Chapter 3

Hydrodynamics

3.1 The equations of ideal hydrodynamics

In this section, the equations of ideal hydrodynamics are derived under the central assumption that the mean-free path for the particles of a fluid is infinitely small compared to all other relevant length scales. Starting point of the derivation is the Boltzmann equation from kinetic theory, moments of which are formed in a relativistically invariant way to show that the ideal hydrodynamical equations can be expressed as four-divergences of the matter-current density and of the energy-momentum tensor. The corresponding equations (3.33) are the first main result. These relativistically invariant or covariant equations are then reformulated in three-dimensional form, leading to the set of three equations (3.61) for ideal hydrodynamics: One each for the conservation of mass, momentum, and energy.

3.1.1 Particle current density and energy-momentum tensor

Even though the one-particle phase-space distribution function $f(\vec{x}, \vec{p}, t)$ is defined such that its integral over momentum space,

$$\int \mathrm{d}^3 p\, f\,(t, \vec{x}, \vec{p}\,) = n\,(t, \vec{x}\,) \tag{3.1}$$

is the spatial number density of particles, it is useful for more general considerations to derive an integral measure in momentum space that allows the construction of relativistically invariant or covariant quantities. In order to do so, let us expand the six-dimensional phase space to an eight-dimensional, extended phase space by adding time and energy as dimensions. This extended phase space is then spanned by the position and momentum four-vectors, (x^μ, p^μ), instead by their three-dimensional analogs, $(\vec{x}, \vec{p}\,)$. We denote the phase-space density in this extended phase space by $\tilde{f}(x^\mu, p^\mu)$.

Since the four components of the energy-momentum four-vector p^μ are related by the relativistic energy-momentum relation (1.66), real particles must be confined to a subspace of the extended phase space identified by the condition

$$\left(p^0\right)^2 = \vec{p}^{\,2} + m^2 c^2 \;, \tag{3.2}$$

Caution The delta distribution in (3.3) ensures that the particle system is on the hypersurface in four-dimensional momentum space defined by the energy-momentum relation (3.2). In quantum field theory, this hypersurface is called the energy shell, and the delta distribution ensures that the system is "on shell". ◀

and the condition that their total energy be positive semi-definite, $p^0 \geq 0$. At a fixed time $ct = x^0$, we must thus be able to return to the phase-space distribution function $f(t, \vec{x}, \vec{p})$ by integrating

$$\int \mathrm{d}p^0 \, \tilde{f}(x^\mu, p^\mu) \, \delta_\mathrm{D}\left[\left(p^0\right)^2 - \vec{p}^{\,2} - m^2c^2\right] \Theta\left(p^0\right) = f(t, \vec{x}, \vec{p}) \ , \tag{3.3}$$

where the Heaviside step function $\Theta(p^0)$ ensures that the energy is non-negative.

We now use property

$$\delta_\mathrm{D}\left[g(x)\right] = \sum_i \frac{1}{|g'(x_i)|} \delta_\mathrm{D}(x_i) \tag{3.4}$$

of the Dirac delta distribution, where the sum extends over all roots x_i of $g(x)$ in the relevant domain. In the case of (3.3), $g(x)$ represents the relativistic energy-momentum relation. It has two roots in total, one of them positive, hence

$$\delta_\mathrm{D}\left[\left(p^0\right)^2 - \vec{p}^{\,2} - m^2c^2\right] = \frac{1}{2p^0} \delta_\mathrm{D}\left(p^0 - p^0_E\right) \ , \tag{3.5}$$

where p^0_E on the right-hand side is related to the particle energy by $cp^0_E = E$. Returning with this result to the integral in (3.3), we see that we can write

$$\begin{aligned} &\int \mathrm{d}^4p \, \tilde{f}(x^\mu, p^\mu) \, \delta_\mathrm{D}\left[\left(p^0\right)^2 - \vec{p}^{\,2} - m^2c^2\right] \Theta\left(p^0\right) \\ &\qquad = \frac{c}{2} \int \frac{\mathrm{d}^3p}{E} \tilde{f}\left(x^\mu, p^0 = p^0_E, \vec{p}\right) \ . \end{aligned} \tag{3.6}$$

A further integration over d^4x must return the total number of particles,

$$\begin{aligned} N &= \int \mathrm{d}^4x \mathrm{d}^4p \, \tilde{f}(x^\mu, p^\mu) \, \delta_\mathrm{D}\left[\left(p^0\right)^2 - \vec{p}^{\,2} - m^2c^2\right] \Theta\left(p^0\right) \\ &= c \int \mathrm{d}^4x \int \frac{\mathrm{d}^3p}{2E} \tilde{f}\left(x^\mu, p^0 = p^0_E, \vec{p}\right) \end{aligned} \tag{3.7}$$

which must be Lorentz invariant. The four-dimensional volume elements d^4x and d^4p are both relativistically invariant because Lorentz transforms have unit determinant. Since the Dirac-delta distribution and the Heaviside step function in (3.7) are manifestly Lorentz invariant, we conclude that the distribution function $\tilde{f}$ in the extended phase-space must be Lorentz invariant as well. The second equality in (3.7) then shows that d^3p/E is a Lorentz-invariant integral measure for integrations over three-dimensional momentum space. The one-particle distribution function $\tilde{f}(x^\mu, p^\mu)$ in extended phase space, constrained by the condition $p^0 = p^0_E = E/c$, can be identified with the distribution function $f(t, \vec{x}, \vec{p})$ in ordinary phase space, which is therefore also a Lorentz invariant.

Armed with this important insight, we now define two Lorentz-covariant quantities, a four-vector

$$J^\alpha(t, \vec{x}) := c \int \frac{\mathrm{d}^3p}{E} f(t, \vec{x}, \vec{p}) \ p^\alpha \tag{3.8}$$

and a rank-2 tensor

$$T^{\alpha\beta}(t, \vec{x}) := c^2 \int \frac{\mathrm{d}^3p}{E} f(t, \vec{x}, \vec{p}) \ p^\alpha p^\beta \ . \tag{3.9}$$

With $p^0 = E/c$ and the further relations $p^i = \gamma m v^i = E v^i / c^2$, we can write the components of J^α as

$$J^0(t,\vec{x}) = n(t,\vec{x}) \ , \quad J^i = \frac{1}{c}\int \mathrm{d}^3p\, f(t,\vec{x},\vec{p})\ \dot{x}^i = \frac{n(t,\vec{x})\left\langle \dot{x}^i \right\rangle}{c} \ , \tag{3.10}$$

where we have used in the final step that arbitrary properties Q of the system considered can be averaged over momenta by the operation

$$\langle Q\rangle(t,\vec{x}) = \frac{\int \mathrm{d}^3p\, Q\, f(t,\vec{x},\vec{p})}{\int \mathrm{d}^3p\, f(t,\vec{x},\vec{p})} = \frac{\int \mathrm{d}^3p\, Q\, f(t,\vec{x},\vec{p})}{n(t,\vec{x})} \ . \tag{3.11}$$

The quantity $\langle \dot{x}^i \rangle$ introduced in (3.10) above is therefore the i component of the velocity averaged over all particles near position $\vec{x}$ at time t. We denote this mean velocity by

$$\vec{v} = \vec{v}(t,\vec{x}) = \left\langle \dot{\vec{x}} \right\rangle (t,\vec{x}) \tag{3.12}$$

and write the four-vector J^α as

$$J^\alpha = \frac{n(t,\vec{x})}{c}\begin{pmatrix} c \\ \vec{v} \end{pmatrix} \ . \tag{3.13}$$

It characterises the particle current density.

Turning now to the tensor components $T^{\alpha\beta}$, we find by using $p^0 = E/c = \gamma m c$ and $p^i = \gamma m \dot{x}^i$ that

$$T^{00} = mc^2 \int \mathrm{d}^3p\, f(t,\vec{x},\vec{p})\ \gamma = mn(t,\vec{x})\, c^2 \langle\gamma\rangle = \rho(t,\vec{x})\, c^2\langle\gamma\rangle \ , \tag{3.14}$$

where the mass density $\rho(t,\vec{x}) = mn(t,\vec{x})$ was identified, further

$$T^{0i} = \rho(t,\vec{x})\, c\left\langle \gamma\dot{x}^i \right\rangle \quad \text{and} \quad T^{ij} = \rho(t,\vec{x})\left\langle \gamma\dot{x}^i\dot{x}^j \right\rangle \ . \tag{3.15}$$

Their meaning becomes perhaps most evident in the non-relativistic limit. Then, we can Taylor-expand the Lorentz factor γ to lowest order,

$$\gamma \approx 1 + \frac{\beta^2}{2} \ , \quad \langle\gamma\rangle \approx 1 + \frac{1}{2c^2}\left\langle \dot{\vec{x}}^2 \right\rangle \ , \tag{3.16}$$

and the time-time element T^{00} turns into

$$T^{00} \approx \rho c^2 + \frac{\rho}{2}\left\langle \dot{\vec{x}}^2 \right\rangle \ , \tag{3.17}$$

which is the sum of the rest-mass and the kinetic energy densities of the particle ensemble near position $\vec{x}$ at time t. In this way, the tensor $T^{\alpha\beta}$ turns out to be the energy-momentum tensor of the ensemble.

To third order in v/c, we can approximate the time-space components of the energy-momentum tensor by

$$T^{0i} \approx \rho c v^i + \frac{\rho}{2c}\left\langle \dot{\vec{x}}^2\dot{x}^i \right\rangle \ , \quad T^{ij} \approx \rho\left\langle \dot{x}^i\dot{x}^j \right\rangle \ . \tag{3.18}$$

The first term in T^{0i} is the rest-energy current density, while the expression

$$\frac{\rho}{2}\left\langle \dot{\vec{x}}^2\dot{x}^i \right\rangle =: q^i \tag{3.19}$$

in the second term is the mean flow of kinetic energy.

3.1.2 Collisional invariants and the fluid approximation

We now return to the Boltzmann equation (1.155) and exclude external, macroscopic forces for now. This allows us to set $\dot{\vec{p}} = 0$ and write

$$\partial_t f(t, \vec{x}, \vec{p}) + \dot{\vec{x}} \cdot \vec{\nabla} f(t, \vec{x}, \vec{p}) = C[f] \,. \tag{3.20}$$

Our next concern is the collision term on the right-hand side, which is yet unspecified.

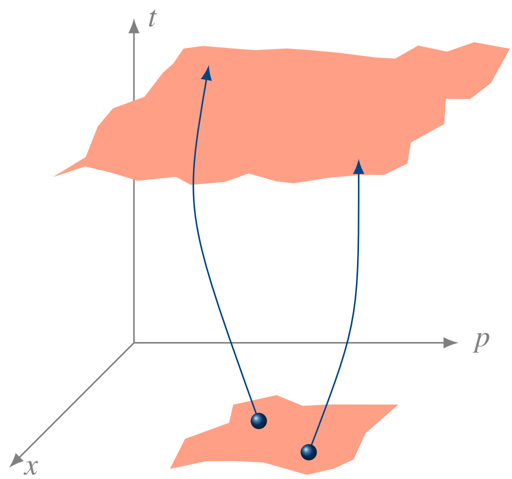

Figure 3.1 Illustration of Liouville's theorem: Trajectories of classical particles are not lost in phase space.

Recall how Boltzmann's equation was derived earlier from Liouville's equation (cf. Figure 3.1). We closed the BBGKY hierarchy by the assumption that the two-particle distribution function could be factorised into one-particle contributions. In other words, collisions between fluid particles were restricted to two-body collisions of otherwise independent particles. We can make substantial progress now by limiting our consideration to collisional invariants. These are defined to be quantities whose sum is conserved in each of these two-body collisions. If the particles can be treated as unstructured, solid bodies without internal degrees of freedom, then the particle number, their total energy and momentum can be considered conserved. Summing over many particles undergoing many collisions, none of these collisional invariants can be changed. We can thus expect that the integrals

$$\int \mathrm{d}^3 p \, C[f] \quad \text{and} \quad \int \mathrm{d}^3 p \, C[f] p^\mu \tag{3.21}$$

must vanish if their integration domains in momentum-space are chosen such that many collisions are contained. To make this possible is the essential motivation for the basic assumption underlying hydrodynamics.

A fluid in the sense of hydrodynamics is an ensemble of many particles whose mean-free path λ is very short compared to all other relevant length scales. Let the overall scale of the system be L, and the scale on which the system's

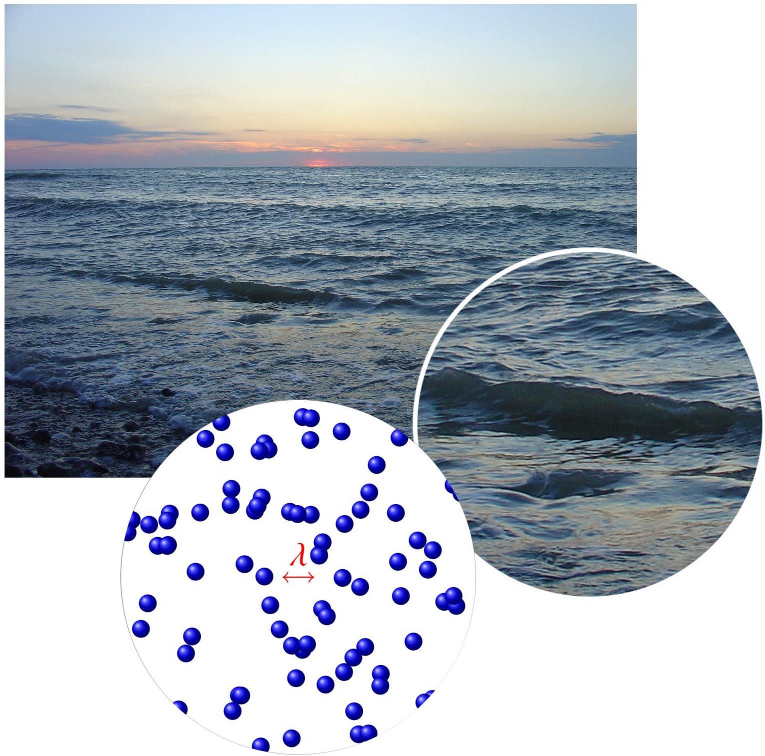

Figure 3.2 Illustration of the fundamental assumption of hydrodynamics: Collections of particles can be treated as a fluid if their mean-free path λ is very much smaller than the typical scale l on which macroscopic properties change, which is in turn much smaller than the overall scale L of the system.

macroscopic physical properties are to be determined be l. Then, for the system to be a fluid, it must be possible to establish the hierarchy of scales

$$\lambda \ll l \ll L\,. \tag{3.22}$$

A swimming pool sets a good example (see also Figure 3.2). For the overall scale, we can take the smallest of its three dimensions length, width and depth, which will be of the order of a metre. If we want to describe the flow of the water in the pool, we need to know its physical properties, such as its local flow velocity, on a length scale of perhaps a millimetre. Under normal conditions, a cubic millimetre of water will weigh 10^{-3} g. Since the mass of a single water molecule is 18 atomic mass units or $3 \cdot 10^{-23}$ g, there are $\sim 3 \cdot 10^{19}$ water molecules in each cubic millimetre, with a mean inter-particle separation of $\sim 3 \cdot 10^{-8}$ cm. The mean-free path is certainly smaller than this, so the hydrodynamical conditions are clearly satisfied very comfortably.

Given this fundamental assumption underlying hydrodynamics, we may safely assert that even a small spatial subvolume of the fluid will contain very many particles. They undergo frequent two-body collisions, in each of which five collisional invariants are conserved: the total particle number, the energy and the momentum. Any individual two-particle collision may or may not change the number of particles in a given phase-space cell. Averaging over an increasing number of collisions, however, the net change in the number of particles, their energies and momenta will decrease since all of these quantities must be conserved. The fundamental assumption of hydrodynamics assures that an average over very many collisions is possible even if the volume is small over which the average is extended.

We can thus conclude that, by the assumption (3.22) defining a fluid, the five integrals

$$\int \mathrm{d}^3p\, C[f] \quad \text{and} \quad \int \mathrm{d}^3p\, C[f]p^\mu \tag{3.23}$$

over the collision term all vanish.

We now return to the force-free Boltzmann equation (3.20) and take its lowest-order moments by carrying out the integrals given in (3.23). The lowest-order moment is

$$\partial_t n\,(t,\vec{x}) + \int \mathrm{d}^3p\, \dot{\vec{x}} \cdot \vec{\nabla} f(t,\vec{x},\vec{p}\,) = 0\ . \tag{3.24}$$

Since $\vec{v}$ and $\vec{x}$ are independent, the spatial gradient applied to $f(t,\vec{x},\vec{p}\,)$ can be pulled out of the integral, giving

$$\partial_t n\,(t,\vec{x}) + \vec{\nabla} \cdot \int \mathrm{d}^3p\, \dot{\vec{x}} f\,(t,\vec{x},\vec{p}\,) = 0\ . \tag{3.25}$$

Comparing this equation with (3.10), we see that we can rewrite it in terms of the four-vector J^α for the particle current density in the very simple, manifestly covariant and Lorentz-invariant form

$$\partial_\alpha J^\alpha = 0\ . \tag{3.26}$$

Next, we form the higher order moments of the force-free Boltzmann equation. This means that we multiply it with p^μ and integrate over d^3p. Beginning with p^0, we first find

$$\partial_t \int \mathrm{d}^3p\, f\,(t,\vec{x},\vec{p}\,)\ p^0 + \partial_i \int \mathrm{d}^3p\, f\,(t,\vec{x},\vec{p}\,)\ \dot{x}^i p^0 = 0\ . \tag{3.27}$$

Recalling $p^0 = E/c$ and $\dot{x}^i = p^i c^2/E$, further using $\partial_t = c\partial_0$, we can bring this equation into the form

$$c^2\partial_0 \int \frac{\mathrm{d}^3p}{E} f\,(t,\vec{x},\vec{p}\,)\ p^0 p^0 + c^2\partial_i \int \frac{\mathrm{d}^3p}{E} f\,(t,\vec{x},\vec{p}\,)\ p^0 p^i = 0\ . \tag{3.28}$$

Here, we can identify the time-time and time-space components of the energy-momentum tensor defined in (3.9) and bring (3.28) into the covariant form

$$\partial_\mu T^{0\mu} = 0\ . \tag{3.29}$$

Finally, we multiply the force-free Boltzmann equation with p^j to obtain

$$\partial_t \int \mathrm{d}^3p\, f\,(t,\vec{x},\vec{p}\,)\ p^j + \partial_i \int \mathrm{d}^3p\, f\,(t,\vec{x},\vec{p}\,)\ \dot{x}^i p^j = 0\ . \tag{3.30}$$

Again, we insert a factor $1 = cp^0/E$ into the first term and use $\dot{x}^i = p^i c^2/E$ in the second to write this equation as

$$c^2\partial_0 \int \frac{\mathrm{d}^3 p}{E} f(t, \vec{x}, \vec{p})\ p^0 p^j + c^2\partial_i \int \frac{\mathrm{d}^3 p}{E} f(t, \vec{x}, \vec{p})\ p^i p^j = 0\ , \tag{3.31}$$

which can be summarised as

$$\partial_\mu T^{j\mu} = 0\ . \tag{3.32}$$

We thus arrive at the very important and intuitive result that, under the fundamental assumption of hydrodynamics, the zeroth- and first-order moments of the force-free Boltzmann equation can be written as

$$\boxed{\partial_\mu J^\mu = 0\ , \quad \partial_\mu T^{\mu\nu} = 0}\ , \tag{3.33}$$

with the four-vector J^μ of the particle-current density and the energy-momentum tensor $T^{\mu\nu}$ of the particle ensemble. These five equations express the conservation of particles, energy and momentum and can already be seen as one form of the hydrodynamical equations.

Recall the assumptions their derivation was based upon. Besides the fundamental assumption (3.22) of hydrodynamics, we made use of five collisional invariants to argue that the momentum-space integrals over the collision term $C[f]$ should vanish. These were the total particle number, their energies and momenta. If any of these assumptions is violated, the conservation equations (3.33) cannot hold any longer. For example, the particle number may change in collisions if particles combine to form molecules. The (kinetic) energy need not be conserved if internal degrees of freedom in the particles can be excited in collisions. Under such circumstances, one needs to return to the collisional Boltzmann equation and work out the collision term explicitly.

The manifestly Lorentz-covariant equations (3.33) can easily be ported into General Relativity. We simply need to replace the partial by covariant derivatives,

$$\nabla_\mu J^\mu = 0\ , \quad \nabla_\mu T^{\mu\nu} = 0 \tag{3.34}$$

to find the fundamental equations of generally-relativistic hydrodynamics.

3.1.3 The equations of ideal hydrodynamics

We now insert the specific expressions (3.13) for the components of the particle-current density J^α as well as the non-relativistic approximations (3.17) and (3.18) for the components of the energy-momentum tensor $T^{\mu\nu}$ into the general conservation equations (3.33). For the particle-current density, we find immediately

$$\partial_t n(t, \vec{x}) + \vec{\nabla} \cdot [n(t, \vec{x})\,\vec{v}] = 0\ . \tag{3.35}$$

Multiplying with the particle mass m turns the number density $n(t, \vec{x})$ into the mass density $\rho(t, \vec{x})$, which then satisfies the equation

$$\partial_t \rho + \vec{\nabla} \cdot (\rho\vec{v}) = 0\ . \tag{3.36}$$

This is the continuity equation, or the equation for mass conservation: The local density ρ changes with time by the divergence of the matter current density $\rho\vec{v}$.

