Chapter 4

Fundamentals of Plasma Physics and Magnetohydrodynamics

4.1 Collision-less Plasmas

We begin this chapter on plasma physics and magnetohydrodynamics with a discussion of Debye shielding, showing that charges embedded in a plasma have a Yukawa- rather than a Coulomb potential with a characteristic length scale, the Debye length (4.11), which can be combined with the mean thermal velocity of the plasma particles to derive the plasma frequency (4.16).

4.1.1 Shielding and the Debye length

Plasmas are gases whose particles are charged. They typically occur when the kinetic energies of the gas particles exceed the ionisation energy of the atomic species they are composed of. The atoms are then partially or fully ionised. Unless the positive and the negative charges, i.e. the ions and the electrons, are separated by macroscopic electric fields, the plasma is macroscopically neutral. In subvolumes of a plasma whose linear dimensions are much larger than the typical inter-particle separation, there are thus on average as many negative as positive charges.

For many purposes, a plasma can be treated as a single fluid. This is possible if not only the mean-free path for collisions between the plasma particles is much smaller than any macroscopically relevant length scale, but if also the interactions between the positive and the negative charges are fast enough for them not to separate on a macroscopic scale. Sometimes, however, the positive and the negative charges need to be treated as two interacting fluids.

In principle, the electromagnetic interaction between the positive and the negative charges has an infinite range because the Coulomb force falls off like the inverse-squared distance from a charge. The Coulomb interaction between its particles is thus the fundamental difference between plasma physics and the hydrodynamics of neutral fluids. A treatment of plasmas as fluids, however, requires that collisions between the plasma particles be random, short-ranged and fast, such that equilibrium can locally be quickly established. This is possible despite the Coulomb interaction because the existence of two different types of charge allows shielding on a characteristic length scale which we first want to work out.

Let the plasma consist of electrons of charge $-e$ and ions of charge Ze. The spatial number densities of the electrons and the ions be n_e and n_i, respectively. We begin with a macroscopically neutral plasma with negative charge density $-en_\mathrm{e}$ and positive charge density Zen_i. For the plasma to be neutral, the number densities must be related by

$$Zen_\mathrm{i} = en_\mathrm{e} \quad \Rightarrow \quad n_\mathrm{i} = \frac{n_\mathrm{e}}{Z} \; . \tag{4.1}$$

Suppose now we place a point charge q at the (arbitrary) origin into the plasma. This point charge will displace the positive and negative plasma charges to some degree and thereby change their number densities locally. If q is positive, as we shall assume without loss of generality, n_i will be lowered in its immediate neighbourhood, while n_e will be slightly increased there compared to the equilibrium densities of the electrons and the ions given by (4.1). The local imbalance between the positive and the negative charge distributions will create an electrostatic potential Φ different from zero.

The thermal motion of the plasma particles will counteract their displacement. In presence of an electrostatic potential, the particles will rearrange such as to minimise their potential energies $Ze\Phi$ and $-e\Phi$. The unperturbed equilibrium densities $\bar{n}_\mathrm{e}$ for the negative charges and $\bar{n}_\mathrm{i} = \bar{n}_\mathrm{e}/Z$ for the positive charges will thus be modified by a Boltzmann factor and read

$$n_\mathrm{i} = \frac{\bar{n}_\mathrm{e}}{Z} \exp\left(\frac{Ze\Phi}{k_\mathrm{B}T}\right) \; , \quad n_\mathrm{e} = \bar{n}_\mathrm{e} \exp\left(-\frac{e\Phi}{k_\mathrm{B}T}\right) \; . \tag{4.2}$$

The potential Φ is determined by the Poisson equation

$$\begin{aligned} \vec{\nabla}^2\Phi &= 4\pi\left(Zen_\mathrm{i} - en_\mathrm{e}\right) + 4\pi q\delta_\mathrm{D}\left(\vec{x}\right) \\ &= 4\pi\bar{n}_\mathrm{e}e\left[\exp\left(\frac{Ze\Phi}{k_\mathrm{B}T}\right) - \exp\left(\frac{-e\Phi}{k_\mathrm{B}T}\right)\right] + 4\pi q\delta_\mathrm{D}\left(\vec{x}\right) \end{aligned} \tag{4.3}$$

together with the boundary condition that $\Phi \to 0$ far away from the point charge. On the right-hand side of the Poisson equation, the point charge q appears at the coordinate origin in addition to the plasma charges. If q is not too large and the plasma is not very cold, it is appropriate to assume

$$\frac{e\Phi}{k_\mathrm{B}T} \ll 1 \; , \quad \frac{Ze\Phi}{k_\mathrm{B}T} \ll 1 \tag{4.4}$$

and to Taylor-expand the exponentials in (4.3) to first order. This leads us to the approximate Poisson equation

$$\vec{\nabla}^2\Phi = 4\pi(Z+1)\,\frac{\bar{n}_\mathrm{e}e^2}{k_\mathrm{B}T}\,\Phi + 4\pi q\delta_\mathrm{D}\left(\vec{x}\right) \; , \tag{4.5}$$

which is most easily solved after transforming it into Fourier space. Introducing the *Debye wave number*

$$k_D^2 = 4\pi \frac{\bar{n}_e e^2}{k_B T} , \tag{4.6}$$

we can write the Fourier-transformed Poisson equation as

$$\hat{\Phi}(k) = -\frac{4\pi q}{(Z+1)k_D^2 + k^2} . \tag{4.7}$$

Caution The approximate Poisson equation (4.5) is an inhomogeneous Helmholtz equation. ◄

This is straightforwardly transformed back into real space because $\hat{\Phi}(k)$ depends on the wave number only, but not on the direction of the wave vector. The angular integrations in the inverse Fourier transform first give

$$\begin{aligned} \Phi(r) &= 2\pi \int_0^\infty \frac{k^2 \mathrm{d}k}{(2\pi)^3} \hat{\Phi}(k) \int_{-1}^1 \mathrm{d}\cos\theta \, \mathrm{e}^{\mathrm{i}kr\cos\theta} \\ &= 4\pi \int_0^\infty \frac{k^2 \mathrm{d}k}{(2\pi)^3} \hat{\Phi}(k) \frac{\sin(kr)}{kr} . \end{aligned} \tag{4.8}$$

With the help of the definite integral

$$\int_0^\infty \mathrm{d}x \frac{x \sin(\alpha x)}{\beta^2 + x^2} = \frac{\pi}{2} \mathrm{e}^{-\alpha\beta} , \tag{4.9}$$

the remaining k integration in (4.9) over the potential (4.7) can now directly be carried out to give

$$\Phi(r) = -\frac{q}{r} \exp\left(-\sqrt{Z+1} k_D r\right) . \tag{4.10}$$

?

How could you prove (4.9)?

By the presence of the plama charges, the Coulomb potential of the point charge q is thus changed to a Yukawa potential (Figure 4.1) which decreases exponentially on the typical length scale

$$\lambda_D = k_D^{-1} = \left(\frac{k_B T}{4\pi \bar{n}_e e^2}\right)^{1/2} . \tag{4.11}$$

Notice that λ_D does not depend on the charge q placed into the plasma! This is the *Debye length*, which gives the characteristic length scale for the shielding of an arbitrary charge in an otherwise neutral plasma (Figure 4.2a).

A plasma is called *ideal* if it contains many particles in a volume given by the cubed Debye length. Then the interaction energy between positive and negative charges is small compared to their thermal energy; in other words, the electrostatic interactions affect the thermal motion of the plasma particles only very weakly. To see this, we compare the mean potential energy $Ze^2/\bar{r}$ of an electron in the Coulomb field of an ion with its kinetic energy $3k_B T/2$ in thermal equilibrium.

The mean separation $\bar{r}$ between the particles is determined by

$$\frac{4\pi}{3} \bar{r}^3 \bar{n}_e \approx 1 \quad \Rightarrow \quad \bar{r} \approx \left(\frac{3}{4\pi \bar{n}_e}\right)^{1/3} . \tag{4.12}$$

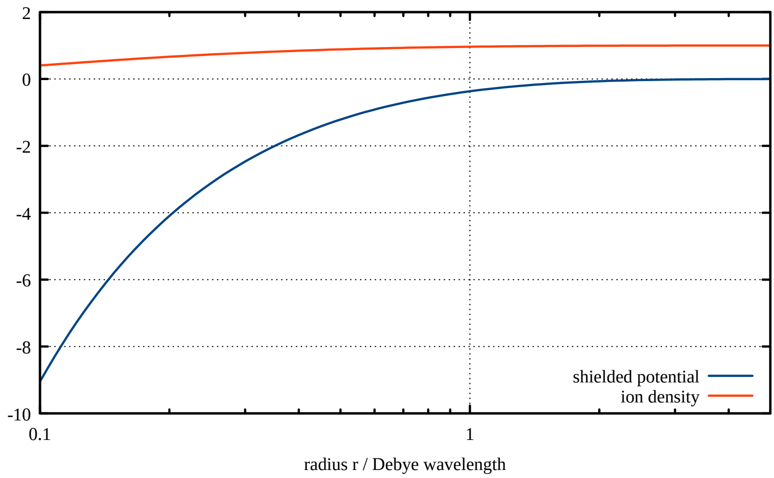

Figure 4.1 A point charge in a plasma rearranges the surrounding charges. Its electrostatic potential is shielded and thus exponentially cut off.

The ratio between the electrostatic interaction energy and the thermal energy is thus (Figure 4.2b)

$$\frac{Ze^2}{k_\mathrm{B}T}\left(\frac{4\pi\bar{n}_\mathrm{e}}{3}\right)^{1/3} = Z\frac{4\pi\bar{n}_\mathrm{e}e^2}{3k_\mathrm{B}T}\left(\frac{3}{4\pi\bar{n}_\mathrm{e}}\right)^{2/3} = \frac{Z}{3}\frac{\bar{r}^2}{\lambda_\mathrm{D}^2}\;, \tag{4.13}$$

which is much less than unity if the Debye length greatly exceeds the mean inter-particle separation $\bar{r}$. For an ideal plasma, we can thus assume $\bar{r} \ll \lambda_\mathrm{D}$ by definition.

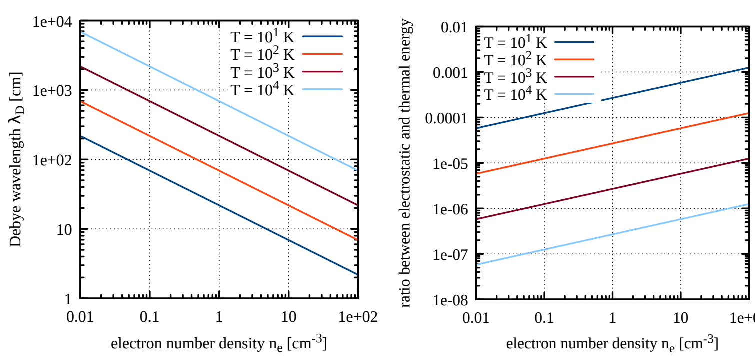

Figure 4.2 The Debye wavelength λ_D in centimetres (left panel) and the ratio between the electrostatic and the thermal energies (right panel) are shown as functions of the plasma temperature T and the mean electron number density $\bar{n}_\mathrm{e}$.

4.1.2 The plasma frequency

By the equipartition theorem from statistical physics, the mean-squared thermal velocity of the plasma electrons in one spatial direction is

$$\left\langle v^2 \right\rangle = \frac{k_\mathrm{B}T}{m_\mathrm{e}} \tag{4.14}$$

if they are in thermal equilibrium with a plasma at temperature T. A plasma electron thus moves by the Debye length in a mean time interval of

$$t_\mathrm{D} = \frac{\lambda_\mathrm{D}}{\langle v^2 \rangle^{1/2}} = \sqrt{\frac{k_\mathrm{B}T}{4\pi\bar{n}_\mathrm{e}e^2}\frac{m_\mathrm{e}}{k_\mathrm{B}T}} = \sqrt{\frac{m_\mathrm{e}}{4\pi\bar{n}_\mathrm{e}e^2}} \,. \tag{4.15}$$

This sets the time scale on which the thermal motion of the electrons can compensate charge displacements by shielding. This characteristic reaction time t_D of the plasma can be transformed into a characteristic frequency,

$$\omega_\mathrm{p} = \frac{1}{t_\mathrm{D}} = \sqrt{\frac{4\pi\bar{n}_\mathrm{e}e^2}{m_\mathrm{e}}} \approx 5.6 \cdot 10^4\,\mathrm{Hz}\left(\frac{\bar{n}_\mathrm{e}}{\mathrm{cm}^{-3}}\right)^{1/2} \,, \tag{4.16}$$

called the *plasma frequency*. External changes applied to the plasma with frequencies higher than the plasma frequency, for example by an incident electromagnetic wave, are too fast for the plasma particles to adapt and rearrange, while changes with lower frequency can be accommodated. With the Debye length λ_D and the plasma frequency ω_p, we now have two essential parameters at hand for describing plasmas.

Problems

1. Solve the Poisson equation (4.5) directly, i.e. without transforming into Fourier space. *Hint*: Solve the homogeneous equation first, introducing spherical polar coordinates. Then solve the inhomogeneous equation by variation of constants. For solving the homogeneous equation, try the ansatz

$$\Phi(r) = \Phi_0(\alpha r)^n \exp(-\alpha r) \tag{4.17}$$

and see whether a suitable exponent n can be found.

4.2 Electromagnetic Waves in Media

This section is a recollection of electrodynamics in media, introducing the dielectric displacement $\vec{D}$, the polarisation $\vec{P}$ and the dielectric tensor ε. For later convenience, we decompose in (4.37) the dielectric tensor into its components parallel and perpendicular to the propagation direction of electromagnetic waves and define the longitudinal and transverse dielectricities in (4.40).

4.2.1 Polarisation and dielectric displacement

An important part of plasma physics concerns the propagation of electromagnetic waves through plasmas. There, Maxwell's vacuum equations in vacuum no longer hold because the plasma as a medium reacts to the presence of electromagnetic fields, and thereby alters them. While the homogeneous Maxwell equations remain unchanged, the inhomogeneous equations change due to the appearance of charges and currents that only appear because the external electromagnetic fields act on the charges and possible magnetic dipoles of the medium.

Recall that electric and magnetic fields are defined by forces on test charges and test-current loops. Such test systems are idealisations whose own, intrinsic fields are so small that the fields can be considered unchanged that are supposed to be measured. The electric force experienced by a test charge embedded in a medium is expressed by the dielectric displacement $\vec{D}$ instead of the electric field $\vec{E}$, while a test-current loop in the medium experiences a magnetic force expressed by the magnetic field strength $\vec{H}$ instead of the magnetic field $\vec{B}$.

Considering the linearity of Maxwell's equations, it is natural to assume that $\vec{D}$ and $\vec{H}$ be linearly related to $\vec{E}$ and $\vec{B}$, respectively. We thus adopt the common linear relations

$$\vec{D} = \varepsilon\vec{E} \;, \quad \vec{B} = \mu\vec{H} \;, \tag{4.18}$$

where the dielectricity ε and the magnetic permeability μ appear. In the Gaussian system of units, $\vec{D}$ and $\vec{H}$ are defined such that ε and μ of the vacuum are unity. Even though the relations (4.18) appear very simple, several complications may arise. First, ε and μ may depend on space and time. Second, they may be tensors if the medium has preferred spatial directions imprinted, such as the principal axes of a crystal or the magnetic field lines in a magnetised plasma. Then, not only the magnitudes of $\vec{D}$ and $\vec{H}$ may differ from those of $\vec{E}$ and $\vec{B}$, but also their directions if the principal axes of the ε and μ tensors are misaligned with $\vec{E}$ and $\vec{B}$.

?

Can you construct an (artificial) dielectric tensor ε such that $\vec{D}$ is perpendicular to $\vec{E}$?

In what follows, we shall generally consider astrophysical media whose particles have no relevant magnetic moments. Then, external magnetic fields will neither be dimished nor strengthened by macroscopically aligned, microscopic magnetic dipoles, the medium will not respond to the presence of an external magnetic field, and we can identify $\vec{B}$ with $\vec{H}$, adopting $\mu = 1$. We shall thus only consider the response of the medium to external electric fields in the following.

External electric fields $\vec{E}$ polarise media, i.e. they slightly displace the positive and negative charges of the microscopic constituents of these media. To lowest, but sufficient order in a multipole expansion, these charge displacements can be described by electric dipole moments. When macroscopically averaged over scales large compared to the individual particles, but small compared to the overall dimensions of the complete medium, these microscopic dipoles can be linearly superposed to form the macroscopic polarisation $\vec{P}$, which counteracts the external electric field $\vec{E}$. The divergence of the polarisation corresponds to a polarised charge density $\rho_{\rm pol}$, defined by

$$\rho_{\rm pol} = -\vec{\nabla} \cdot \vec{P} \;. \tag{4.19}$$

Inside the medium, this polarised charged density must be added to any free charge density ρ that may additionally be present. The Maxwell equation for an electric field in vacuum, $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$, must now be augmented by the polarised charge density $\rho_{\rm pol}$,

$$\vec{\nabla}\cdot\vec{E} = 4\pi\left(\rho + \rho_{\rm pol}\right) = 4\pi\rho - 4\pi\vec{\nabla}\cdot\vec{P}\ . \tag{4.20}$$

As discussed before, the dielectric displacement, $\vec{D} \equiv \vec{E} + 4\pi\vec{P}$, is introduced as an auxiliary field describing the response of the medium to an external electric field. Its sources are the free charges only,

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho\ , \tag{4.21}$$

excluding any charges $\rho_{\rm pol}$ that are exclusively caused by polarisation of otherwise neutral microscopic particles.

Charge conservation, expressed by the continuity equation for the charge density, must also apply to the polarised charge density,

$$\frac{\partial\rho_{\rm pol}}{\partial t} + \vec{\nabla}\cdot\vec{j}_{\rm pol} = 0\ . \tag{4.22}$$

If we substitute (4.19) here, we find the current density $j_{\rm pol}$ caused by changes of the polarisation with time,

$$\vec{\nabla}\cdot\left[-\frac{\partial\vec{P}}{\partial t} + \vec{j}_{\rm pol}\right] = 0 \quad \text{or} \quad \vec{j}_{\rm pol} = \frac{\partial\vec{P}}{\partial t}\ , \tag{4.23}$$

where the final step assumes that $j_{\rm pol}$ is curl-free. This current density needs to be added to the current density $\vec{j}$ of the free charges. The induction equation then reads

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\left(\vec{j} + \vec{j}_{\rm pol}\right) = \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t} + 4\pi\frac{\partial\vec{P}}{\partial t}\right) + \frac{4\pi}{c}\vec{j} \tag{4.24}$$

and can be brought into the form

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\partial_t\vec{D} + \frac{4\pi}{c}\vec{j}\ : \tag{4.25}$$

In a medium, it is not the time derivative of the electric field, but of the dielectric displacement that creates the curl of the magnetic field together with the current. Equations (4.21) and (4.25) replace the previous inhomogeneous Maxwell equations for the divergence of the electric field $\vec{E}$ and the curl of the magnetic field $\vec{B}$.

In the following, we shall assume macroscopically neutral media in which the free charge density vanishes, $\rho = 0$. This does not necessarily imply that there could be no macroscopic currents. In fact, the free current density $\vec{j}$ may differ from zero. We shall further assume that the microscopic constituents of the media have no net magnetic moment, allowing us to neglect any distinction between $\vec{B}$ and $\vec{H}$. In such media, Maxwell's equations then read

$$\vec{\nabla}\cdot\vec{D} = 0\ , \quad \dot{\vec{D}} + 4\pi\vec{j} = c\vec{\nabla}\times\vec{H}\ , \quad \dot{\vec{B}} = -c\vec{\nabla}\times\vec{E}\ , \quad \vec{\nabla}\cdot\vec{B} = 0\ , \tag{4.26}$$

augmented by the linear relations

$$\vec{D} = \varepsilon\vec{E}\ , \quad \vec{j} = \sigma\vec{E} \tag{4.27}$$

between $\vec{D}$, $\vec{E}$ and $\vec{j}$, where the dielectricity ε and the conductivity σ may be tensors. For the time being, we shall neglect free currents, setting $\vec{j} = 0$. Later on, in magnetohydrodynamics, we shall have to take them into account.

4.2.2 Structure of the dielectric tensor

The dielectric displacement $\vec{D}$ may be delayed with respect to the external field $\vec{E}$, and it may propagate through the medium. The dielectric displacement $\vec{D}(t,\vec{x})$ at a time t and a place $\vec{x}$ may thus depend on the electric field $\vec{E}(t',\vec{x}')$ at an earlier time $t' < t$ and another place $\vec{x}'$. The dielectric displacement would then be determined by a convolution of the electric field with a suitable kernel because they must still be related in a linear way. Since convolutions are multiplications in Fourier space, it is thus reasonable to begin with a multiplicative ansatz in Fourier space.

Second, the medium itself may have a preferred direction. This could be a crystal axis or the local direction of a magnetic field. Then, $\vec{D}$ does no longer need to be colinear with $\vec{E}$, but may point into a different direction. Such a change of orientation can be expressed by introducing a dielectric tensor instead of a dielectric constant.

