor. Magnetic fields add two types of force to a plasma, one due to gradients in the magnetic pressure, the other due to the bending of field lines. Through the induction equation, the fluid flow acts back on the magnetic field. Ambipolar diffusion was introduced as an effect arising from non-ideal coupling between a plasma a neutral fluid, and the battery mechanism was mentioned as an example for how magnetic fields can be generated. The very rich field of magneto-hydrodynamic stability analysis could only be discussed to the level of Alfvén waves and the fast and slow hydromagnetic modes.

Stellar dynamics finally has the same root as hydrodynamics, namely the Liouville or Boltzmann equations, depending on what degree of interaction between particles is to be included. Three main differences to hydrodynamics arise: The long-range gravitational interaction between the particles cannot be shielded, the absence of collisions allows anisotropic particle orbits and thus an anisotropic pressure, and self-gravitating systems have a negative heat capacity and thus no stable equilibrium state. Again, the rich subject of stability analyses of self-gravitating systems could only briefly be touched.

Equally important as recognising the foundations of theoretical astrophysics is becoming aware of their limitations due to idealising assumptions. Some care was taken to clearly specify the assumptions made. If this book can enable its readers to ask and address their own questions on the theory of astrophysical phenomena, it has reached its goal.