In the conservation equation $\partial_\mu T^{\mu\nu} = 0$, the time component, $\nu = 0$, selects the energy-conservation equation, while momentum conservation is expressed by its spatial components, $\nu = i$. With the non-relativistic approximations for T^{00} and T^{0i} and T^{ij} derived in (3.17) and (3.18), we find

$$c^{-1}\partial_t\left(\rho c^2 + \frac{\rho}{2}\left\langle\dot{\vec{x}}^2\right\rangle\right) + \vec{\nabla}\cdot\left(\rho c\vec{v} + \frac{\vec{q}}{c}\right) = 0 \tag{3.37}$$

for the conservation of the energy density, and

$$c^{-1}\partial_t\left(\rho c\vec{v} + \frac{\vec{q}}{c}\right) + \vec{\nabla}\cdot\left(\rho\left\langle\dot{\vec{x}}\otimes\dot{\vec{x}}\right\rangle\right) = 0 \tag{3.38}$$

for momentum conservation. Recall that the vector $\vec{q}$ is the current density of the kinetic energy, defined in (3.19). We can re-arrange the energy-conservation equation (3.37) to read

$$c\left[\partial_t\rho + \vec{\nabla}\cdot(\rho\vec{v})\right] + c^{-1}\left[\partial_t\left(\frac{\rho}{2}\left\langle\dot{\vec{x}}^2\right\rangle\right) + \vec{\nabla}\cdot\vec{q}\right] = 0\;. \tag{3.39}$$

By the continuity equation (3.36), the first term in brackets vanishes, which expresses the fact that mass conservation implies the conservation of rest-mass energy. The energy-conservation equation is thus simplified to

$$\partial_t\left(\frac{\rho}{2}\left\langle\dot{\vec{x}}^2\right\rangle\right) + \vec{\nabla}\cdot\vec{q} = 0\;. \tag{3.40}$$

Comparing terms in the momentum-conservation equation (3.38), we see that the current density of the kinetic energy $\vec{q}$ is smaller by a factor of order v^2/c^2 compared to the current density $\rho c^2\vec{v}$ of the rest-energy density. We can thus safely neglect it in our non-relativistic approximation and write momentum conservation as

$$\partial_t\left(\rho v^i\right) + \vec{\nabla}\cdot\left(\rho\left\langle\dot{\vec{x}}\otimes\dot{\vec{x}}\right\rangle\right) = 0\;. \tag{3.41}$$

Having arrived at this point, we split up the microscopic velocities $\dot{\vec{x}}$ into the mean macroscopic velocity $\vec{v}$ of the fluid flow and a random velocity $\vec{u}$ about the mean,

$$\dot{\vec{x}} = \vec{v} + \vec{u}\;. \tag{3.42}$$

As $\vec{v}$ has been defined as the average over $\dot{\vec{x}}$, the average of $\vec{u}$ must vanish by definition. The average over the squared microscopic velocity is therefore

$$\left\langle\dot{\vec{x}}^2\right\rangle = \vec{v}^2 + \left\langle\vec{u}^2\right\rangle\;, \tag{3.43}$$

which allows us to split up the kinetic energy density into a macroscopic part $\rho v^2/2$ and a microscopic or internal part $\rho\langle u^2\rangle/2$. If this internal kinetic energy density is of thermal origin, we can identify it with the thermal energy density

$$\varepsilon = \frac{\rho}{2}\left\langle u^2\right\rangle = \frac{3}{2}nk_\mathrm{B}T\;. \tag{3.44}$$

The kinetic-energy current density $\vec{q}$ has been introduced as the average

$$\vec{q} = \frac{\rho}{2}\left\langle\dot{\vec{x}}^2\dot{\vec{x}}\right\rangle \tag{3.45}$$

in (3.19). Splitting the microscopic velocities as in (3.42), we can write

$$\left\langle \dot{\vec{x}}^2 \dot{\vec{x}} \right\rangle = \left\langle \left(v^2 + 2\vec{v} \cdot \vec{u} + u^2 \right) (\vec{v} + \vec{u}\,) \right\rangle = v^2 \vec{v} + \left\langle u^2 \right\rangle \vec{v} + 2 \left\langle \vec{u} \otimes \vec{u}\,\right\rangle \vec{v} \tag{3.46}$$

because all terms must vanish in which components of $\vec{u}$ appear linearly. Thus, the kinetic-energy current density is

$$\vec{q} = \frac{\rho}{2} \left(v^2 + \left\langle u^2 \right\rangle \right) \vec{v} + \rho \left\langle \vec{u} \otimes \vec{u}\,\right\rangle \vec{v} = \left(\frac{\rho}{2} v^2 + \varepsilon \right) \vec{v} + \rho \left\langle \vec{u} \otimes \vec{u}\,\right\rangle \vec{v}\,. \tag{3.47}$$

The first two terms are the current densities of the macroscopic and the internal kinetic energies, and the meaning of the third term remains to be clarified.

We finally study the stress-energy tensor $\bar{T}$ with elements T^{ij},

$$\bar{T} = \rho \left\langle \dot{\vec{x}} \otimes \dot{\vec{x}} \right\rangle = \rho \left\langle (\vec{v} + \vec{u}\,) \otimes (\vec{v} + \vec{u}\,) \right\rangle = \rho \left(\vec{v} \otimes \vec{v} + \left\langle \vec{u} \otimes \vec{u}\,\right\rangle \right) \;, \tag{3.48}$$

where we have used once more that all terms linear in $\vec{u}$ must average to zero. The average $\langle \vec{u} \otimes \vec{u}\,\rangle$ appears again. In the rest frame of the macroscopic fluid flow, $\vec{v} = 0$. The trace of the stress-energy tensor is then three times the pressure of the fluid,

$$\rho \,\mathrm{Tr} \left\langle \vec{u} \otimes \vec{u}\,\right\rangle = \rho \left\langle u^2 \right\rangle = 3P\,. \tag{3.49}$$

If the fluid is microscopically isotropic, the random velocity components u^i must be independent, hence $\langle u^i u^j \rangle = 0$ for $i \neq j$ and

$$\rho \left\langle u^i u^i \right\rangle = \frac{\rho}{3} \,\mathrm{Tr} \left\langle \vec{u} \otimes \vec{u}\,\right\rangle = P\,. \tag{3.50}$$

Combining these arguments, we can write

$$\vec{q} = \left(\frac{\rho}{2} v^2 + \varepsilon + P \right) \vec{v} \quad \text{and} \quad \bar{T} = \rho \vec{v} \otimes \vec{v} + P \mathbb{1}_3\,. \tag{3.51}$$

With these results, we can now bring the momentum-conservation equation (3.41) into the form

$$\partial_t (\rho \vec{v}\,) + \vec{\nabla} \cdot (\rho \vec{v} \otimes \vec{v}\,) + \vec{\nabla} P = 0\,. \tag{3.52}$$

Once more, we can re-group terms suitably to identify and remove the two terms representing mass conservation,

$$\left[\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}\,) \right] \vec{v} + \rho \left[\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v}\, \right] + \vec{\nabla} P = \rho \left(\partial_t + \vec{v} \cdot \vec{\nabla} \right) \vec{v} + \vec{\nabla} P = 0\,. \tag{3.53}$$

Momentum conservation is thus expressed by Euler's equation

$$\rho \left(\partial_t + \vec{v} \cdot \vec{\nabla} \right) \vec{v} + \vec{\nabla} P = 0\,. \tag{3.54}$$

The differential operator in parentheses is the total time derivative,

$$\partial_t + \vec{v} \cdot \vec{\nabla} = \partial_t + \frac{\partial \vec{x}}{\partial t} \cdot \frac{\partial}{\partial \vec{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\,. \tag{3.55}$$

Equation (3.54) thus simply states that ideal fluids are accelerated by pressure gradients,

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\vec{\nabla} P\,, \tag{3.56}$$

in absence of external, macroscopic forces.

We finally turn to the energy-conservation equation (3.40). With our results (3.43) and (3.51), it becomes

$$\partial_t\left(\frac{\rho}{2}v^2+\varepsilon\right)+\vec{\nabla}\cdot\left[\left(\frac{\rho}{2}v^2+\varepsilon+P\right)\vec{v}\right]=0\ . \tag{3.57}$$

Expanding the derivatives and re-grouping terms, we can identify those terms here that must vanish due to mass conservation and momentum conservation,

$$\frac{v^2}{2}\left[\partial_t\rho+\vec{\nabla}\cdot(\rho\vec{v})\right]+\frac{\rho}{2}\left(\partial_t+\vec{v}\cdot\vec{\nabla}\right)v^2+\partial_t\varepsilon+\vec{\nabla}\cdot[(\varepsilon+P)\vec{v}]=0\ . \tag{3.58}$$

By mass conservation, the first term in brackets vanishes. By momentum conservation, the second term in parentheses is

$$\frac{\rho}{2}\left(\partial_t+\vec{v}\cdot\vec{\nabla}\right)v^2=\rho\vec{v}\cdot\left(\partial_t+\vec{v}\cdot\vec{\nabla}\right)\vec{v}=-\vec{v}\cdot\vec{\nabla}P=-\vec{\nabla}\cdot(P\vec{v})+P\vec{\nabla}\cdot\vec{v}\ . \tag{3.59}$$

With this identification, the energy-conservation equation shrinks to

$$\partial_t\varepsilon+\vec{\nabla}\cdot(\varepsilon\vec{v})+P\vec{\nabla}\cdot\vec{v}=0\ . \tag{3.60}$$

Again, this has a very intuitive interpretation: The internal energy density changes locally not only by the current density $\varepsilon\vec{v}$, but also by the pressure-volume work $P\vec{\nabla}\cdot\vec{v}$ that the fluid has to exert against its surroundings. If the velocity field is divergent, $\vec{\nabla}\cdot\vec{v}>0$, the fluid expands, and part of its internal energy must be used for working against the pressure of its surroundings. Conversely, if $\vec{\nabla}\cdot\vec{v}<0$, the velocity field is convergent, the fluid is compressed, and its surroundings increase its internal energy by pressure-volume work.

Summarising, our final set of equations for ideal hydrodynamics reads

$$\begin{aligned}\partial_t\rho+\vec{\nabla}\cdot(\rho\vec{v})=0\ ,\\ \partial_t\vec{v}+\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}+\frac{\vec{\nabla}P}{\rho}=0\ ,\\ \partial_t\varepsilon+\vec{\nabla}\cdot(\varepsilon\vec{v})+P\vec{\nabla}\cdot\vec{v}=0\ .\end{aligned} \tag{3.61}$$

They express mass, momentum, and energy conservation in a very intuitive way. They are five equations for the mass density ρ, the internal energy density ε, the pressure P, and the velocity $\vec{v}$, which are six quantities in total. The set (3.61) of equations thus needs to be complemented by an equation of state that relates the pressure to the density, $P=P(\rho)$. The second equation, describing momentum conservation, is often called Euler's equation.

With a slight rearrangement in the energy-conservation equation, we can identify the total time derivative of the energy density,

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}+(\varepsilon+P)\vec{\nabla}\cdot\vec{v}=0\ . \tag{3.62}$$

From the point of view of thermodynamics, this is quite intuitive since the sum of the internal energy density ε and the pressure P is the enthalpy per unit volume, or the enthalpy density h,

$$h=\varepsilon+P\ . \tag{3.63}$$

Energy conservation can thus also be expressed by

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + h\vec{\nabla}\cdot\vec{v} = 0\ , \tag{3.64}$$

which is the first law of thermodynamics at given pressure.

If external, macroscopic forces are present, such as the gravitational force, the momentum-conservation equation must be augmented by the corresponding force densities. Let Φ be the Newtonian gravitational potential, its negative gradient $-\vec{\nabla}\Phi$ is the gravitational force per unit mass. It can be added to the right-hand side of the momentum-conservation equation to yield

$$\partial_t\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} + \frac{\vec{\nabla}P}{\rho} = -\vec{\nabla}\Phi\ . \tag{3.65}$$

It is sometimes useful to write the complete set of equations (3.61) in terms of total time derivatives. It then reads

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\vec{\nabla}\cdot\vec{v} = 0\ , \quad \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \frac{\vec{\nabla}P}{\rho} = -\vec{\nabla}\Phi\ , \quad \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + h\vec{\nabla}\cdot\vec{v} = 0\ . \tag{3.66}$$

Problems

1. The energy-momentum tensor is defined as

$$T^{\mu\nu} \equiv c^2 \int \frac{\mathrm{d}^3p}{E(p)}\, p^\mu p^\nu\, f(\vec{x},\vec{p},t)\ , \tag{3.67}$$

where $(p^\mu) = (E/c, \vec{p})^T$ is the four-momentum, E the energy, and $f(\vec{x},\vec{p},t)$ the one-particle phase-space density distribution. While the energy density is $\varepsilon = T^{00}$, the pressure is given by one third of the stress-energy tensor's trace, hence $P = (1/3)\sum_{i=1}^3 T^{ii}$.

   (a) Determine $T^{\mu\nu}$ for a single particle of mass m with trajectory $\vec{x}_0(t)$ and momentum $\vec{p}_0(t)$. Compare to the energy momentum tensor of an ideal fluid.

   (b) Determine $T^{\mu\nu}$ for a photon of frequency ω with trajectory $\vec{x}_0(t)$.

   (c) How is the energy density related to the pressure in the two cases discussed?

2. The hydrodynamical equations describing mass conservation, momentum conservation, and energy conservation for an ideal fluid are

$$\partial_t\rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0\ , \tag{3.68}$$

$$\partial_t\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\frac{\vec{\nabla}P}{\rho}\ , \tag{3.69}$$

$$\partial_t\varepsilon + \vec{\nabla}\cdot(\varepsilon\vec{v}) = -P\vec{\nabla}\cdot\vec{v}\ . \tag{3.70}$$

respectively.

(a) Show using equation (3.70) that an isothermal ideal fluid, i.e. a fluid with constant temperature $T(x,t) = T_0$, is also incompressible, $\vec{\nabla} \cdot \vec{v} = 0$.

(b) Show that for a spherically symmetric and isothermal flow of an ideal gas, equations (3.68) through (3.70) simplify to

$$\partial_t \rho + v \partial_r \rho = 0 \;, \quad \partial_t v - \frac{2v^2}{r} = -c_s^2 \partial_r \ln \rho \;, \tag{3.71}$$

where $c_s \equiv k_B T_0 / m$ is a characteristic thermal speed.

3.2 Relativistic Hydrodynamics

This section is a detour from the main track of this book in so far as General Relativity is otherwise avoided. Yet, it is an irresistible temptation to show how generally-relativistic, ideal hydrodynamics emerges simply if the partial derivatives in the covariant conservation equations (3.33) are replaced by covariant derivatives, and Poisson's equation by (the appropriate limit of) Einstein's field equation. The first main result are the relativistic versions (3.81) and (3.82) of the continuity and Euler equations. In the limit of weak gravitational fields, the relativistic generalisations (3.95) of these equations are derived. Together with the gravitational field equation in the same limit, the final set of hydrodynamical equations is given by (3.106). Perturbative analysis then yields the linear, second-order evolution equation (3.116) for the fluid density.

3.2.1 Hydrodynamic Equations

We shall now derive the ideal hydrodynamic equations from the generally-relativistic equation of local energy conservation. We do this for one specific reason. In the preceding section, we have derived the equations of ideal hydrodynamics by taking appropriate moments of the Boltzmann equation. In that derivation, it has become clear how ideal hydrodynamics builds upon the fluid approximation, and how viscosity and other transport processes such as heat conduction arise if the ideal-fluid approximation is gradually released. Yet, that derivation does not easily allow incorporating the main repercussions of General Relativity in hydrodynamics, which arise because pressure has inertia and contributes as a source to the gravitational field. Therefore, we give this relativistic derivation of the hydrodynamical equations here, borrowing from the differential-geometric formalism of General Relativity without detailed explanation, and contrasting the generally-relativistic hydrodynamic equations at the end with their Newtonian analoga. Our main motivation is that sometimes fluids occur in astrophysics which either move relativistically or whose pressure is comparable to their energy density. In both cases, the classical Newtonian hydrodynamical equations are suspect, and their relativistic counterparts should be used instead.

Readers unfamiliar with general relativity might wish to skip the following subsections, returning when the equations of relativistic hydrodynamics will be summarised and compared to the Newtonian equations.

We begin with the equation of local energy-momentum conservation,

$$\nabla_\nu T^{\mu\nu} = 0 \,, \tag{3.72}$$

which states that the covariant four-divergence of the energy-momentum tensor T has to vanish. This is an immediate consequence of Einstein's field equations. By the second contracted Bianchi identity, the covariant divergence of the Einstein tensor G vanishes identically, so the covariant divergence of the energy-momentum tensor needs to vanish as well.

At this level, we only need to specify that the covariant derivative ∇ is a bi-linear map of (tangent) vectors $(x, y) \in TM$ to a manifold M into the real numbers,

$$\nabla : TM \times TM \to \mathbb{R} \,, \quad (x, y) \mapsto \nabla_x y \,, \tag{3.73}$$

satisfying the Leibniz (product) rule,

$$\nabla_x (fy) = \mathrm{d}f(x) y + f \nabla_x y \,, \tag{3.74}$$

with functions f.

Caution The connection conventionally used in general relativity is specified by two further conditions: it is supposed to be symmetric (torsion-free) and metric-compatible ($\nabla g = 0$). ◂

In a coordinate basis of tangent space, the covariant derivatives are uniquely represented by the Christoffel symbols. More generally, in an arbitrary basis $\{e_\mu\}$ of tangent space, the covariant derivative is defined by the connection 1-forms,

$$\nabla_x e_\mu = \omega^\nu{}_\mu (x) e_\nu \,. \tag{3.75}$$

We now choose to insert the energy-momentum tensor of an ideal fluid,

$$T = \left(\rho c^2 + p\right) u \otimes u - p g \,, \tag{3.76}$$

which is spanned by the only two tensors available in relativistically flowing ideal fluid, namely the tensor product of the four-velocity u with itself and the metric tensor g. The local fluid properties are given by the density ρ and the pressure p measured by the observer flowing with the four-velocity u. Writing the energy-momentum tensor as in (3.76) implies that the four-velocity u must be dimension-less, and thus be measured in units of the light speed c. The components of the energy-momentum tensor T, without specifying the basis vectors yet, are

$$T^{\mu\nu} = \left(\rho c^2 + p\right) u^\mu u^\nu - p g^{\mu\nu} \,. \tag{3.77}$$

Inserting these into the local conservation equation (3.72) gives

$$u^\mu \nabla_u \left(\rho c^2 + p\right) + u^\mu \left(\rho c^2 + p\right) \nabla \cdot u + \left(\rho c^2 + p\right) \nabla_u u^\mu + \nabla^\mu p = 0 \tag{3.78}$$

if we specify the covariant derivative ∇ as usual to be metric, requiring $\nabla g = 0$.

We now project equation (3.78) first on the local time direction by contracting it with the (dual) four-velocity u_μ, and then on the three-space perpendicular to the four-velocity. By their construction, these projections will yield the time and space components of the local conservation equation (3.72), which generalise the continuity and Euler equations.

By definition of the proper time τ, the four-velocity must be normalised by

$$\langle u, u \rangle = u_\mu u^\mu = -1 \,. \tag{3.79}$$

In particular, this normalisation condition implies that

$$0 = \nabla_u \left(u_\mu u^\mu\right) = 2u_\mu \nabla_u u^\mu \ . \tag{3.80}$$

Taking (3.79) and (3.80) into account, contracting (3.78) with u_μ gives the relativistic continuity equation

$$\nabla_u \left(\rho c^2\right) + \left(\rho c^2 + p\right) \nabla \cdot u = 0 \ , \tag{3.81}$$

while its spatial projection by contraction with the projection tensor $\pi_{\alpha\mu} = g_{\alpha\mu} + u_\alpha u_\mu$ yields the relativistic Euler equation

$$\left(\rho c^2 + p\right) \nabla_u u_\alpha + \nabla_\alpha p + u_\alpha \nabla_u p = 0 \ . \tag{3.82}$$

It can easily be seen that $\pi_{\alpha\mu}$ is a projection tensor perpendicular to the four velocity since it maps the four-velocity to zero,

$$\pi_{\alpha\mu} u^\mu = \left(g_{\alpha\mu} + u_\alpha u_\mu\right) u^\mu = u_\alpha - u_\alpha = 0 \ . \tag{3.83}$$

?

Projection tensors π (or, more generally, projections) need to be idempotent, i.e. they need to satisfy $\pi^2 = \pi$. Why is this so? Show that $\pi = g + u \otimes u$ is indeed idempotent. Written in terms of tensor components, show that $\pi_\alpha{}^\beta \pi_{\beta\mu} = \pi_{\alpha\mu}$.

Equations (3.81) and (3.82) form the basis for the following calculations. What do they mean?

The continuity equation (3.81) begins with the covariant derivative of ρc^2 in the direction of the local four-velocity. This generalises the time derivative of the matter density ρ in the continuity equation in three ways. First, the derivative with respect to the coordinate time t is replaced by a derivative with respect to proper time; second, the partial derivative is replaced by a covariant derivative; and third, the matter density is replaced by the energy density ρc^2. The second term in the continuity equation generalises the divergence of the velocity field to the four-divergence of the four-velocity, multiplied with the energy density plus the pressure rather than the density alone: The relativistic continuity equation automatically contains the contribution of pressure-volume work to energy conservation.

The Euler equation starts with the four-acceleration, i.e. the covariant derivative of the four-velocity into the direction of the local four velocity itself. The prefactor $(\rho c^2 + p)$ shows the inertia of pressure. The second term is the pressure gradient, while the third term adds a proper time derivative of the pressure times the flow velocity.