Taking these two arguments together, we begin with

$$\hat{\vec{D}}\left(\omega,\vec{k}\right) = \hat{\varepsilon}\left(\omega,\vec{k}\right)\,\hat{\vec{E}}\left(\omega,\vec{k}\right)\ , \tag{4.28}$$

defining the *dielectric tensor* $\hat{\varepsilon}(\omega,\vec{k})$ as a function of frequency ω and wave vector $\vec{k}$. Since the fields must remain real in configuration space $(t,\vec{x})$, the dielectric tensor must satisfy the symmetry relation

$$\hat{\varepsilon}\left(-\omega,-\vec{k}\right) = \hat{\varepsilon}^*\left(\omega,\vec{k}\right) \tag{4.29}$$

in Fourier space $(\omega,\vec{k})$.

If the medium, in our case the plasma, does not imprint a specific direction, the only vector that can be used to span the tensor $\hat{\varepsilon}^i_{\ j}$ is the wave vector $\vec{k}$ of an incoming electromagnetic wave itself. We introduce a unit vector $\hat{k}$ in the direction of $\vec{k}$ by $\hat{k} = \vec{k}/k$ and begin with the ansatz

$$\hat{\varepsilon}\left(\omega,\vec{k}\right) = \hat{A}\mathbb{1}_3 + \hat{B}\,\hat{k}\otimes\hat{k} \tag{4.30}$$

with functions $\hat{A}(\omega,k)$ and $\hat{B}(\omega,k)$ that may depend on the frequency ω and the wave number k. That is, we linearly compose the dielectric tensor $\hat{\varepsilon}$ of the unit tensor and the tensor $\hat{k}\otimes\hat{k}$, the only basis tensors we have available here (Figure 4.3). Notice that the tensor $\hat{\varepsilon}$ defined this way is symmetric. In particular, this means that $\vec{D}$ and $\vec{E}$ can be multiplied in a symmetric scalar product,

$$\hat{\vec{D}}\cdot\hat{\vec{E}} = \hat{\vec{E}}^\top\hat{\varepsilon}\hat{\vec{E}}\ . \tag{4.31}$$

If we had reasons to believe that $\hat{\varepsilon}$ should contain an antisymmetric part, this could be supplied by adding a contribution proportional to the Levi-Civita tensor, e.g. of the form

$$\varepsilon_{ijk}\hat{k}^k \ . \tag{4.32}$$

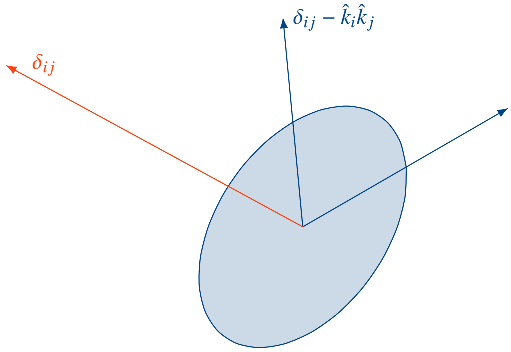

Figure 4.3 The direction of the wave vector $\vec{k}$ itself defines the structure of the dielectric tensor, if no other preferred directions are defined in the plasma.

Obviously, since $\vec{k}\cdot\hat{k} = \vec{k}^2/k = k$, the expression

$$\left(\hat{k}\otimes\hat{k}\right)\vec{k} =: \pi_{\|}\vec{k} = \vec{k} \tag{4.33}$$

is parallel to $\vec{k}$, while

$$\left(\mathbb{1}_3 - \hat{k}\otimes\hat{k}\right)\vec{k} =: \pi_{\perp}\vec{k} = \vec{k} - \vec{k} = 0 \tag{4.34}$$

vanishes and is thus perpendicular to $\vec{k}$. The tensors $\pi_{\|}$ and $\pi_{\perp}$ with the components

$$\pi_{\|} = \hat{k}\otimes\hat{k} \quad \text{and} \quad \pi_{\perp} = \mathbb{1}_3 - \hat{k}\otimes\hat{k} = \mathbb{1}_3 - \pi_{\|} \tag{4.35}$$

are generally convenient *projectors* for vector components parallel and perpendicular to the direction $\hat{k}$ satisfying

$$\pi_{\|}^2 = \pi_{\|}\ , \quad \pi_{\perp}^2 = \pi_{\perp}\ , \quad \pi_{\|}\pi_{\perp} = 0 = \pi_{\perp}\pi_{\|}\ . \tag{4.36}$$

?

As remarked before, projections need to be idempotent, in the present case $\pi_{\|}^2 = \pi_{\|}$ and $\pi_{\perp}^2 = \pi_{\perp}$. Confirm and interpret equations (4.36) geometrically.

We can use them we split the tensor $\hat{\varepsilon}$ into a transversal and a longitudinal part,

$$\hat{\varepsilon} = \hat{\varepsilon}_{\perp}\pi_{\perp} + \hat{\varepsilon}_{\|}\pi_{\|}\ , \tag{4.37}$$

where the transversal and the longitudinal dielectricities $\hat{\varepsilon}_{\perp}$ and $\hat{\varepsilon}_{\|}$ were defined. These are related to the functions $\hat{A}$ and $\hat{B}$ introduced in (4.30) above by $\hat{A} = \hat{\varepsilon}_{\perp}$ and $\hat{B} = \hat{\varepsilon}_{\|} - \hat{\varepsilon}_{\perp}$. Of course, $\hat{\varepsilon}_{\perp}$ and $\hat{\varepsilon}_{\|}$ are generally functions of ω and k which also need to satisfy the symmetry condition (4.29),

$$\hat{\varepsilon}_{\perp,\|}(-\omega, k) = \hat{\varepsilon}^*_{\perp,\|}(\omega, k)\ . \tag{4.38}$$

Contracting (4.37) with either of the projection tensors $\pi^{\perp}$ or $\pi^{\|}$, and using that their traces are

$$\mathrm{Tr}\,\pi_{\perp} = 2 \quad \text{and} \quad \mathrm{Tr}\,\pi_{\|} = 1\ , \tag{4.39}$$

we can project the longitudinal and transverse dielectricities out of the dielectric tensor,

$$\hat{\varepsilon}_\perp = \frac{1}{2}\,\mathrm{Tr}\,\pi_\perp\hat{\varepsilon} \quad \text{and} \quad \hat{\varepsilon}_\parallel = \mathrm{Tr}\,\pi_\parallel\hat{\varepsilon}\ . \tag{4.40}$$

Notice explicitly that we have neglected in this decomposition of the dielectric tensor $\hat{\varepsilon}_{ij}(\omega, \vec{k}\,)$ that preferred macroscopic directions may exist in the plasma, e.g. due to magnetic fields ordered on large scales. If they exist, they must also be built into the dielectric tensor.

?

Why is there a factor of $1/2$ in the expression for $\hat{\varepsilon}_\perp$ in (4.40)?

Problems

1. In media, the electric and magnetic fields $\vec{E}$ and $\vec{B}$ in Maxwell's inhomogeneous equations need to be replaced according to (4.18), while Maxwell's homogeneous equations remain unchanged.

   (a) From Maxwell's equations in media and Ohm's law $\vec{j} = \sigma\vec{E}$, derive the telegraph equation

   $$\vec{\nabla}^2\vec{E} - \frac{\varepsilon\mu}{c^2}\partial_t^2\vec{E} = \frac{4\pi\sigma\mu}{c^2}\partial_t\vec{E}\ . \tag{4.41}$$

   Assume plane-wave solutions for $\vec{E}$ and derive the dispersion relation for waves in media.

   (b) From the equation of motion

   $$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{e}{m}\vec{E} - \frac{\vec{v}}{\tau} \tag{4.42}$$

   for the electrons, containing a damping term with a characteristic collision time τ, derive an equation for the current density $\vec{j}$. Assume harmonic time dependence of $\vec{E}$ and $\vec{j}$ and identify the conductivity

   $$\sigma = \frac{ne^2}{m}\frac{\tau}{1 - \mathrm{i}\omega\tau}\ . \tag{4.43}$$

   (c) Combine the results from the preceding subproblems, assume $\mu = 1$ and $\omega\tau \gg 1$ and identify the plasma frequency ω_p. What does the limit $\omega\tau \gg 1$ mean?

4.3 Dispersion Relations

We proceed in this section by deriving the general expression (4.51) for the dispersion relation of electromagnetic waves in a plasma, which we split into the two dispersion relations (4.53) and (4.54) for transverse and longitudinal waves. By a perturbative analysis of the one-particle phase-space distribution of the plasma charges and its evolution equation, we derive the models (4.69) and (4.69) for the longitudinal and the transverse dielectricity. We conclude by deriving the Landau damping rate (4.87) of longitudinal waves.

4.3.1 General form of the dispersion relations

The dielectric tensor determines which kinds of electromagnetic wave can propagate through the plasma. The conditions for propagating waves are given by dispersion relations, which relate the frequency ω to the wave vector $\vec{k}$. Recall that the dispersion relation for electromagnetic waves in vacuum is $c^2\omega^2 = \vec{k}^2$. Based on our ansatz for the dielectric tensor and its decomposition into a longitudinal and a transversal part, we shall now derive the dispersion relations for electromagnetic waves propagating through a plasma.

We begin as usual by decomposing the incoming waves into plane waves with a phase factor $\exp[\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)]$. When applied to plane waves, Maxwell's equations (4.26) in a medium read

$$\begin{aligned} \vec{k}\times\hat{\vec{E}} &= \frac{\omega\hat{\vec{B}}}{c}\;, \quad \vec{k}\times\hat{\vec{B}} = -\frac{\omega\hat{\vec{D}}}{c}\;, \\ \vec{k}\cdot\hat{\vec{D}} &= 0\;, \quad \vec{k}\cdot\hat{\vec{B}} = 0\;, \end{aligned} \tag{4.44}$$

if we neglect any free charge densities and currents. Combining the curl of the first equation (4.44) with the second yields

$$\vec{k}\times\left(\vec{k}\times\hat{\vec{E}}\right) = \frac{\omega}{c}\,\vec{k}\times\hat{\vec{B}} = -\frac{\omega^2}{c^2}\hat{\vec{D}}\;. \tag{4.45}$$

If we expand the double vector product, we see that the dielectric displacement vector must satisfy the equation

$$\frac{\omega^2}{c^2}\hat{\vec{D}} = k^2\hat{\vec{E}} - \vec{k}\left(\vec{k}\cdot\hat{\vec{E}}\right)\;. \tag{4.46}$$

We now introduce the dielectric tensor by substituting $\hat{\vec{D}} = \hat{\varepsilon}\hat{\vec{E}}$ and find

$$\frac{\omega^2}{c^2}\hat{\varepsilon}\hat{\vec{E}} = k^2\hat{\vec{E}} - \left(\vec{k}\otimes\vec{k}\right)\hat{\vec{E}} = k^2\left(\mathbb{1}_3 - \hat{k}\otimes\hat{k}\right)\hat{\vec{E}} \tag{4.47}$$

or, after dividing by k^2 and rearranging,

$$\left(\mathbb{1}_3 - \hat{k}\otimes\hat{k} - \frac{\omega^2}{c^2k^2}\,\hat{\varepsilon}\right)\hat{\vec{E}} = \left(\pi_\perp - \frac{\omega^2}{c^2k^2}\,\hat{\varepsilon}\right)\hat{\vec{E}} = 0\;. \tag{4.48}$$

This is a linear equation with the expression in parentheses representing a square, 3×3 matrix. Equation (4.48) has non-trivial solutions $\hat{\vec{E}}\neq 0$ if and only if the determinant of this matrix vanishes,

$$\det\left(\pi_\perp - \frac{\omega^2}{c^2k^2}\,\hat{\varepsilon}\right) = 0\;. \tag{4.49}$$

This is the general form of the dispersion relation between the frequency ω and the wave vector $\vec{k}$ for electromagnetic waves that can propagate through the plasma.

It is now convenient to insert the decomposition (4.37) of the dielectric tensor and to rewrite the matrix in (4.48) such that its transverse and longitudinal components are grouped together. This results in

$$\left[\left(1 - \frac{\omega^2}{c^2k^2}\hat{\varepsilon}_\perp\right)\pi_\perp - \frac{\omega^2}{c^2k^2}\hat{\varepsilon}_\parallel\pi_\parallel\right]\hat{\vec{E}} = 0 \tag{4.50}$$

and the form

$$\det\left[\left(1 - \frac{\omega^2}{c^2k^2}\hat{\varepsilon}_\perp\right)\pi_\perp - \frac{\omega^2}{c^2k^2}\hat{\varepsilon}_\|\pi_\|\right] = 0 \tag{4.51}$$

of the dispersion relation.

4.3.2 Transversal and longitudinal waves

We expect that the last condition (4.51) defines more than one dispersion relation because the dielectric tensor $\hat{\varepsilon}$ has a longitudinal and a transversal part. In contrast to electromagnetic waves in vacuum, which are exclusively transversal, longitudinal as well as transversal electromagnetic waves may occur in media.

For transversal waves, the projection of $\hat{\vec{E}}$ on $\vec{k}$ vanishes, $\pi_\|\hat{\vec{E}} = 0$, while $\pi_\perp\hat{\vec{E}} = \hat{\vec{E}}$. The matrix equation (4.50) then reduces to the simpler equation

$$\left(1 - \frac{\omega^2}{c^2k^2}\,\hat{\varepsilon}_\perp\right)\hat{\vec{E}} = 0\ , \tag{4.52}$$

which implies the dispersion relation

$$\omega^2 = \frac{c^2k^2}{\hat{\varepsilon}_\perp}\ . \tag{4.53}$$

Caution Note that, according to the dispersion relation (4.53), the refractive index

$$n_\perp = \hat{\varepsilon}_\perp^{1/2}$$

for transversal electromagnetic waves can be assigned to a magnetised plasma. ◂

This recovers the usual result that transversal electromagnetic waves propagate in a medium with a reduced velocity $c\hat{\varepsilon}_\perp^{-1/2}$.

For longitudinal waves, $\pi_\perp\hat{\vec{E}} = 0$ and $\pi_\|\hat{\vec{E}} = \hat{\vec{E}}$, and the matrix equation (4.50) is reduced to

$$\frac{\omega^2}{c^2k^2}\hat{\varepsilon}_\|\hat{\vec{E}} = 0\ . \tag{4.54}$$

Generally, this requires that the longitudinal dielectricity itself must vanish, $\hat{\varepsilon}_\| = 0$. In order to understand this condition, we first need to determine the form of the longitudinal and transversal dielectricities, $\hat{\varepsilon}_\|$ and $\hat{\varepsilon}_\perp$.

4.3.3 Longitudinal and transversal dielectricities

In order to determine the form of $\hat{\varepsilon}_\|$ and $\hat{\varepsilon}_\perp$, we invoke the collision-less Boltzmann equation to study the response of the plasma particles to the incoming electromagnetic wave. We neglect the motion of the ions because of their lower velocities and concentrate on the plasma electrons. Before the electromagnetic wave arrives, the phase space density is assumed to have attained an equilibrium value f_0 which is then slightly perturbed by the wave,

$$f = f_0 + \delta f\ . \tag{4.55}$$

This expresses our expectation that sufficiently weak fields $\vec{E}$ and $\vec{B}$ will perturb the phase-space distribution function only by a small amount away from the equilibrium distribution f_0. Inserting the perturbation ansatz (4.55) into the

Boltzmann equation, subtracting the pure equilibrium terms and dropping terms of second order in the perturbation then gives

$$\frac{\partial \delta f}{\partial t} + \vec{v} \cdot \vec{\nabla} \delta f - e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \cdot \frac{\partial f_0}{\partial \vec{p}} = 0 \,, \tag{4.56}$$

where $\vec{v}$ and $\vec{p}$ are the equilibrium plasma velocity and momentum. For a locally isotropic distribution f_0, we must further have

$$\frac{\partial f_0}{\partial \vec{p}} \parallel \vec{v} \tag{4.57}$$

because no other preferred directions can be present. Thus

$$\left(\vec{v} \times \vec{B}\right) \cdot \frac{\partial f_0}{\partial \vec{p}} = 0 \,, \tag{4.58}$$

and Boltzmann's equation in linear approximation shrinks to

$$\frac{\partial \delta f}{\partial t} + \vec{v} \cdot \vec{\nabla} \delta f = e\vec{E} \cdot \frac{\partial f_0}{\partial \vec{p}} \,. \tag{4.59}$$

We now decompose the incoming electric field into plane waves with a phase factor $\exp[\mathrm{i}(\vec{k} \cdot \vec{x} - \omega t)]$ and assume that the phase-space distribution will respond in form of plane waves with equal phase. Then, (4.59) turns into the algebraic equation

$$-\mathrm{i}\omega \delta f + \mathrm{i}\vec{v} \cdot \vec{k} \delta f = e\vec{E} \cdot \frac{\partial f_0}{\partial \vec{p}} \,, \tag{4.60}$$

which can be solved for the perturbation δf of the phase-space distribution,

$$\delta f = -\frac{\mathrm{i}e\vec{E}}{\vec{k} \cdot \vec{v} - \omega} \cdot \frac{\partial f_0}{\partial \vec{p}} \,. \tag{4.61}$$

If the equilibrium distribution f_0 is locally homogeneous, isotropic and stationary, charge and current densities are exclusively caused by the perturbations δf of f_0. Therefore, the polarised charge density $\rho_{\rm pol}$ and the polarised current density $\vec{j}_{\rm pol}$ are

$$\rho_{\rm pol} = -e \int \mathrm{d}^3 p \, \delta f \,, \quad \vec{j}_{\rm pol} = -e \int \mathrm{d}^3 p \, \delta f \vec{v} \,. \tag{4.62}$$

These quantities are then also proportional to the same phase factor $\exp[\mathrm{i}(\vec{k} \cdot \vec{x} - \omega t)]$, and the polarisation equations (4.19) and (4.23) can be written in the form

$$\mathrm{i}\vec{k} \cdot \hat{\vec{P}} = -\hat{\rho}_{\rm pol} \,, \quad -\mathrm{i}\omega \hat{\vec{P}} = \hat{\vec{j}}_{\rm pol} \,. \tag{4.63}$$

We take the second of these equations and insert δf from (4.61) to find first

$$-\mathrm{i}\omega \hat{\vec{P}} = \mathrm{i}e^2 \int \mathrm{d}^3 p \, \frac{\vec{v}}{\vec{k} \cdot \vec{v} - \omega} \frac{\partial f_0}{\partial \vec{p}} \cdot \hat{\vec{E}} \,. \tag{4.64}$$

Since at the same time we have to satisfy the general relation

$$4\pi \hat{\vec{P}} = \hat{\vec{D}} - \hat{\vec{E}} = (\hat{\varepsilon} - \mathbb{1}_3) \, \hat{\vec{E}} \,, \tag{4.65}$$

we can directly read the dielectric tensor $\hat{\varepsilon}$ off (4.64),

$$\hat{\varepsilon} = \mathbb{1}_3 - \frac{4\pi e^2}{\omega} \int \mathrm{d}^3 p \, \frac{\vec{v}}{\vec{k} \cdot \vec{v} - \omega} \otimes \frac{\partial f_0}{\partial \vec{p}} \,. \tag{4.66}$$

By means of the projection tensors, we can now project out the longitudinal and transversal components of the dielectric tensor, as shown in (4.40). In this way, we first find the longitudinal dielectricity

$$\hat{\varepsilon}_{\parallel} = \mathrm{Tr}\, \pi_{\parallel} \hat{\varepsilon} = 1 - \frac{4\pi e^2}{\omega k^2} \int \mathrm{d}^3 p \, \frac{\vec{k} \cdot \vec{v}}{\vec{k} \cdot \vec{v} - \omega} \frac{\partial f_0}{\partial \vec{p}} \cdot \vec{k} \,. \tag{4.67}$$

Noticing that

$$\frac{\vec{k} \cdot \vec{v}}{\vec{k} \cdot \vec{v} - \omega} = 1 + \frac{\omega}{\vec{k} \cdot \vec{v} - \omega} \,, \tag{4.68}$$

and integrating once by parts, we can bring this expression into the form

$$\hat{\varepsilon}_{\parallel} = 1 - \frac{4\pi e^2}{k^2} \int \mathrm{d}^3 p \, \frac{1}{\vec{k} \cdot \vec{v} - \omega} \frac{\partial f_0}{\partial \vec{p}} \cdot \vec{k} \,. \tag{4.69}$$

The transverse dielectricity is

$$\hat{\varepsilon}_{\perp} = \frac{1}{2} \mathrm{Tr}\, \pi_{\perp} \hat{\varepsilon} = 1 - \frac{2\pi e^2}{\omega} \int \mathrm{d}^3 p \, \frac{1}{\vec{k} \cdot \vec{v} - \omega} \frac{\partial f_0}{\partial \vec{p}_{\perp}} \cdot \vec{v}_{\perp} \,, \tag{4.70}$$

where the perpendicular velocity $\vec{v}_{\perp}$ and the perpendicular momentum $\vec{p}_{\perp}$ are defined by

$$\vec{v}_{\perp} = \pi_{\perp} \vec{v} = \vec{v} - \left(\hat{k} \cdot \vec{v}\right) \hat{k} \,, \quad \vec{p}_{\perp} = m \vec{v}_{\perp} \,. \tag{4.71}$$

?

Beginning with (4.66), confirm the expressions (4.69) and (4.69) for the transverse and longitudinal dielectric tensors.

4.3.4 Landau Damping

Before we proceed to calculate the longitudinal and the transversal dielectricities for the special, but frequent case of a thermal plasma, we consider longitudinal waves in particular and identify an interesting damping process.