3.2.2 Hydrodynamics in a Weak Gravitational Field

We now proceed to specialise the generally-relativistic continuity and Euler equations, (3.81) and (3.82), to weak gravitational fields. In any metric theory of gravity, in the weak-field limit, the line element can be expressed by means of the two Bardeen potentials ϕ, ψ as

$$\mathrm{d}s^2 = -(1 + 2\phi)\, c^2 \mathrm{d}t^2 + (1 + 2\psi)\, \mathrm{d}\vec{x}^2 \ . \tag{3.84}$$

Both potentials are given in units of c^2, thus dimension-less, and they are assumed to be small, $\phi, \psi \ll 1$. For simplicity, we further take the potentials to be time-independent, $\dot{\phi} = 0 = \dot{\psi}$. The line element (3.84) suggests introducing the dual basis

$$\theta^0 = (1+\phi)c\mathrm{d}t\ , \quad \theta^i = (1+\psi)\mathrm{d}x^i \tag{3.85}$$

and its orthonormal basis

$$e_0 = (1-\phi)c^{-1}\partial_t\ , \quad e_i = (1-\psi)\partial_i\ . \tag{3.86}$$

By this choice of the (dual) basis, the components of the metric become Minkowskian, $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. By means of Cartan's first structure equation, the dual basis $\{\theta^\mu\}$ implies the connection forms

$$\omega^0{}_i = \phi_i\theta^0\ , \quad \omega^i{}_j = \psi_j\theta^i - \psi_i\theta^j\ . \tag{3.87}$$

Now, in a coordinate basis, the four-velocity is

$$u = \tilde{u}^\mu\partial_\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\partial_\mu\ . \tag{3.88}$$

From the line element (3.84), we can read off the proper-time element

$$\mathrm{d}\tau = \left[(1+2\phi) - (1+2\psi)\vec{\beta}^2\right]^{1/2} c\mathrm{d}t \approx \left(1+\phi-\frac{\vec{\beta}^2}{2}\right)c\mathrm{d}t\ , \tag{3.89}$$

valid to first and relevant order in ϕ and $\vec{\beta}^2$. Here, as usual, $\vec{\beta} = \vec{v}/c = \dot{\vec{x}}/c$, and the dot abbreviates the derivative with respect to the *coordinate* time. Thus, to the same order in ϕ and β, the four-velocity is

$$u = \left(1-\phi+\frac{\vec{\beta}^2}{2}\right)\frac{\mathrm{d}x^\mu}{c\mathrm{d}t}\partial_\mu = \left(1-\phi+\frac{\vec{\beta}^2}{2}\right)\left(\partial_0 + \beta^i\partial_i\right) \tag{3.90}$$

in the coordinate basis. Its components in the basis $\{e_\mu\}$ introduced in (3.85) are then determined by $u^\mu = \theta^\mu(u)$ or, again to first order in ϕ and $\vec{v}^2$,

$$u^0 = 1 + \frac{\beta^2}{2}\ , \quad u^i = \beta^i\ . \tag{3.91}$$

Using now the expressions

$$\nabla_\mu f = e_\mu f\ , \quad \nabla_u f = u^\mu\nabla_\mu f = \left(1-\phi+\frac{\beta^2}{2}\right)c^{-1}\dot{f} + \beta^i\partial_i f \tag{3.92}$$

for arbitrary scalar functions f and

$$\nabla_v u^\mu = \mathrm{d}u^\mu(v) + u^\nu\omega^\mu{}_\nu(v) \tag{3.93}$$

for the component μ of the covariant derivative of a vector u into the direction v, we can finally bring the hydrodynamic equations (3.81) and (3.82) into the form

$$\begin{aligned}
\left(1-\phi+\frac{\beta^2}{2}\right)\dot{\rho}c + \left(\vec{\beta}\cdot\vec{\nabla}\right)\rho c^2 + \left(\rho c^2 + p\right)\left[\vec{\nabla}\cdot\vec{\beta} + c^{-1}\partial_t\left(\frac{\beta^2}{2}\right)\right] &= 0\ ,\\
\left(\rho c^2 + p\right)\partial_t\left(\frac{\beta^2}{2}\right) + \beta^2\dot{p} + c\vec{\beta}\cdot\vec{\nabla}p &= 0\ ,\\
\left(\rho c^2 + p\right)\left[c^{-1}\dot{\vec{\beta}} + \left(\vec{\beta}\cdot\vec{\nabla}\right)\vec{\beta} + \vec{\nabla}\phi\right] + (1-\psi)\vec{\nabla}p + c^{-1}\dot{p}\vec{\beta} + \vec{\beta}\left(\vec{\beta}\cdot\vec{\nabla}\right)p &= 0\ .
\end{aligned} \tag{3.94}$$

In these equations, $\vec{\nabla}$ with a vector arrow is now specialised to be the ordinary gradient operator in three-dimensional, Euclidean space. The second of these equations, which is the time component of the Euler equation, shows that the term $\vec{\beta}\cdot\vec{\nabla}p$ is of order β^2, thus $\psi\vec{\nabla}p$ is of order $\beta^3 \approx 0$ because the potential ψ is itself of order β^2. The continuity equation, to linear order in β, and the Euler equation, to quadratic order in v, are thus

$$\dot{\rho}c + \left(\vec{\beta}\cdot\vec{\nabla}\right)\rho c^2 + \left(\rho c^2 + p\right)\vec{\nabla}\cdot\vec{\beta} = 0 \ ,$$
$$\left(\rho c^2 + p\right)\left[c^{-1}\dot{\vec{\beta}} + \left(\vec{\beta}\cdot\vec{\nabla}\right)\vec{\beta} + \vec{\nabla}\phi\right] + \vec{\nabla}p + c^{-1}\dot{p}\vec{\beta} = 0 \ . \tag{3.95}$$

Notice that, reassuringly, all terms in both these equations have the dimension [energy density]/[length].

3.2.3 Gravitational Field Equation

To linear order in ϕ and ψ, the curvature 2-forms implied by the connection 1-forms (3.79) through Cartan's second structure equation are

$$\Omega^0{}_i = \phi_{ij}\theta^j \wedge \theta^0 \ , \quad \Omega^i{}_j = \psi_{jk}\theta^k \wedge \theta^i - \psi_{ik}\theta^k \wedge \theta^j \ . \tag{3.96}$$

From them, the components of the Ricci tensor can be found via

$$R_{\mu\nu} = \Omega^\alpha{}_\mu\left(e_\alpha, e_\nu\right) \ . \tag{3.97}$$

With (3.96), they are

$$R_{00} = \vec{\nabla}^2\phi \ , \quad R_{0i} = 0 \ , \quad R_{ij} = -(\phi+\psi)_{ij} - \delta_{ij}\vec{\nabla}^2\psi \ . \tag{3.98}$$

The Ricci scalar is

$$R = R^\mu{}_\mu = -2\vec{\nabla}^2\left(\phi + 2\psi\right) \ , \tag{3.99}$$

and thus the components of the Einstein tensor become

$$G_{00} = -2\vec{\nabla}^2\psi \ , \quad G_{0i} = 0 \ , \quad G_{ij} = -(\phi+\psi)_{ij} + \delta_{ij}\vec{\nabla}^2(\phi+\psi) \ . \tag{3.100}$$

?

Beginning from the components (3.98) of the Ricci tensor, confirm by your own calculation that the Einstein tensor has the components (3.100).

With (3.100), the time-time component of the field equations gives

$$-\vec{\nabla}^2\psi = \frac{4\pi G}{c^4}\left[\rho c^2 + \beta^2\left(\rho c^2 + p\right)\right] \ , \tag{3.101}$$

while the spatial trace of the field equations yields

$$\vec{\nabla}^2(\phi+\psi) = \frac{4\pi G}{c^4}\left[3p + \beta^2\left(\rho c^2 + p\right)\right] \ . \tag{3.102}$$

The sum of the latter two equations gives the generalised Poisson equation

$$\vec{\nabla}^2\phi = \frac{4\pi G}{c^4}\left[\rho c^2 + 3p + 2\beta^2\left(\rho c^2 + p\right)\right] \ . \tag{3.103}$$

The trace of the field equations is

$$\vec{\nabla}^2\phi + 2\vec{\nabla}^2\psi = \frac{4\pi G}{c^4}\left(3p - \rho c^2\right) \ , \tag{3.104}$$

and their off-diagonal components require

$$-(\phi+\psi)_{ij} = \frac{8\pi G}{c^4}\left(\rho c^2 + p\right)\beta_i\beta_j \ . \tag{3.105}$$

3.2.4 The Combined Set of Equations

Thus, to lowest relevant order in ϕ, ψ and $\beta^2 = v^2/c^2$, the combined hydrodynamic and gravitational equations are

$$\dot\rho + \left(\vec v \cdot \vec\nabla\right)\rho + \left(\rho + \frac{p}{c^2}\right)\vec\nabla \cdot \vec v = 0 \ ,$$
$$\dot{\vec v} + \left(\vec v \cdot \vec\nabla\right)\vec v = -\vec\nabla\Phi - \frac{\vec\nabla p + \dot p \vec v/c^2}{\rho + p/c^2} \ ,$$
$$\vec\nabla^2\Phi = 4\pi G\left(\rho + \frac{3p}{c^2}\right) \ . \tag{3.106}$$

They generalise the Newtonian hydrodynamic equations

$$\dot\rho + \left(\vec v \cdot \vec\nabla\right)\rho + \rho\vec\nabla \cdot \vec v = 0 \ ,$$
$$\dot{\vec v} + \left(\vec v \cdot \vec\nabla\right)\vec v = -\vec\nabla\Phi - \frac{\vec\nabla p}{\rho} \ ,$$
$$\vec\nabla^2\Phi = 4\pi G\rho \ , \tag{3.107}$$

where $\Phi = c^2\phi$ is the Newtonian gravitational potential in physical units. Comparing (3.106) and (3.107), one clearly sees the pressure-volume work in the continuity equation, the inertia of the mass-density equivalent p/c^2 of the pressure in the Euler equation and the contribution of $3p/c^2$ to the source of the gravitational field. Notice also the additional force term $\propto \dot p\vec v/c^2$ in the Euler equation.

3.2.5 Perturbative Analysis

Let us now continue with a perturbative analysis of the set of equations (3.106). As usual, we assume that a smooth background solution is already given, which is indicated by a subscript 0. We thus have a set of fields $(\rho_0, p_0, \vec v_0, \phi_0)$ which separately satisfy Eqs. (3.106). They are perturbed by small deviations $(\delta\rho, \delta p, \delta\vec v, \delta\phi)$. The equations will be linearised in these perturbations, meaning that terms will be dropped that are of quadratic or higher order in the perturbations.

We transform into a coordinate system comoving with the unperturbed flow, which allows us to set $\vec v_0 = 0$. We assume that the perturbations are small compared to the overall length scale of the unperturbed solution, hence gradients of the background solution can be neglected. Finally, we assume that the fluid has a polytropic equation of state,

$$p = \bar p\left(\frac{\rho}{\bar\rho}\right)^\gamma \ , \tag{3.108}$$

where $(\bar\rho, \bar p)$ are arbitrary reference values for the density and the pressure and γ is the adiabatic index of the fluid. Since the squared sound speed is

$$c_\mathrm{s}^2 = \frac{\partial p}{\partial\rho} \tag{3.109}$$

at constant entropy, the pressure fluctuations can be written as $\delta p = c_s^2 \delta\rho$. We express the density fluctuation by the dimension-less density contrast

$$\delta = \frac{\delta\rho}{\rho_0} \;, \quad \delta\rho = \rho_0\delta \;, \tag{3.110}$$

which allows to write the pressure fluctuation as

$$\delta p = c_s^2 \rho_0 \delta \;. \tag{3.111}$$

Substituting $\rho = \rho_0 + \delta\rho = \rho_0(1+\delta)$ and $\vec{v} = \vec{v}_0 + \delta\vec{v} = \delta\vec{v}$ into the continuity equation gives, to lowest order in the perturbations,

$$\dot{\rho}_0 = 0 \;, \tag{3.112}$$

and to first order

$$\dot{\delta} + \left(1 + \frac{p_0}{\rho_0 c^2}\right) \vec{\nabla} \cdot \delta\vec{v} = 0 \;. \tag{3.113}$$

By the polytropic equation-of-state, (3.112) also implies $\dot{p}_0 = 0$. Then, to first order in the perturbations, Euler's equation and the generalised Poisson equation are reduced to

$$\begin{aligned} \delta\dot{\vec{v}} &= -\vec{\nabla}\delta\Phi - \frac{c_s^2 \vec{\nabla}\delta}{1 + \frac{p_0}{\rho_0 c^2}} \;, \\ \vec{\nabla}^2 \delta\Phi &= 4\pi G \rho_0 \delta \left(1 + \frac{3c_s^2}{c^2}\right) \;. \end{aligned} \tag{3.114}$$

Taking the time derivative of the continuity equation (3.113) and the divergence of the Euler equation from (3.114) transforms these equations into

$$\begin{aligned} \ddot{\delta} + \left(1 + \frac{p_0}{\rho_0 c^2}\right) \vec{\nabla} \cdot \delta\dot{\vec{v}} &= 0 \;, \\ \vec{\nabla} \cdot \delta\dot{\vec{v}} &= -\vec{\nabla}^2 \delta\Phi - \frac{c_s^2 \vec{\nabla}^2 \delta}{1 + \frac{p_0}{\rho_0 c^2}} \;. \end{aligned} \tag{3.115}$$

Eliminating the divergence of the peculiar acceleration, $\vec{\nabla} \cdot \delta\dot{\vec{v}}$, between these equations and inserting the generalised Poisson equation from (3.114) then leads to the evolution equation

$$\ddot{\delta} - 4\pi G \rho_0 \delta \left(1 + \frac{3c_s^2}{c^2}\right)\left(1 + \frac{p_0}{\rho_0 c^2}\right) - c_s^2 \vec{\nabla}^2 \delta = 0 \tag{3.116}$$

for the density contrast δ. In the non-relativistic limit, when the sound speed c_s is small compared to the light speed c and the pressure p_0 is negligible compared to the rest-energy density $\rho_0 c^2$, this linear evolution equation for the density contrast shrinks to

$$\ddot{\delta} - 4\pi G \rho_0 \delta - c_s^2 \vec{\nabla}^2 \delta = 0 \;. \tag{3.117}$$

Example: Without gravity

Some special cases should now be instructive and illustrate why we went through this analysis here. Let us first ignore gravity completely. Then, the second term in (3.116) disappears altogether because it originates from gravity alone. Ignoring gravity can formally be expressed by setting the Newtonian gravitational constant to zero, $G = 0$, and thus suppress all gravitational coupling. Then, the density fluctuations δ are found to obey the wave equation

$$\Box\delta = 0 \,, \tag{3.118}$$

where the sound speed c_s appears as the characteristic velocity in the d'Alembert operator. The density contrast then undergoes ordinary sound waves. ◄

Example: With gravity on a non-relativistic background

If gravity is switched back on, but the background remains non-relativistic, Eq. (3.116) simplifies to

$$\ddot{\delta} - 4\pi G\rho_0\delta - c_\mathrm{s}^2\vec{\nabla}^2\delta = 0 \,. \tag{3.119}$$

If we expand δ into plane waves, the Laplacian is replaced by the negative square of the wave number k, and δ obeys

$$\ddot{\delta} - \left(4\pi G\rho_0 - c_\mathrm{s}^2k^2\right)\delta = 0 \,. \tag{3.120}$$

This is an ordinary oscillator equation, with

$$c_\mathrm{s}^2k^2 - 4\pi G\rho_0 = \omega^2 \tag{3.121}$$

taking the role of the squared frequency. If $\omega^2 > 0$, i.e. if k is larger than the so-called Jeans wave number

$$k_\mathrm{J} = \left(\frac{4\pi G\rho_0}{c_\mathrm{s}^2}\right)^{1/2} \,, \tag{3.122}$$

the solutions oscillate like sound waves, satisfying the dispersion relation

$$\omega = c_\mathrm{s}\sqrt{k^2 - k_\mathrm{J}^2} \,. \tag{3.123}$$

Otherwise, if k is smaller than the Jeans wave number, there is an exponentially growing and an exponentially decaying mode of the density fluctuations. ◄

Example: With gravity on a relativistic background

If, finally, the background fluid is relativistic, we have the full equation

$$\ddot{\delta} - \left[4\pi G\rho_0\left(1 + \frac{3c_s^2}{c^2}\right)\left(1 + \frac{p_0}{\rho_0 c^2}\right) - c_s^2 k^2\right]\delta = 0 \tag{3.124}$$

for plane waves of wave number k. Suppose, for example, we have a plasma tightly coupled to a dominating photon gas like in the early universe. Then, the fluid is relativistic, $p_0 \approx \rho_0 c^2/3$ and $c_s^2 \approx c^2/3$, and

$$\left(1 + \frac{3c_s^2}{c^2}\right)\left(1 + \frac{p_0}{\rho_0 c^2}\right) \approx \frac{8}{3}\,. \tag{3.125}$$

The Jeans wave number then changes to

$$k_J = \left(\frac{32\pi G\rho_0}{3c_s^2}\right)^{1/2} = \left(\frac{32\pi G\rho_0}{c^2}\right)^{1/2}\,, \tag{3.126}$$

which is typically much smaller than for a non-relativistic fluid. Acoustically oscillating perturbations are thus possible in a much wider range of scales in a relativistic than in a non-relativistic fluid, and growth or decay of perturbations is possible only for very large perturbations. ◂

3.3 Viscous hydrodynamics

So far, we have considered ideal fluids, whose particles have a negligibly small mean free path. In this section, we shall loosen this approximation and allow a very small, but finite mean free path. The fluid particles can now move relative to the mean flow and transport fluid properties by small distances, in particular mass, momentum and energy. The transport of momentum causes friction and energy dissipation, the transport of energy gives rise to heat conduction. The first important result is the diffusive extension of the energy-momentum tensor (3.143) which can then be introduced into the conservation equation to derive the Navier-Stokes equation (3.148) and the energy-conservation equation (3.155) containing heat flow and dissipation. Finally, we introduce gravitational forces into the equations of viscous hydrodynamics and derive the tensor virial theorem (3.189).

3.3.1 Diffusion of particles, momentum and internal energy

Previously, we have assumed that our fluid is ideal, that is, that the mean-free path λ is negligibly small. We have used this implicitly when we set the momentum-space integrals over the collision term to zero. If we cannot neglect the mean-free path any more, we must take into account that particles may move over small, but non-vanishing distances and thereby carry their physical properties with them. In that way, transport phenomena occur over small distance scales.

Let us begin with a simple example. Suppose there is a homogeneous ideal fluid, into which we place a screen of the small cross-sectional area $\mathrm{d}A$. For definiteness, we set up a coordinate system such that the screen is perpendicular to one of the coordinate axes, say the x axis, which it may intersect at the coordinate origin. This screen and the coordinate system may flow with the mean fluid velocity.

Particles will move by random motion from one side of the screen to the other. Let $n(0)$ be their mean number density at the location of the screen, and the screen be small enough for us to neglect any change of the number density across the screen. If the particle number density behind the screen is the same as in front of the screen, the same number of particles will cross the screen per unit time in the positive as in the negative x direction, and the net number of particles flowing through the screen will be zero.

Now let us gradually relax this stationary situation by imagining a number-density gradient along the x direction (cf. Fig. 3.3). Then, there will be fewer particles behind than before the screen, and even though their random velocities in the $\pm x$ directions will be the same, more particles will flow down than up the gradient. Let $\bar{u}$ be a characteristic velocity of the particles. Since their random velocities $\vec{u}$ will average to zero, $\bar{u}$ could be set to the root mean-square velocity, $\bar{u} = \langle \vec{u}^2 \rangle^{1/2}$. How exactly $\bar{u}$ and $\langle \vec{u}^2 \rangle^{1/2}$ relate depends on the velocity distribution of the particles, which is however irrelevant for our purposes.

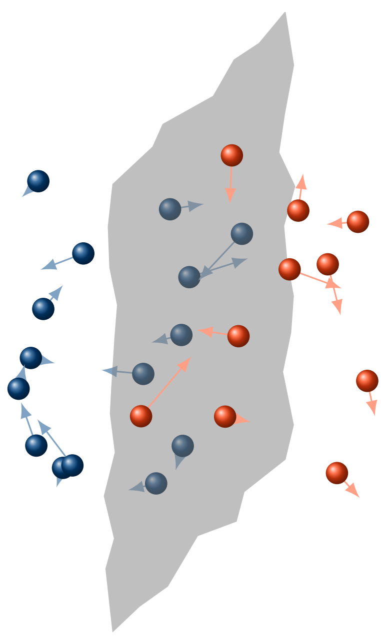

Figure 3.3 Particle diffusion: If there are more particles on one side of the imagined screen than on the other, such as there are more blue than red particles in this example, particles will effectively diffuse from the denser region to the less dense.

Then if the particle velocities are randomly oriented, the velocity in the x direction, perpendicular to the screen, will be of order $\bar{u}/\sqrt{3}$. Since approximately half of the particles will move into the positive x direction, the number of particles N moving through the screen in either direction per unit time is

$$\frac{\mathrm{d}N}{\mathrm{d}t} \approx -\frac{\bar{u}}{2\sqrt{3}}\mathrm{d}A\,[n(x+\lambda) - n(x-\lambda)] \ , \tag{3.127}$$

where λ is the mean free path of the particles. If λ is finite, but small, we can replace the difference in particle number densities by a derivative to find the particle current density

$$j_\mathrm{p} = \frac{\mathrm{d}N}{\mathrm{d}A\mathrm{d}t} \approx -\frac{\bar{u}\lambda}{\sqrt{3}}\frac{\partial n}{\partial x} \ . \tag{3.128}$$

In three dimensions, the derivative with respect to x is replaced by the gradient,

$$\vec{j}_\mathrm{p} = -\frac{\bar{u}\lambda}{\sqrt{3}}\vec{\nabla}n \ . \tag{3.129}$$

Gradients in particle number densities drive particle diffusion. Inserting this current together with the particle number density into the continuity equation for the particles gives Fick's (second) law for diffusive particle transport,

$$\partial_t n + \vec{\nabla}\cdot\vec{j}_\mathrm{p} = 0 \quad \Rightarrow \quad \partial_t n = \vec{\nabla}\cdot\left(D\vec{\nabla}n\right) \ , \quad D = \frac{\bar{u}\lambda}{\sqrt{3}} \ . \tag{3.130}$$

Recall that the expression given here for the diffusion coefficient D has been heuristically derived. More precise definitions can be given if the probability distribution of the random velocities is known.

Let us now apply the same approach to momentum and energy transport. Consider how particles transport a velocity component v^i diffusively into the x direction. If v^i changes with x, the velocity component v^i of the particles diffusing towards the positive x direction differs from the v^i that the particles transport towards the negative x direction. By essentially the same argument that led to (3.127), we find the current density component $(j_\mathrm{v})^i_x$ of v^i

$$(j_\mathrm{v})^i_x = -\frac{n\bar{u}\lambda}{\sqrt{3}}\frac{\partial v^i}{\partial x} \ . \tag{3.131}$$

?

Derive (3.131) in a way similar to the derivation of (3.128).

The diffusive transport of the velocity component v^i into the spatial direction x_j can accordingly be described by the rank-2 tensor

$$(j_\mathrm{v})^i{}_j = -\frac{n\bar{u}\lambda}{\sqrt{3}}\frac{\partial v^i}{\partial x^j} \ . \tag{3.132}$$

This suggests that the stress-energy tensor $\bar{T}_\mathrm{d}$ describing diffusive momentum transport should be proportional to the tensor of spatial velocity derivatives,

$$\bar{T}_\mathrm{d} \propto -\left(\vec{\nabla}\otimes\vec{v}\right)^\top \ , \tag{3.133}$$

with a proportionality constant giving the right-hand side the appropriate dimension of a momentum current density.

Energy transport by diffusion is easily completed. Completely analogously to the previous derivations, we find the diffusive current density of the internal energy

$$\vec{q}_\varepsilon = -\frac{n\bar{u}\lambda}{\sqrt{3}}\vec{\nabla}\varepsilon \ . \tag{3.134}$$

We can express the gradient of the internal energy by a temperature gradient and obtain

$$\vec{q}_\varepsilon = -\frac{n\bar{u}\lambda}{\sqrt{3}}\frac{\mathrm{d}\varepsilon}{\mathrm{d}T}\vec{\nabla}T = -\frac{n\bar{u}\lambda c_\mathrm{v}}{\sqrt{3}}\vec{\nabla}T = -\kappa\vec{\nabla}T \ . \tag{3.135}$$

For the second equality, we have inserted the heat capacity c_v at constant volume, and the last equality defines the heat conductivity κ.

The diffusive stress-energy tensor requires further consideration. While the velocity-gradient tensor $\vec{\nabla} \otimes \vec{v}$ may be asymmetric, the stress-energy tensor should be symmetric. This suggests to assume that the diffusive stress-energy tensor should be set proportional to the symmetric part of $\vec{\nabla} \otimes \vec{v}$, or

$$\bar{T}_\mathrm{d} \propto -\left[\left(\vec{\nabla}\otimes\vec{v}\right) - \left(\vec{\nabla}\otimes\vec{v}\right)^\top\right] \ . \tag{3.136}$$

This is reasonable also because of the following consideration. If a system of particles rotates like a solid body of angular velocity $\vec{\omega}$, i.e. with the velocity field

$$\vec{v} = \vec{\omega}\times\vec{r} \ , \quad v^j = \varepsilon^j{}_{kl}\,\omega^k x^l \ , \tag{3.137}$$

no momentum transport should occur. The derivatives of the velocity components (3.137) are

$$\partial_i v^j = \partial_i\left(\varepsilon^j{}_{kl}\,\omega^k x^l\right) = \varepsilon^j{}_{kl}\,\omega^k\delta^l_i = \varepsilon^j{}_{ki}\,\omega^k \ , \tag{3.138}$$

which is manifestly antisymmetric because of the antisymmetry of the Levi-Civita symbol. Excluding momentum-transport effects in systems rotating like solid bodies thus also argues for setting the diffusive stress-energy tensor proportional to the symmetrised velocity-gradient tensor.

It is further often convenient to distinguish between divergent or convergent flows, for which

$$\vec{\nabla}\cdot\vec{v} = \partial_i v^i = \mathrm{Tr}\left(\vec{\nabla}\otimes\vec{v}\right) \neq 0 \ , \tag{3.139}$$

and so-called shear flows, for which the trace vanishes,

$$\mathrm{Tr}\left(\vec{\nabla}\otimes\vec{v}\right) = 0 \ . \tag{3.140}$$

We thus split up the symmetrised velocity-gradient tensor into a trace-free part

$$\left(\vec{\nabla}\otimes\vec{v}\right) + \left(\vec{\nabla}\otimes\vec{v}\right)^\top + \frac{2}{3}\vec{\nabla}\cdot\vec{v}\,\mathbb{1}_3 \tag{3.141}$$

and a diagonal part carrying the trace,

$$\vec{\nabla}\cdot\vec{v}\,\mathbb{1}_3 \ , \tag{3.142}$$

and assemble the diffusive stress-energy tensor from these two contributions separately,

$$\bar{T}_\mathrm{d} = -\eta\left[\left(\vec{\nabla}\otimes\vec{v}\right) + \left(\vec{\nabla}\otimes\vec{v}\right)^\top - \frac{2}{3}\vec{\nabla}\cdot\vec{v}\,\mathbb{1}_3\right] - \zeta\vec{\nabla}\cdot\vec{v}\,\mathbb{1}_3 \ . \tag{3.143}$$

The two constants η and ζ introduced here represent the viscosity of the fluid.

?

Explain the factor of $2/3$ in the trace-free part (3.141) of the velocity-gradient tensor.

The components of the stress-energy tensor must have the dimension of a momentum current density, hence

$$\left[T_{\mathrm{d}}^{ij}\right] = \frac{\mathrm{g\,cm}}{\mathrm{s}}\frac{1}{\mathrm{cm^2\,s}} = \frac{\mathrm{g}}{\mathrm{cm\,s^2}}\ . \tag{3.144}$$

Since the velocity gradient components have the dimension s^{-1}, the viscosity constants must have the dimension

$$[\eta] = \frac{\mathrm{g}}{\mathrm{cm\,s}} = [\zeta]\ . \tag{3.145}$$

3.3.2 The equations of viscous hydrodynamics

The diffusion of fluid particles cannot affect mass conservation, so the continuity equation (3.36) for the mass density must remain unchanged. However, the preceding considerations of diffusive particle, energy and momentum transport have shown that we have to augment the stress-energy tensor of an ideal fluid by the diffusive stress-energy tensor,

$$\bar{T} \to \bar{T} + \bar{T}_{\mathrm{d}}\ . \tag{3.146}$$

Since momentum conservation is expressed by the spatial components $\partial_\mu T^{\mu i} = 0$ of the conservation equation $\partial_\mu T^{\mu\nu} = 0$, the additional, diffusive part of the stress-energy tensor creates the further terms

$$\begin{aligned} \vec{\nabla}\cdot\bar{T}_{\mathrm{d}} &= -\eta\left[\vec{\nabla}^2\vec{v} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right) - \frac{2}{3}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)\right] - \zeta\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right) \\ &= -\eta\vec{\nabla}^2\vec{v} - \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right) \end{aligned} \tag{3.147}$$

in the momentum-conservation equation (3.54). With those terms, it turns into the Navier-Stokes equation

$$\rho\left(\partial_t + \vec{v}\cdot\vec{\nabla}\right)\vec{v} + \vec{\nabla}P = \eta\vec{\nabla}^2\vec{v} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)\ . \tag{3.148}$$

?

Compare the Navier-Stokes equation (3.148) to the Euler equation (3.54) and discuss (with yourself or somebody else) the physical meaning of the difference between the two.

In the energy-conservation equation, we must first of all take the diffusive transport of the internal energy into account, thus

$$\vec{q} \to \vec{q} + \vec{q}_\varepsilon = \vec{q} - \kappa\vec{\nabla}T \tag{3.149}$$

needs to be substituted in (3.40). However, this is not all, since the diffusive momentum-current density corresponds to a force per unit area, or a pressure. This force, times the flow velocity, is the internal work carried out per unit time by the diffusing particles on the fluid itself; in other words, it is the energy per unit time dissipated by friction. The current density of this friction work is the flow velocity times the force per unit area,

$$\vec{q}_{\mathrm{fr}} = -\bar{T}_{\mathrm{d}}\vec{v}\,, \tag{3.150}$$

which must also be added to the energy current density $\vec{q}$. Thus,

$$\vec{q} \to \vec{q} - \kappa\vec{\nabla}T - \bar{T}_{\mathrm{d}}\vec{v} \tag{3.151}$$

must be replaced in (3.40). Deriving the final form for the energy-conservation equation, we must finally recall that the momentum-conservation equation has also changed. We used it before to bring the energy-conservation equation into the form (3.60), subtracting the momentum-conservation equation, multiplied with the flow velocity, from the energy-conservation equation. We thus have to subtract a further term $(\vec{\nabla} \cdot \bar{T}_{\mathrm{d}}) \cdot \vec{v}$ from the energy-conservation equation. Summing up, the right-hand side of the energy-conservation equation must now be replaced by the terms

$$\vec{\nabla} \cdot \left(\kappa \vec{\nabla} T + \bar{T}_{\mathrm{d}} \vec{v}\right) - \left(\vec{\nabla} \cdot \bar{T}_{\mathrm{d}}\right) \cdot \vec{v} = \vec{\nabla} \cdot \left(\kappa \vec{\nabla} T\right) + \mathrm{Tr}\left[\bar{T}_{\mathrm{d}}^{\top} \left(\vec{\nabla} \otimes \vec{v}\right)\right] . \tag{3.152}$$

Since the diffusive stress-energy tensor is symmetric, any antisymmetric part of the velocity-gradient tensor $\vec{\nabla} \otimes \vec{v}$ would be cancelled in its contraction with $\bar{T}_{\mathrm{d}}$, hence we can just as well write

$$\mathrm{Tr}\left[\bar{T}_{\mathrm{d}}^{\top} \left(\vec{\nabla} \otimes \vec{v}\right)\right] = \mathrm{Tr}\left(\bar{T}_{\mathrm{d}}^{\top} Dv\right) , \tag{3.153}$$

where Dv abbreviates the symmetrised velocity-gradient tensor,

$$Dv := \frac{1}{2}\left[\left(\vec{\nabla} \otimes \vec{v}\right) + \left(\vec{\nabla} \otimes \vec{v}\right)^{\top}\right] . \tag{3.154}$$

The energy-conservation equation for a viscous fluid then reads

$$\partial_t \varepsilon + \vec{\nabla} \cdot (\varepsilon \vec{v}) + P \vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot \left(\kappa \vec{\nabla} T\right) + \mathrm{Tr}\left(\bar{T}_{\mathrm{d}}^{\top} Dv\right) . \tag{3.155}$$

This intuitive equation shows that temperature gradients cause diffusive heat transport, and viscosity creates heat by friction. Together with the unchanged continuity equation (3.36) for the density ρ, the Navier-Stokes equation (3.148) and the energy-conservation equation (3.155) are the fundamental equations for viscous hydrodynamics.

3.3.3 Entropy

It is instructive to translate the energy-conservation equation (3.155) to an equation explicitly containing the fluid entropy. For doing so, we introduce the internal energy and the entropy per unit mass, $\tilde{\varepsilon}$ and $\tilde{s}$, respectively, by defining

$$\varepsilon = \tilde{\varepsilon}\rho , \quad s = \tilde{s}\rho , \tag{3.156}$$

This enables us to bring the left-hand side of (3.155) into the form

$$\partial_t \varepsilon + \vec{\nabla} \cdot (\varepsilon \vec{v}) + P \vec{\nabla} \cdot \vec{v} = \partial_t (\tilde{\varepsilon}\rho) + \vec{\nabla} \cdot (\tilde{\varepsilon}\rho \vec{v}) + P \vec{\nabla} \cdot \vec{v} . \tag{3.157}$$

Subtracting the continuity equation leaves us with

$$\partial_t \varepsilon + \vec{\nabla} \cdot (\varepsilon \vec{v}) + P \vec{\nabla} \cdot \vec{v} = \rho \left(\partial_t + \vec{v} \cdot \vec{\nabla}\right) \tilde{\varepsilon} + P \vec{\nabla} \cdot \vec{v} = \rho \frac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}t} + P \vec{\nabla} \cdot \vec{v} . \tag{3.158}$$

The volume per unit mass, $\tilde{V}$, is the reciprocal density, $\tilde{V} = \rho^{-1}$, hence

$$\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} = -\rho^{-2} \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho^{-2} \left(\partial_t + \vec{v} \cdot \vec{\nabla}\right) \rho = \rho^{-1} \vec{\nabla} \cdot \vec{v} , \tag{3.159}$$

where we have used the continuity equation once more in the final step. Solving for the velocity divergence,

$$\vec{\nabla} \cdot \vec{v} = \rho \frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} , \tag{3.160}$$

we can write (3.158) as

$$\partial_t \varepsilon + \vec{\nabla} \cdot (\varepsilon \vec{v}) + P\vec{\nabla} \cdot \vec{v} = \rho \left(\frac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}t} + P \frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} \right) . \tag{3.161}$$

?

What is the physical meaning of the expression $\rho T \mathrm{d}\tilde{s}$?

By the first law of thermodynamics, $T\mathrm{d}\tilde{s} = \mathrm{d}\tilde{\varepsilon} + P\mathrm{d}\tilde{V}$, we can finally identify

$$\partial_t \varepsilon + \vec{\nabla} \cdot (\varepsilon \vec{v}) + P\vec{\nabla} \cdot \vec{v} = \rho T \frac{\mathrm{d}\tilde{s}}{\mathrm{d}t} \tag{3.162}$$

and write the energy-conservation equation (3.155) as an equation for the total time derivative of the specific entropy,

$$\rho T \frac{\mathrm{d}\tilde{s}}{\mathrm{d}t} = \vec{\nabla} \cdot \left(\kappa \vec{\nabla} T \right) + \mathrm{Tr}\left(\bar{T}_{\mathrm{d}}^{\top} D v \right) . \tag{3.163}$$

This shows explicitly how heat conduction and viscous friction change the entropy. In absence of transport processes, $\kappa = 0 = \eta = \zeta$, the specific entropy is conserved. In particular, flows of ideal fluids are isentropic.

3.3.4 Fluids in a gravitational field

From a consistent, generally-relativistic point of view, fluids in a gravitational field should be treated starting from the covariant, local energy-momentum conservation laws (3.33). The covariant derivative would then automatically take care of gravitational forces. Here, in our non-relativistic, Newtonian approach, we have to add gravitational fields by hand to the fluid equations. We shall do so by deriving the energy-momentum tensor of the free gravitational field, whose space-space components can then be added to the stress-energy tensor T^{ij} of the fluid.

In a specially-relativistic theory for a scalar field ϕ characterised by a Lagrange density $\mathcal{L}(\phi, \partial_\mu \phi)$, the energy-momentum tensor $T^\mu{}_\nu$ of the field is given by the Legendre transform

$$T^\mu{}_\nu = \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \delta^\mu_\nu \mathcal{L} ; \tag{3.164}$$

cf. (1.104) and the explanation given there. The Lagrange density

$$\mathcal{L} = \frac{1}{8\pi G} \partial_\mu \phi \partial^\mu \phi + \phi \rho \tag{3.165}$$

serves our purposes because its Euler-Lagrange equation,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 , \tag{3.166}$$

reproduces Poisson's equation if the potential ϕ does not change very rapidly with time,

$$-\partial_0 \partial^0 \phi = c^{-2} \partial_t^2 \phi \ll \partial_i \partial^i \phi . \tag{3.167}$$

Then, we can approximate the d'Alembert by the Laplace operator and find the familiar field equation

$$\vec{\nabla}^2\phi = 4\pi G\rho \tag{3.168}$$

relating the potential to the density, i.e. the Poisson equation.

We thus take

$$\mathcal{L}_{\text{free}} = \frac{1}{8\pi G}\partial_\mu\phi\partial^\mu\phi \tag{3.169}$$

as the Lagrange density of the free, Newtonian gravitational field and find the energy-momentum tensor

$$\left(T^\mu{}_\nu\right)_{\text{grav}} = \frac{1}{4\pi G}\left(\partial_\nu\phi\partial^\mu\phi - \frac{1}{2}\delta^\mu_\nu\partial_\alpha\phi\partial^\alpha\phi\right) \tag{3.170}$$

for it. Its stress-energy tensor is then

$$\bar{T}_{\text{grav}} = \frac{1}{4\pi G}\left[\vec{\nabla}\phi\otimes\vec{\nabla}\phi - \frac{1}{2}\left(\vec{\nabla}\phi\right)^2\mathbb{1}_3\right] , \tag{3.171}$$

again neglecting the time derivative of ϕ compared to its spatial derivatives. This gravitational stress-energy tensor must now be introduced into the equations for momentum and energy conservation.

The momentum-conservation equation must be augmented by the divergence of $\bar{T}_{\text{grav}}$,

$$\vec{\nabla}\cdot\bar{T}_{\text{grav}} = \frac{1}{4\pi G}\left(\vec{\nabla}^2\phi\right)\vec{\nabla}\phi = \rho\vec{\nabla}\phi , \tag{3.172}$$

where the Poisson equation (3.168) was used in the last step. With this additional specific force term, the Navier-Stokes equation becomes

$$\rho\left(\partial_t + \vec{v}\cdot\vec{\nabla}\right)\vec{v} + \vec{\nabla}P = -\rho\vec{\nabla}\Phi + \eta\vec{\nabla}^2\vec{v} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right) . \tag{3.173}$$

Caution Of course, we could have guessed the additional gravitational-force term $-\rho\vec{\nabla}\Phi$ in the Navier-Stokes immediately since it simply expresses the gravitational-force density. Returning to the stress-energy tensor of the gravitational field and taking its divergence emphasises the common origin of all force terms in the Euler or Navier-Stokes equations. ◀

In most applications, the stress-energy tensor for the free gravitational field is integrated over the entire volume of a body. If the boundary surface of the integration volume is chosen large enough, we can use Gauss' law to add or subtract arbitrary divergences from $\bar{T}_{\text{grav}}$ without changing the volume integral over it. This allows us to modify the expression for the stress-energy tensor to bring it into more familiar forms that can more easily be interpreted. We shall use the sign $\simeq$ here to express that two expressions for T^{ij}_{grav} differ only by a divergence.

Let us begin with the expression (3.171) and write

$$\begin{aligned}\bar{T}_{\text{grav}} &= \frac{\vec{\nabla}\phi\otimes\vec{\nabla}\phi}{4\pi G} - \frac{\mathbb{1}_3}{8\pi G}\vec{\nabla}\cdot\left(\phi\vec{\nabla}\phi\right) + \frac{\mathbb{1}_3}{2}\phi\rho \\ &\simeq \frac{\vec{\nabla}\phi\otimes\vec{\nabla}\phi}{4\pi G} + \frac{\mathbb{1}_3}{2}\phi\rho .\end{aligned} \tag{3.174}$$

The trace of the final expression is

$$\begin{aligned}\mathrm{Tr}\,\bar{T}_{\text{grav}} &= \frac{1}{4\pi G}\left(\vec{\nabla}\phi\right)^2 + \frac{3}{2}\phi\rho = \frac{1}{4\pi G}\vec{\nabla}\cdot\left(\phi\vec{\nabla}\phi\right) - \rho\phi + \frac{3}{2}\rho\phi \\ &\simeq \frac{1}{2}\rho\phi .\end{aligned} \tag{3.175}$$

If we rather begin with the expression $\vec{x} \otimes \bar{T}$ for an arbitrary, not necessarily gravitational stress-energy tensor, we can write

$$-\vec{x} \otimes \left(\vec{\nabla} \cdot \bar{T}\right) = -\vec{\nabla} \cdot \left(\vec{x} \otimes \bar{T}\right) + \bar{T} \simeq \bar{T} \ . \quad (3.176)$$

Applying this result to the gravitational stress-energy tensor, and identifying its divergence (3.172), we find

$$\bar{T}_{\rm grav} \simeq -\rho \vec{x} \otimes \vec{\nabla}\phi \ . \quad (3.177)$$

This leads to Chandrasekhar's expression for the gravitational potential energy, which is often used in stellar dynamics,

$$\int {\rm d}^3 x\, \bar{T}_{\rm grav} =: U = -\int {\rm d}^3 x\, \rho \vec{x} \otimes \vec{\nabla}\phi \ . \quad (3.178)$$

From our previous result (3.175), we can further infer that the volume integral over the trace of $\bar{T}_{\rm grav}$ is

$$\int {\rm d}^3 x\ {\rm Tr}\, \bar{T}_{\rm grav} = {\rm Tr}\, U = \frac{1}{2} \int {\rm d}^3 x\, \rho\phi \ . \quad (3.179)$$

Comparing (3.178) and (3.179), we find the useful equality

$$\frac{1}{2} \int {\rm d}^3 x\, \rho\phi = -\int {\rm d}^3 x\, \rho \vec{x} \cdot \vec{\nabla}\phi \ . \quad (3.180)$$

3.3.5 The tensor virial theorem

We can now derive an important generalisation of the virial theorem from classical mechanics, which is typically derived there for point particles on bounded orbits. We begin with the inertial tensor of a body, defined by

$$I = \int {\rm d}^3 x\, \rho \vec{x} \otimes \vec{x} \ . \quad (3.181)$$

Integrating over a fixed volume, the position vectors $\vec{x}$ do not depend on time. The total time derivative of I is

$$\frac{{\rm d}I}{{\rm d}t} = \left(\partial_t + \vec{v} \cdot \vec{\nabla}\right) \int {\rm d}^3 x\, \rho \vec{x} \otimes \vec{x} = \int {\rm d}^3 x\, (\partial_t \rho) \vec{x} \otimes \vec{x} \quad (3.182)$$

because the volume integral does not depend on $\vec{x}$. The continuity equation allows us to continue

$$\begin{aligned} \int {\rm d}^3 x\, (\partial_t \rho) \vec{x} \otimes \vec{x} &= -\int {\rm d}^3 x\, \vec{\nabla} \cdot (\rho \vec{v})\, \vec{x} \otimes \vec{x} \\ &= -\int {\rm d}^3 x\, \vec{\nabla} \cdot (\rho \vec{x} \otimes \vec{x} \otimes \vec{v}) + \int {\rm d}^3 x\, \rho\, (\vec{x} \otimes \vec{v} + \vec{v} \otimes \vec{x}) \\ &= \int {\rm d}^3 x\, \rho\, (\vec{x} \otimes \vec{v} + \vec{v} \otimes \vec{x}) \ . \end{aligned} \quad (3.183)$$

?