Because of the pole at $\vec{k} \cdot \vec{v} = \omega$, the longitudinal dielectricity $\hat{\varepsilon}_{\parallel}$ has a real and an imaginary part. The latter is responsible for damping of the incoming waves because it leads to an imaginary frequency. As we shall see, this damping process dissipates the incoming electromagnetic energy. To begin with, we note that the energy dissipation Q rate has two contributions, one from the damping of the electromagnetic waves and the associated decrease of the electromagnetic field energy density, and another from the Ohmic heating,

$$Q = \frac{\partial}{\partial t} \left(\frac{\vec{E}^2}{8\pi} \right) + \vec{E} \cdot \vec{j}_{\mathrm{pol}} \,. \tag{4.72}$$

In absence of any current of the free charges, the current density is solely the polarisation current defined by the continuity equation (4.22) and related to the polarisation change by (4.23). We can thus continue calculating the dissipation as

$$Q = \frac{\vec{E} \cdot \dot{\vec{E}}}{4\pi} + \vec{E} \cdot \dot{\vec{P}} = \frac{\vec{E}}{4\pi} \left(\dot{\vec{E}} + 4\pi \dot{\vec{P}} \right) = \frac{\vec{E} \cdot \dot{\vec{D}}}{4\pi} \,. \tag{4.73}$$

Notice that the electromagnetic field, the dielectric displacement and the polarisation do not wear hats here: They are to be taken as functions of $\vec{x}$ and t here.

We now consider the contribution of an individual plane wave characterised by frequency ω and wave number $\vec{k}$ to the dissipation Q, i.e. we insert

$$\vec{E} = \hat{\vec{E}}\,\mathrm{e}^{\mathrm{i}\left(\vec{k}\cdot\vec{x}-\omega t\right)}\;, \quad \vec{D} = \hat{\vec{D}}\,\mathrm{e}^{\mathrm{i}\left(\vec{k}\cdot\vec{x}-\omega t\right)} \tag{4.74}$$

into (4.73). The dissipation Q resulting therefrom will thus be the dissipation *per Fourier mode* $(\omega, \vec{k})$. Since we should finally arrive at a real expression for Q, we replace $\vec{E}$ by $(\vec{E}+\vec{E}^*)/2$ to obtain its real part, and likewise for $\vec{D}$ by $(\vec{D}+\vec{D}^*)/2$. Furthermore, since $\hat{\vec{D}} = \hat{\varepsilon}_{\|}\hat{\vec{E}}$ for longitudinal waves, we can write

$$\hat{\vec{D}} = \hat{\varepsilon}_{\|}\hat{\vec{E}} \quad \to \quad \frac{1}{2}\left(\hat{\varepsilon}_{\|}\hat{\vec{E}} + \hat{\varepsilon}_{\|}^*\hat{\vec{E}}^*\right)\;. \tag{4.75}$$

Inserting these expressions for $\vec{E}$ and $\vec{D}$ into Q from (4.73) gives

$$Q = -\frac{\mathrm{i}\omega}{16\pi}\left(\vec{E}+\vec{E}^*\right)\left(\hat{\varepsilon}_{\|}\vec{E} - \hat{\varepsilon}_{\|}^*\vec{E}^*\right)\;, \tag{4.76}$$

where the minus sign in the second factor comes from the change in sign in the phase factor $\exp[\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)]$ due to the complex conjugation of the dielectric displacement. Averaging the dissipation (4.76) over time eliminates the products $\vec{E}\cdot\vec{E}$ and $\vec{E}^*\cdot\vec{E}^*$ because they vary with the phase factor like $\exp(-2\mathrm{i}\omega t)$, while the mixed terms become independent of time. Thus, the time-averaged dissipation is

$$\langle Q\rangle = -\frac{\mathrm{i}\omega}{16\pi}\left(\hat{\varepsilon}_{\|}\hat{\vec{E}}\cdot\hat{\vec{E}}^* - \hat{\varepsilon}_{\|}^*\hat{\vec{E}}^*\cdot\hat{\vec{E}}\right) = -\frac{\mathrm{i}\omega}{16\pi}\left(\hat{\varepsilon}_{\|} - \hat{\varepsilon}_{\|}^*\right)\left|\hat{\vec{E}}\right|^2\;. \tag{4.77}$$

The remaining expression in brackets is twice the imaginary part of the longitudinal dielectricity $\hat{\varepsilon}_{\|}$,

$$\hat{\varepsilon}_{\|} - \hat{\varepsilon}_{\|}^* = 2\mathrm{i}\,\mathrm{Im}\,\hat{\varepsilon}_{\|}\;, \tag{4.78}$$

and thus we find

$$\langle Q\rangle = \frac{\omega}{8\pi}\,\mathrm{Im}\,\hat{\varepsilon}_{\|}\left|\hat{\vec{E}}\right|^2\;. \tag{4.79}$$

The imaginary part of $\hat{\varepsilon}_{\|}$ can be obtained from (4.71). In order to avoid the pole in the integrand there, we shift it away from the real axis by a small amount δ,

$$\frac{1}{\vec{k}\cdot\vec{v}-\omega} \to \frac{1}{\vec{k}\cdot\vec{v}-\omega-\mathrm{i}\delta}\,, \tag{4.80}$$

and then take the imaginary value

$$\mathrm{Im}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega-\mathrm{i}\delta} = \mathrm{Im}\,\frac{\vec{k}\cdot\vec{v}-\omega+\mathrm{i}\delta}{\left(\vec{k}\cdot\vec{v}-\omega\right)^2+\delta^2} = \frac{\delta}{\left(\vec{k}\cdot\vec{v}-\omega\right)^2+\delta^2}\;. \tag{4.81}$$

In the limit $\delta \to 0$, this turns into π times a Dirac delta function,

$$\lim_{\delta\to 0}\frac{\delta}{\left(\vec{k}\cdot\vec{v}-\omega\right)^2+\delta^2} = \pi\delta_{\mathrm{D}}\left(\vec{k}\cdot\vec{v}-\omega\right)\;. \tag{4.82}$$

This can be seen by verifying that the limit satisfies the two defining criteria of a δ function,

$$\lim_{\delta\to 0} \frac{\delta}{\pi\,(x^2+\delta^2)} = 0 \quad (x \neq 0) \tag{4.83}$$

and

$$\lim_{\delta\to 0} \int_{-R}^{R} \frac{\delta}{\pi\,(x^2+\delta^2)} \mathrm{d}x = \frac{2}{\pi} \lim_{\delta\to 0} \arctan \frac{R}{\delta} = 1 \; . \tag{4.84}$$

Without loss of generality, we now rotate the coordinate frame such that the x axis aligns with the wave vector $\vec{k}$. We further integrate the one-particle phase-space distribution function f_0 over the momentum components p_y and p_z and define

$$\bar{f}(p_x) = \int \mathrm{d}p_y \int \mathrm{d}p_z\, f_0\,(\vec{p}) \; . \tag{4.85}$$

The imaginary part of the longitudinal dielectricity $\hat{\varepsilon}_{\|}$ can then be written as

$$\begin{aligned} \mathrm{Im}\,\hat{\varepsilon}_{\|} &= -\frac{4\pi^2 e^2}{k^2} \int \mathrm{d}p_x\, k \frac{\mathrm{d}\bar{f}}{\mathrm{d}p_x}\, \delta_{\mathrm{D}}\,(kv_x - \omega) \\ &= -\frac{4\pi^2 e^2 m_{\mathrm{e}}}{k^2} \left.\frac{\mathrm{d}\bar{f}}{\mathrm{d}p_x}\right|_{p_x=\omega m/k} , \end{aligned} \tag{4.86}$$

and the mean dissipation rate turns into

$$\langle Q \rangle = -\left|\hat{\vec{E}}\right|^2 \frac{\pi m e^2 \omega}{k^2} \left.\frac{\mathrm{d}\bar{f}}{\mathrm{d}p_x}\right|_{p_x=\omega m/k} . \tag{4.87}$$

This is *Landau damping*, which is caused by the fact that electrons which are slightly faster than the wave are slowed down, electrons which are slightly slower than the wave are accelerated, and since the velocity distribution is typically monotonically decreasing, more electrons need to be accelerated than decelerated, and thus the wave loses energy.

Problems

1. Landau damping in a thermal plasma:

    (a) Evaluate the mean energy dissipation rate $\langle Q \rangle$ due to Landau damping for waves propagating through a thermal plasma, assuming (as will be shown in the next section) that longitudinal waves with the plasma frequency can propagate.

    (b) Estimate a time scale for Landau damping of a longitudinal wave in a thermal plasma, depending on its wavelength.

4.4 Electromagnetic Waves in Thermal Plasmas

In this section, we evaluate the microscopic models for the longitudinal and the transverse dielectricities from the preceding section for a thermal plasma. This leads us to the expressions (4.101) and (4.102), giving the two dielectricities in terms of the plasma dispersion function. From appropriate expansions of this function, approximations to the dispersion relations for frequencies low and high compared to the plasma frequency are derived. The section ends with a discussion of dispersion and damping of transversal waves propagating through a plasma, leading to the definition (4.115) of the dispersion measure.

4.4.1 Longitudinal and transversal dielectricities

In a thermal plasma with temperature T, the equilibrium phase-space distribution f_0 of the electrons can be assumed to be a Maxwellian,

$$f_0(\vec{p}) = \frac{\bar{n}}{(2\pi\sigma^2)^{3/2}} \mathrm{e}^{-p^2/(2\sigma^2)}\;, \quad \sigma = \sqrt{m_\mathrm{e} k_\mathrm{B} T}\;. \tag{4.88}$$

With this specific choice, the integrals in the longitudinal and the transversal dielectricities, (4.69) and (4.70) respectively, can be worked out.

Without loss of generality, we can first conveniently rotate the coordinate frame such that $\vec{k}$ points into the positive $\vec{x}$ direction. Then, $\vec{v}_\perp$ falls into the y-z plane. Since the derivative of f_0 with respect to any momentum component p_i is

$$\frac{\partial f_0(\vec{p})}{\partial p_i} = -\frac{p_i}{\sigma^2} f_0(\vec{p})\;, \tag{4.89}$$

the remaining integrals in (4.69) and (4.70) read

$$\int \mathrm{d}^3 p\, \vec{k}\cdot\frac{\partial f_0}{\partial \vec{p}}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega} = -\int \mathrm{d}^3 p\,\frac{k p_x}{\sigma^2}\, f_0(\vec{p})\,\frac{1}{k p_x/m-\omega} \tag{4.90}$$

and

$$\int \mathrm{d}^3 p\, \vec{v}_\perp\cdot\frac{\partial f_0}{\partial \vec{p}_\perp}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega} = -\frac{1}{m}\int \mathrm{d}^3 p\,\frac{p_y^2+p_z^2}{\sigma^2}\, f_0(\vec{p})\,.\frac{1}{k p_x/m-\omega}\;. \tag{4.91}$$

The integrations over p_y and p_z in (4.91) and (4.91) can immediately be carried out, resulting in

$$\int \mathrm{d}^3 p\, \vec{k}\cdot\frac{\partial f_0}{\partial \vec{p}}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega} = -\frac{\bar{n}}{(2\pi\sigma^2)^{1/2}}\,\frac{k}{\sigma^2}\int_{-\infty}^{\infty} \mathrm{d}p_x\,\frac{p_x\,\mathrm{e}^{-p_x^2/(2\sigma^2)}}{k p_x/m-\omega} \tag{4.92}$$

and

$$\int \mathrm{d}^3 p\, \vec{v}_\perp\cdot\frac{\partial f_0}{\partial \vec{p}_\perp}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega} = -\frac{2\bar{n}}{(2\pi\sigma^2)^{1/2} m}\int_{-\infty}^{\infty} \mathrm{d}p_x\,\frac{\mathrm{e}^{-p_x^2/(2\sigma^2)}}{k p_x/m-\omega}\;. \tag{4.93}$$

To simplify the remaining integrals in (4.92) and (4.93), we introduce the dimension-less momentum component

$$t := \frac{p_x}{\sqrt{2}\sigma} \tag{4.94}$$

and the dimension-less frequency

$$y = \frac{m\omega}{\sqrt{2}k\sigma} = \frac{\omega}{\omega_\mathrm{th}} \quad \text{with} \quad \omega_\mathrm{th} := \frac{\sqrt{2}k\sigma}{m} = k\sqrt{\frac{2k_\mathrm{B}T}{m}} \tag{4.95}$$

to bring (4.92) and (4.93) into the forms

$$\int d^3p\,\vec{k}\cdot\frac{\partial f_0}{\partial\vec{p}}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega} = \frac{\bar{n}}{(2\pi\sigma^2)^{1/2}}\,\frac{\sqrt{2}m}{\sigma}\int_{-\infty}^{\infty}dt\,\frac{t\,e^{-t^2}}{y-t} \tag{4.96}$$

and

$$\int d^3p\,\vec{v}_\perp\cdot\frac{\partial f_0}{\partial\vec{p}_\perp}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega} = \frac{2\bar{n}}{(2\pi\sigma^2)^{1/2}k}\int_{-\infty}^{\infty}dt\,\frac{e^{-t^2}}{y-t}\,. \tag{4.97}$$

?

Since the integrands in (4.96) and (4.97) are singular at $t = y$, it remains to be shown that the integrals exist at all. How could you achieve this?

Before we can gain further insight into expressions like (4.96) and (4.97), we need to carefully evaluate the integrals appearing there because the integrands have a pole on the t axis at $t = y$. These integrals can be solved by continuing the integrand into the complex plane, $t \to z \in \mathbb{C}$, and then using the residue theorem from the theory of complex functions. Before doing so, we rewrite the integral in (4.96) as

$$\int_{-\infty}^{\infty}dz\,\frac{z\,e^{-z^2}}{y-z} = -\int_{-\infty}^{\infty}dz\,\frac{(z-y+y)\,e^{-z^2}}{z-y} = -\sqrt{\pi}+y\int_{-\infty}^{\infty}\frac{dz\,e^{-z^2}}{y-z}\,. \tag{4.98}$$

The remaining integral is given by the so-called *Faddeeva function* $w(z)$ which, for a positive imaginary part of z, has the integral representation

$$w(z) = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{z-t}dt\,. \tag{4.99}$$

If we choose the imaginary part of y to be arbitrarily small, we may thus express the integral from (4.96) by

$$\int_{-\infty}^{\infty}dt\,\frac{t\,e^{-t^2}}{y-t} = -\sqrt{\pi}-i\pi y\,w(y) = -\sqrt{\pi}\,[1+y\,Z(y)] \quad \text{with}$$
$$Z(y) := i\,\sqrt{\pi}w(y)\,. \tag{4.100}$$

?

Based on the definitions of the Debye wavelength and the plasma frequency, confirm the expressions (4.101) and (4.102) for $\hat{\varepsilon}_\parallel$ and $\hat{\varepsilon}_\perp$.

The function $Z(y)$ defined here is called *plasma dispersion function*.

Returning with (4.100) into (4.96) and inserting the result into (4.69), we can now write the longitudinal dielectricity (4.69) as

$$\hat{\varepsilon}_\parallel = 1+\frac{1+y\,Z(y)}{\lambda_D^2k^2} = 1+\frac{\omega_p^2}{\omega^2}\,2y^2\,[1+y\,Z(y)]\,, \tag{4.101}$$

where the Debye wavelength λ_D as defined in (4.11) was identified in the first step and the plasma frequency ω_p from (4.16) as well as the dimension-less frequency y from (4.95) in the second. Similarly, identifying the Faddeeva function (4.99) in (4.97), using the definition (4.100) of the plasma dispersion function and inserting these expressions into (4.70), we find the simple expression

$$\hat{\varepsilon}_\perp = 1+\frac{\omega_p^2}{\omega^2}\,y\,Z(y) \tag{4.102}$$

for the transversal dielectricity. Here, we have again inserted the plasma frequency ω_p defined in (4.16) and used the definition (4.95) of the dimension-less frequency y. We have thus reduced the longitudinal and transverse dielectricities of a thermal plasma to the plasma dispersion function $Z(y)$, with the scaled frequency y defined by (4.95).

Series expansions of $Z(y)$ are useful for practical calculations (Figure 4.4). For small $|y| \ll 1$,

$$y\,Z(y) \approx -2y^2\left(1 - \frac{2y^2}{3} + \frac{4x^4}{15} - \ldots\right) + \mathrm{i}\sqrt{\pi}\,y\left(1 - y^2\right) \tag{4.103}$$

while for large $|y| \gg 1$,

$$y\,Z(y) \approx -1 - \frac{1}{2y^2} - \frac{3}{4y^4} + \ldots\,. \tag{4.104}$$

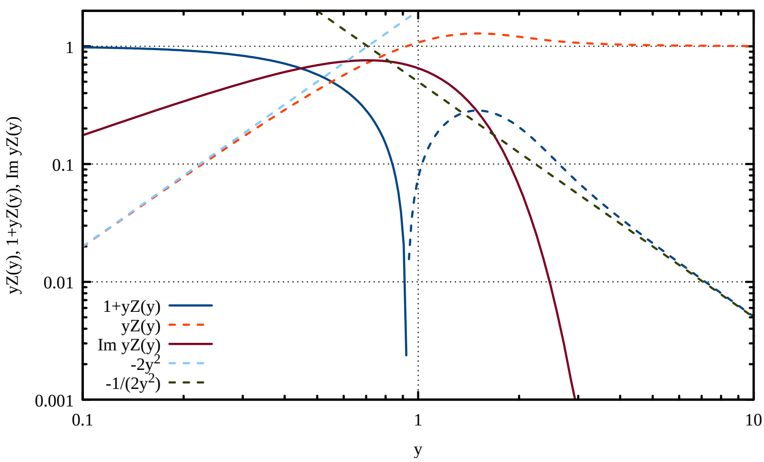

Figure 4.4 The absolute value of the plasma dispersion function $Z(y)$ as a function of y is plotted along the real axis, together with the approximations given in (4.103) and (4.104).

Before we begin discuss the results (4.101) and (4.102) in more detail, we recall the definition (4.95) of y and introduce the plasma frequency ω_p from (4.16) and the Debye wave number k_D from (4.6) into it,

$$y = \frac{\omega m}{\sqrt{2}k\sigma} = \frac{\omega}{\omega_\mathrm{p}}\frac{k_\mathrm{D}}{k}\frac{\omega_\mathrm{p}}{k_\mathrm{D}}\frac{m}{\sqrt{2}\sigma} = \frac{\tilde{\omega}}{\sqrt{2}\tilde{k}}\,, \tag{4.105}$$

inserting the definition (4.88) of σ in the final step. Quantities with a tilde now refer to plasma units,

$$\tilde{\omega} = \frac{\omega}{\omega_\mathrm{p}}\,,\quad \tilde{k} = \frac{k}{k_\mathrm{D}}\,. \tag{4.106}$$

In terms of these quantities, we can write the dielectricities (4.101) and (4.102) as

$$\hat{\varepsilon}_\parallel = 1 + \frac{1 + y\,Z(y)}{\tilde{k}^2}\,,\quad \hat{\varepsilon}_\perp = 1 + \frac{y\,Z(y)}{\tilde{\omega}^2}\,. \tag{4.107}$$

With the series expansions (4.103) and (4.104) of the plasma dispersion function, we arrive at the formal expressions for $\hat{\varepsilon}_\parallel$ and $\hat{\varepsilon}_\perp$ listed in Tab. 4.1.

?

Verify the entries in Tab. 4.1.