Can you confirm the calculation shown in (3.183)?

The second absolute time derivative of the inertial tensor is thus

$$\frac{{\rm d}^2 I}{{\rm d}t^2} = \int {\rm d}^3 x\, \left[\partial_t (\rho \vec{v}) \otimes \vec{x} + \vec{x} \otimes \partial_t (\rho \vec{v})\right] \ . \quad (3.184)$$

Now, we use (3.176) and take advantage of momentum conservation, $\partial_0 T^{0i} + \partial_j T^{ij} = 0$, to replace the divergence of the stress-energy tensor by the time derivative of the energy-current density $T^{0j} = \rho c v^j$,

$$T^{ij} \simeq x^i \partial_0 T^{0j} = x^i \partial_t \left(\rho v^j\right) \ . \tag{3.185}$$

Symmetrising this expression,

$$\bar{T} = \frac{1}{2} \left[\vec{x} \otimes \partial_t \left(\rho \vec{v}\right) + \partial_t \left(\rho \vec{v}\right) \otimes \vec{x} \right] \ , \tag{3.186}$$

and inserting the result into (3.184), we can finally write

$$\frac{1}{2} \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \int \mathrm{d}^3 x \, \bar{T} \ . \tag{3.187}$$

For a perfect fluid in a gravitational field, the stress-energy tensor reads

$$\bar{T} = \rho \vec{v} \otimes \vec{v} + P \, \mathbb{1}_3 + \bar{T}_{\mathrm{grav}} \ . \tag{3.188}$$

We integrate this over the complete volume of the fluid, use (3.178) and find

$$\frac{1}{2} \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \int \mathrm{d}^3 x \, \rho \vec{v} \otimes \vec{v} + \mathbb{1}_3 \int \mathrm{d}^3 x \, P + U \ . \tag{3.189}$$

This is the tensor virial theorem for a pefect fluid in its most general form. If the system is stable, the left-hand side vanishes, and a relation between the kinetic-energy tensor

$$K = \frac{1}{2} \int \mathrm{d}^3 x \, \rho \vec{v} \otimes \vec{v} \, , \tag{3.190}$$

the potential-energy tensor U^{ij} and the volume-integrated pressure remains,

$$2K = -\mathbb{1}_3 \int \mathrm{d}^3 x \, P - U \ . \tag{3.191}$$

3.3.6 Transformation to cylindrical or spherical coordinates

It is convenient in many applications of hydrodynamics to use coordinates other than Cartesian ones, in particular when systems with axial or spherical symmetry are to be studied. Then, of course, the spatial differential operators need to be transformed accordingly, but there is one more aspect of the transformation that needs to be taken into accout.

In cylindrical coordinates (r, φ, z), the basis vectors expressed in Cartesian coordinates are

$$\hat{e}_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \, , \quad \hat{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \, , \quad \hat{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ . \tag{3.204}$$

Since the position vector is $\vec{x} = r\hat{e}_r + z\hat{e}_z$, the velocity is

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r + \dot{z}\hat{e}_z = \dot{r}\hat{e}_r + r\dot{\varphi}\hat{e}_\varphi + \dot{z}\hat{e}_z \ , \tag{3.205}$$

Example: Virial theorem applied to a homogeneous sphere

To illustrate the power of the virial theorem to find out about the equilibrium state of a perfect fluid in a gravitational field, suppose we have a homogeneous sphere of density ρ, mass M and radius R which is macroscopically at rest, $v^i = 0$. The kinetic energy tensor vanishes, $K^{ij} = 0$. The fluid is assumed to have an ideal equation of state,

$$P = \frac{\rho}{m} k_\mathrm{B} T \ , \tag{3.192}$$

with a constant temperature throughout. Then,

$$\int \mathrm{d}^3 x\, P = \frac{M}{m} k_\mathrm{B} T \ . \tag{3.193}$$

By (3.179), the trace of the potential-energy tensor is

$$\begin{aligned} \mathrm{Tr}\, U &= \frac{1}{2} \int \mathrm{d}^3 x\, \rho\phi = -\frac{4\pi G\rho}{2} \int_0^R \frac{M(r)}{r} r^2 \mathrm{d}r = -\frac{3G}{10} \left(\frac{4\pi}{3}\right)^2 \rho^2 R^5 \\ &= -\frac{3}{10} \frac{GM^2}{R} \ . \end{aligned} \tag{3.194}$$

The trace of the tensor virial theorem (3.191) thus implies the relation

$$\frac{k_\mathrm{B} T}{m} = \frac{1}{10} \frac{GM}{R} \tag{3.195}$$

between the mass, the radius and the temperature of the sphere in equilibrium. Its so-called virial radius is

$$R = \frac{1}{10} \frac{GMm}{k_\mathrm{B} T} \ . \tag{3.196}$$

Suppose now that the sphere is rotating slowly like a solid body. The rotation needs to be slow to ensure that the body can still be assumed to be spherical. With a constant angular velocity $\vec{\omega}$, the velocity field is

$$\vec{v} = \vec{\omega} \times \vec{r} \ , \quad \vec{v}^2 = \omega^2 r^2 \sin^2\theta \tag{3.197}$$

if we arrange the z axis of the coordinate system to be parallel to the angular velocity $\vec{\omega}$ and θ is the usual polar angle. The trace of the kinetic-energy tensor (3.190) is

$$\begin{aligned} \mathrm{Tr}\, K &= \frac{2\pi}{2} \rho\omega^2 \int_0^R r^4 \mathrm{d}r \int_0^\pi \sin^2\theta \sin\theta \mathrm{d}\theta \\ &= \frac{\pi}{5} \rho\omega^2 R^5 \int_{-1}^1 \left(1 - \mu^2\right) \mathrm{d}\mu \\ &= \frac{M}{5} \omega^2 R^2 \ . \end{aligned} \tag{3.198}$$

The trace of the tensor virial theorem (3.191) now gives the cubic equation

$$\frac{2}{5} \omega^2 R^3 + \frac{3}{m} k_\mathrm{B} T R - \frac{3GM}{10} = 0 \ . \tag{3.199}$$

◀

Example: Virial theorem applied to a slowly rotating, fln shomogeneous sphere

For a slowly rotating sphere, R will deviate little from the virial radius (3.196) of the sphere at rest, which we now call R_0 to write

$$R = R_0 + \delta R = R_0 \left(1 + \frac{\delta R}{R_0}\right) . \tag{3.200}$$

To lowest order in the small quantities ω^2 and δR, (3.200) can be approximated by

$$\frac{2}{5}\omega^2 R_0^3 + \frac{3}{m} k_\mathrm{B} T R_0 \left(1 + \frac{\delta R}{R_0}\right) - \frac{3GM}{10} = 0 . \tag{3.201}$$

With R_0 from (3.196), we can further simplify this equation to

$$\delta R = -\frac{4}{3}\frac{\omega^2 R_0^4}{GM} . \tag{3.202}$$

Therefore, if the temperature of the fluid in the rotating sphere is the same as in the non-rotating sphere, its virial radius is reduced because the centrifugal force partly stabilises the body against gravity, allowing the body to be smaller. We can even set $T = 0$ in (3.199) and find

$$R = \left(\frac{3GM}{4\omega^2}\right)^{1/3} \tag{3.203}$$

for the radius of a stable, self-gravitating, rotating sphere. ◂

where $\dot{\hat{e}}_r = \dot{\varphi}\hat{e}_\varphi$ was inserted. We read off the velocity components

$$v_r = \dot{r}\ , \quad v_\varphi = r\dot{\varphi}\ , \quad v_z = \dot{z} \tag{3.206}$$

and write the acceleration as

$$\vec{a} = \dot{v}_r\hat{e}_r + \dot{v}_\varphi\hat{e}_\varphi + \dot{v}_z\hat{e}_z + v_r\dot{\hat{e}}_r + v_\varphi\dot{\hat{e}}_\varphi\ . \tag{3.207}$$

Since the time derivatives of the unit vectors $\hat{e}_r$ and $\hat{e}_\varphi$ are

$$\dot{\hat{e}}_r = \frac{v_\varphi}{r}\hat{e}_\varphi\ , \quad \dot{\hat{e}}_\varphi = -\frac{v_\varphi}{r}\hat{e}_r\ , \tag{3.208}$$

we can immediately identify the acceleration components

$$a_r = \dot{v}_r - \frac{v_\varphi^2}{r}\ , \quad a_\varphi = \dot{v}_\varphi + \frac{v_r v_\varphi}{r}\ , \quad a_z = \dot{v}_z\ . \tag{3.209}$$

Therefore, the components of the acceleration cannot simply be written as time derivatives of the velocity, but acquire additional terms. The expressions (3.209) imply that, in cylinder coordinates, the components in (r, φ, z) direction of the total time derivative on the left-hand side of Euler's equation needs to be augmented as

$$\mathrm{d}_t v_r \to \mathrm{d}_t v_r - \frac{v_\varphi^2}{r}\ , \quad \mathrm{d}_t v_\varphi \to \mathrm{d}_t v_\varphi + \frac{v_r v_\varphi}{r}\ , \quad \mathrm{d}_t v_z \to \mathrm{d}_t v_z\ . \tag{3.210}$$

?

Convince yourself by your own calculation of the expressions (3.208) and (3.213) for the time derivatives of the unit vectors.

In much the same way, we proceed for spherical polar coordinates (r, θ, φ), for which the basis vectors are

$$\hat{e}_r = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}\ , \quad \hat{e}_\theta = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix}\ , \quad \hat{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}\ . \tag{3.211}$$

Since $\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta + \dot{\varphi}\sin\theta\hat{e}_\varphi$, the components of the velocity $\vec{v} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$ are

$$v_r = \dot{r}\ , \quad v_\theta = r\dot{\theta}\ , \quad v_\varphi = r\dot{\varphi}\sin\theta\ . \tag{3.212}$$

We can thus write the time-derivatives of the unit vectors as

$$\begin{aligned} \dot{\hat{e}}_r &= \frac{v_\theta}{r}\hat{e}_\theta + \frac{v_\varphi}{r}\hat{e}_\varphi\ , \quad \dot{\hat{e}}_\theta = -\frac{v_\theta}{r}\hat{e}_r + \frac{v_\varphi}{r}\cot\theta\hat{e}_\varphi\ , \\ \dot{\hat{e}}_\varphi &= -\frac{v_\varphi}{r}\left(\hat{e}_r + \cot\theta\hat{e}_\theta\right) \end{aligned} \tag{3.213}$$

and immediately identify the components

$$\begin{aligned} a_r &= \dot{v}_r - \frac{v_\theta^2 + v_\varphi^2}{r}\ , \quad a_\theta = \dot{v}_\theta + \frac{v_r v_\theta}{r} - \frac{v_\varphi^2}{r}\cot\theta\ , \\ a_\varphi &= \dot{v}_\varphi + \frac{v_r v_\varphi}{r} + \frac{v_\theta v_\varphi}{r}\cot\theta \end{aligned} \tag{3.214}$$

of the acceleration.

In spherical coordinates, then, the left-hand side of Euler's equation needs to be transformed as

$$\begin{aligned} \mathrm{d}_t v_r &\to \mathrm{d}_t v_r - \frac{v_\theta^2 + v_\varphi^2}{r} \\ \mathrm{d}_t v_\theta &\to \mathrm{d}_t v_\theta + \frac{v_r v_\theta}{r} - \frac{v_\varphi^2}{r}\cot\theta \\ \mathrm{d}_t v_\varphi &\to \mathrm{d}_t v_\varphi + \frac{v_r v_\varphi}{r} + \frac{v_\theta v_\varphi}{r}\cot\theta\ . \end{aligned} \tag{3.215}$$

The total time derivatives in the transformations (3.210) and (3.215) remain formally unchanged,

$$\mathrm{d}_t v_i = \left(\partial_t + \vec{v}\cdot\vec{\nabla}\right) v_i\ , \tag{3.216}$$

but the gradient operator $\vec{\nabla}$ needs to be expressed in the respective coordinate basis.

Example: Hydrodynamic equations in cylinder coordinates

To give one specific example, we express the continuity and Euler equations for ideal hydrodynamics in cylinder coordinates (r, φ, z). Since the gradient and the divergence are

$$\vec{\nabla} f = \hat{e}_r\partial_r + \frac{\hat{e}_\varphi}{r}\partial_\varphi + \hat{e}_z\partial_z \quad \text{and} \quad \vec{\nabla}\cdot\vec{f} = \frac{1}{r}\partial_r(r f_r) + \frac{1}{r}\partial_\varphi f_\varphi + \partial_z f_z\ , \tag{3.217}$$

the continuity equation transforms to

$$\partial_t\rho + \frac{1}{r}\partial_r(r\rho v_r) + \frac{1}{r}\partial_\varphi(\rho v_\varphi) + \partial_z(\rho v_z) = 0\ , \tag{3.218}$$

while the components of Euler's equation turn into

$$\begin{aligned} \partial_t v_r + \left(\vec{v}\cdot\vec{\nabla}\right) v_r - \frac{v_\varphi^2}{r} &= -\partial_r\left(\frac{P}{\rho} + \phi\right)\ , \\ \partial_t v_\varphi + \left(\vec{v}\cdot\vec{\nabla}\right) v_\varphi + \frac{v_r v_\varphi}{r} &= -\frac{1}{r}\partial_\varphi\left(\frac{P}{\rho} + \phi\right)\ , \\ \partial_t v_z + \left(\vec{v}\cdot\vec{\nabla}\right) v_z &= -\partial_z\left(\frac{P}{\rho} + \phi\right)\ , \end{aligned} \tag{3.219}$$

with the representation of $\vec{\nabla}$ to be taken from (3.217). ◄

Problems

1. Young stars often form in the centre of a thin accretion disk whose height is much smaller than its radius. If the mass of the central object M is much larger than the disk's mass, the gas particles move on approximately Keplerian orbits which are almost circular.

   (a) What is the velocity v of a gas particle as a function of the radius r? Determine also the divergence of the velocity field.

(b) Calculate the components of the velocity tensor

$$v_{ij} = \frac{1}{2}\left(\partial_j v_i - \partial_i v_j\right) \tag{3.220}$$

for the Keplerian disk.

3.4 Flows under specific circumstances

In this section, the hydrodynamical equations are applied to a variety of different flows. We begin with a perturbative analysis to derive the equation (3.226) for sound waves, identifying the expression (3.228) for the sound speed. Following the introduction of polytropic equations of state, we discuss hydrostatic equilibrium configurations and derive the Lane-Emden equation (3.259). Vorticity and circulation are defined next in the derivation of Kelvin's circulation theorem (3.280). Then, we demonstrate Bernoulli's law (3.286) for stationary flows by integration of Euler's equation and apply it to Bondi's problem of spherical accretion, leading to the relations (3.308) between velocity and radius in polytropic or isothermal flows. Next, we extend Bernoulli's law to non-stationary, but irrotational flows in (3.315). Viscous flows are briefly discussed at the end of the section. We begin with the diffusion of vorticity (3.317), define the Reynolds number (3.321) and conclude with viscous flow through a pipe, leading to the Hagen-Poiseulle law (3.329).

3.4.1 Sound waves

We begin with an ideal fluid for which we assume that a solution of the hydrodynamical equations is already given. This solution may consist of functions ρ_0, $\vec{v}_0$ and P_0, with the subscript 0 indicating that these functions are considered as a fixed, given, so-called background solution. We transform into the rest frame of this background solution and can thus assume $\vec{v}_0 = 0$. Then, we perturb this solution by small amounts $\delta\rho$, $\delta\vec{v}$ and δP, insert the perturbed solution

$$\rho = \rho_0 + \delta\rho\ , \quad \vec{v} = \delta\vec{v}\ , \quad P = P_0 + \delta P \tag{3.221}$$

into the continuity- and Euler equations and keep only terms up to first order in the perturbations. This procedure, which is typical for a perturbative analysis, results in

$$\partial_t(\rho_0 + \delta\rho) + \vec{\nabla}\cdot(\rho_0\delta\vec{v}) = 0\ , \quad \partial_t\delta\vec{v} + \frac{\vec{\nabla}(P_0 + \delta P)}{\rho_0 + \delta\rho} = 0\ . \tag{3.222}$$

Typically, the background solution is smooth on the length scale of the perturbations. If we may assume this, we can neglect gradients of ρ_0 and P_0 as well as $\partial_t\rho_0$ and continue writing

$$\partial_t\delta\rho + \rho_0\vec{\nabla}\cdot\delta\vec{v} = 0\ , \quad \partial_t\delta\vec{v} + \frac{\vec{\nabla}\delta P}{\rho_0} = 0\ . \tag{3.223}$$

We further relate the gradient of the pressure perturbation to the gradient of the density perturbation by

$$\vec{\nabla}\delta P = \frac{\partial P}{\partial \rho}\vec{\nabla}\delta\rho =: c_s^2\vec{\nabla}\delta\rho \,, \tag{3.224}$$

introducing the abbreviation c_s^2 for the partial derivative of the pressure with respect to the density. Equations (3.223) then become

$$\partial_t\delta\rho + \rho_0\vec{\nabla}\cdot\delta\vec{v} = 0 \,, \quad \rho_0\partial_t\delta\vec{v} + c_s^2\vec{\nabla}\delta\rho = 0 \,. \tag{3.225}$$

Taking a further time derivative of the first equation and the divergence of the second equation allows us to eliminate the velocity perturbation altogether and express the density perturbation as

$$\partial_t^2\delta\rho - c_s^2\vec{\nabla}^2\delta\rho = 0 \,. \tag{3.226}$$

This is a d'Alembert equation for the density contrast,

$$\Box\delta\rho = 0 \,, \tag{3.227}$$

in which c_s appears as the characteristic velocity. The solutions of (3.227) are linear density waves, accompanied by waves in the velocity perturbation. Such waves are sound waves, and

$$c_s = \left(\frac{\partial P}{\partial \rho}\right)^{1/2} \tag{3.228}$$

is the sound speed. The derivative in (3.228) has to be taken at constant entropy.

?

Why would the sound speed (3.228) have to be determined at constant entropy? Is this necessarily so? What assumption may enter here?

The solutions of the d'Alembert equation can be expanded into plane, mono-"chromatic" waves. Let

$$\delta\rho = a\mathrm{e}^{\mathrm{i}\left(\vec{k}\cdot\vec{x}-\omega t\right)} \,, \quad \delta\vec{v} = \vec{b}\mathrm{e}^{\mathrm{i}\left(\vec{k}\cdot\vec{x}-\omega t\right)} \tag{3.229}$$

be such waves with wave vector $\vec{k}$ and frequency ω for the density and velocity perturbations. Inserting them into the d'Alembert equation gives the dispersion relation

$$k^2 = \frac{\omega^2}{c_s^2} \tag{3.230}$$

familiar from electrodynamics, but with the sound speed in place of the light speed. The second equation (3.225), however, gives

$$\omega\rho_0\vec{b} = c_s^2 a\vec{k} \,. \tag{3.231}$$

The amplitude $\vec{b}$ of the velocity perturbation is thus oriented with the wave vector $\vec{k}$, showing that $\delta\vec{v}$ is longitudinal.

3.4.2 Polytropic equation of state

We have noticed earlier that the equations of hydrodynamics are a set of five equations (one each for the conservation of the mass density, its internal energy and each of its momentum components) for six quantities, namely the density, the pressure, the internal energy or temperature of the fluid and its macroscopic velocity. One equation is missing. Typically, an equation of state is chosen for this purpose, that is an equation relating the pressure to the other fluid properties, such as the density and the temperature.

In astrophysics, it is frequently appropriate to assume the so-called polytropic relation between pressure and density,

$$P(\rho) = P_0 \left(\frac{\rho}{\rho_0}\right)^\gamma , \tag{3.232}$$

which can be derived for any fluid under adiabatic conditions. To see this, consider the first law of thermodynamics, $\delta Q = \mathrm{d}E + P\mathrm{d}V$. If no heat is exchanged, $\delta Q = 0$, and

$$\mathrm{d}E = c_\mathrm{v}\mathrm{d}T = -P\mathrm{d}V . \tag{3.233}$$

The enthalpy is obtained from the internal energy by the Legendre transform

$$H = E + PV , \quad \mathrm{d}H = \mathrm{d}E + P\mathrm{d}V + V\mathrm{d}P . \tag{3.234}$$

?

Why are infinitesimal changes of the internal energy and the enthalpy given by $\mathrm{d}E = c_\mathrm{v}\mathrm{d}T$ and $\mathrm{d}H = c_\mathrm{p}\mathrm{d}T$, respectively, with c_v and c_p being the heat capacities at constant volume or pressure?

Under adiabatic conditions, therefore,

$$\mathrm{d}H = c_\mathrm{p}\mathrm{d}T = V\mathrm{d}P . \tag{3.235}$$

If we now divide (3.235) by (3.233), the temperature differential $\mathrm{d}T$ cancels, and we find

$$\frac{c_\mathrm{p}}{c_\mathrm{v}} = \gamma = -\frac{V}{P}\frac{\mathrm{d}P}{\mathrm{d}V} , \tag{3.236}$$

where γ is defined to be the adiabatic index. Separating variables leads immediately to

$$\frac{\mathrm{d}P}{P} = -\gamma\frac{\mathrm{d}V}{V} , \tag{3.237}$$

or $P \propto V^{-\gamma} \propto \rho^\gamma$, which is already the polytropic relation (3.232). Notice in particular that we have nowhere used the assumption of an ideal gas. The entire derivation is based on the adiabatic condition that the fluid does not exchange heat with its environment. If we can additionally treat the fluid as an ideal gas, we have $PV \propto T$ and conclude

$$PV^\gamma = (PV)V^{\gamma-1} \propto TV^{\gamma-1} = \text{const.} \tag{3.238}$$

For an ideal gas, the polytropic relation (3.232) thus implies

$$T = T_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} . \tag{3.239}$$

The sound speed in a polytropic fluid is easily derived. We have to take the derivative of the pressure with respect to the density at constant entropy, but the polytropic relation has already been derived assuming that entropy is constant. It is therefore justified to write

$$c_\mathrm{s}^2 = \frac{\partial P}{\partial\rho} = \gamma\frac{P}{\rho} = c_\mathrm{s0}^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} . \tag{3.240}$$

For the enthalpy, we begin from (3.235) to derive the enthalpy per unit mass, $\tilde{h}$. Since the volume per unit mass is simply ρ^{-1}, we must have

$$\tilde{h} = \int \frac{\mathrm{d}P}{\rho} = \frac{\gamma}{\gamma - 1}\frac{P}{\rho} = \frac{c_\mathrm{s}^2}{\gamma - 1} \,. \tag{3.241}$$

The relations (3.240) and (3.241) are frequently used and often very convenient in discussions of astrophysical fluid flows.

Let us briefly remark on entropy here before continuing with the discussion of hydrodynamical flows under specific circumstances. The first law of thermodynamics states

$$T\mathrm{d}S = \mathrm{d}E + P\mathrm{d}V = c_\mathrm{v}\mathrm{d}T + P\mathrm{d}V = c_\mathrm{v}\mathrm{d}T + \mathrm{d}(PV) - V\mathrm{d}P \,, \tag{3.242}$$

where c_v is again the heat capacity at constant volume. Dividing by T, using the equation of state $PV = Nk_\mathrm{B}T$ for an ideal gas and the relation

$$c_\mathrm{p} - c_\mathrm{v} = Nk_\mathrm{B} \tag{3.243}$$

between the heat capacities c_p and c_v at constant pressure and constant volume, respectively, we transform (3.242) into

$$\mathrm{d}S = c_\mathrm{p}\frac{\mathrm{d}T}{T} - (c_\mathrm{p} - c_\mathrm{v})\frac{\mathrm{d}P}{P} \,. \tag{3.244}$$

Recalling the adiabatic index $\gamma = c_\mathrm{p}/c_\mathrm{v}$, we have

$$\mathrm{d}S = c_\mathrm{v}\left[\gamma\frac{\mathrm{d}T}{T} - (\gamma - 1)\frac{\mathrm{d}P}{P}\right] \,, \tag{3.245}$$

from which we can infer the derivatives

$$\left(\frac{\partial S}{\partial T}\right)_P = \gamma\frac{c_\mathrm{v}}{T} \quad \text{and} \quad \left(\frac{\partial S}{\partial P}\right)_T = -(\gamma - 1)\frac{c_\mathrm{v}}{P} \tag{3.246}$$

for the entropy with respect to T at constant P, and vice versa. We shall need these relations in the derivation of the convective instability below.

Caution Recall the Maxwell relation

$$\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$$

which, when evaluated for an ideal gas, results in

$$\left(\frac{\partial S}{\partial P}\right)_T = -\frac{Nk_\mathrm{B}}{P} = -\frac{c_\mathrm{p} - c_\mathrm{v}}{P} \,.$$

◀

From the ideal gas equation written in the form

$$T = \frac{PV}{Nk_\mathrm{B}} = \frac{P}{\rho}\frac{\bar{m}}{k_\mathrm{B}} \tag{3.247}$$

with the mean particle mass $\bar{m}$, we immediately infer that

$$\frac{\mathrm{d}T}{T} = \frac{\mathrm{d}P}{P} - \frac{\mathrm{d}\rho}{\rho} \,, \tag{3.248}$$

and insert this expression into (3.244) to find

$$\mathrm{d}S = c_\mathrm{v}\frac{\mathrm{d}P}{P} - c_\mathrm{p}\frac{\mathrm{d}\rho}{\rho} \,. \tag{3.249}$$

The derivatives of the entropy with respect to P at constant ρ and vice versa are thus

$$\left(\frac{\partial S}{\partial P}\right)_\rho = \frac{c_\mathrm{v}}{P} \quad \text{and} \quad \left(\frac{\partial S}{\partial \rho}\right)_P = -\frac{c_\mathrm{p}}{\rho} \,. \tag{3.250}$$

We shall return to these relations in the discussion of the thermal instability.

Finally, it is instructive to conclude from (3.249) that the entropy as a function of pressure and density is

$$S(P,\rho) = c_\mathrm{v} \ln\left[\frac{P}{P_0}\left(\frac{\rho_0}{\rho}\right)^\gamma\right] . \tag{3.251}$$

For a polytropic gas with $P \propto \rho^\gamma$, the entropy is manifestly constant, as it should be by construction.

3.4.3 Hydrostatic equilibrium

We begin our study of hydrodynamical flows under specific, generally simplifying conditions with a fluid in hydrostatic equilibrium. In a static situation, the flow velocity vanishes, $\vec{v} = 0$, and the Navier-Stokes equation (3.148) shrinks to

$$\vec{\nabla} P = -\rho\vec{\nabla}\Phi . \tag{3.252}$$

Taking the curl of this equation, we immediately see that

$$\vec{\nabla}\rho \times \vec{\nabla}\Phi = 0 \tag{3.253}$$

because the curl of a gradient vanishes identically. The gradients of the gravitational potential and of the density must therefore be parallel to each other, which means that the equipotential surfaces, i.e. the surfaces of constant potential, must also be the surfaces of constant density. In hydrostatic equilibrium, the shape of the fluid body thus adapts to the shape of the gravitational potential.

Taking the divergence of the hydrostatic equation, we can use Poisson's equation to write

$$\vec{\nabla} \cdot \left(\frac{\vec{\nabla} P}{\rho}\right) = -4\pi G\rho . \tag{3.254}$$

Once an equation of state is chosen for the fluid, i.e. a relation between the pressure P and the density ρ, this equation determines the configuration of the fluid density in its own gravitational field. Let us suppose that the pressure satisfies the polytropic relation, and restrict the discussion to spherically-symmetric configurations. Then,

$$\frac{1}{r^2}\partial_r\left(r^2\frac{\partial_r P}{\rho}\right) = \frac{c_{\mathrm{s}0}^2}{r^2}\partial_r\left[r^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\partial_r\rho\right] = -4\pi G\rho . \tag{3.255}$$

Instead of the adiabatic index, we now introduce the polytropic index n by defining

$$\gamma - 1 = \frac{1}{n} . \tag{3.256}$$

Moreover, we introduce a function θ to describe the density as

$$\frac{\rho}{\rho_0} = \theta^n , \tag{3.257}$$

define a characteristic radius

$$r_0 = \left(\frac{nc_{\mathrm{s}0}^2}{4\pi G\rho_0}\right)^{1/2} \tag{3.258}$$

and use that to introduce the dimension-less radial coordinate $x = r/r_0$. These operations leave (3.255) in the dimension-less form

$$\frac{1}{x^2}\partial_x\left(x^2\partial_x\theta\right) = -\theta^n \ , \tag{3.259}$$

which is called the Lane-Emden equation.

Given a polytropic index n, it can be solved with the boundary conditions $\partial_x\theta = 0$ and $\theta = 1$ at $x = 0$ to return the density profile of a polytropic, self-gravitating gas sphere. Expanding the differential operator in (3.259), the Lane-Emden equation reads

$$\theta'' + \frac{2}{x}\theta' + \theta^n = 0 \ . \tag{3.260}$$

?

Independently carry out all steps leading from the hydrostatic equation (3.252) to the Lane-Emden equation (3.259).

Example: Solutions of the Lane-Emden equation

Analytic solutions for the Lane-Emden equation exist for $n = 0$, $n = 1$ and $n = 5$. For $n = 0$, direct integration of (3.259) results in

$$\theta = -\frac{x^2}{6} - \frac{A}{x} + B \tag{3.261}$$

with two integration constants A and B. The boundary conditions require $A = 0$ for the solution to remain regular at the centre and $B = 1$ for θ to reach unity there. Thus,

$$\theta(x) = 1 - \frac{x^2}{6} \tag{3.262}$$

for $n = 0$. For $n = 1$, (3.260) is a spherical Bessel differential equation of order zero, which is solved by spherical Bessel function

$$\theta(x) = j_0(x) = \frac{\sin x}{x} \ . \tag{3.263}$$

Numerical solutions for the Lane-Emden equation with adiabatic indices $\gamma = 5/3$ (polytropic index $n = 3/2$) or $\gamma = 4/3$ ($n = 3$) are often used to model the internal structure of white dwarfs or other stars (Figure 3.4). ◄

Another interesting and illustrative example for systems in hydrostatic equilibrium is the case of a gas filled into a spherical gravitational potential well caused by the dominant dark matter. If the gas mass is overall negligible, the gravitational potential is given independently, and the gas just responds to it. This requires us to separate the gas density $\rho_{\rm gas}$ from the dark-matter density $\rho_{\rm DM}$ in the hydrostatic equation, thus

$$\frac{1}{r^2}\partial_r\left(\frac{r^2}{\rho_{\rm gas}}\partial_r P\right) = -4\pi G\rho_{\rm DM} \ . \tag{3.264}$$

With the equation of state for an ideal gas,

$$P = \frac{\rho_{\rm gas}}{m}k_{\rm B}T \ , \tag{3.265}$$

where m is the (mean) mass of a gas particle, we find by integrating once

$$\frac{r^2}{\rho_{\rm gas}}\frac{k_{\rm B}}{m}\partial_r\left(\rho_{\rm gas}kT\right) = -4\pi G\int_0^r r'^2{\rm d}r'\,\rho_{\rm DM} = -GM_{\rm DM}(r) \ , \tag{3.266}$$

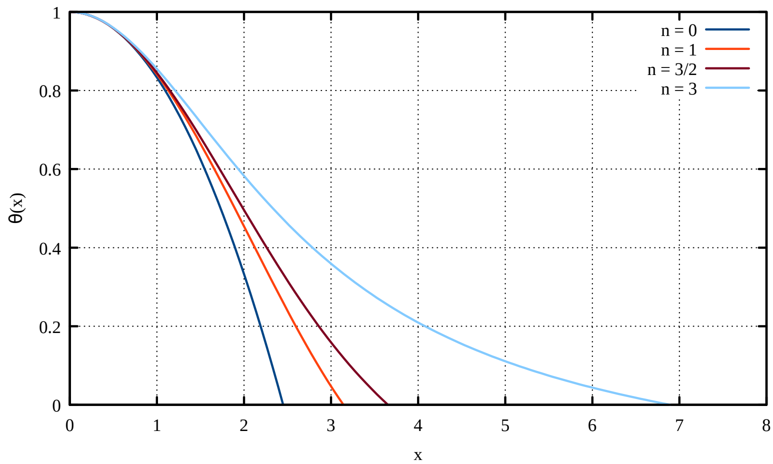

Figure 3.4 Solutions of the Lane-Emden equation are shown for different choices of the polytropic index. The curves are displayed up to their first root only.

where $M_{\rm DM}(r)$ is the dark-matter mass enclosed by a sphere of radius r. Solving this equation for the dark-matter mass shows how it is related to the logarithmic gradients of temperature and gas density,

$$M_{\rm DM}(r) = -\frac{rk_{\rm B}T}{mG}\left(\frac{\mathrm{d}\ln\rho_{\rm gas}}{\mathrm{d}\ln r}+\frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}\right)\ . \tag{3.267}$$

This equation is often applied to find mass estimates for galaxy clusters. There, the two logarithmic gradients can be inferred from X-ray observations of the hot intracluster gas.

3.4.4 Vorticity and Kelvin's circulation theorem

We shall now give up the hydrostatic assumption, but still neglect any dissipative effects, such as viscous friction and heat conduction. In the Navier-Stokes equation (3.148), we therefore set $\eta = 0 = \zeta$, and $\kappa = 0$ in the energy-conservation equation. We then also know from (3.163) that entropy is conserved under such circumstances. Momentum conservation is then expressed by Euler's equation

$$\partial_t\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} + \frac{\vec{\nabla}P}{\rho} + \vec{\nabla}\Phi = 0\ . \tag{3.268}$$

The identity

$$\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) = \vec{\nabla}\left(\frac{v^2}{2}\right) - \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} \tag{3.269}$$

enables us to replace the convective velocity derivative $(\vec{v}\cdot\vec{\nabla})\vec{v}$ in (3.268) to obtain

$$\partial_t\vec{v} - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) = -\vec{\nabla}\left(\frac{v^2}{2}\right) - \frac{\vec{\nabla}P}{\rho} - \vec{\nabla}\Phi\ . \tag{3.270}$$

The curl of the velocity,

$$\vec{\Omega} := \vec{\nabla}\times\vec{v}\ , \tag{3.271}$$

is called the *vorticity* of the flow. If we take the curl of Euler's equation in its form (3.270), we find the evolution equation for the vorticity

$$\partial_t \vec{\Omega} = \vec{\nabla} \times \left(\vec{v} \times \vec{\Omega} \right) + \frac{\vec{\nabla}\rho \times \vec{\nabla} P}{\rho^2} \tag{3.272}$$

since the curl of the gradients vanishes identically. If the pressure P is a function of ρ only, as for example in a polytropic fluid, the gradients of P and ρ must align because then

$$\vec{\nabla} P = \frac{\mathrm{d}P}{\mathrm{d}\rho} \vec{\nabla} \rho \quad \Rightarrow \quad \vec{\nabla} P \times \vec{\nabla} \rho = 0 \ . \tag{3.273}$$

For such *barotropic* fluids, the vorticity equation simplifies to

$$\frac{\partial \Omega}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{\Omega} \right) \ . \tag{3.274}$$

?

What does $\partial_t \vec{\Omega} = 0$ imply for the solution(s) of the vorticity equation (3.274) for barotropic fluids?

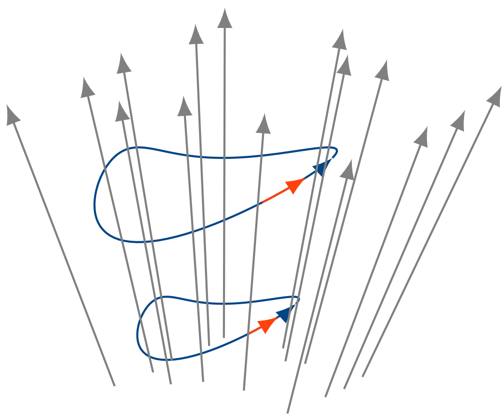

Figure 3.5 Illustration of Kelvin's circulation theorem: The circulation of the velocity field in an inviscid fluid is conserved.

Having derived an evolution equation for the vorticity, we now consider the so-called *circulation*, which is the line integral over the velocity along closed curves swimming with the fluid flow,

$$\Gamma := \oint_C \vec{v} \cdot \mathrm{d}\vec{l} \ . \tag{3.275}$$

We are interested in the total change with time of the circulation embedded into the flow (Figure 3.5). We must therefore take into consideration that the contour C is deformed by the flow. The total time derivative of Γ consists of the change of the velocity field within the contour, plus the change of the contour itself. For a more transparent notation, we write the infinitesimal path length $\mathrm{d}\vec{l}$ as a difference $\delta\vec{r}$ of the position vectors pointing at the beginning and the end of $\mathrm{d}\vec{l}$. Accordingly, we write

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint_C \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \cdot \mathrm{d}\vec{l} + \oint_C \vec{v} \cdot \frac{\mathrm{d}\delta\vec{r}}{\mathrm{d}t} \ . \tag{3.276}$$

In the first term on the right-hand side, we expand the total time derivative of the velocity into

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \partial_t\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = \partial_t\vec{v} + \vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v}\times\Omega\ . \tag{3.277}$$

The line integral suggests taking the curl and applying Stokes' law. The curl of (3.277) is

$$\frac{\mathrm{d}\vec{\Omega}}{\mathrm{d}t} = \partial_t\vec{\Omega} - \vec{\nabla}\times\left(\vec{v}\times\vec{\Omega}\right) = 0 \tag{3.278}$$

according to the vorticity equation (3.274), hence the first term of the total time derivative (3.276) of the circulation vanishes. The second term is

$$\oint_C \vec{v}\cdot\frac{\mathrm{d}\delta\vec{r}}{\mathrm{d}t} = \oint_C \vec{v}\cdot\delta\vec{v} = \oint_C \vec{\nabla}\left(\frac{v^2}{2}\right)\cdot\mathrm{d}\vec{l} = 0\ , \tag{3.279}$$

which also vanishes because an integral along a closed loop of a gradient field must vanish. The circulation is thus conserved in a barotropic, ideal fluid,

$$\boxed{\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0}\ , \tag{3.280}$$

which is Kelvin's circulation theorem.

3.4.5 Bernoulli's constant

If the fluid is not static, but the flow is stationary, all partial derivatives with respect to time will vanish. In such cases, flow lines can be introduced as the integral curves of the velocity field. Quite obviously, the flow lines must obey the equations

$$\frac{\mathrm{d}x}{v_x} = \mathrm{d}t = \frac{\mathrm{d}y}{v_y} = \frac{\mathrm{d}z}{v_z}\ . \tag{3.281}$$

In ideal fluids, we have seen that the specific entropy $\tilde{s}$ is constant because energy dissipation and heat flows do not occur. For a stationary flow, $\partial_t\tilde{s} = 0$ and

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}t} = \partial_t\tilde{s} + \left(\vec{v}\cdot\vec{\nabla}\right)\tilde{s} = \left(\vec{v}\cdot\vec{\nabla}\right)\tilde{s} = 0\ , \quad \text{thus} \quad \left(\vec{v}\cdot\vec{\nabla}\right)\tilde{s} = 0 \tag{3.282}$$

The specific entropy must therefore be constant along flow lines. Moreover, we have seen in (3.241) that the specific enthalpy per unit mass satisfies

$$\mathrm{d}\tilde{h} = \frac{\mathrm{d}P}{\rho} \tag{3.283}$$

under adiabatic conditions.

For stationary flows, $\partial_t\vec{v} = 0$, and Euler's equation in its form (3.270) implies

$$\frac{1}{2}\vec{\nabla}\left(\vec{v}^2\right) - \vec{v}\times\vec{\Omega} = -\frac{\vec{\nabla}P}{\rho} - \vec{\nabla}\Phi\ . \tag{3.284}$$

Let us now multiply (3.284) with the fluid velocity $\vec{v}$ to obtain the change of its terms with time along flow lines. The term containing the vector product

$\vec{v} \times \vec{\Omega}$ then vanishes because it is perpendicular to $\vec{v}$. The remaining terms can be combined under the gradient,

$$\vec{v} \cdot \vec{\nabla} \left(\frac{v^2}{2} + \tilde{h} + \Phi \right) = 0 \ . \tag{3.285}$$

This reveals that the term in parentheses,

$$\frac{v^2}{2} + \tilde{h} + \Phi =: B = \text{constant along flow lines} \tag{3.286}$$

must be a constant B along flow lines, which is called Bernoulli's constant. We have thus proven Bernoulli's very important and intuitive law for ideal flows. It shows that the specific kinetic energy of the flow, $v^2/2$, is not only balanced by the specific potential energy in the gravitational field, but also by the specific enthalpy. For example, if a gas flow is expanding as it propagates into a surrounding medium and against a gravitational field, the enthalpy term takes into account that the gas will have to exert pressure-volume work against the surrounding medium and thereby cool.