If multiple particle species need to be taken into account in addition to the electrons, the individual dielectricities are summed over all species according to

$$\hat{\varepsilon} - 1 = \sum_{\text{species}\,i}(\hat{\varepsilon}_i - 1)\,. \tag{4.108}$$

Table 4.1 Limiting cases for the longitudinal and transverse dielectricities, $\hat{\varepsilon}_{\|}$ and $\hat{\varepsilon}_{\perp}$, for small and large values of $|y|$.

dielectricity	$|y| \ll 1, \tilde{\omega} \ll \tilde{k}$	$|y| \gg 1, \tilde{k} \ll \tilde{\omega}$
$\hat{\varepsilon}_{\|}$	$1+\frac{1}{\tilde{k}^2}\left(1-\frac{\tilde{\omega}^2}{\tilde{k}^2}\right)+\mathrm{i}\sqrt{\frac{\pi}{2}}\frac{\tilde{\omega}}{\tilde{k}^3}$	$1-\frac{1}{\tilde{\omega}^2}\left(1+\frac{3\tilde{k}^2}{\tilde{\omega}^2}\right)$
$\hat{\varepsilon}_{\perp}$	$1-\frac{1}{\tilde{k}^2}\left(1-\frac{1}{3}\frac{\tilde{\omega}^2}{\tilde{k}^2}\right)+\mathrm{i}\sqrt{\frac{\pi}{2}}\frac{1}{\tilde{k}\tilde{\omega}}$	$1-\frac{1}{\tilde{\omega}^2}\left(1+\frac{\tilde{k}^2}{\tilde{\omega}^2}\right)$

4.4.2 Dispersion Measure and Dispersion Relations

The dispersion relation for transversal waves was given by (4.53). We combine it with the relation

$$\omega_{\mathrm{p}}^2 = \frac{k_{\mathrm{B}}T}{m}k_{\mathrm{D}}^2 = \left\langle v^2\right\rangle k_{\mathrm{D}}^2 \ . \tag{4.109}$$

between the plasma frequency and the Debye wavenumber and introduce the root-mean square velocity from (4.14). Defining a mean-squared beta factor by

$$\beta^2 := \frac{1}{c^2}\left\langle v^2\right\rangle \ , \tag{4.110}$$

we can write the dispersion relation (4.53) in dimension-less form as

$$\beta^2\tilde{\omega}^2 = \frac{\tilde{k}^2}{\hat{\varepsilon}_{\perp}} \tag{4.111}$$

in plasma units. For high frequencies $\tilde{\omega} \gg \tilde{k}$, we can approximate the transverse dielectricity fom Tab. 4.1 by $\hat{\varepsilon}_{\perp} \approx 1 - \tilde{\omega}^{-2}$ and find the dispersion relation

$$\tilde{\omega}^2 = \frac{\tilde{k}^2}{\beta^2} + 1 \quad \text{or} \quad \tilde{k} = \beta\sqrt{\tilde{\omega}^2 - 1} \ . \tag{4.112}$$

Since the group velocity of such transversal waves is

$$c_{\mathrm{g}} = \frac{\partial\omega}{\partial k} = \frac{\omega_{\mathrm{p}}}{k_{\mathrm{D}}}\frac{\partial\tilde{\omega}}{\partial\tilde{k}} = \frac{c}{\beta}\frac{\tilde{k}}{\tilde{\omega}} = c\sqrt{1-\frac{1}{\tilde{\omega}^2}} \ , \tag{4.113}$$

the propagation time of such waves is

$$\Delta t_{\omega} = \int\frac{\mathrm{d}l}{c_{\mathrm{g}}} \approx \int\frac{\mathrm{d}l}{c}\left(1+\frac{\omega_{\mathrm{p}}^2}{2\omega^2}\right) = \frac{L}{c} + \frac{2\pi e^2}{mc\omega^2}\int\mathrm{d}l\,n \ , \tag{4.114}$$

where the integral over the electron density along the light path,

$$\int\mathrm{d}l\,n \equiv \mathrm{DM} \ , \tag{4.115}$$

is called the *dispersion measure*. In the last steps, we have used $\omega \gg \omega_{\mathrm{p}}$ to approximate the square root in the group velocity (4.113).

For lower frequencies, we need to take one more order in $\tilde{k}/\tilde{\omega}$ into account. Taking the expression for $\hat{\varepsilon}_{\perp}$ from the second row and the second column of

Tab. 4.1 and inserting it into the dimension-less dispersion relation (4.111), we find first

$$\beta^2\tilde{\omega}^2\left[1-\frac{1}{\tilde{\omega}^2}\left(1+\frac{\tilde{k}^2}{\tilde{\omega}^2}\right)\right]=\tilde{k}^2 \tag{4.116}$$

or, solving for the wavenumber $\tilde{k}$,

$$\tilde{k}=\beta\tilde{\omega}\sqrt{\frac{\tilde{\omega}^2-1}{\tilde{\omega}^2+\beta^2}}\,. \tag{4.117}$$

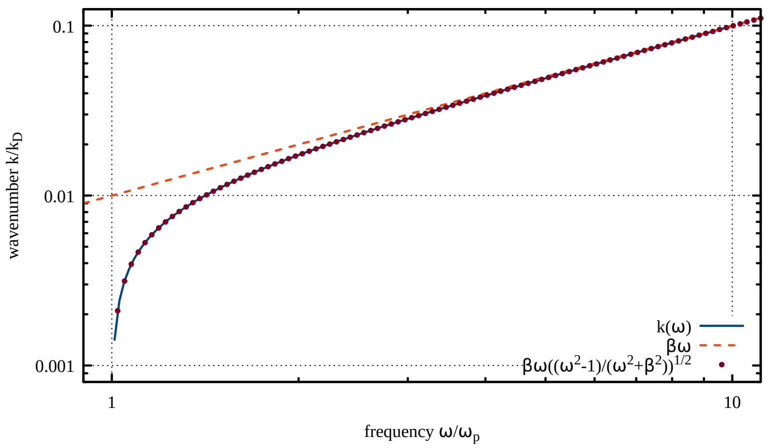

Figure 4.5 The dispersion relation for transverse waves is shown together with the approximations $\tilde{k}=\beta\tilde{\omega}$ and (4.117).

?

Solve the dispersion relation (4.116) for the frequency $\tilde{\omega}$.

Example: Ionosphere of the Earth

In the ionosphere of the Earth, $n\approx 10^6\,\mathrm{cm^{-3}}$. Assuming a temperature of $T=273\,\mathrm{K}$, the Debye length and the root-mean square thermal electron velocity are

$$\lambda_D\approx 0.11\,\mathrm{cm}\quad\text{and}\quad v_e=\sqrt{\frac{k_BT}{m}}\approx 6.43\cdot 10^6\,\mathrm{cm\,s^{-1}}\,; \tag{4.118}$$

see also (4.14). Thus, the Debye wavenumber is $k_D=9.1\,\mathrm{cm^{-1}}$, the β factor of the electrons is $\beta=2.1\cdot 10^{-4}$, and the plasma frequency is

$$\omega_p=\frac{v_e}{\lambda_D}\approx 58.5\,\mathrm{MHz}\,. \tag{4.119}$$

Transversal Gigahertz waves, for example, have $\tilde{\omega}\gg 1$, thus a transversal dielectricity of $\hat{\varepsilon}_\perp\approx 1$ and wavelengths of

$$\lambda=\frac{2\pi}{k}\approx 2\pi\frac{c}{\omega} \tag{4.120}$$

like in vacuum. Approaching the plasma frequency from above, the wavenumber falls below its vacuum value, and thus the waves become longer than in vacuum. ◂

The dispersion relation for longitudinal waves requires $\hat{\varepsilon}_{\parallel} = 0$, as was shown in (4.54) above. Assuming high frequencies,

the entry in the first row and the second column in Tab. 4.1 gives

$$\tilde{\omega}^2 = 1 + \frac{3\tilde{k}^2}{\tilde{\omega}^2} \tag{4.121}$$

or, solving for $\tilde{\omega}$,

$$\tilde{\omega}_{\pm}^2 = \frac{1}{2}\left(1 \pm \sqrt{1 + 12\tilde{k}^2}\right) . \tag{4.122}$$

Only the positive branch is meaningful here. In the high-frequency limit applied, we require $\tilde{\omega} \gg \tilde{k}$, thus $\tilde{\omega} \approx 1$ and $\tilde{k} \ll 1$. This allows us to approximate

$$\tilde{\omega} \approx 1 + \frac{3}{2}\tilde{k}^2 . \tag{4.123}$$

Such longitudinal waves thus have frequencies slightly higher than the plasma frequency and very large wavelengths.

Problems

1. Derive the phase and the group velocities of longitudinal and transverse electromagnetic waves in a thermal plasma in the high-frequency limit.

2. Derive the series expansions (4.103) and (4.104) of the plasma dispersion function in the limits $y \ll 1$ and $y \gg 1$.

3. Radio waves propagating through a thermal plasma in a deep gravitational well experience two kinds of time delay: the delay (4.114) due to the electromagnetic dispersion and the so-called Shapiro delay

$$\Delta t_{\mathrm{Shapiro}} = -\frac{2}{c^3}\int \mathrm{d}l\,\Phi , \tag{4.124}$$

due to generally-relativistic time dilation, where Φ is the Newtonian gravitational potential. Estimate the relative magnitude of both time delays.

4.5 The Magnetohydrodynamic Equations

In this section, we introduce the assumptions of magnetohydrodynamics and derive the induction equation (4.141) for the evolution of the magnetic field in a plasma. Magnetic forces on the plasma appear in the extension (4.145) of Euler's equation, and we show in (4.161) how the energy current density has to be modified in presence of a magnetic field. The comparison of magnetic advection and diffusion leads to the definition of the magnetic Reynolds number in (4.169).

4.5.1 Assumptions

Magnetohydrodynamics is the theory of how magnetised plasmas move. It is built upon several assumptions which go significantly beyond hydrodynamics. They begin with the fact that plasmas consist of ions and electrons which should in the simplest case be described as two fluids coupled to each other rather than a single fluid, as in hydrodynamics. At this point, recall that the constitutive assumption of hydrodynamics was that the mean-free path of the fluid particles is very small compared to all other relevant length scales occurring in the system. In ideal hydrodynamics, the mean-free path is infinitely short. Giving up this idealisation, but still assuming that the mean-free path is very short, gives rise to effects based on particle transport, such as viscosity and diffusion.

The common, greatly simplifying assumption in magnetohydrodynamics is that the ions and the electrons are so tightly coupled to each other by their electrodynamic interaction that they can indeed be treated as a single fluid. The central assumption of hydrodynamics is then applied in addition, namely that the mean-free path of the plasma particles, ions and electrons alike, is very small.

If, however, there is no net motion of the electrons with respect to the ions, then there is no separation of charges, no net electric current, and thus neither an electric nor a magnetic field. For magnetohydrodynamics, therefore, we need to assume that there is in fact a small drift velocity $\vec{v}_{\rm drift}$ between the electrons and ions,

$$\vec{v}_{\rm drift} = \vec{v}_{\rm e} - \vec{v}_{\rm i} \;, \tag{4.125}$$

causing an electric current $\vec{j}$ of free charges, which can sustain a magnetic field. As in non-ideal, viscous hydrodynamics, the strict idealisation of two fluids of opposite charge infinitely tightly coupled to each other is slightly loosened here.

A final, common assumption is that the plasma flows non-relativistically, allowing us to neglect terms of higher than linear order in v/c, where v is the flow velocity.

More quantitatively, we thus arrive at the following assumptions: We first transform into the rest frame of the plasma, defined as the frame locally co-moving with the mean velocity of the two or more plasma components. Quantities in this rest frame are primed.

The plasma is macroscopically neutral, but as it consists of charged particles, even small drift velocities can create substantial currents. This is expressed by supposing that the time component of the current-density four-vector j'^{μ} observed in this rest frame be negligibly small compared to the spatial components of the current density,

$$c\rho' \ll \left|\vec{j}'\right| \;. \tag{4.126}$$

As usual in electrodynamics, the spatial current density $\vec{j}'$ itself is assumed to be related to the electric field $\vec{E}'$ through Ohm's law by the conductivity σ,

$$\vec{j}' = \sigma\vec{E}' \;. \tag{4.127}$$

The plasma is assumed to be an ideal or near-ideal conductor such that even weak electric fields can be responsible for significant currents. By its definition, the conductivity must have the dimension

$$[\sigma] = \text{time}^{-1} \ . \tag{4.128}$$

Accordingly, very high conductivity means that the time scale needed by the plasma charges to respond to changes in the electromagnetic fields is very small. This allows us to neglect the displacement current compared to the charge current,

$$\frac{\partial \vec{E}'}{\partial t} \ll \vec{j}' \ , \tag{4.129}$$

because any change in the electric field will immediately (or at least on a very short time scale of order σ^{-1}) lead to a substantial charge current. Maxwell's equations for the magnetic field thus simplify to read

$$\vec{\nabla} \cdot \vec{B}' = 0 \ , \quad \vec{\nabla} \times \vec{B}' = \frac{4\pi}{c}\vec{j}' \tag{4.130}$$

in the rest frame of the plasma. This rest frame and the observer's laboratory frame are related by a Lorentz transform as given in (1.26). However, by the assumption of non-relativistic plasma flow relative to the laboratory frame, we can expand the Lorentz factor γ to lowest order in $\beta = v/c$, i.e. we can adopt $\gamma \approx 1$. Then, the Lorentz transform of the four-current gives

$$c\rho' = c\rho - \vec{\beta} \cdot \vec{j} \ , \quad \vec{j}' = \vec{j} - \vec{\beta} c\rho \ . \tag{4.131}$$

Since we have assumed $c\rho' \ll |\vec{j}'|$ in the plasma's rest frame, we also have $c\rho \ll |\vec{j}|$ in the laboratory frame due to the non-relativistic plasma flow, $\beta \ll 1$. Since we must further obey the Maxwell equations

$$\vec{\nabla} \cdot \vec{E}' = 4\pi\rho' \ , \quad \vec{\nabla} \times \vec{B}' = \frac{4\pi}{c}\vec{j}' \ , \tag{4.132}$$

this also implies $|\vec{E}| \ll |\vec{B}|$. Accordingly, the assumptions of (non-relativistic) magnetohydrodynamics imply the conditions

$$c\rho \ll \left|\vec{j}\right| \ , \quad \left|\vec{E}\right| \ll \left|\vec{B}\right| \ , \quad \left|\frac{\partial \vec{E}}{\partial t}\right| \ll \left|\vec{j}\right| \ , \quad \beta = \frac{\left|\vec{v}\right|}{c} \ll 1 \ . \tag{4.133}$$

?

Confirm expressions (4.131) by carrying out a Lorentz transform in the appropriate limit.

On the basis of these relations, we can now derive the equations of magnetohydrodynamics.

4.5.2 The induction equation

We begin with Ohm's law

$$\vec{j} \approx \vec{j}' = \sigma \vec{E}' \tag{4.134}$$

and relate the electric field $\vec{E}'$ in the plasma's rest frame to the electric field $\vec{E}$ in the laboratory frame by the Lorentz transform (1.87) in the limit $\gamma \approx 1$. We can thus insert

$$\vec{E}' = \vec{E} + \vec{\beta} \times \vec{B} \tag{4.135}$$

into (4.134) and to solve for $\vec{E}$ to obtain

$$\vec{E} = \frac{\vec{j}}{\sigma} - \vec{\beta} \times \vec{B} \,. \tag{4.136}$$

Next, we put this result into the induction equation

$$\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla} \times \vec{E} \tag{4.137}$$

and find

$$\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla} \times \left(\frac{\vec{j}}{\sigma} - \vec{\beta} \times \vec{B} \right) \tag{4.138}$$

for the evolution of the magnetic field. At the same time, we need to satisfy Ampère's law with vanishing displacement current,

$$\vec{j} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B} \,, \tag{4.139}$$

which enables us to eliminate the current density from the induction equation. Using the identity

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B} \right) = \vec{\nabla} \left(\vec{\nabla} \cdot \vec{B} \right) - \vec{\nabla}^2 \vec{B} \tag{4.140}$$

for the double curl and Maxwell's equation $\vec{\nabla} \cdot \vec{B} = 0$ for the divergence of the magnetic field, we find

$$\frac{\partial \vec{B}}{\partial t} = \frac{c^2}{4\pi\sigma} \vec{\nabla}^2 \vec{B} + \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) \tag{4.141}$$

if we further assume that the conductivity is spatially constant, $\vec{\nabla}\sigma = 0$. This *induction equation* determines the evolution of the magnetic field embedded into a plasma flow with the velocity $\vec{v}$.

?

What do the two terms on the right-hand side of the induction equation (4.141) mean, i.e. what physical effects do they encode?

4.5.3 Euler's equation

The induction equation tells us how the magnetic field changes in response to the plasma flow. In addition, we need equations for the back-reaction of the magnetic field on the plasma flow. We have to expect that a magnetised plasma flows differently than a neutral fluid because the magnetic field acts on the charged particles through the Lorentz force.

Notice, however, that the continuity equation for the mass density ρ will remain unchanged,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \,, \tag{4.142}$$

because mass conservation must not be affected by the presence of the magnetic field. Euler's equation, which describes the conservation of momentum or, more precisely, the transport of the specific momentum density, must be modified by the presence of the Lorentz force. In absence of electric fields, the force exerted on a charge e by the magnetic field $\vec{B}$ is

$$\frac{e}{c} \vec{v} \times \vec{B} \,. \tag{4.143}$$

Multiplying this expression with the number density n of charges will turn it into the Lorentz force density, i.e. into the combined Lorentz force on all charges in a unit volume. Noticing that the product $ne\vec{v}$ is the current density $\vec{j}$, and eliminating the current density once more by Ampère's law (4.139), we find that the magnetic force density on the plasma must be

$$\frac{1}{4\pi}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}\,, \tag{4.144}$$

and this term must be added to Euler's equation, which now reads

$$\rho\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\vec{\nabla}P + \frac{1}{4\pi}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B} \tag{4.145}$$

in absence of gravitational forces. By means of the identity

$$\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B} = \left(\vec{B}\cdot\vec{\nabla}\right)\vec{B} - \frac{1}{2}\vec{\nabla}\left(\vec{B}^2\right)\,, \tag{4.146}$$

the Lorentz force density acting on the plasma can be cast into the very intuitive form

$$\frac{1}{4\pi}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B} = \frac{1}{4\pi}\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B} - \frac{1}{8\pi}\vec{\nabla}\left(\vec{B}^2\right)\,. \tag{4.147}$$

The first term specifies how $\vec{B}$ changes along $\vec{B}$, i.e. it quantifies the tension of the magnetic field lines, which obviously tend to be as straight as possible. The second term is the gradient of the magnetic energy density and augments the pressure gradient in Euler's equation. We thus find that the magnetic field acts in two ways on the plasma flow: It resists motions that bend and compress the field (Figure 4.6).

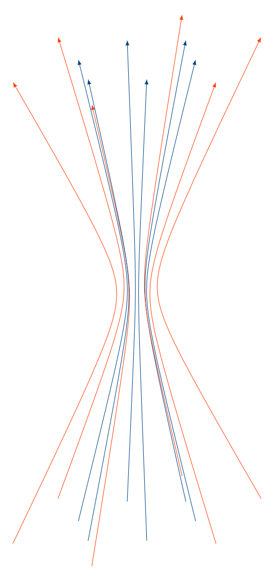

Figure 4.6 A magnetic field exert two kinds of force on a plasma: The field lines tend to straighten due to the $(\vec{B}\cdot\vec{\nabla})\vec{B}$ term, and they tend to reduce the magnetic pressure due to the $\vec{\nabla}(\vec{B}^2)$ term in the magnetic Euler equation.

We have seen in normal, viscous hydrodynamics that Euler's equation can be written in the manifestly conservative form

$$\partial_t (\rho\vec{v}) + \vec{\nabla} \cdot \bar{T} = 0 , \tag{4.148}$$

where the stress-energy tensor

$$\bar{T} = \rho\vec{v} \otimes \vec{v} + P\mathbb{1}_3 + \bar{T}_\mathrm{d} \tag{4.149}$$

occurs. It contains the diffusive contribution T_d, given in (3.143), which reads

$$\bar{T}_\mathrm{d} = -\eta\left[\left(\vec{\nabla} \otimes \vec{v}\right) + \left(\vec{\nabla} \otimes \vec{v}\right)^\top - \frac{2}{3}\vec{\nabla} \cdot \vec{v}\,\mathbb{1}_3\right] - \zeta\vec{\nabla} \cdot \vec{v}\,\mathbb{1}_3 . \tag{4.150}$$

In presence of a magnetic field, the stress-energy tensor must be augmented by a magnetic contribution,

$$\bar{T} \to \bar{T} + \bar{T}_\mathrm{m} , \tag{4.151}$$

whose components are given in (1.111),

$$\bar{T}_\mathrm{m} = -\frac{1}{4\pi}\left(\vec{B} \otimes \vec{B} - \frac{\vec{B}^2}{2}\,\mathbb{1}_3\right) . \tag{4.152}$$

The stress-energy tensor of the ideal fluid (3.51), a the diffusive part (3.143) for the viscous fluid and the contribution by the magnetic field (4.152) are thus simply added.

Together with an equation of state, $P = P(\rho)$, the induction equation (4.141), the continuity equation (4.142) and Euler's equation (4.145) determine both the plasma flow and the evolution of the magnetic field embedded into it. These are two scalar and two vector equations, thus eight equations, for the eight unknowns ρ, P, $\vec{v}$ and $\vec{B}$. If the magnetic field is known, the current follows from Ampère's law (4.139), and the electric field is finally given by (4.136).

4.5.4 Energy and entropy

The evolution equation (3.163) for the specific entropy $\tilde{s}$ per unit mass, which read

$$\rho T\frac{\mathrm{d}\tilde{s}}{\mathrm{d}t} = \vec{\nabla} \cdot \left(\kappa\vec{\nabla}T\right) + \mathrm{Tr}\left(\bar{T}_\mathrm{d}Dv\right) , \tag{4.153}$$

in ordinary, viscous hydrodynamics of neutral fluids, now needs to be augmented by the entropy production through the release of Ohmic heat.

?

Entropy production by which physical process does the new term on the right-hand side of (4.155) describe?