Example: The faucet

Bernoulli's law, together with the equation of continuity, are very powerful tools to study stationary fluid flows. Let us begin with water flowing from a faucet, accelerated by gravity (Figure 3.6). Everyday experience tells us that the diameter of the water shrinks as it falls. How exactly does the diameter depend on the height, and why?
Bernoulli's law tells us that the quantity

$$\frac{v^2}{2} + \tilde{h} + \Phi = \text{const} \tag{3.287}$$

along the flow lines of the water. Here, we can replace the gravitational potential by $\Phi = gz$ if z points vertically upwards, where g is the local gravitational acceleration. The pressure is set by the atmospheric pressure surrounding the water, the density can be assumed to be constant. Bernoulli's law then tells us that the water accelerates as it falls according to $v^2 = v_0^2 + 2g(h - z)$ if it is initially at rest at the height h.
To evaluate the continuity equation for a stationary flow, $\partial_t \rho = 0$, we integrate the divergence $\vec{\nabla} \cdot (\rho \vec{v}) = 0$ over an infinitesimally thin cylinder with cross section A whose axis is aligned with the water. Gauss' law then implies that

$$\rho v A = \text{const} = \rho v_0 A_0 \ , \tag{3.288}$$

from which we conclude that

$$A = \frac{A_0 v_0}{\sqrt{v_0^2 + 2g(h - z)}} \ . \tag{3.289}$$

The cross section of the water decreases as it falls from $z = h$ to $z = 0$. ◀

Example: The Laval nozzle

Completely analogous is the discussion of gas flowing through a nozzle whose cross section first decreases, then increases along the gas flow. For definiteness, the x axis of the coordinate system may point into the direction of the gas flow, and the cross section $A(x)$ is given. Continuity now demands

$$\rho v A(x) = \rho_0 v_0 A_0 \ , \tag{3.290}$$

while Bernoulli's law requires

$$\frac{v^2}{2} + \frac{c_s^2 - c_{s0}^2}{\gamma - 1} = \frac{v_0^2}{2} \ . \tag{3.291}$$

Dividing by the squared initial sound speed c_{s0}^2 gives the dimension-less equation

$$\frac{u^2}{2} + \frac{\alpha^{\gamma-1} - 1}{\gamma - 1} = \frac{u_0^2}{2} \ , \tag{3.292}$$

where the dimension-less density $\alpha := \rho/\rho_0$ was introduced. A similar operation brings the continuity equation into the form

$$\alpha u A = u_0 A_0 \ . \tag{3.293}$$

Let us now take the total differentials of both equations (3.292) and (3.293). This leads us to

$$u\mathrm{d}u + \alpha^{\gamma-2}\mathrm{d}\alpha = 0 \ , \quad \frac{\mathrm{d}\alpha}{\alpha} + \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}A}{A} = 0 \ . \tag{3.294}$$

Eliminating $\mathrm{d}\alpha$ between these two equations leaves us with the equation

$$u\mathrm{d}u\left(1 - \frac{\alpha^{\gamma-1}}{u^2}\right) = u\mathrm{d}u\left(1 - \frac{1}{\mathcal{M}^2}\right) = \alpha^{\gamma-1}\frac{\mathrm{d}A}{A} \ , \tag{3.295}$$

where we have identified the squared local Mach number $\mathcal{M}^2 = u^2/\alpha^{\gamma-1}$. As long as the flow remains subsonic, $\mathcal{M} < 1$, the left-hand side is negative. The flow will continue to accelerate, $u\mathrm{d}u > 0$, if the cross section of the nozzle decreases, $\mathrm{d}A < 0$. This agrees with everyday experience: A gas flow through a narrowing pipe accelerates. However, the sign changes once the flow becomes supersonic, $\mathcal{M} > 1$. Then, for $u\mathrm{d}u$ to remain positive, the cross section of the nozzle must *increase*, $\mathrm{d}A > 0$! Otherwise, once the sonic point is reached, the gas will decelerate in narrowing nozzle. If the situation is arranged such that the sound speed is reached at the narrowest point of the nozzle, the flow will continue accelerating. This is the principle of the Laval nozzle, which is used for example in rocket engines (Figure 3.7). ◂

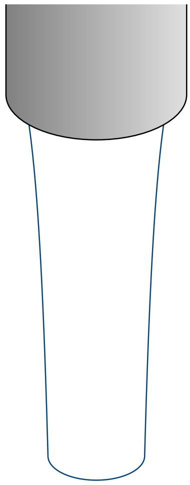

Figure 3.6 Water running from a faucet has a cross section determined by Bernoulli's law.

3.4.6 Bondi accretion

Completely analogous to the discussion of the faucet and the Laval nozzle is Bondi's accretion problem. The situation is as follows: A star or another point-like gravitating body of mass M is placed into a formerly homogeneous, extended gas cloud of density ρ_0 and pressure P_0. Driven by gravity, the gas will flow towards the star. How does it flow, and how much gas per unit time will the star accrete? Again, Bernoulli's law and the continuity equation provide the complete answer.

For a stationary, spherically-symmetric flow, the continuity equation reads

$$\frac{1}{r^2}\partial_r\left(r^2\rho v\right) = 0 \quad \Rightarrow \quad r^2\rho v = \text{const.} \tag{3.296}$$

The constant has the dimension $\mathrm{g\,s^{-1}}$ and therefore corresponds to the accretion rate, i.e. the rate at which matter flows onto the star. If we multiply (3.296) with 4π, we obtain the mass per unit time $\dot{M}$ flowing through the complete spherical surface,

$$4\pi r^2\rho v = -\dot{M}\ , \tag{3.297}$$

where the minus sign is introduced to express that the mass is flowing towards the star.

Bernoulli's law reads

$$\frac{v^2}{2} + \frac{c_\mathrm{s}^2 - c_\mathrm{s0}^2}{\gamma - 1} - \frac{GM}{r} = 0 \tag{3.298}$$

because the gas is assumed to be at rest far away from the star. This equation holds for adiabatic gas which can be treated as a polytrope. If the gas is isothermal and ideal rather than polytropic, its enthalpy per unit mass is

$$\tilde{h} = \int\frac{\mathrm{d}P}{\rho} = \frac{k_\mathrm{B}T}{m}\int\frac{\mathrm{d}\rho}{\rho} = c_\mathrm{s0}^2\ln\left(\frac{\rho}{\rho_0}\right)\ , \tag{3.299}$$

Figure 3.7 An example for a de Laval nozzle is the Vulcain-II engine of an Ariane 5 rocket (Wikipedia, Creative Commons License)

and Bernoulli's law becomes

$$\frac{v^2}{2} + c_{s0}^2 \ln\left(\frac{\rho}{\rho_0}\right) - \frac{GM}{r} = 0 \qquad (3.300)$$

instead. We now divide both versions of Bernoulli's law by the unperturbed, squared sound speed c_{s0}^2, introduce the dimension-less velocity $u = v/c_{s0}$, the density $\alpha = \rho/\rho_0$, the so-called Bondi-radius

$$r_B = \frac{GM}{c_{s0}^2} \qquad (3.301)$$

and the dimension-less radius $x := r/r_B$. These substitutions leave our two versions of Bernoulli's equations in the convenient, dimension-less forms

$$\frac{u^2}{2} + \frac{\alpha^{\gamma-1} - 1}{\gamma - 1} - \frac{1}{x} = 0 , \quad \frac{u^2}{2} + \ln\alpha - \frac{1}{x} = 0 . \qquad (3.302)$$

The same substitutions turn the continuity equation into

$$x^2 \alpha u = \mu , \quad \mu := -\frac{\dot{M}}{4\pi r_B^2 \rho_0 c_{s0}} . \qquad (3.303)$$

The parameter μ is the accretion rate in units of the so-called Bondi accretion rate,

$$\dot{M}_B = 4\pi r_B^2 \rho_0 c_{s0} . \qquad (3.304)$$

We now have two equations, the continuity equation (3.303) and Bernoulli's law (3.302), for the two functions α and u. Eliminating α between them leaves one equation for the velocity u,

$$\frac{u^2}{2} + \frac{\left(\frac{\mu}{x^2 u}\right)^{\gamma-1} - 1}{\gamma - 1} - \frac{1}{x} = 0 \,, \quad \frac{u^2}{2} + \ln\left(\frac{\mu}{x^2 u}\right) - \frac{1}{x} = 0 \,. \tag{3.305}$$

But what accretion rates are possible? Does the flow turn supersonic somewhere? And if so, what happens? In order to see this, let us take complete differentials of the continuity equation,

$$\frac{2\mathrm{d}x}{x} + \frac{\mathrm{d}\alpha}{\alpha} + \frac{\mathrm{d}u}{u} = 0 \,, \tag{3.306}$$

and of Bernoulli's law,

$$u\mathrm{d}u + \alpha^{\gamma-1}\frac{\mathrm{d}\alpha}{\alpha} + \frac{\mathrm{d}x}{x^2} = 0 \,, \quad u\mathrm{d}u + \frac{\mathrm{d}\alpha}{\alpha} + \frac{\mathrm{d}x}{x^2} = 0 \,, \tag{3.307}$$

and eliminate $\mathrm{d}\alpha/\alpha$ between them. For the polytropic gas, the squared sound speed is $c_\mathrm{s}^2 = c_{\mathrm{s}0}^2\alpha^{\gamma-1}$ according to (3.240), while it is constant $c_\mathrm{s}^2 = c_{\mathrm{s}0}^2$ for the isothermal gas. This leads to

$$u\mathrm{d}u\left(1 - \frac{1}{\mathcal{M}^2}\right) = \begin{cases} \dfrac{\mathrm{d}x}{x}\left(2\alpha^{\gamma-1} - \dfrac{1}{x}\right) & \text{polytropic} \\ \dfrac{\mathrm{d}x}{x}\left(2 - \dfrac{1}{x}\right) & \text{isothermal} \end{cases} \,, \tag{3.308}$$

where we have once more identified the squared Mach number $\mathcal{M}$ as before in (3.295) for the Laval nozzle.

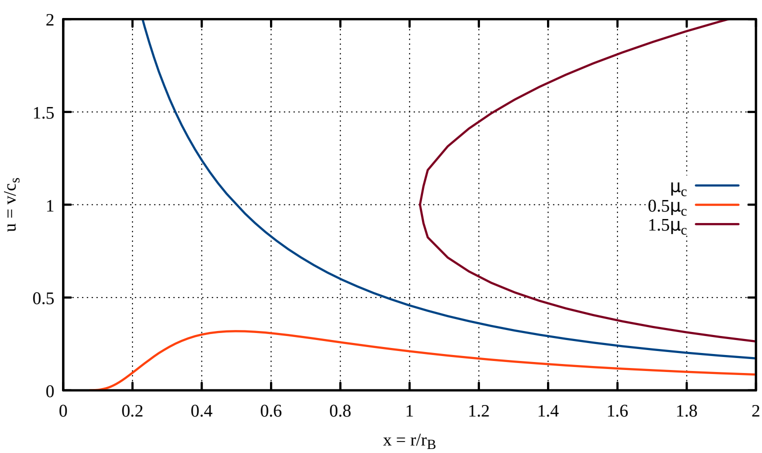

Figure 3.8 The radial velocity is shown as a function of radius for isothermal Bondi accretion. Velocity curves are given for three different accretion rates: the critical accretion rate μ_c in units of the Bondi accretion rate as well as 50 % more or less. The curve for $\mu_\mathrm{c} > 1$ is mathematically possible, but physically excluded because it corresponds to two velocities at the same radius.

This equation shows that there exists a critical radius, $x_\mathrm{c} = 1/2$ in the isothermal and $x_\mathrm{c} = \alpha^{1-\gamma}/2$ in the polytropic case, where the right-hand side vanishes. The left-hand side must then also vanish, which is either possible if the flow comes to a halt there, $u = 0$, if the velocity reaches a maximum, $\mathrm{d}u = 0$, or if the flow turns supersonic, $\mathcal{M} = 1$. What exactly happens, depends on the accretion rate (Figure 3.8). If the flow turns supersonic at the critical radius, $u^2 = \alpha^{\gamma-1}$ in the polytropic and $u = 1$ in the isothermal case, we can solve Bernoulli's equation (3.302) for the dimension-less density α there, obtaining

$$\alpha = \left(\frac{2}{5-3\gamma}\right)^{1/(\gamma-1)} \quad \text{(polytropic)}\,, \quad \alpha = \mathrm{e}^{3/2} \quad \text{(isothermal)}\,. \tag{3.309}$$

The continuity equation finally gives the critical accretion rate,

$$\mu_\mathrm{c} = \frac{1}{4}\left(\frac{2}{5-3\gamma}\right)^{(5-3\gamma)/(2(\gamma-1))}\,, \quad \mu_\mathrm{c} = \frac{\mathrm{e}^{3/2}}{4}\,. \tag{3.310}$$

For accretion rates smaller than μ_c, the flow speed reaches a subsonic maximum at the critical radius, corresponding to a gentle accretion flow that is everywhere subsonic. For accretion rates higher than μ_c, the solution is mathematically possible, but not physically: as Fig. 3.8 shows, the velocity then becomes double-valued where it exists, while two velocities at the same radius cannot exist in a fluid.

?

What would happen to the preceding calculation if the accretion rate $-\dot{M}$ from (3.297) would be set negative? What physical situation would this corresponed to?

3.4.7 Bernoulli's law for irrotational, non-stationary flows

We have derived Bernoulli's law for stationary flows before. It can be generalised to some degree for irrotational flows. For those, $\vec{\nabla} \times \vec{v} = \vec{\Omega} = 0$, which allows us to introduce a velocity potential ψ such that $\vec{v} = \vec{\nabla}\psi$. Euler's equation can then be written in the form

$$\partial_t \vec{\nabla}\psi + \vec{\nabla}\left(\frac{v^2}{2}\right) + \frac{\vec{\nabla}P}{\rho} + \vec{\nabla}\Phi = 0\,. \tag{3.311}$$

For adiabatic flows,

$$\vec{\nabla}\tilde{h} = \frac{\vec{\nabla}P}{\rho}\,, \tag{3.312}$$

hence we can infer from (3.311) that the function

$$\partial_t\psi + \frac{v^2}{2} + \tilde{h} + \Phi = B(t) \tag{3.313}$$

must be a function of time only. Since the velocity is given by a spatial gradient of ψ, we can gauge the velocity potential such that the right-hand side of (3.313) vanishes,

$$\psi \to \psi + \int \mathrm{d}t\, B(t) \tag{3.314}$$

and simplify (3.313) to Bernoulli's law for non-stationary, but irrotational flows,

$$\boxed{\partial_t\psi + \frac{v^2}{2} + \tilde{h} + \Phi = 0}\,. \tag{3.315}$$

3.4.8 Diffusion of vorticity

Let us now turn to simple examples of viscous flows. We begin with the Navier-Stokes equation (3.148) in the form

$$\partial_t\vec{v} + \vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v}\times\vec{\Omega} = \frac{1}{\rho}\left[-\vec{\nabla}P + \eta\vec{\nabla}^2\vec{v} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)\right] \tag{3.316}$$

and take its curl. For simplicity, we assume that the flow is incompressible, $\vec{\nabla}\rho = 0$, and find

$$\partial_t\vec{\Omega} - \vec{\nabla}\times\left(\vec{v}\times\vec{\Omega}\right) = \nu\vec{\nabla}^2\vec{\Omega}\ , \tag{3.317}$$

where the kinematic viscosity

$$\nu := \frac{\eta}{\rho} \tag{3.318}$$

was introduced. Equation (3.317) relates a first-order partial time derivative to a second-order spatial derivative and is thus a diffusion equation for the vorticity. It shows how vorticity diffuses away in presence of viscosity.

3.4.9 The Reynolds Number

The kinematic viscosity has the dimension

$$\frac{\mathrm{g}}{\mathrm{cm\,s}}\,\frac{\mathrm{cm}^3}{\mathrm{g}} = \frac{\mathrm{cm}^2}{\mathrm{s}}\ , \tag{3.319}$$

that is, it is squared length over time. Suppose we scale all lengths with a typical length scale L, all velocities with a typical velocity V and all times with a time scale L/V in the vorticity equation (3.317). The expressions occuring would then scale as

$$\partial_t \to \frac{L}{V}\partial_t\ , \quad \partial_x \to L\partial_x\ , \quad \vec{v} \to \frac{\vec{v}}{V}\ , \quad \vec{\Omega} \to \vec{\Omega}\,\frac{L}{V}\ , \quad \nu \to \frac{\nu}{LV}\ , \tag{3.320}$$

such that all terms in (3.317) would be scaled by L^2/V^2 and thus become dimension-less. Therefore, if we characterise the viscosity by the dimension-less number

$$\frac{\nu}{LV} =: \frac{1}{\mathcal{R}}\ , \tag{3.321}$$

nothing in (3.317) reminds of the dimensions and the velocity of the flow. This shows that flows with the same Reynolds number $\mathcal{R}$ are scale-free. If lengths and velocities in a flow are stretched by factors L and V, respectively, the flow remains the same if the viscosity is simultaneously scaled by LV. The Reynolds number thus classifies such self-similar solutions of the flow equations. The transition to ideal fluids is characterised by $\mathcal{R} \to \infty$.

3.4.10 Hagen-Poiseulle flow

As one instructive example for a viscous flow, let us consider a viscous fluid running under the influence of a pressure gradient through a long, straight pipe.

Long means that its length L is much larger than its Radius R. We turn the coordinate system such that the symmetry axis of the pipe coincides with the x axis. The velocity $\vec{v}$ will then only have an x component which will itself only depend on the y and z coordinates perpendicular to the pipe. Since there are no other components of $\vec{v}$, we write $\vec{v} = v(y, z)\hat{e}_x$.

For a stationary flow, the continuity equation requires

$$\partial_x(\rho v) = v\partial_x\rho + \rho\partial_x v = 0 \,, \tag{3.322}$$

from where we read off that the density ρ will not depend on x either.

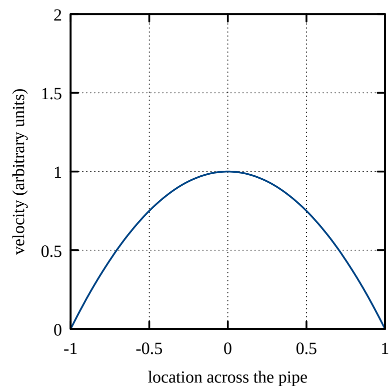

Figure 3.9 Illustration of the parabolic velocity profile across a pipe for Hagen-Poiseulle flow.

The Navier-Stokes equation shrinks to

$$\partial_x P = \eta\left(\partial_y^2 + \partial_z^2\right)v \,, \quad \partial_y P = 0 = \partial_z P \tag{3.323}$$

because the partial time derivative $\partial_t\vec{v}$ vanishes for a stationary flow, $(\vec{v}\cdot\vec{\nabla})v = v\partial_x v = 0$ since v does not depend on x, and $\vec{\nabla}\cdot\vec{v} = 0$ for the same reason. The second equation tells us that P is constant on planes perpendicular to the pipe. Since the right-hand side of the first equation cannot depend on x, neither can $\partial_x P$, hence $\partial_x P$ is constant along the pipe,

$$\partial_x P = \frac{\Delta P}{L} \,, \tag{3.324}$$

if ΔP is the pressure gradient applied between the ends of the pipe. Transforming the two-dimensional Laplacian in the first equation (3.323) to plane polar coordinates and taking into account that the flow must be symmetric about the symmetry axis of the pipe, we find the equation

$$\frac{\Delta P}{L\eta} = \frac{1}{r}\partial_r\left(r\partial_r v\right) \,, \tag{3.325}$$

which can easily be integrated to determine the velocity profile

$$v(r) = \frac{\Delta P}{4L\eta}r^2 + A\ln r + B \,, \tag{3.326}$$

with two integration constants A and B. We must require that $\upsilon(r) = 0$ at the wall of the pipe at $r = R$ and that $\upsilon(r)$ remains regular at $r = 0$. This can be achieved by setting $A = 0$ and

$$B = -\frac{\Delta P}{4L\eta}R^2 , \tag{3.327}$$

which leaves us with the parabolic velocity profile

$$\upsilon(r) = \frac{\Delta P}{4L\eta}\left(r^2 - R^2\right) \tag{3.328}$$

across the pipe (Figure 3.9). The amount of mass flowing through the pipe per unit time is

$$\dot{M} = 2\pi \int_0^R r\mathrm{d}r\,\rho\upsilon(r) = \frac{\pi\Delta P\rho R^4}{8L\eta} , \tag{3.329}$$

which is the Hagen-Poiseulle law: The mass of a viscous fluid flowing through a pipe per unit time is proportional to the squared cross section of the pipe.

Problems

1. A cylinder that contains an incompressible fluid rotates with constant angular velocity $\vec{\omega} = \omega\hat{e}_z$ in the gravitational field of the Earth, characterised by the gravitational acceleration $\vec{g} = -g\hat{e}_z$.

   (a) Use Euler's equation of momentum conservation

   $$\partial_t\vec{\upsilon} + \left(\vec{\upsilon}\cdot\vec{\nabla}\right)\vec{\upsilon} = -\frac{\vec{\nabla}P}{\rho} - \vec{\nabla}\Phi \tag{3.330}$$

   to derive differential equations for each velocity component that contain the angular velocity ω and the gravitational acceleration g. Why does the continuity equation

   $$\partial_t\rho + \vec{\nabla}\cdot(\rho\vec{\upsilon}) = 0 \tag{3.331}$$

   not yield any additional information?

   (b) Determine a function $P(r, \varphi, z)$ from these differential equations. What does the surface of the rotating fluid look like?

2. Jets are large directed outflows of material and a common astrophysical phenomenon. They can be observed under various circumstances, e.g. together with young T Tauri stars and the accretion onto a black hole in the centre of an active galaxy. Here, we want to examine some basic properties of a jet. Assume that a stationary jet has its origin on the surface of a spherical star with mass M and radius R, has initially the velocity υ_0 and the cross-sectional area A_0. The outflowing material has a polytropic equation-of-state, $P = P_0(\rho/\rho_0)^\gamma$, where γ is the adiabatic index, and the entropy stays constant, $\mathrm{d}s = 0$ along flow lines. The gas surrounding the star is assumed to be adiabatic, i.e. the pressure drops exponentially with the distance r from the surface, $P(r) = P_0\exp(-r/h)$, where h is the pressure scale height and P_0 the pressure at the surface.

(a) Determine the specific enthalpy $\tilde{h}$ per unit mass as a function of P and ρ.

(b) Use Bernoulli's equation,

$$\frac{v^2}{2} + \tilde{h} + \Phi = \text{const.}\,, \tag{3.332}$$

along flow lines, to determine $v(r)$.