Per unit time, the induction current $\vec{j}'$ in the rest frame of the fluid dissipates the energy

$$\vec{j}' \cdot \vec{E}' = \frac{\vec{j}'^2}{\sigma} \approx \frac{\vec{j}^2}{\sigma} = \frac{c^2}{16\pi^2\sigma}\left(\vec{\nabla} \times \vec{B}\right)^2 , \tag{4.154}$$

where we have used Ohm's law in the first, $\vec{j}' \approx \vec{j}$ in the second and Ampère's law (4.139) in the last steps. The resulting expression must be added to the right-hand side of the entropy equation, giving

$$\rho T\frac{\mathrm{d}\tilde{s}}{\mathrm{d}t} = \vec{\nabla} \cdot \left(\kappa\vec{\nabla}T\right) + \mathrm{Tr}\left(\bar{T}_\mathrm{d}Dv\right) + \frac{c^2}{16\pi^2\sigma}\left(\vec{\nabla} \times \vec{B}\right)^2 . \tag{4.155}$$

If we need to express energy conservation by the specific energy density ε instead of the specific entropy density s, we start from the energy conservation equation of viscous hydrodynamics and augment it in a completely analogous way. First, the energy density of the magnetic field, $\vec{B}^2/(8\pi)$, must be added to the kinetic and thermal energy density of the fluid,

$$\frac{\rho}{2}\vec{v}^2 + \varepsilon \to \frac{\rho}{2}\vec{v}^2 + \varepsilon + \frac{\vec{B}^2}{8\pi} \,. \tag{4.156}$$

Next, the Poynting vector of the electromagnetic field,

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B} \,, \tag{4.157}$$

must be added to the energy current density $\vec{q}$. In ordinary, viscous hydrodynamics, its components were given by (3.51) and (3.151)

$$\vec{q} = \rho\left(\frac{\vec{v}^2}{2} + \tilde{h}\right)\vec{v} - \kappa\vec{\nabla}T - \bar{T}_\mathrm{d}\vec{v} \,. \tag{4.158}$$

Using (4.136) and Ampère's law (4.139) once more, the Poyting vector can be expressed by

$$\vec{S} = \frac{c^2}{16\pi^2\sigma}\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} - \frac{1}{4\pi}\left(\vec{v} \times \vec{B}\right) \times \vec{B} \,, \tag{4.159}$$

which we can rearrange by expanding the vector products into

$$\vec{S} = \frac{c^2}{16\pi^2\sigma}\left[\left(\vec{B} \cdot \vec{\nabla}\right)\vec{B} - \frac{1}{2}\vec{\nabla}\vec{B}^2\right] - \frac{1}{4\pi}\left[\left(\vec{B} \cdot \vec{v}\right)\vec{B} - \vec{B}^2\,\vec{v}\right] \tag{4.160}$$

Thus, the energy current density in a viscous, magnetised plasma has the components

$$\begin{aligned}\vec{q} &= \rho\left(\frac{\vec{v}^2}{2} + \tilde{h}\right)\vec{v} - \kappa\vec{\nabla}T - \bar{T}_\mathrm{d}\vec{v} \\ &- \frac{c^2}{16\pi^2\sigma}\left[\left(\vec{B} \cdot \vec{\nabla}\right)\vec{B} - \frac{1}{2}\vec{\nabla}\vec{B}^2\right] - \frac{1}{4\pi}\left[\left(\vec{B} \cdot \vec{v}\right)\vec{B} - \vec{B}^2\,\vec{v}\right] \,.\end{aligned} \tag{4.161}$$

Each of these terms has an intuitive physical meaning: The first term in parentheses is the transport of kinetic energy and enthalpy with the fluid flow, where the enthalpy appears instead of the kinetic energy to take any pressure-volume work into account that the fluid may have to exert. The following two terms describe energy loss by heat conduction and by viscous friction. The next term in brackets is multiplied with the inverse conductivity and thus disappears if the plasma is ideally conducting, $\sigma \to \infty$. The first term in brackets is the magnetic tension, the second is the gradient of the internal energy of the magnetic field. In the final bracket, the first term quantifies how the magnetic field changes along the flow lines of the fluid, and the final term is the transport of the magnetic energy with the fluid.

4.5.5 Incompressible flows

For incompressible flows with $\vec{\nabla} \cdot \vec{v} = 0$, the magnetohydrodynamic equations simplify somewhat. First, expanding the curl of the vector product in the induction equation (4.141) and using $\vec{\nabla} \cdot \vec{v} = 0$ in addition to $\vec{\nabla} \cdot \vec{B} = 0$, we find

$$\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} = \frac{\partial \vec{B}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{B} = \left(\vec{B} \cdot \vec{\nabla}\right) \vec{v} + \frac{c^2}{4\pi\sigma} \vec{\nabla}^2 \vec{B} , \tag{4.162}$$

and Euler's equation becomes

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\frac{1}{\rho} \vec{\nabla} \left(P + \frac{\vec{B}^2}{8\pi}\right) + \frac{1}{4\pi\rho} \left(\vec{B} \cdot \vec{\nabla}\right) \vec{B} + \nu \vec{\nabla}^2 \vec{v} , \tag{4.163}$$

where $\nu = \eta/\rho$ is the specific viscosity per unit mass. Moreover, the diffusive stress-energy tensor T_d in the energy-conservation equation simplifies to read

$$\bar{T}_\mathrm{d} = -\eta \left[\left(\vec{\nabla} \otimes \vec{v}\right) + \left(\vec{\nabla} \otimes \vec{v}\right)^\top\right] = -2\eta Dv , \tag{4.164}$$

where the symmetrised velocity-gradient tensor Dv from (3.154) was inserted. This enables us to bring the viscous dissipation term in the energy-conservation equation into the simple form

$$\mathrm{Tr}\left[\bar{T}_\mathrm{d}^\top \left(\vec{\nabla} \otimes \vec{v}\right)\right] = \frac{1}{2} \mathrm{Tr}\left(\bar{T}_\mathrm{d}^\top Dv\right) = -\eta \ \mathrm{Tr}\left(Dv^\top Dv\right) . \tag{4.165}$$

4.5.6 Magnetic advection and diffusion

Two terms determine the temporal change of the magnetic field in the induction equation (4.141),

$$\vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) \quad \text{and} \quad \frac{c^2}{4\pi\sigma} \vec{\nabla}^2 \vec{B} . \tag{4.166}$$

The first term, $\vec{\nabla} \times (\vec{v} \times \vec{B})$, determines the transport of the magnetic field with the fluid flow. It is called *advection term*. Its order-of-magnitude is

$$\frac{vB}{L} , \tag{4.167}$$

if L is a typical length scale characterising the plasma flow. The second term, proportional to $\vec{\nabla}^2 \vec{B}$, determines the diffusion of the magnetic field due to the finite conductivity of the plasma. If the conductivity is ideally large, $\sigma \to \infty$, the diffusion coefficient vanishes, showing that magnetic fields cannot move with respect to an ideally conducting plasma.

?

Expand the curl of the vector product $\vec{v} \times \vec{B}$ and verify the meaning of this term noted in the text.

The diffusion term has the order of magnitude

$$\frac{c^2}{4\pi\sigma} \frac{B}{L^2} . \tag{4.168}$$

The order-of-magnitude ratio between the advection and diffusion terms,

$$\frac{\text{advection}}{\text{diffusion}} = \frac{4\pi\sigma}{c^2} \frac{L^2}{B} \frac{vB}{L} = \boxed{\frac{4\pi\sigma v L}{c^2} \equiv \mathcal{R}_\mathrm{M}} , \tag{4.169}$$

is called the *magnetic Reynolds number*. Obviously, the magnetic-field diffusion can be neglected if $\mathcal{R}_\mathrm{M} \gg 1$, and the induction equation simplifies to

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) = 0 \; . \tag{4.170}$$

In absence of diffusion, the magnetic field is said to be "frozen" into the plasma. The physical reason for this is that, if the conductivity is very high, $\sigma \to \infty$, each motion of the magnetic field with respect to the plasma immediately induces strong current densities which counter-act their origin, i.e. the motion of the field. This is a typical case in astrophysical plasmas.

In the opposite limit, $\mathcal{R}_\mathrm{M} \ll 1$, which occurs if the conductivity is small, the induction equation turns into the pure diffusion equation

$$\frac{\partial \vec{B}}{\partial t} = \frac{c^2}{4\pi\sigma} \vec{\nabla}^2 \vec{B} \; . \tag{4.171}$$

Transforming this equation into Fourier space immediately shows that Fourier modes of wave number k solve this equation if their frequency is

$$\omega_\mathrm{diff} = \mathrm{i}\frac{c^2 k^2}{4\pi\sigma} \; . \tag{4.172}$$

This means that magnetic field modes with wavelength $\lambda = 2\pi k^{-1}$ must decay exponentially on the diffusion time scale

$$\tau_\mathrm{diff} = \frac{2\pi}{\mathrm{Im}\,\omega_\mathrm{diff}} \approx 2\sigma \frac{\lambda^2}{c^2} \; , \tag{4.173}$$

which is directly proportional to the conductivity σ: The lower the conductivity is, the faster the magnetic field decays by diffusion. Plasmas in the laboratory are typically characterised by $\mathcal{R}_\mathrm{M} \ll 1$, while astrophysical plasmas typically have $\mathcal{R}_\mathrm{M} \gg 1$.

Problems

1. Specialise the expression (4.144) for the magnetic force density to the case of a magnetic field confined to the *x*-*y* plane or parallel to the *z* axis.

2. Consider a general diffusion equation for a function $f(t, \vec{x})$,

$$\frac{\partial f}{\partial t} = C \vec{\nabla}^2 f \; , \tag{4.174}$$

and show that it is solved by a convolution of an initial function $f_0(\vec{x})$ with a Gaussian. How does the width of this Gaussian evolve in time?

4.6 Generation of Magnetic Fields

In this section, we briefly touch the vast subject of how magnetic fields can be generated. We show how magnetic fields can be generated if electrons and ions are not ideally tightly coupled to each other and derive the modified induction equation (4.185) that now contains a source term given by any misalignment between the gradients of the electron pressure and the particle number density.

The induction equation (4.141) contains no source term: Both terms on its right-hand side, which together determine the time evolution of $\vec{B}$, are linear in $\vec{B}$. The equation can therefore only describe how existing magnetic fields change, but if $\vec{B} = 0$ initially, this remains so. This is a consequence of the assumption that ions and electrons are ideally (tightly) coupled to each other. Should this not be the case, the flows of the electrons and of the ions need to be considered separately, most notably with different velocities $\vec{v}_\mathrm{e}$ and $\vec{v}_\mathrm{i}$. Then, two separate Euler equations must hold for the electrons and the ions,

$$\begin{aligned} n_\mathrm{e} m_\mathrm{e} \frac{\mathrm{d}\vec{v}_\mathrm{e}}{\mathrm{d}t} &= -\vec{\nabla} P_\mathrm{e} - n_\mathrm{e} e \left(\vec{E} + \frac{\vec{v}_\mathrm{e}}{c} \times \vec{B} \right) - n_\mathrm{e} m_\mathrm{e} \vec{\nabla}\Phi \;, \\ n_\mathrm{i} m_\mathrm{i} \frac{\mathrm{d}\vec{v}_\mathrm{i}}{\mathrm{d}t} &= -\vec{\nabla} P_\mathrm{i} + n_\mathrm{i} e \left(\vec{E} + \frac{\vec{v}_\mathrm{i}}{c} \times \vec{B} \right) - n_\mathrm{i} m_\mathrm{i} \vec{\nabla}\Phi \;, \end{aligned} \tag{4.175}$$

which are coupled by common electromagnetic fields $\vec{E}$ and $\vec{B}$ and by the gravitational potential Φ. We divide these equations by $n_\mathrm{e} m_\mathrm{e}$ and $n_\mathrm{i} m_\mathrm{i}$, respectively, and subtract the second from the first to find the evolution of the relative velocity

$$\frac{\mathrm{d}\,(\vec{v}_\mathrm{e} - \vec{v}_\mathrm{i})}{\mathrm{d}t} = -\frac{\vec{\nabla} P_\mathrm{e}}{n_\mathrm{e} m_\mathrm{e}} + \frac{\vec{\nabla} P_\mathrm{i}}{n_\mathrm{i} m_\mathrm{i}} - \frac{e}{m_\mathrm{e}} \left(\vec{E} + \frac{\vec{v}_\mathrm{e}}{c} \times \vec{B} \right) - \frac{e}{m_\mathrm{i}} \left(\vec{E} + \frac{\vec{v}_\mathrm{i}}{c} \times \vec{B} \right) \;. \tag{4.176}$$

Since the ion mass m_i is much larger than the electron mass m_e, but $n_\mathrm{e} = n_\mathrm{i} \equiv n$, equation (4.176) can be approximated by

$$\frac{\mathrm{d}\,(\vec{v}_\mathrm{e} - \vec{v}_\mathrm{i})}{\mathrm{d}t} = -\frac{\vec{\nabla} P_\mathrm{e}}{n m_\mathrm{e}} - \frac{e}{m_\mathrm{e}} \left(\vec{E} + \frac{\vec{v}_\mathrm{e}}{c} \times \vec{B} \right) \;. \tag{4.177}$$

The terms containing the ion mass may be neglected because, if a relative velocity difference is to be maintained, it must be due to the lower inertia and thus the higher mobility of the electrons.

The last equation must be augmented by a phenomenological collision term through which different electron and ion velocities can be justified or produced in the first place. Introducing a collision time τ, we simply write

$$\frac{\mathrm{d}\,(\vec{v}_\mathrm{e} - \vec{v}_\mathrm{i})}{\mathrm{d}t} = -\frac{\vec{\nabla} P_\mathrm{e}}{n m_\mathrm{e}} - \frac{e}{m_\mathrm{e}} \left(\vec{E} + \frac{\vec{v}_\mathrm{e}}{c} \times \vec{B} \right) - \frac{\vec{v}_\mathrm{e} - \vec{v}_\mathrm{i}}{\tau} \;. \tag{4.178}$$

The net current density of the electrons and the ions together is

$$\vec{j} = e n_\mathrm{i} \vec{v}_\mathrm{i} - e n_\mathrm{e} \vec{v}_\mathrm{e} = e n (\vec{v}_\mathrm{i} - \vec{v}_\mathrm{e}) \;, \tag{4.179}$$

where we have implicitly assumed singly-charged ions. This is not a severe restriction at all because whatever the ion charge Z is, we only need to make sure that the plasma is electrically neutral by satisfying (4.1).

For a stationary situation, the relative drift velocity between electrons and ions must be constant,

$$\frac{\mathrm{d}\,(\vec{v}_\mathrm{i} - \vec{v}_\mathrm{e})}{\mathrm{d}t} = 0 \;, \tag{4.180}$$

which implies by (4.179) a constant total electric current density $\vec{j}$. In this situation, we can solve the drift equation (4.178) for the electric field $\vec{E}$ and eliminate the drift velocity by the current density $\vec{j}$ to find

$$\vec{E} = -\frac{\vec{\nabla}P_\mathrm{e}}{en_\mathrm{e}} - \frac{\vec{v}_\mathrm{e}}{c} \times \vec{B} + \frac{m_\mathrm{e}\vec{j}}{ne^2\tau} \ . \tag{4.181}$$

What does this electric field mean? If the last term on the right-hand side was missing, (4.181) would say that an equilibrium situation required an electric field which, when combined with the existing magnetic field, creates a Lorentz force balancing the pressure-gradient force. The phenomenological collision term on the right-hand side adds friction between the electrons and the ions. Seen from the perspective of the electrons, (4.181) now means that the particle collisions hinder the motion of the electrons and thereby enhance the electric field required for equilibrium.

By Faraday's law (1.97), this electric field determines the time evolution of the magnetic field,

$$\frac{\partial\vec{B}}{\partial t} = -c\vec{\nabla}\times\vec{E} = \frac{c}{e}\vec{\nabla}\times\frac{\vec{\nabla}P_\mathrm{e}}{n} + \vec{\nabla}\times\left(\vec{v}_\mathrm{e}\times\vec{B}\right) - \frac{m_\mathrm{e}c}{e^2\tau}\vec{\nabla}\times\frac{\vec{j}}{n} \ . \tag{4.182}$$

Now, since the curl of the pressure gradient vanishes identically, $\vec{\nabla}\times\vec{\nabla}P_\mathrm{e}$, we can re-write the first term on the right-hand side as

$$\vec{\nabla}\times\frac{\vec{\nabla}P_\mathrm{e}}{n} = -\vec{\nabla}P_\mathrm{e}\times\frac{\vec{\nabla}n}{en^2} \ . \tag{4.183}$$

?

In what kind of situations does the vector product $\vec{\nabla}P_\mathrm{e}\times\vec{\nabla}n$ between the pressure and number-density gradients not vanish? Construct examples.

The curl of the current density can be rewritten using Ampère's law, making use of the fundamental assumption of magnetohydrodynamics that displacement currents can be neglected; recall (4.182). We can then further conclude that

$$\vec{\nabla}\times\frac{\vec{j}}{n} = \frac{c}{4\pi n}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) + \vec{j}\times\frac{\vec{\nabla}n}{n^2} \ . \tag{4.184}$$

If the electric current is flowing along the gradient in the number density of the electrons, which is generally a reasonable assumption, the latter term vanishes identically, and we obtain the modified induction equation

$$\frac{\partial\vec{B}}{\partial t} = \frac{c^2}{4\pi\sigma}\vec{\nabla}^2\vec{B} + \vec{\nabla}\times\left(\vec{v}_\mathrm{e}\times\vec{B}\right) - \frac{c}{en^2}\left(\vec{\nabla}P_\mathrm{e}\times\vec{\nabla}n\right) \ , \tag{4.185}$$

where we have made use of the definition

$$\sigma \equiv \frac{ne^2\tau}{m_\mathrm{e}} \tag{4.186}$$

of the conductivity. Compared to (4.141), this new induction equation is augmented by an inhomogeneity in terms of the magnetic field, given by the term on the right-hand side containing the gradient of the number density,

$$-\frac{c}{en^2}\left(\vec{\nabla}P_\mathrm{e}\times\vec{\nabla}n\right) \ , \tag{4.187}$$

which now appears as a source of the magnetic field. It shows that magnetic fields can be created if there is a gradient in the number density of the electrons which is misaligned with the pressure gradient. Mechanisms like this for *creating* magnetic fields are called *battery mechanisms*.

Problems

1. The condition for an inhomogeneity of the form (4.187) to occur in the induction equation can be given in the simple form

$$\vec{\nabla} \times \left(\frac{\vec{\nabla} P}{\rho} \right) \neq 0 \,, \tag{4.188}$$

if the electron pressure and the electron number density are supposed to be proportional to the total gas pressure P and the gas density ρ.

(a) Set up Euler's equation for an ideal fluid in hydrostatic equilibrium, ignoring magnetic forces, but adding the centrifugal force appearing if the fluid is rotating with an angular velocity $\vec{\Omega}$ about the z axis.

(b) Can magnetic fields build up in a rotating object in hydrostatic equilibrium? If so, under which conditions on the rotation? How is the magnetic field oriented that is generated this way?

(c) Suppose the rotation is uniform, but the fluid is chemically inhomogeneous such that the relation between the electron density and the matter density changes. Can magnetic fields be built up now?

4.7 Ambipolar Diffusion

This section discusses what happens if plasma and neutral gas are mixed: The neutral gas moves freely relative to the magnetic field, but remains coupled to the plasma by particle collisions. The first main result is the expression (4.208) for the density of the friction force between the plasma and the neutral gas. Introducing the effect of the friction into the induction equation yields the evolution equation (4.216) for the magnetic field, showing how the friction causes diffusion of the magnetic field.

4.7.1 Velocity-averaged scattering cross section

Suppose we now have a partially ionised medium, which can be seen as a mixture of neutral particles and plasma. As we have discussed before, the magnetic field can then be thought of being "frozen into" the plasma. Collisions between the plasma and neutral particles then create a friction force between the plasma and the neutral fluid, which causes the magnetic field to diffuse with respect to the plasma even if the plasma's conductivity is ideal. This diffusion process is called "ambipolar".

In order to work out this friction force, we first need a cross section σ for the collisions, or, more conveniently, the velocity-averaged cross section $\langle \sigma v \rangle$. We adopt two limiting cases for it, one for small and one for high relative velocities

$$v = \left| \vec{v}_\mathrm{i} - \vec{v}_\mathrm{n} \right| \tag{4.189}$$

during the interaction of the collision partners, i.e. the ions ("i") and the neutral gas particles ("n").

If v is very large, we may approximate the cross section by its geometrical value. If r_i and r_n are the effective radii of the ions and the neutral particles, respectively, we can then replace the cross section by a disk whose radius is the sum of the two radii,

$$\sigma = \pi(r_\mathrm{i} + r_\mathrm{n})^2 \,, \tag{4.190}$$

implying the velocity-averaged cross section

$$\langle \sigma v \rangle = \langle v \rangle \sigma = \langle v \rangle \pi (r_\mathrm{i} + r_\mathrm{n})^2 \,. \tag{4.191}$$

?

What would the cross section for the Coulomb scattering of an electron with an ion be, or between two ions?

If v is sufficiently small, the ion's charge can polarise the neutral particle and thereby enlarge the interaction cross section due to the electromagnetic interaction. While the electric field of an ion with charge Ze is the Coulomb field

$$\vec{E}_\mathrm{i} = \frac{Ze}{r^2}\hat{e}_r \,, \tag{4.192}$$

it appears reasonable to assume that the electric field of the polarised neutral particle is the dipole field

$$\vec{E}_\mathrm{n} = \frac{3(\vec{p}\cdot\hat{e}_r)\hat{e}_r - \vec{p}}{r^3} = -\vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right) \tag{4.193}$$

of the polarised dipole moment $\vec{p}$. We assume that the dipole moment responds linearly to the ion's electric field,

$$\vec{p} = \alpha\vec{E}_\mathrm{i} = \frac{Z\alpha e}{r^2}\hat{e}_r \,, \tag{4.194}$$

with a parameter α quantifying the polarisability of the neutral particles.

This induced dipole field of the neutral particle exerts the force

$$\vec{F} = Ze\vec{E}_\mathrm{n} = -Ze\vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right) = -Z^2e^2\alpha\vec{\nabla}\left(\frac{1}{r^4}\right) \tag{4.195}$$

on the ion, whose potential is evidently

$$V = \frac{Z^2e^2\alpha}{r^4} \,. \tag{4.196}$$

Since this is a central potential, Noether's theorems imply that motion under its influence conserves angular momentum. We can then characterise the motion of the ion past the neutral particle by the minimum separation r_0, the impact parameter b and the velocity v_∞ at infinity. Angular-momentum conservation requires

$$\mu v_\infty b = \mu r_0 v_0 \,, \tag{4.197}$$

where the reduced mass

$$\mu \equiv \frac{m_\mathrm{i} m_\mathrm{n}}{m_\mathrm{i} + m_\mathrm{n}} \tag{4.198}$$

occurs because we are treating a two-body problem. Energy conservation further demands

$$\frac{\mu}{2}v_\infty^2 = \frac{\mu}{2}v_0^2 - \frac{\alpha Z^2e^2}{r_0^4} \,. \tag{4.199}$$

Eliminating now the velocity v_0 at closest approach by angular-momentum conservation (4.197), we obtain a quadratic equation for the squared minimum separation r_0^2,

$$r_0^4 - b^2 r_0^2 + \frac{\alpha Z^2 e^2}{\mu v_\infty^2} = 0 \;, \tag{4.200}$$

which has the two solutions

$$r_{0,\pm}^2 = \frac{b^2}{2} \pm \sqrt{\frac{b^4}{4} - \frac{\alpha Z^2 e^2}{\mu v_\infty^2}} \;. \tag{4.201}$$

Both roots are mathematically possible, but only $r_{0,+}^2$ is physically relevant because in the limiting case of vanishing coupling α, the minimum radius must equal the impact parameter, $r_0 = b$, since the ion is then not scattered at all. The minimum separation r_0 will itself be smallest if the root in (4.201) vanishes and the impact parameter satisfies

$$b_0 = \left(\frac{4\alpha Z^2 e^2}{\mu v_\infty^2}\right)^{1/4} \;. \tag{4.202}$$

Since the force between the ion and the neutral particle decreases very steeply with increasing r, by far the strongest effect occurs for close encounters. Thus, we estimate the cross section as

$$\sigma = \pi b_0^2 = \frac{2\pi Z e}{v_\infty} \sqrt{\frac{\alpha}{\mu}} \;. \tag{4.203}$$

Obviously, the velocity-averaged cross section $\langle \sigma v_\infty \rangle$ is independent of the asymptotic velocity v_∞, and we find

$$\langle \sigma v_\infty \rangle = 2\pi Z e \sqrt{\frac{\alpha}{\mu}} \;. \tag{4.204}$$

4.7.2 Friction force and diffusion coefficient

A single collision between an ion and a neutral particle transfers the momentum

$$\left|\Delta \vec{p}\right| = \mu \left|\vec{v}_\mathrm{i} - \vec{v}_\mathrm{n}\right| \tag{4.205}$$

between the two. Since the scattering rate per volume is $n_\mathrm{i} n_\mathrm{n} \langle \sigma v_\infty \rangle$, the momentum transfer per unit time and volume is

$$\vec{f}_\mathrm{friction} = n_\mathrm{i} n_\mathrm{n} \langle \sigma v_\infty \rangle \mu \left(\vec{v}_\mathrm{i} - \vec{v}_\mathrm{n}\right) \;, \tag{4.206}$$

which corresponds to the spatial density of a friction force. With

$$\rho_\mathrm{i} \rho_\mathrm{n} = n_\mathrm{i} m_\mathrm{i} n_\mathrm{n} m_\mathrm{n} = (m_\mathrm{i} + m_\mathrm{n}) n_\mathrm{i} n_\mathrm{n} \mu \;, \tag{4.207}$$

this can be cast into the form

$$\boxed{\vec{f}_\mathrm{friction} = \gamma \rho_\mathrm{i} \rho_\mathrm{n} \left(\vec{v}_\mathrm{i} - \vec{v}_\mathrm{n}\right)} \;, \tag{4.208}$$

where the friction coefficient

$$\gamma \equiv \frac{\langle \sigma v_\infty \rangle}{m_\mathrm{i} + m_\mathrm{n}} \tag{4.209}$$

appears. In the two limiting cases discussed above, those of very small or very large relative velocities, we find

$$\gamma = \frac{\pi(r_\mathrm{i} + r_\mathrm{n})^2}{m_\mathrm{i} + m_\mathrm{n}} \left|\vec{v}_\mathrm{i} - \vec{v}_\mathrm{n}\right| \quad \text{or}$$
$$\gamma = \frac{2\pi Ze}{m_\mathrm{i} + m_\mathrm{n}} \sqrt{\frac{\alpha}{\mu}} \,. \tag{4.210}$$

A stationary situation can be established if this friction force between the ions and the neutral particles is balanced by the Lorentz force on the net electric current density caused by the motion of the plasma particles with the magnetic field. Since the Lorentz force density is

$$\vec{f}_\mathrm{L} = \frac{\vec{j} \times \vec{B}}{c} = \frac{1}{4\pi}\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} \,, \tag{4.211}$$

where Ampère's law was used in the second step, we find the relation

$$\vec{v}_\mathrm{d} \equiv \vec{v}_\mathrm{i} - \vec{v}_\mathrm{n} = \frac{\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B}}{4\pi\gamma\rho_\mathrm{i}\rho_\mathrm{n}} \tag{4.212}$$

between the drift velocity of the ions relative to the neutral particles by equating the Lorentz force density (4.211) to the friction force density (4.208). In a magnetic field with characteristic length scale L, the drift velocity thus has the order of magnitude

$$v_\mathrm{d} \approx \frac{B^2}{4\pi\gamma\rho_\mathrm{i}\rho_\mathrm{n} L} \,. \tag{4.213}$$

A magnetic field which can be assumed to be "frozen" into the flow of an ideally conducting plasma must satisfy the induction equation

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \left(\vec{B} \times \vec{v}_\mathrm{i}\right) = 0 \tag{4.214}$$

without the diffusion term arising from a finite conductivity. In order to calculate the diffusion of the magnetic field relative to the neutral particles, we transform into the rest frame of the neutral flow, where $\vec{v}_\mathrm{n} = 0$, allowing us to replace $\vec{v}_\mathrm{i}$ by the drift velocity $\vec{v}_\mathrm{d}$. This leads to the remarkable equation

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \left(\vec{B} \times \frac{\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B}}{4\pi\gamma\rho_\mathrm{i}\rho_\mathrm{n}}\right) = 0 \tag{4.215}$$

for the magnetic field in the rest-frame of the neutral gas. Using $\vec{\nabla} \cdot \vec{B} = 0$ and introducing the magnetic pressure $P_B = \vec{B}^2/(8\pi)$, it can be brought into the form

$$\frac{\partial \vec{B}}{\partial t} + \frac{1}{\gamma\rho_\mathrm{i}\rho_\mathrm{n}} \left[\vec{\nabla}\left(\vec{B} \cdot \vec{\nabla} P_B\right) - \vec{\nabla}^2 \left(P_B \vec{B}\right)\right] = 0 \,. \tag{4.216}$$

The second term has the order of magnitude DB, where D corresponds to a diffusion coefficient

$$D \approx \frac{P_B}{\gamma\rho_\mathrm{i}\rho_\mathrm{n}} \,, \tag{4.217}$$

which approximately equals the estimate (4.213) for the drift velocity $\vec{v}_\mathrm{d}$, times the length scale L of the magnetic field.

Problems

1. Verify that (4.216) follows from (4.215).

4.8 Waves in magnetised cold plasmas

This section deals with electromagnetic waves in cold plasmas. Since the magnetic field imprints a preferred direction into the plasma which charges are gyrating around, the dielectric tensor is most conveniently split into contributions parallel, perpendicular and helical to the magnetic field, shown in (4.242). The dispersion relations (4.264) for electromagnetic waves are found to depend on the angle between their propagation direction and the magnetic field. We illustrate in (4.276) that waves polarised along their propagation direction are generally damped. Waves polarised transverse to their propagation direction are found to be eigenstates of the dielectric tensor only if they propagate along the magnetic field, in which case Faraday rotation occurs which is quantified by the rotation measure (4.284).

4.8.1 The dielectric tensor

We now proceed to study the propagation of electromagnetic waves in a magnetised plasma in which random particle motion is negligible, whose temperature is thus low, and which can in this sense be considered as cold. The equation of motion of an electron in such a plasma with embedded magnetic field $\vec{B}_0$ is then exclusively determined by the Lorentz force.

In a magnetised plasma irradiated by electromagnetic radiation, the Lorentz force is caused by the magnetic field embedded into the plasma, now called $\vec{B}_0$ to avoid confusion, together with the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, of the incoming electromagnetic wave. The fields $\vec{E}$ and $\vec{B}$ of the electromagnetic wave are of similar magnitude. For non-relativistic motion of the plasma electrons, the magnetic part of the Lorentz force contributed by the electromagnetic wave can thus be neglected. Therefore, the Lorentz force of the combined fields has the electric part $-e\vec{E}$ of the electromagnetic wave and the magnetic part $-e\vec{\beta} \times \vec{B}_0$ of the magnetic field embedded into the plasma. The equation of motion for a plasma electron is then

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\frac{e}{m}\left(\vec{E} + \vec{\beta} \times \vec{B}_0\right) . \tag{4.218}$$

We now assume that the amplitude of the electric field $\vec{E}$ depends harmonically on time, $\vec{E}(t, \vec{x}) = \vec{E}(\vec{x})\mathrm{e}^{-\mathrm{i}\omega t}$. Likewise, we expand the velocity into an position-dependent amplitude and a harmonic time dependence, $\vec{v}(t, \vec{x}) = \vec{v}(\vec{x})\mathrm{e}^{-\mathrm{i}\omega t}$. Then, each of these monochromatic velocity modes is determined by

$$-\mathrm{i}\omega\vec{v} = -\frac{e}{m}\left(\vec{E} + \vec{\beta} \times \vec{B}_0\right) \tag{4.219}$$

due to the equation of motion (4.218). For the following calculations, it will be enormously helpful to introduce the matrix $\hat{\mathcal{B}}$ with the components

$$\hat{\mathcal{B}}_{ij} := \varepsilon_{ijk}\hat{b}^k , \tag{4.220}$$

where $\hat{b}$ is the unit vector in the direction of the magnetic field, $\vec{B}_0 = B_0\hat{b}$. This allows us to cast the linear system of equations (4.219) into the matrix form

$$\left(\mathrm{i}\omega\mathbb{1}_3 - \frac{eB_0}{mc}\hat{\mathcal{B}}\right)\vec{v} := M\vec{v} = \frac{e}{m}\vec{E} , \tag{4.221}$$

where we have separated the amplitude B_0 of the magnetic field from its unit direction vector $\hat{b}$. Identifying now the non-relativistic Larmor frequency

$$\omega_\mathrm{L} \equiv \frac{eB_0}{mc} , \tag{4.222}$$

see (2.60), we can abbreviate the matrix M implicitly defined in (4.221) by

$$M = \mathrm{i}\omega\mathbb{1}_3 - \omega_\mathrm{L}\hat{\mathcal{B}} = \mathrm{i}\omega\left(\mathbb{1}_3 + \mathrm{i}w_\mathrm{L}\hat{\mathcal{B}}\right) , \tag{4.223}$$

where $w_\mathrm{L} = \omega_\mathrm{L}/\omega$ is the Larmor frequency divided by the frequency of the incoming electromagnetic radiation. As we move on, the identities

$$\det\left(\mathbb{1}_3 + a\hat{\mathcal{B}}\right) = 1 + a^2 , \quad \hat{\mathcal{B}}^2 = \hat{b}\otimes\hat{b} - \mathbb{1}_3 \quad \text{and} \quad \left(\hat{b}\otimes\hat{b}\right)\hat{\mathcal{B}} = 0 \tag{4.224}$$

will turn out to be most convenient. The first of these immediately gives

$$\det M = -\mathrm{i}\omega^3\left(1 - w_\mathrm{L}^2\right) = -\mathrm{i}\omega\left(\omega^2 - \omega_\mathrm{L}^2\right) , \tag{4.225}$$

while the second and the third will greatly simplify inverting the matrix M.

?

Verify the identities (4.224) by direct calculation.

The inverse of M should be a linear combination of the three matrices $\mathbb{1}_3$, $\hat{b}\otimes\hat{b}$, and $\hat{\mathcal{B}}$ we have available here. We thus try the ansatz

$$M^{-1} = A\mathbb{1}_3 + B\hat{b}\otimes\hat{b} + C\hat{\mathcal{B}} , \tag{4.226}$$

which must satisfy

$$\left(A\mathbb{1}_3 + B\hat{b}\otimes\hat{b} + C\hat{\mathcal{B}}\right)\left(\mathrm{i}\omega\mathbb{1}_3 + \omega_\mathrm{L}\hat{\mathcal{B}}\right) = \mathbb{1}_3 . \tag{4.227}$$

Multiplying out the sums in the parentheses, using the identities (4.224) and grouping terms takes us immediately to

$$A\mathrm{i}\omega - C\omega_\mathrm{L} = 1 , \quad B\mathrm{i}\omega + C\omega_\mathrm{L} = 0 \quad \text{and} \quad C\mathrm{i}\omega + A\omega_\mathrm{L} = 0 . \tag{4.228}$$

The sum of the first and the second of these equations implies

$$\mathrm{i}\omega(A + B) = 1 , \tag{4.229}$$

while multiplying the first with $\mathrm{i}\omega_\mathrm{L}$ and the third with ω and adding these two results in

$$C = \frac{w_\mathrm{L}}{\omega\left(1 - w_\mathrm{L}^2\right)} . \tag{4.230}$$

?

Determine the coefficients (A, B, C) from (4.226) in your own, independent calculation. Confirm that the matrix M^{-1} found in (4.232) indeed inverts the matrix M defined in (4.221).

Substituting backwards and identifying $1 - w_\mathrm{L}^2 = \mathrm{i}\omega^{-3}\det M$ from (4.225) gives

$$A = -\frac{\omega^2}{\det M} , \quad B = Aw_\mathrm{L}^2 \quad \text{and} \quad C = -\mathrm{i}Aw_\mathrm{L} . \tag{4.231}$$

Thus, the matrix M is inverted by

$$M^{-1} = \frac{\omega^2}{\det M}\left(-\mathbb{1}_3 + w_\mathrm{L}^2\hat{b}\otimes\hat{b} - \mathrm{i}w_\mathrm{L}\hat{\mathcal{B}}\right) . \tag{4.232}$$

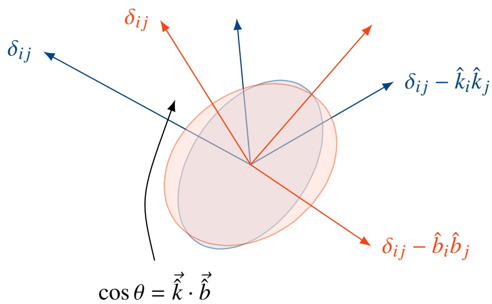

Figure 4.7 Electromagnetic waves in a magnetised plasma experience two preferred directions which are usually not aligned.

With this result, we can invert (4.221) to obtain the velocity

$$\vec{v} = \frac{e}{m} M^{-1} \vec{E} = \frac{e}{\mathrm{i}m\omega\left(1 - w_\mathrm{L}^2\right)} \left(-\mathbb{1}_3 + w_\mathrm{L}^2 \hat{b} \otimes \hat{b} - \mathrm{i}w_\mathrm{L}\hat{\mathcal{B}}\right) . \tag{4.233}$$

This electron velocity gives us the polarised current density $\vec{j}_\mathrm{pol}$, from which we can find the polarisation and the dielectricity tensor.

As usual, the polarisation is given by $4\pi\vec{P} = \vec{D} - \vec{E}$, and its derivative with respect to time is the polarised current density

$$\vec{j}_\mathrm{pol} = -en_\mathrm{e}\vec{v} = \frac{\partial \vec{P}}{\partial t} . \tag{4.234}$$

Extending the assumed harmonic time dependence of the incoming electromagnetic radiation to the polarisation, we adopt $\vec{P}(t, \vec{x}) = \vec{P}(\vec{x})\mathrm{e}^{-\mathrm{i}\omega t}$. Then, the previous equation shows that the electromagnetic fields and the velocity must be related by

$$-\mathrm{i}\omega\vec{P} = -\frac{\mathrm{i}\omega}{4\pi}\left(\vec{D} - \vec{E}\right) = -en_\mathrm{e}\vec{v} . \tag{4.235}$$

Solving the last equation in this chain for the dielectric displacement $\vec{D}$, and substituting the velocity vector from (4.233), we find

$$\vec{D} = \frac{4\pi e n_\mathrm{e}}{\mathrm{i}\omega}\vec{v} + \vec{E} = \vec{E} - \frac{w_\mathrm{p}^2}{1 - w_\mathrm{L}^2}\left(\mathbb{1}_3 - w_\mathrm{L}^2 \hat{b} \otimes \hat{b} - \mathrm{i}w_\mathrm{L}\hat{\mathcal{B}}\right)\vec{E} , \tag{4.236}$$

where we have identified the plasma frequency

$$\omega_\mathrm{p} = \sqrt{\frac{4\pi e^2 n_\mathrm{e}}{m}} \tag{4.237}$$

and introduced its dimension-less form $w_\mathrm{p} = \omega_\mathrm{p}/\omega$. Recalling that the dielectric tensor is defined by the linear relation $\vec{D} = \varepsilon\vec{E}$, we can directly read it off (4.236) and obtain

$$\varepsilon = \left(1 - \frac{w_\mathrm{p}^2}{1 - w_\mathrm{L}^2}\right)\mathbb{1}_3 + \frac{w_\mathrm{p}^2 w_\mathrm{L}^2}{1 - w_\mathrm{L}^2}\,\hat{b} \otimes \hat{b} + \frac{\mathrm{i}w_\mathrm{p}^2 w_\mathrm{L}}{1 - w_\mathrm{L}^2}\,\hat{\mathcal{B}} . \tag{4.238}$$

This result for the dielectric tensor can be further decomposed into components parallel and perpendicular to the magnetic field, and an additional, antisymmetric contribution. To this end, we define the parallel and perpendicular projection operators (Figure 4.7),

$$\pi_\parallel := \hat{b} \otimes \hat{b} \,, \quad \pi_\perp := \mathbb{1}_3 - \hat{b} \otimes \hat{b} \,, \tag{4.239}$$

and contract ε with them, finding

$$\varepsilon_\parallel = \mathrm{Tr}\,(\pi_\parallel \varepsilon) = 1 - w_\mathrm{p}^2 \,, \quad \varepsilon_\perp = \frac{1}{2}\,\mathrm{Tr}\,(\pi_\perp \varepsilon) = 1 - \frac{w_\mathrm{p}^2}{1 - w_\mathrm{L}^2} \,. \tag{4.240}$$

?

Can you confirm the expressions (4.240) for the longitudinal and the transverse dielectricity?

Finally abbreviating the antisymmetric amplitude by

$$g = \frac{w_\mathrm{p}^2 w_\mathrm{L}}{1 - w_\mathrm{L}^2} \,, \tag{4.241}$$

we can bring the dielectricity tensor into the compact form

$$\varepsilon = \left(1 - w_\mathrm{p}^2\right)\pi_\parallel + \left(1 - \frac{w_\mathrm{p}^2}{1 - w_\mathrm{L}^2}\right)\pi_\perp + \mathrm{i}\frac{w_\mathrm{p}^2 w_\mathrm{L}}{1 - w_\mathrm{L}^2}\,\hat{\mathcal{B}} \,. \tag{4.242}$$

4.8.2 Contribution by ions

If ions need to be taken into consideration, the parallel, perpendicular and antisymmetric dielectricity components change according to

$$\begin{aligned} \varepsilon_\perp - 1 &\to (\varepsilon_\perp - 1)_\mathrm{e} + (\varepsilon_\perp - 1)_\mathrm{i} \,, \\ \varepsilon_\parallel - 1 &\to (\varepsilon_\parallel - 1)_\mathrm{e} + (\varepsilon_\parallel - 1)_\mathrm{i} \,, \\ g &\to g_\mathrm{e} + g_\mathrm{i} \,, \end{aligned} \tag{4.243}$$

where the plasma and the Larmor frequencies of the electrons and the ions have to be distinguished. The Larmor frequency of ions with charge Ze is

$$\omega_\mathrm{L,i} = \frac{ZeB_0}{m_\mathrm{i}c} = f\omega_\mathrm{L,e} \ll \omega_\mathrm{L,e} \quad \text{with} \quad f \equiv \frac{Zm_\mathrm{e}}{m_\mathrm{i}} \,, \tag{4.244}$$

much smaller than the Larmor frequency of the electrons. The squared plasma frequency of the ions is

$$\omega_\mathrm{p,i}^2 = \frac{4\pi Z^2 e^2 n_\mathrm{i}}{m_\mathrm{i}} = \frac{4\pi Z e^2 n_\mathrm{e}}{m_\mathrm{i}} = f\omega_\mathrm{p,e}^2 \,, \tag{4.245}$$

where we have used in the second step that the plasma is supposed to be neutral, $Zen_\mathrm{i} = en_\mathrm{e}$. Therefore, the ratio between the plasma frequencies of the ions and the electrons is

$$\frac{\omega_\mathrm{pi}}{\omega_\mathrm{pe}} = f^{1/2} \ll 1 \,, \tag{4.246}$$

also much less than unity.

The contribution of the ions to the longitudinal dielectricity $\varepsilon_{\|}$ thus turns out to be negligible. However, their contribution to the transverse dielectricity $\varepsilon_{\perp}$ and the antisymmetric dielectricity g is not necessarily small. The ratio

$$\left|\left(\frac{w_{\mathrm{p,i}}^2}{1-w_{\mathrm{L,i}}^2}\right)\left(\frac{w_{\mathrm{p,e}}^2}{1-w_{\mathrm{L,e}}^2}\right)^{-1}\right| = f\left|\frac{1-w_{\mathrm{L,e}}^2}{1-f^2w_{\mathrm{L,e}}^2}\right| \tag{4.247}$$

is of order unity if

$$\left|\frac{1-w_{\mathrm{L,e}}^2}{1-f^2w_{\mathrm{L,e}}^2}\right| \approx \frac{1}{f} \tag{4.248}$$

holds. Searching for a solution of this approximate equation, it turns out that we need to choose the negative branch of the modulus on the left-hand side because otherwise w_{L}^2 turned out negative. Then, the approximate equation (4.248) demands

$$-f\left(1-w_{\mathrm{L,e}}^2\right) \approx 1-f^2w_{\mathrm{L,e}}^2\;, \tag{4.249}$$

showing that ions can contribute substantially to the transverse dielectricity if the Larmor frequency of the electrons is much larger than the frequency ω of the electromagnetic radiation,

$$w_{\mathrm{L,e}} = \frac{\omega_{\mathrm{L,e}}}{\omega} \lesssim \frac{1}{\sqrt{f}}\;. \tag{4.250}$$

In a fully analogous way, we can see that the contributions of ions and electrons to g are comparable if the radiation frequency ω is suitably small compared to $\omega_{\mathrm{L,e}}$,

$$w_{\mathrm{L}}^2 \lesssim \frac{1}{2f^2}\;. \tag{4.251}$$

Thus, radiation with sufficiently low frequency, $\omega \lesssim \sqrt{f}\omega_{\mathrm{L,e}}$, will feel the ion contribution to the transverse dielectricity, and it will feel the ion contribution to the antisymmetric dielectricity if $\omega \lesssim \sqrt{2}f\omega_{\mathrm{L,e}}$. For a pure hydrogen plasma, $f \approx 5.6\cdot 10^{-4}$. Ions then become important for the transverse dielectricity for frequencies $\omega \lesssim 0.02\omega_{\mathrm{L,e}}$, and for the antisymmetric dielectricity g if $\omega \lesssim 8\cdot 10^{-4}\omega_{\mathrm{L,e}}$.

4.8.3 Dispersion relations in a cold, magnetised plasma

With the dielectric tensor (4.239) in presence of a magnetic field, we return to the general dispersion relation (4.49)

$$\det\left(\mathbb{1}_3 - \hat{k}\otimes\hat{k} - \frac{\omega^2}{k^2c^2}\hat{\varepsilon}\right) = 0\;. \tag{4.252}$$

Through the dielectric tensor, this dispersion relation contains the projectors $\pi_{\|}$ and $\pi_{\perp}$ parallel and perpendicular to the magnetic field, while the first two terms together are the projector

$$\tilde{\pi}_{\perp} = \mathbb{1}_3 - \hat{k}\otimes\hat{k} \tag{4.253}$$

perpendicular to the propagation direction of the incoming electromagnetic field, which we now mark with a tilde to distinguish it from the projectors relative to the magnetic field. It is advantageous to express $\tilde{\pi}_\perp$ in terms of the projectors $\pi_{||}$ and $\pi_\perp$. To this end, we write

$$\tilde{\pi}_\perp = A\pi_{||} + B\pi_\perp \ , \tag{4.254}$$

apply $\pi_{||}$ and $\pi_\perp$ from the left and take the trace to find

$$A = \mathrm{Tr}\,(\pi_{||}\tilde{\pi}_\perp) \quad \text{and} \quad B = \frac{1}{2}\,\mathrm{Tr}\,(\pi_\perp\tilde{\pi}_\perp) \ . \tag{4.255}$$

Caution Notice (and convince yourself) that $\mathrm{Tr}(\hat{b}\otimes\hat{b}) = \hat{b}^2 = 1$ and

$$\mathrm{Tr}\left[\left(\hat{b}\otimes\hat{b}\right)\left(\hat{k}\otimes\hat{k}\right)\right] = \left(\hat{b}\cdot\hat{k}\right)^2 = \cos^2\theta \ .$$

◀

The remaining traces are easily worked out. To do so, we introduce the angle θ between the direction $\hat{k}$ of the wave propagation and $\hat{b}$ of the magnetic field by $\cos\theta = \hat{k}\cdot\hat{b}$.

This gives

$$A = \mathrm{Tr}\,(\pi_{||}\tilde{\pi}_\perp) = \mathrm{Tr}\left(\hat{b}\otimes\hat{b}\right)\left(\mathbb{1}_3 - \hat{k}\otimes\hat{k}\right) = 1 - \cos^2\theta = \sin^2\theta \tag{4.256}$$

and

$$B = \frac{1}{2}\,\mathrm{Tr}\,(\pi_\perp\tilde{\pi}_\perp) = \frac{1}{2}\,\mathrm{Tr}\left(\mathbb{1}_3 - \hat{b}\otimes\hat{b}\right)\left(\mathbb{1}_3 - \hat{k}\otimes\hat{k}\right) = \frac{1+\cos^2\theta}{2} \ , \tag{4.257}$$

allowing us to write the projector $\tilde{\pi}_\perp$ perpendicular to the wave vector as the linear combination

$$\tilde{\pi}_\perp = \sin^2\theta\,\pi_{||} + \frac{1+\cos^2\theta}{2}\,\pi_\perp \tag{4.258}$$

of the projectors $\pi_\perp$ and $\pi_{||}$ relative to the magnetic field. In much the same way, the parallel projector $\tilde{\pi}_{||}$ relative to the wave vector can be expanded as

$$\tilde{\pi}_{||} = \cos^2\theta\,\pi_{||} + \frac{\sin^2\theta}{2}\,\pi_\perp \ . \tag{4.259}$$

?

Perform the calculations yourself that lead to the representations (4.258) and (4.259) of the perpendicular and parallel projectors with respect to $\hat{k}$.

Introducing (4.258) into (4.252) and abbreviating further $\omega^2/(k^2c^2) =: w^2$ enables us to write the dispersion relation as

$$\det\left\{\left(\sin^2\theta - w^2\varepsilon_{||}\right)\pi_{||} + \left[\frac{1}{2}\left(1+\cos^2\theta\right) - w^2\varepsilon_\perp\right]\pi_\perp - \mathrm{i}w^2 g\hat{\mathcal{B}}\right\} = 0 \ . \tag{4.260}$$

Since the determinant is invariant under orthogonal transformations, we can rotate the coordinate frame such that $\hat{b}$ points into the $\hat{e}_z$ direction such that $\hat{b}^i = \delta^i_3$. Under this choice, which does not affect generality, the dispersion relation (4.260) simplifies to

$$\det\begin{pmatrix} \frac{1}{2}\left(1+\cos^2\theta\right) - w^2\varepsilon_\perp & -\mathrm{i}w^2 g & 0 \\ \mathrm{i}w^2 g & \frac{1}{2}\left(1+\cos^2\theta\right) - w^2\varepsilon_\perp & 0 \\ 0 & 0 & \sin^2\theta - w^2\varepsilon_{||} \end{pmatrix} = 0 \ . \tag{4.261}$$

The solutions can directly be read off this expression now. They are

$$\sin^2\theta = w^2\varepsilon_{||} \quad \text{and} \quad \frac{1+\cos^2\theta}{2} = w^2\left(\varepsilon_\perp \pm g\right) \ , \tag{4.262}$$

showing that the propagation of electromagnetic waves through a magnetised plasma depends on the propagation direction of the waves relative to the magnetic field. Inserting the expressions (4.240) and (4.241) for the parallel, transverse, and antisymmetric dielectricities, the dispersion relations (4.262) can finally be cast into the forms

$$w^2\left(1 - w_\mathrm{p}^2\right) = \sin^2\theta \quad \text{and} \quad w^2\left(1 - \frac{w_\mathrm{p}^2}{1 \pm w_\mathrm{L}}\right) = \frac{1 + \cos^2\theta}{2} \tag{4.263}$$

or, when written in terms of the frequencies,

$$\omega^2 - \omega_\mathrm{p}^2 = k^2c^2\sin^2\theta \quad \text{and} \quad \omega^2\left[1 - \frac{\omega_\mathrm{p}^2}{\omega(\omega \pm \omega_\mathrm{L})}\right] = \frac{1 + \cos^2\theta}{2}\,k^2c^2 \ . \tag{4.264}$$

The first dispersion relation (4.264) is a second-order polynomial with one real root. It applies to waves with an electric field vector polarised parallel to the magnetic field. The second dispersion relation is a fourth-order polynomial with four real roots which belong to two branches split by the Larmor frequency of gyration in the magnetic field (Figure 4.8).

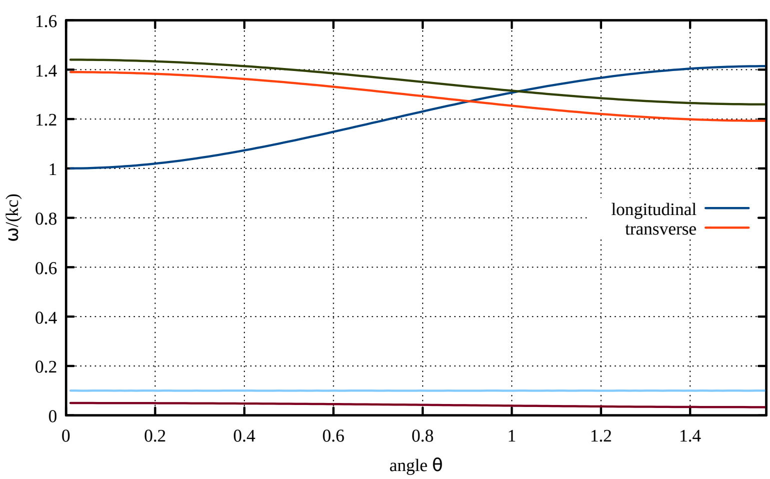

Figure 4.8 Dispersion relations for electromagnetic waves propagating through a magnetised, cold plasma.

4.8.4 Longitudinal and transverse waves

Having derived the dispersion relations for waves polarised relative to the magnetic field, it is now very interesting and instructive to repeat this derivation for waves polarised longitudinally or transversally to their propagation direction. For this purpose, we need to return to the original propagation equation

$$\left(\tilde{\pi}_\perp - w^2\varepsilon\right)\hat{\vec{E}} = 0\,, \tag{4.265}$$

but now expand the projectors $\pi_\|$ and $\pi_\perp$ relative to the magnetic field in terms of the projectors $\tilde{\pi}_\|$ and $\tilde{\pi}_\perp$ parallel and perpendicular to the propagation direction

$\hat{k}$. By a procedure entirely analogous to the reverse expansion of $\tilde{\pi}_\perp$ performed above, we find expressions (4.258) and (4.259) with the projectors $\tilde{\pi}$ and π interchanged,

$$\pi_\perp = \sin^2\theta\,\tilde{\pi}_\parallel + \frac{1+\cos^2\theta}{2}\,\tilde{\pi}_\perp \quad \text{and} \quad \pi_\parallel = \cos^2\theta\,\tilde{\pi}_\parallel + \frac{\sin^2\theta}{2}\,\tilde{\pi}_\perp \tag{4.266}$$

and accordingly decompose the propagation equation (4.265) as

$$\left\{\left(1 - \frac{w^2}{2}\left[\varepsilon_\parallel \sin^2\theta + \varepsilon_\perp\left(1+\cos^2\theta\right)\right]\right)\tilde{\pi}_\perp - w^2\left(\varepsilon_\parallel\cos^2\theta + \varepsilon_\perp\sin^2\theta\right)\tilde{\pi}_\parallel - \mathrm{i}w^2 g\,\hat{\mathcal{B}}\right\}\hat{\vec{E}} = 0\;. \tag{4.267}$$

We turn the coordinate system such that the wave propagates into the z direction, $\hat{k} = \hat{e}_z$, and that the unit vector $\hat{b}$ in field direction falls into the $x - z$ plane,

$$\hat{b} = \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}\;. \tag{4.268}$$

The propagation condition (4.267) can then be cast into the matrix form

$$\begin{pmatrix} A & -\mathrm{i}C\cos\theta & 0 \\ \mathrm{i}C\cos\theta & A & \mathrm{i}C\sin\theta \\ 0 & -\mathrm{i}C\sin\theta & -B \end{pmatrix}\begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0 \tag{4.269}$$

with the abbreviations

$$A = 1 - \frac{w^2}{2}\left[\varepsilon_\parallel\sin^2\theta + \varepsilon_\perp\left(1+\cos^2\theta\right)\right]\;,\quad B = w^2\left(\varepsilon_\parallel\cos^2\theta + \varepsilon_\perp\sin^2\theta\right) \quad \text{and} \quad C = w^2 g\;. \tag{4.270}$$

?

Verify that, for the orientation of the coordinate frame defined in the text, (4.267) turns into (4.269) with the coefficients (4.270).

Longitudinal waves. Consider now a longitudinally polarised wave, $\hat{E}_x = 0 = \hat{E}_y$. According to (4.270), its dispersion relation is $B - \mathrm{i}C\cos\theta = 0$ or

$$\varepsilon_\parallel\cos^2\theta + \varepsilon_\perp\sin^2\theta - \mathrm{i}g\sin\theta = 0\;. \tag{4.271}$$

An imaginary part appears if the wave does not propagate in or against the direction of the magnetic field, $\theta = 0$ or $\theta = \pi$. Then, the frequency is typically complex, and the longitudinally polarised wave is expected to be damped. Let us have a closer look into this. If $\sin\theta = 0$, the dispersion relation simplifies to $\varepsilon_\parallel = 0$ or $\omega = \omega_\mathrm{p}$ according to (4.240). Such waves can propagate if they have the plasma frequency. Suppose now $\sin\theta \neq 0$, but let the Larmor frequency be much smaller than the plasma frequency, allowing us to approximate

$$\varepsilon_\perp \approx 1 - w_\mathrm{p}^2 = \varepsilon_\parallel\;,\quad g \approx w_\mathrm{p}^2 w_\mathrm{L} \tag{4.272}$$

to linear order in w_L. We assume that the frequency ω can then be approximated as $\omega = \omega_\mathrm{p} + \delta\omega$ with a small and possibly complex correction $\delta\omega$. To first order in $\delta\omega$ and ω_L, the dielectricities are

$$\varepsilon_\perp \approx \varepsilon_\parallel = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2} = 1 - \frac{1}{\left(1 + \delta\omega/\omega_\mathrm{p}\right)^2} \approx 2\frac{\delta\omega}{\omega_\mathrm{p}} \tag{4.273}$$

and

$$g = \frac{\omega_\mathrm{p}^2\omega_\mathrm{L}}{\omega^3} = \frac{\omega_\mathrm{L}}{\omega_\mathrm{p}\left(1 + \delta\omega/\omega_\mathrm{p}\right)^3} \approx \frac{\omega_\mathrm{L}}{\omega_\mathrm{p}}\left(1 - 3\frac{\delta\omega}{\omega_\mathrm{p}}\right) . \tag{4.274}$$

With these expressions, the dispersion relation (4.271) turns into

$$2\frac{\delta\omega}{\omega_\mathrm{p}} - \mathrm{i}\frac{\omega_\mathrm{L}}{\omega_\mathrm{p}}\left(1 - 3\frac{\delta\omega}{\omega_\mathrm{p}}\right)\sin\theta = 0 , \tag{4.275}$$

which has the solution

$$\delta\omega \approx \frac{\mathrm{i}}{2}\omega_\mathrm{L}\sin\theta \tag{4.276}$$

to first order in the Larmor frequency ω_L. This imaginary part damps the longitudinally polarised waves.

Transverse waves. Beginning instead with transversally polarised waves, $\hat{E}_z = 0$, we see immediately that field vectors in the x-y plane cannot be eigenvectors of the matrix in (4.269) unless $\sin\theta = 0$: An initially transversally polarised field vector $\hat{E}$ immediately acquires a longitudinal component $\hat{E}_z = \mathrm{i}\hat{E}_y\, C\sin\theta$. For transverse waves to remain transverse, let us therefore assume that the wave vector is aligned with the magnetic field, $\sin\theta = 0$. Then, the propagation condition (4.269) shrinks to

$$\begin{pmatrix} 1 - w^2\varepsilon_\perp & -\mathrm{i}w^2 g \\ \mathrm{i}w^2 g & 1 - w^2\varepsilon_\perp \end{pmatrix}\begin{pmatrix} \hat{E}^1 \\ \hat{E}^2 \end{pmatrix} = 0 . \tag{4.277}$$

Using the second dispersion relation from (4.262) with $\cos\theta = \pm 1$, we can eliminate

$$w^2 = \frac{1}{\varepsilon_\perp \pm g} \tag{4.278}$$

and bring the condition (4.277) into the simple form

$$\frac{g}{\varepsilon_\perp \pm g}\begin{pmatrix} \pm 1 & -\mathrm{i} \\ \mathrm{i} & \pm 1 \end{pmatrix}\begin{pmatrix} \hat{E}^1 \\ \hat{E}^2 \end{pmatrix} = 0 . \tag{4.279}$$

This equation immediately shows that the electric-field components must satisfy

$$\hat{E}^1 = \mp\mathrm{i}\hat{E}^2 \quad \text{or} \quad \left(\hat{E}^1\right)^2 + \left(\hat{E}^2\right)^2 = 1 , \tag{4.280}$$

which characterises circularly polarised light: The two remaining components of the electric field are determined such that the electric-field vector lies on a circle. Since the multiplication with the imaginary unit i corresponds to a rotation by $\pi/2$ in the plane transversal to the propagation direction of the wave, $E^1 = \mp\mathrm{i}E^2$ describe right- and left-circularly polarised light.

4.8.5 Faraday rotation

The preceding discussion of transversal waves propagating parallel to the magnetic field thus leads us to the conclusion that the two branches of the second dispersion relation from (4.263) describe the propagation of left- and right-circularly polarised waves which propagate differently because they obey different dispersion relations. The left-circular polarisation state propagates through the magnetised plasma in a different way than the right-circular polarisation state. Qualitatively, this is not surprising because the motion of the electrons in the magnetised plasma has a fixed sense of rotation: The Lorentz force (1.146) on negatively charged particles requires them to spiral counter-clockwise around the magnetic field lines, seen in the direction of the field lines themselves.

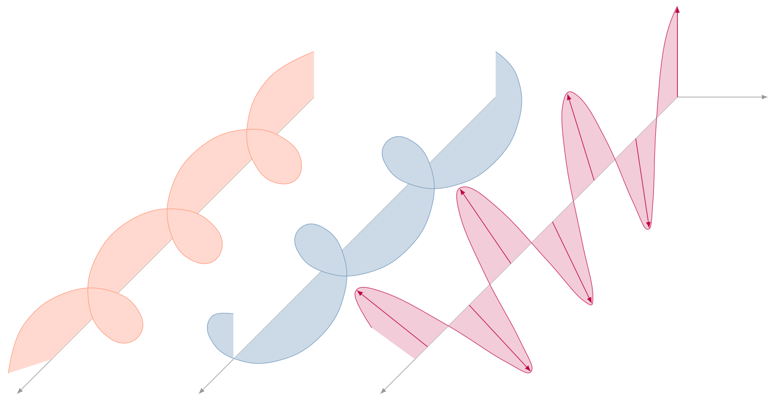

Figure 4.9 The superposition of a left- and a right-circularly polarised wave with a slight phase difference is a plane-polarised wave with a rotating polarisation direction.

Linearly polarised light can be decomposed into left- and right-circularly polarised modes of equal intensity and constant phase difference. If the two circularly-polarised modes now travel through a magnetised plasma at different phase velocities, their phase difference changes as they travel. The polarisation direction of linearly polarised light is then rotated. This effect is called *Faraday rotation* (Figure 4.9). In the high-frequency limit, when the frequency of the electromagnetic wave is much higher than both the Larmor and the plasma frequencies, $\omega \gg \omega_\mathrm{L}$ and $\omega \gg \omega_\mathrm{p}$, we can approximate the second dispersion relation from (4.264) for $\cos\theta = \pm 1$ by

$$k_\pm^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_\mathrm{p}^2}{\omega(\omega \pm \omega_\mathrm{L})}\right] \approx \frac{\omega^2}{c^2}\left[1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\left(1 \mp \frac{\omega_\mathrm{L}}{\omega}\right)\right] \tag{4.281}$$

in a first step for which only $\omega \gg \omega_\mathrm{L}$ is required. The condition $\omega \gg \omega_\mathrm{p}$ allows us to continue by taking a square root to first order Taylor approximation,

$$k_\pm \approx \frac{\omega}{c}\left[1 - \frac{\omega_\mathrm{p}^2}{2\omega^2}\left(1 \mp \frac{\omega_\mathrm{L}}{\omega}\right)\right] = \left(\frac{\omega}{c} - \frac{\omega_\mathrm{p}^2}{2\omega c}\right) \pm \frac{\omega_\mathrm{p}^2\omega_\mathrm{L}}{2\omega^2 c} \equiv k_0 \pm \Delta k\ . \tag{4.282}$$

The first term, k_0, corresponds to a wave vector in the unmagnetised plasma, while the second term, Δk, is responsible for the phase shift between the left- and right-circularly polarised states. This phase shift causes the direction of linear polarisation by an angle

$$\psi = \int \Delta k \mathrm{d}z = \int \frac{\omega_\mathrm{p}^2\omega_\mathrm{L}}{2\omega^2 c}\mathrm{d}z = \int \frac{4\pi e^2 n_\mathrm{e}}{m}\frac{eB}{mc}\frac{\mathrm{d}z}{2\omega^2 c} = \frac{2\pi e^3}{m^2c^2\omega^2}\int \mathrm{d}z\, n_\mathrm{e} B\ , \tag{4.283}$$

where we have inserted the explicit expressions for the plasma and Larmor frequencies. According to this result, in the high-frequency limit, the Faraday rotation is proportional to ω^{-2} or, equivalently, to the squared wave length λ^2. The expression

$$\int \mathrm{d}z\, n_\mathrm{e} B \equiv \mathrm{RM} \tag{4.284}$$

is called the *rotation measure*.

Faraday rotation is an important diagnostic for astrophysical magnetic fields. If a source of linearly polarised light, such as a radio source emitting synchrotron radiation, shines through a magnetised plasma in its foreground, the plane of linear polarisation rotates by different amounts at different frequencies. If the polarisation direction can be measured in two or more frequency bands, the rotation measure can be determined and thus a line-of-sight integral over the magnetic field strength parallel to the line-of-sight, weighted by the electron density. Assumptions then need to be made on the orientation of the magnetic field and on the electron density, under which the field strength can then be estimated.

Problems

1. Return to the dispersion relations (4.264) for electromagnetic waves in a magnetised plasma and consider their limit for very weak fields. Derive approximate dispersion relations for this case and discuss their physical meaning.

2. Since Faraday rotation is only sensitive to the line-of-sight component $B_\parallel$ of the magnetic field, it can only measure a net magnetic field remaining after cancellation of sections along the line-of-sight where the field is pointing towards and away from the observer.

   (a) What is the expectation value of the rotation measure created by a randomly magnetised, homogeneous medium?

   (b) What is the variance of the distribution of rotation measures obtained along many different lines-of-sight through the randomly magnetised medium if the energy density in the magnetic field is U_B?

   (c) If the magnetic field is not completely random, but has a correlation function $\xi_B(r)$ given by

   $$\xi_B(r)\delta_{ij} = \left\langle B_i\,(\vec{x})\,B_j\,(\vec{x}+\vec{r})\right\rangle \;, \tag{4.285}$$

   what correlation function of the rotation measure is observed?

   (d) Suppose a magnetised medium of thickness L can be modelled as composed of subvolumes with a characteristic linear dimension λ, carrying magnetic fields of identical strength B_0 but random orientation. How will the variance of the observed distribution of Faraday rotations depend on λ and L?

4.9 Hydromagnetic Waves

In this section, a linear perturbation analysis of the equations of ideal, inviscid magnetohydrodynamics is carried out, allowing us to identify different modes of hydromagnetic waves. The result of linearising the equations in the perturbations is the general dispersion relation (4.301), which can be specialised to identify the dispersion relation (4.304) for Alfvén waves and to infer the existence of fast and slow hydromagnetic waves with the sound speeds (4.309).

4.9.1 Linearised perturbation equations

Now we consider, in a way very similar to the treatment of sound waves in a neutral fluid, the propagation of waves in a magnetised plasma. For simplicity, we assume that dissipation and heat conduction are unimportant, $\zeta = \eta = \kappa = 0$, and that the conductivity be infinite, $\sigma^{-1} = 0$. Then, the combined equations of this ideal, inviscid specialisation of magnetohydrodynamics read

$$\vec\nabla\cdot\vec B = 0\,,\quad \frac{\partial\vec B}{\partial t} = \vec\nabla\times\left(\vec v\times\vec B\right)\,,\quad \frac{\partial\rho}{\partial t} + \vec\nabla\cdot(\rho\vec v) = 0\,,$$
$$\frac{\partial\vec v}{\partial t} + \left(\vec v\cdot\vec\nabla\right)\vec v = -\frac{\vec\nabla P}{\rho} + \frac{1}{4\pi\rho}\left(\vec\nabla\times\vec B\right)\times\vec B\ : \tag{4.286}$$

Besides mass conservation and Maxwell's equation $\vec\nabla\cdot\vec B = 0$, the magnetic field must satisfy the induction equation and Euler's equation must contain the back-reaction of the magnetic field on the plasma flow. The energy conservation equation is not relevant for the following considerations. We now proceed as usual in a perturbation analysis. We begin by assuming that an equilibrium solution for the magnetic field and the plasma quantities exists,

$$\vec B_0\,,\quad \rho_0\,,\quad P_0\,,\quad \vec v_0 = 0\,, \tag{4.287}$$

which we indicate by the subscript 0. Setting the equilibrium velocity to zero is not a severe restriction because it means that we transform into a coordinate system comoving with the equilibrium plasma flow. This equilibrium solution is then perturbed by small amounts

$$\delta\vec B\,,\quad \delta\rho\,,\quad \delta P\,,\quad \delta\vec v \tag{4.288}$$

in all variables. In absence of dissipation, entropy has to be conserved along flow lines. We further assume isentropic flow, thus $s =$ const. everywhere in the flow.

Then, we proceed by linearising the ideal magnetohydrodynamic equations. For doing so, we insert the perturbed variables $\vec B_0 + \delta\vec B$, $\vec\rho_0 + \delta\rho$, $P_0 + \delta P$ and $\delta\vec v$ into the equations (4.286) and drop all terms of higher than first order in the perturbations. Moreover, we use the fact that the equilibrium quantities $\vec B_0$, ρ_0, P_0 and $\vec v_0$ are in fact solutions of the equations. In this way, we find from the first three equations (4.286)

$$\vec\nabla\cdot\delta\vec B = 0\,,\quad \frac{\partial\delta\vec B}{\partial t} = \vec\nabla\times\left(\delta\vec v\times\vec B_0\right)\,,\quad \frac{\partial\delta\rho}{\partial t} + \vec\nabla\cdot(\rho_0\delta\vec v) = 0\,. \tag{4.289}$$

Suppose further that the fluctuations in the density, $\delta\rho$, are of much smaller scale than any scale on which the equilibrium density ρ_0 might change. This allows us to assume that the equilibrium density is locally constant, $\rho_0 = \text{const.}$, so that the continuity equation can be simplified to read

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \vec{\nabla} \cdot \delta\vec{v} = 0 \; . \tag{4.290}$$

Finally, also to first order in all perturbations, Euler's equation reads

$$\frac{\partial \delta\vec{v}}{\partial t} = -\frac{\vec{\nabla}\delta P}{\rho_0} + \frac{\left(\vec{\nabla} \times \delta\vec{B}\right) \times \vec{B}_0}{4\pi\rho_0} \; , \tag{4.291}$$

again under the assumption that the equilibrium solution is locally homogeneous, thus $\vec{\nabla} P_0 = 0 = \vec{\nabla} \times \vec{B}_0$. The pressure perturbation δP can further be related to the density perturbation $\delta\rho$ by means of the sound speed c_s of the neutral gas, $\delta P = c_\mathrm{s}^2 \delta\rho$.

As usual in a perturbation analysis, we decompose all of the perturbations, jointly represented by Q, into plane waves,

$$\delta Q \propto \mathrm{e}^{\mathrm{i}\left(\vec{k}\cdot\vec{x} - \omega t\right)} \; , \tag{4.292}$$

which turns the ideal magnetohydrodynamic equations into a set of algebraic equations. These are

$$\vec{k} \cdot \delta\vec{B} = 0 \; , \quad \omega\delta\vec{B} + \vec{k} \times \left(\delta\vec{v} \times \vec{B}_0\right) = 0 \tag{4.293}$$

for the magnetic field and

$$\omega\delta\rho - \rho_0 \vec{k} \cdot \delta\vec{v} = 0 \; , \quad \omega\delta\vec{v} - \frac{c_\mathrm{s}^2 \delta\rho}{\rho_0}\vec{k} + \frac{\left(\vec{k} \times \delta\vec{B}\right) \times \vec{B}_0}{4\pi\rho_0} = 0 \tag{4.294}$$

for the plasma velocity.

Without loss of generality, we can now rotate the coordinate frame such that $\vec{k}$ points along the positive x axis and that $\vec{B}_0$ falls into the x-y plane. Further, we denote the angle between $\vec{k}$ and $\vec{B}$ with ψ such that $\vec{k} \cdot \vec{B}_0 = kB_0 \cos\psi$. With this choice of coordinates, the vector products in (4.293) and (4.294) become

$$k \times \left(\delta\vec{v} \times \vec{B}_0\right) = kB_0 \begin{pmatrix} 0 \\ \delta v_y \cos\psi - \delta v_x \sin\psi \\ \delta v_z \cos\psi \end{pmatrix} \tag{4.295}$$

and

$$\left(\vec{k} \times \delta\vec{B}\right) \times \vec{B}_0 = kB_0 \begin{pmatrix} -\delta B_y \sin\psi \\ \delta B_y \cos\psi \\ \delta B_z \cos\psi \end{pmatrix} \tag{4.296}$$

Equations (4.293) for the magnetic field now specialise to

$$\delta B_x = 0 \; , \quad \delta B_y = \frac{kB_0}{\omega}\left(\delta v_x \sin\psi - \delta v_y \cos\psi\right) \; , \quad \delta B_z = -\frac{kB_0}{\omega}\delta v_z \cos\psi \; . \tag{4.297}$$

The continuity equation simplifies to

$$\frac{\delta\rho}{\rho_0} = \frac{k}{\omega}\delta v_x \tag{4.298}$$

and allows us to express the density fluctuation $\delta\rho$ by the velocity fluctuation δv_x. This, then, turns the three components of the Euler equations into

$$\delta v_x - \frac{k^2 c_\mathrm{s}^2}{\omega^2}\delta v_x - \frac{kB_0\delta B_y \sin\psi}{4\pi\rho_0\omega} = 0\;, \quad \delta v_{y,z} + \frac{kB_0\delta B_{y,z}\cos\psi}{4\pi\rho_0\omega} = 0\;. \quad (4.299)$$

?

Confirm the expressions given in (4.295) and (4.296) for the double vector products.

Next, we use the equations (4.297) for the magnetic field to eliminate the field fluctuations $\delta\vec{B}$ from the components (4.299) of the Euler equation. On the way, we introduce two velocities, the phase velocity $c_k = \omega/k$ of the plane-wave perturbations (4.292) and the so-called Alfvén velocity c_A through

$$c_\mathrm{A}^2 = \frac{B_0^2}{4\pi\rho_0}\;. \quad (4.300)$$

These definitions allow writing the three components of the Euler equation in the compact matrix form

$$\begin{pmatrix} c_k^2 - c_\mathrm{s}^2 - c_\mathrm{A}^2\sin^2\psi & c_\mathrm{A}^2\sin\psi\cos\psi & 0 \\ c_\mathrm{A}^2\sin\psi\cos\psi & c_k^2 - c_\mathrm{A}^2\cos^2\psi & 0 \\ 0 & 0 & c_k^2 - c_\mathrm{A}^2\cos^2\psi \end{pmatrix}\delta\vec{v} = 0\;. \quad (4.301)$$

Once the velocity perturbations are found from this propagation equation, the magnetic-field perturbations follow from (4.297), the density fluctuation from the continuity equation (4.298), and pressure fluctuations from the density fluctuations by multiplication with the squared sound speed. Since the density perturbations are caused exclusively by the x component of the velocity perturbations δv_x which point, by construction, into the direction of the wave vector, only longitudinal waves are responsible for the density fluctuations.

4.9.2 Alfvén waves

Let us focus on velocity perturbations in $\hat{e}_z$ direction first. For such perturbations, the propagation equation (4.301) requires the dispersion relation

$$c_k^2 = c_\mathrm{A}^2\cos^2\psi \quad (4.302)$$

or, since the phase velocity c_k of the plane-wave perturbation is $c_k = \omega/k$,

$$\omega = c_\mathrm{A}k\cos\psi = c_\mathrm{A}\vec{k}\cdot\hat{b}\;, \quad (4.303)$$

where the Alfvén speed occurs. Comparing with the ordinary sound speed in a gas, the expression (4.300) for the Alfvén speed is very intuitive: The squared Alfvén speed is the pressure of the magnetic field divided by the plasma density, just as the ordinary sound speed c_s is given by the ratio of the gas pressure and the gas density. The phase velocity of these Alfvén waves is

$$\frac{\omega}{k} = c_\mathrm{A}\cos\psi\;, \quad (4.304)$$

while their group velocity is

$$\frac{\partial\omega}{\partial\vec{k}} = c_\mathrm{A}\hat{b}\;. \quad (4.305)$$

We thus see that the Alfvén waves described by (4.303) are wave-like perturbations of the velocity field and the magnetic field transverse to their propagation direction *and* to the unperturbed magnetic field (Figure 4.10). Their group velocity has an absolute value depending only on the ratio of the magnetic pressure and the matter density. While the phase of the wave propagates along the wave vector $\vec{k}$ into the $\hat{e}_x$ direction, the group velocity points into the direction of the magnetic field. Alfvén waves thus transport physical quantities, for example their energy and momentum, along the magnetic field $\vec{B}$, independent of $\vec{k}$. The phase velocity of the Alfvén waves, $c_\mathrm{A} \cos\psi$, depends on the angle between $\vec{k}$ and $\vec{B}$ and vanishes if $\vec{k}$ is transverse to the magnetic field. Such Alfvén waves have a time-independent phase, and their energy propagates with the Alfvén velocity perpendicular to their wave vector. Alfvén wave packets, for example, with $\vec{k} \perp \vec{B}$ would propagate along $\vec{B}$, without changing their phase. If $\vec{k}$ and $\vec{B}$ are aligned, phase and group velocity become equal and point into the same direction.

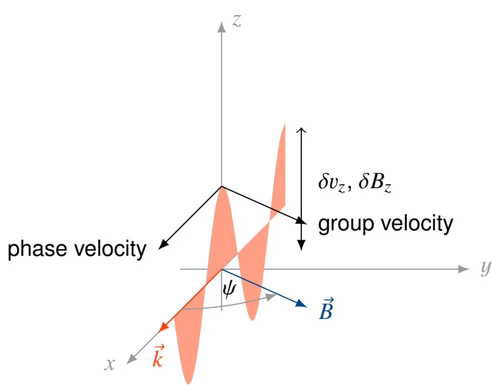

Figure 4.10 Alfvén waves are perturbations of the velocity field and the magnetic field perpendicular to their propagation direction $\vec{k}$ and the magnetic field $\vec{B}$.

Since the velocity perturbations δv_x and δv_y vanish for pure Alfvén waves, no density perturbations are associated with them, and the only component of the magnetic-field perturbation is

$$\delta B_z = -B_0 \cos\psi \frac{\delta v_z}{c_k} . \tag{4.306}$$

The magnetic-field perturbation associated with Alfvén waves is thus antiparallel to the velocity perturbation, transverse to both the wave vector $\vec{k}$ and the magnetic field $\vec{B}_0$, and its amplitude is proportional to the component of the unperturbed magnetic field in the direction of the wave vector.

4.9.3 Slow and fast hydro-magnetic waves

Let us now consider waves described by the x and y components of the propagation equation (4.301),

$$\begin{pmatrix} c_k^2 - c_\mathrm{s}^2 - c_\mathrm{A}^2 \sin^2\psi & c_\mathrm{A}^2 \sin\psi\cos\psi \\ c_\mathrm{A}^2 \sin\psi\cos\psi & c_k^2 - c_\mathrm{A}^2 \cos^2\psi \end{pmatrix} \begin{pmatrix} \delta v_x \\ \delta v_y \end{pmatrix} = 0 . \tag{4.307}$$

The dispersion relation is found requiring that the determinant of the coefficient matrix in this equation vanish, which gives a quadratic equation in the phase velocity c_k,

$$c_k^4 - c_k^2\left(c_{\mathrm{A}}^2 + c_{\mathrm{s}}^2\right) + c_{\mathrm{A}}^2 c_{\mathrm{s}}^2 \cos^2\psi = 0\,. \tag{4.308}$$

Its solutions are

$$c_{k,\pm}^2 = \frac{1}{2}\left[\left(c_{\mathrm{A}}^2 + c_{\mathrm{s}}^2\right) \pm \sqrt{\left(c_{\mathrm{A}}^2 + c_{\mathrm{s}}^2\right)^2 - 4c_{\mathrm{A}}^2 c_{\mathrm{s}}^2 \cos^2\psi}\,\right]\,. \tag{4.309}$$

Thus, a fast and a slow wave mode are possible (Figure 4.11). We analyse their modes in the special cases when the wave vector $\vec{k}$ is either aligned with the magnetic field $\vec{B}_0$, or transverse to it.

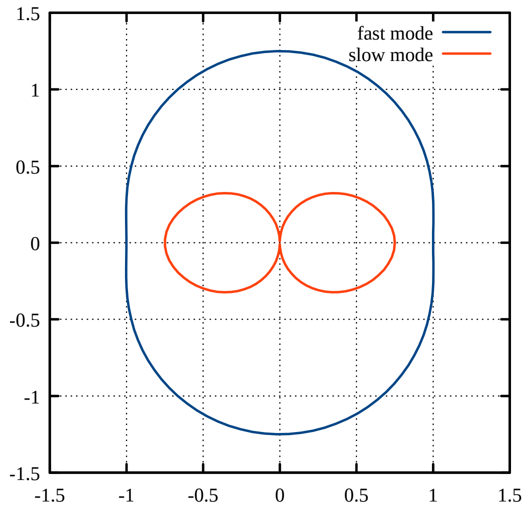

Figure 4.11 Polar velocity diagram of the fast and slow hydromagnetic modes. The polar angle is the angle ψ between the wave vector $\vec{k}$ and the magnetic field $\vec{B}$. The sound and Alfvén speeds are arbitrarily set to $c_{\mathrm{s}} = 1$ and $c_{\mathrm{A}} = 0.75$.

Let us begin with the case $\psi = 0$, when the perturbation propagates with or against the magnetic field. Then, $\cos\psi = 1$ and the dispersion relation becomes

$$c_k^2 = \frac{\omega^2}{k^2} = \frac{1}{2}\left(c_{\mathrm{s}}^2 + c_{\mathrm{A}}^2 \pm \left|c_{\mathrm{s}}^2 - c_{\mathrm{A}}^2\right|\right) = \begin{cases} c_{\mathrm{s}}^2 & \text{or} \\ c_{\mathrm{A}}^2 & \end{cases}\,. \tag{4.310}$$

Accordingly, the fast wave propagates with the faster of the sound and the Alfvén velocities, the slow wave with the slower of these two. The propagation condition (4.301) shows that the wave travelling with the sound speed, the so-called acoustic mode, must have $\delta v_y = 0$ and is therefore logitudinal, while the wave travelling with the Alfvén speed, called the Alfvénic mode, is transversal since $\delta v_x = 0$. The acoustic mode creates density perturbations according to the continuity equation (4.298) while the Alfvénic mode does not because the density perturbations are proportional to δv_x. Similarly, the Alfvénic mode creates a transverse magnetic-field perturbation according to (4.297) while the acoustic mode has no magnetic-field perturbation associated.

For waves perpendicular to the magnetic field, $\vec{B} \perp \vec{k}$ and $\cos\psi = 0$, the phase velocities are

$$c_{k,\pm}^2 = \frac{1}{2}\left(c_\mathrm{s}^2 + c_\mathrm{A}^2 \pm c_\mathrm{s}^2 + c_\mathrm{A}^2\right) = \begin{cases} c_\mathrm{s}^2 + c_\mathrm{A}^2 & \text{or} \\ 0 \end{cases} \tag{4.311}$$

for the fast and the slow hydromagnetic waves. As for the Alfvén waves themselves, the phase velocity of the slow hydromagnetic waves then drops to zero. For $\psi = \pi/2$, the propagation condition (4.301) shows that the fast wave must be longitudinal while the slow wave must be transversal. Then, the fast wave creates density fluctuations and transversal magnetic-field perturbations as shown by (4.297), while the slow mode creates neither of them.

This concludes our brief introduction into the very rich field of magneto-hydrodynamics. Even neglecting any thermal motion of the plasma particles, viscosity or gravity, we found an interesting collection of phenomena, of which the Faraday rotation, the Alfvén waves, and the occurrence of the fast and the slow hydromagnetic waves were the most important.

Problems

1. Return to the ideal magneto-hydrodynamic equations (4.286), add a gravitational field, and assume a static, planar system infinitely extended in the x-y plane. Let the magnetic field be oriented parallel to the plane. For simplicity, assume further that the fluid is isothermal and that the ratio of the magnetic to the thermal pressure is constant.

   (a) Derive and solve an equation for the pressure as a function of distance z above the plane.

   (b) How is the magnetic field structured above the plane?

Suggested further reading: [2, 13, 15, 16, 18]