(c) Use the continuity equation

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{3.333}$$

to determine the cross-sectional area $A(r)$.

3. Assume that a layer of height h of a viscous and incompressible fluid flows down a plane inclined by an angle α relative to the horizontal. The top of the fluid is free and feels the atmospheric pressure P_0. The coordinate system is chosen such that the x-axis is parallel to the velocity vector of the fluid and the z-axis is perpendicular to the plane.

(a) Determine the two equations that the Navier-Stokes equation

$$\rho \mathrm{d}_t \vec{v} = -\vec{\nabla} P + \eta \vec{\nabla}^2 \vec{v} + \left(\frac{\eta}{3} + \zeta\right) \vec{\nabla} \left(\vec{\nabla} \cdot \vec{v}\right) + \rho \vec{g} \tag{3.334}$$

simplifies to, where $\vec{g}$ is the gravitational acceleration.

(b) Solve these two differential equations for $P(z)$ and $v(z)$. What are the appropriate boundary conditions to be chosen for $P(z = h)$, $v(z = 0)$ and $(\mathrm{d}v/\mathrm{d}z)(z = h)$?

3.5 Shock waves

This section deals with the formation and the properties of shock waves. We begin with the method of characteristics for quasi-linear systems of partial differential equations and derive the Riemann invariants (3.359), which are used to explain the steepening of non-linear sound waves. Then, we turn to global properties of shock waves following from conservation laws, finding the Rankine-Hugoniot shock jump conditions (3.376) and (3.377). The velocity (3.388) of the shock itself relative to the flow is derived and used in the derivation of Sedov's solution (3.399) for the outer radius of a strong spherical shock wave.

The hydrodynamical equations are a set of non-linear, partial differential equations which give rise to non-linear phenomena in fluid flows. One important aspect is the formation of shock waves, where the velocity field changes discontinuously. Despite the non-linearity of the equations and some of the phenomena they describe, some statements can be made on characteristic properties of the flow without even solving the hydrodynamical equations. We have seen some examples before, such as Kelvin's circulation theorem and Bernoulli's law. We shall now proceed to show that even strongly non-linear phenomena such as shock waves can be predicted as inevitable, and that some important properties they display can be generally given. For doing so, we restrict our treatment to one-dimensional flows, having in mind fluid flows in pipes, for example. We shall begin with the method of characteristics.

3.5.1 The method of characteristics

In one dimension, for a polytropic, inviscid fluid, the continuity and Euler equations simplify to

$$\partial_t \rho + \rho \partial_x v + v \partial_x \rho = 0 \ ,$$
$$\partial_t v + v \partial_x v + \frac{\partial_x P}{\rho} = 0 \ . \qquad (3.335)$$

The derivative of the pressure can be expressed by the derivative of the density,

$$\partial_x P = c_s^2 \partial_x \rho \ , \qquad (3.336)$$

introducing the sound speed

$$c_s^2 = c_{s0}^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma - 1} \ . \qquad (3.337)$$

Taking the differential of the last equation, we see that the differentials of the sound velocity and of the density are related by

$$2 \frac{\mathrm{d}c_s}{c_s} = (\gamma - 1) \frac{\mathrm{d}\rho}{\rho} \ , \qquad (3.338)$$

which enables us to replace the partial density derivates of the density according to

$$\frac{\partial_t \rho}{\rho} = \frac{2}{\gamma - 1} \frac{\partial_t c_s}{c_s} \ , \quad \frac{\partial_x \rho}{\rho} = \frac{2}{\gamma - 1} \frac{\partial_x c_s}{c_s} \ . \qquad (3.339)$$

Our reduced set of one-dimensional hydrodynamical equations now reads

$$\frac{2}{\gamma - 1} \partial_t c_s + c_s \partial_x v + \frac{2v}{\gamma - 1} \partial_x c_s = 0 \ ,$$
$$\partial_t v + v \partial_x v + \frac{2c_s}{\gamma - 1} \partial_x c_s = 0 \ . \qquad (3.340)$$

They are two partial differential equations for two functions, c_s and v, in two variables, t and x.

It is important to see that these equations are quasi-linear, which means that the highest-order derivatives of the unknown functions occur linearly in them. Due to this property, we can summarise the two equations as

$$\begin{pmatrix} \frac{2}{\gamma - 1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_t c_s \\ \partial_t v \end{pmatrix} + \begin{pmatrix} \frac{2v}{\gamma - 1} & c_s \\ \frac{2c_s}{\gamma - 1} & v \end{pmatrix} \begin{pmatrix} \partial_x c_s \\ \partial_x v \end{pmatrix} = 0 \ . \qquad (3.341)$$

At this point, the method of characteristics sets in.

Suppose, more generally, that we are given a set of n quasi-linear, partial differential equations for the n unknown functions u_j of the two variables x and y. By its quasi-linearity, this set of equations can be brought into the form

$$X_{ij} \partial_x u_j + Y_{ij} \partial_y u_j = Z_i \ , \qquad (3.342)$$

where the Z_i represent possible inhomogeneities of the equations. The method of characteristics consists in finding local directions in the x-y plane into which the partial differential equations can be written as complete differentials, and thus be integrated. Of course, the functions could also depend on more than two independent variables, but we restrict the discussion to this case here for simplicity.

We wish to find differentials $\mathrm{d}\vec{s}$ in the two-dimensional space of independent variables satisfying the conditions

$$\mathrm{d}\vec{s}^\top X = \vec{L}^\top \mathrm{d}x\,, \quad \mathrm{d}\vec{s}^\top Y = \vec{L}^\top \mathrm{d}y \tag{3.343}$$

with the same vector $\vec{L}$ on the right-hand sides, for both matrices X and Y. If we could find such differentials, with a vector $\vec{L}$ yet to be determined, multiplying our set of equations (3.342) with it from the left would result in

$$L_j \left(\partial_x u_j \mathrm{d}x + \partial_y u_j \mathrm{d}y\right) = L_j \mathrm{d}u_j = \mathrm{d}s_i Z_i\;. \tag{3.344}$$

We could then directly integrate these equations, finding

$$L_j u_j = Z_i s_i\;. \tag{3.345}$$

In order to see when we can hope to find such a vector of differentials $\mathrm{d}\vec{s}$, we multiply the first equation (3.343) by $\mathrm{d}y$, the second by $\mathrm{d}x$ and subtract the second from the first to get

$$\mathrm{d}\vec{s}^\top\left(X\mathrm{d}y - Y\mathrm{d}x\right) = 0\;. \tag{3.346}$$

For this set of linear equations to have a non-trivial solution for $\mathrm{d}\vec{s}$, the determinant of the matrix $X\mathrm{d}y - Y\mathrm{d}x$ must vanish,

$$\det\left(X\mathrm{d}y - Y\mathrm{d}x\right) = 0\;. \tag{3.347}$$

This will give us relations between the two differentials $\mathrm{d}y$ and $\mathrm{d}x$ which will define preferred directions in x-y space. The integral curves of the expressions for $\mathrm{d}y/\mathrm{d}x$ are the *characteristics* of the system (3.342) of quasi-linear partial differential equations. The differentials are then found as eigenvectors of the matrix $X\mathrm{d}y - Y\mathrm{d}x$ belonging to the eigenvalue zero. Once they have been found, the vector $\vec{L}$ is given by the two equations (3.343).

Let us apply this method of characteristics now to the set of hydrodynamical equations (3.341). Here, we have the two functions c_s and v in place of the u_1 and u_2, and the two independent variables (t, x) in place of (x, y). The two matrices X and Y are replaced by

$$T = \begin{pmatrix} \frac{2}{\gamma-1} & 0 \\ 0 & 1 \end{pmatrix}\,, \quad X = \begin{pmatrix} \frac{2v}{\gamma-1} & c_\mathrm{s} \\ \frac{2c_\mathrm{s}}{\gamma-1} & v \end{pmatrix}\;. \tag{3.348}$$

The characteristics are defined by the condition

$$\begin{aligned} 0 = \det\left(T\mathrm{d}x - X\mathrm{d}t\right) &= \det\begin{pmatrix} \frac{2(\mathrm{d}x - v\mathrm{d}t)}{\gamma-1} & -c_\mathrm{s}\mathrm{d}t \\ -\frac{2c_\mathrm{s}\mathrm{d}t}{\gamma-1} & \mathrm{d}x - v\mathrm{d}t \end{pmatrix} \\ &= \frac{2}{\gamma-1}\det\begin{pmatrix} \mathrm{d}x - v\mathrm{d}t & -c_\mathrm{s}\mathrm{d}t \\ -c_\mathrm{s}\mathrm{d}t & \mathrm{d}x - v\mathrm{d}t \end{pmatrix}\,, \end{aligned} \tag{3.349}$$

which leads to the quadratic equation

$$\mathrm{d}x^2 - 2v\mathrm{d}x\mathrm{d}t + \left(v^2 - c_\mathrm{s}^2\right)\mathrm{d}t^2 = 0 \,, \tag{3.350}$$

whose two solutions

$$\mathrm{d}x_\pm = (v \pm c_\mathrm{s})\mathrm{d}t \tag{3.351}$$

define the characteristics of the system (3.341) of hydrodynamical equations.

?

What does the condition (3.351) mean geometrically in a space-time diagram?

The differentials ds must be eigenvectors with eigenvalue zero of the difference matrix $T\mathrm{d}x - X\mathrm{d}t$,

$$(\mathrm{d}s_1, \mathrm{d}s_2) \cdot \begin{pmatrix} \frac{2(\mathrm{d}x - v\mathrm{d}t)}{\gamma-1} & -c_\mathrm{s}\mathrm{d}t \\ -\frac{2c_\mathrm{s}\mathrm{d}t}{\gamma-1} & \mathrm{d}x - v\mathrm{d}t \end{pmatrix} = (0, 0) \,. \tag{3.352}$$

In particular, this establishes the relation

$$-c_\mathrm{s}\mathrm{d}t\mathrm{d}s_1 + (\mathrm{d}x - v\mathrm{d}t)\mathrm{d}s_2 = 0 \tag{3.353}$$

between $\mathrm{d}s_1$ and $\mathrm{d}s_2$. On the characteristics, $\mathrm{d}x = \mathrm{d}x_\pm = (v \pm c_\mathrm{s})\mathrm{d}t$, hence $\mathrm{d}s_1$ and $\mathrm{d}s_2$ must agree except for their sign,

$$-c_\mathrm{s}\mathrm{d}s_1 \pm c_\mathrm{s}\mathrm{d}s_2 = 0 \quad \Rightarrow \quad \mathrm{d}s_2 = \pm\mathrm{d}s_1 \,. \tag{3.354}$$

The vector $\vec{L}$ is finally found from one of the equations (3.343) applied to our current situation,

$$(\mathrm{d}s_1, \pm\mathrm{d}s_1)\begin{pmatrix} \frac{2v}{\gamma-1} & c_\mathrm{s} \\ \frac{2c_\mathrm{s}}{\gamma-1} & v \end{pmatrix} = (L_1, L_2)\mathrm{d}x \,, \tag{3.355}$$

which implies

$$L_1\mathrm{d}x = \frac{2(v \pm c_\mathrm{s})}{\gamma - 1}\mathrm{d}s_1 \,, \quad L_2\mathrm{d}x = (c_\mathrm{s} \pm v)\mathrm{d}s_1 \,. \tag{3.356}$$

The ratio between these two components is all we need because $\mathrm{d}\vec{s}$ and $\mathrm{d}\vec{L}$ can only be determined up to a common normalisation factor. The last equation tells us

$$\frac{L_1}{L_2} = \pm\frac{2}{\gamma - 1} \,. \tag{3.357}$$

We arbitrarily set $L_2 = 1$ and return to evaluate (3.344) for our hydrodynamical equations, where the inhomogeneities $Z_i = 0$. This finally leads us to

$$\mathrm{d}v \pm \frac{2}{\gamma - 1}\mathrm{d}c_\mathrm{s} = 0 \,, \tag{3.358}$$

which we can directly integrate to find the two *Riemann invariants*

$$R_\pm = v \pm \frac{2}{\gamma - 1}c_\mathrm{s} \,, \tag{3.359}$$

which are conserved on the plus and minus characteristics, respectively.

3.5.2 Steepening of sound waves

Consider now a hitherto unperturbed fluid at rest, on which a non-linear density perturbation is imprinted at time $t = 0$. Every point within the perturbation can be connected to its unperturbed neighbourhood by means of minus characteristics coming from the positive x region in the past. Those characteristics are straight lines with slope

$$\frac{dx_-}{dt} = v - c_s = -c_{s0} \tag{3.360}$$

because they propagate through unperturbed material at rest. Along these minus characteristics, the Riemann invariant

$$R_- = v - \frac{2c_s}{\gamma - 1} = -\frac{2c_{s0}}{\gamma - 1} \tag{3.361}$$

is conserved. This establishes the relation

$$v = \frac{2(c_s - c_{s0})}{\gamma - 1} \tag{3.362}$$

at every point that can be reached by a minus characteristic coming from unperturbed material, which is every point in the fluid.

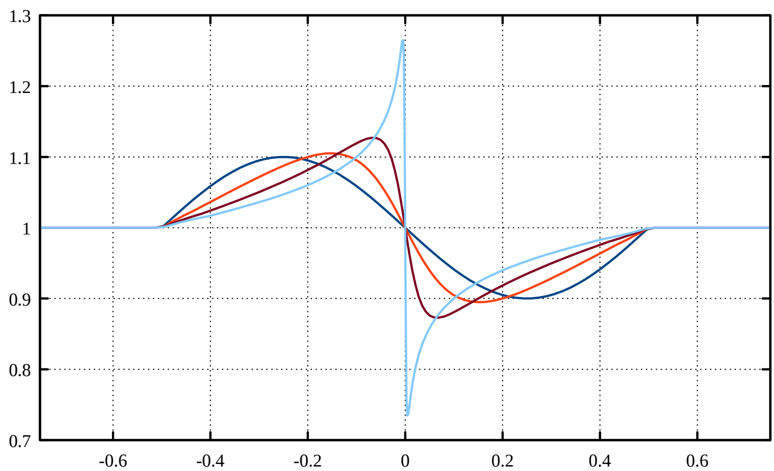

Figure 3.10 Illustration of how a non-linear density wave steepens and ultimately turns into a shock.

At the boundary points of the perturbation, the density is supposed to have dropped to the unperturbed density, and they are considered to be at rest. A plus characteristic attached to the boundary point at $x > 0$ is determined by

$$\frac{dx_+}{dt} = v + c_s = c_{s0} \,, \tag{3.363}$$

because it propagates into unperturbed material. Along this plus characteristic, the Riemann invariant

$$R_+ = v + \frac{2c_s}{\gamma - 1} = \frac{2c_{s0}}{\gamma - 1} \tag{3.364}$$

is conserved. The right boundary point of the perturbation has unperturbed density and retains it as it propagates along the plus characteristic. Now consider the central point of the density perturbation, where the density is highest. Its enhanced density implies a sound speed higher than that of the unperturbed fluid. According to (3.362), overdense points have a velocity $v > 0$. Both, the velocity v and the sound speed c_s, are therefore higher at an overdensity than in the unperturbed fluid. Plus characteristics originating there thus propagate faster than plus characteristics originating from unperturbed points. The plus characteristic of the density peak thus approaches that of the right boundary point of the perturbation. The density peak will catch up with the boundary point and ultimately reach it: The density perturbation steepens and ultimately produces a discontinuity in the density and the velocity because streams of different density and velocity cannot coexist at the same location in a fluid (Figure 3.10).

We have thus shown simply by the method of characteristics that non-linear density perturbations have to steepen and ultimately form discontinuities, or shocks. As generic as our discussion was, as generic is this result: The formation of shocks by steepening of non-linear waves is inevitable in a fluid. It is quite remarkable that we did not have to solve any of the hydrodynamical equations to see this. The method of characteristics was sufficient.

3.5.3 The Rankine-Hugoniot shock jump conditions

Even though at least some of the flow variables may be discontinuous at a shock, three current densities must be conserved across the shock, namely the matter current density $\rho\vec{v}$, the energy current density

$$\vec{q} = \left(\frac{v^2}{2} + \tilde{h}\right)\rho\vec{v} \tag{3.365}$$

and the momentum-current density

$$T^{ij} = \rho v^i v^j + P\delta^{ij} \ . \tag{3.366}$$

We consider now a shock that is perpendicular to the local flow direction. We fix an arbitrary point on the shock surface and locally construct a coordinate system such that the x axis is perpendicular to the shock surface and the y-z plane is tangential to it. On the y-z plane, the three conserved current densities must meet. Identifying with subscripts 1 and 2 quantities on either side of the shock, we must have

$$\begin{aligned} \rho_1 v_1 &= \rho_2 v_2 \ , \\ \left(\frac{v_1^2}{2} + \tilde{h}_1\right)\rho_1 v_1 &= \left(\frac{v_2^2}{2} + \tilde{h}_2\right)\rho_2 v_2 \ , \\ \rho_1 v_1^2 + P_1 &= \rho_2 v_2^2 + P_2 \ . \end{aligned} \tag{3.367}$$

We wish to express the flow variables ρ_2, v_2 and P_2 on one side of the shock by those on the other. For doing so, we first adopt a polytropic equation of state and thereby fix the sound speed

$$c_s^2 = \gamma\frac{P}{\rho} \tag{3.368}$$

and the specific enthalpy per unit mass

$$\tilde{h} = \frac{\gamma}{\gamma - 1}\frac{P}{\rho} = \frac{c_s^2}{\gamma - 1} \ . \tag{3.369}$$

We further introduce the Mach number on the 1-side of the shock,

$$v_1^2 = \mathcal{M}_1^2 c_{s1}^2 \ , \tag{3.370}$$

and the ratios r and q between the density and the pressure values on both sides of the shock,

$$r := \frac{\rho_2}{\rho_1} \ , \quad q := \frac{P_2}{P_1} \ . \tag{3.371}$$

The enthalpy on the 2-side of the shock is then

$$\tilde{h}_2 = \frac{\gamma}{\gamma - 1}\frac{P_2}{\rho_2} = \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1}\frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \tilde{h}_1 \frac{q}{r} = \frac{c_{s1}^2}{\gamma - 1}\frac{q}{r} \ . \tag{3.372}$$

After this preparation, (3.367) can be reduced to

$$\begin{aligned} \frac{\mathcal{M}_1^2}{2} + \frac{1}{\gamma - 1} &= \frac{\mathcal{M}_1^2}{2r^2} + \frac{1}{\gamma - 1}\frac{q}{r} \ , \\ \mathcal{M}_1^2 + \frac{1}{\gamma} &= \frac{\mathcal{M}_1^2}{r} + \frac{q}{\gamma} \ . \end{aligned} \tag{3.373}$$

We multiply the first of these equations with $r(\gamma - 1)$ and the second with γ to find

$$\begin{aligned} \frac{\mathcal{M}_1^2}{2} r(\gamma - 1) + r &= \frac{\mathcal{M}_1^2}{2r}(\gamma - 1) + q \ , \\ \mathcal{M}_1^2 \gamma + 1 &= \frac{\mathcal{M}_1^2}{r}\gamma + q \ , \end{aligned} \tag{3.374}$$

and subtract the first from the second to elimitate q and retain the quadratic equation in r

$$r^2 \left[\mathcal{M}_1^2(\gamma - 1) + 2\right] - 2r\left(\mathcal{M}_1^2 \gamma + 1\right) + \mathcal{M}_1^2(\gamma + 1) = 0 \ , \tag{3.375}$$

which has the two solutions

$$r_+ =: \boxed{r = \frac{\mathcal{M}_1^2(\gamma + 1)}{\mathcal{M}_1^2(\gamma - 1) + 2}} \ , \quad r_- = 1 \ . \tag{3.376}$$

Only the solution r_+ is interesting since r_- corresponds to no density jump at all. We thus set $r = r_+$ and use this to find q from the second equation (3.374),

$$\boxed{q = \frac{2\gamma \mathcal{M}_1^2 - \gamma + 1}{\gamma + 1}} \ . \tag{3.377}$$

?

Repeat the derivation of the jump conditions (3.376) and (3.377) on your own.

The temperature jump can finally be obtained from the equation of state, such as the ideal-gas equation, through

$$\frac{T_2}{T_1} = \frac{P_2}{\rho_2}\frac{\rho_1}{P_1} = \frac{q}{r} \ . \tag{3.378}$$

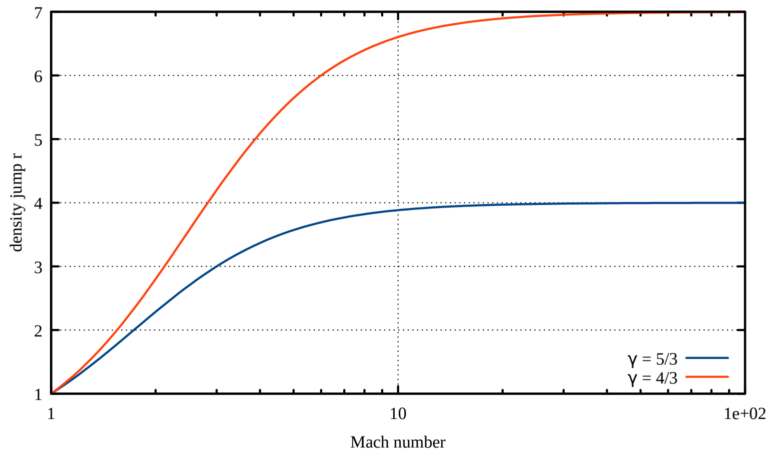

Figure 3.11 On the Rankine-Hugoniot shock jump conditions: The density jump r at a shock is shown as a function of the Mach number upstream the shock for gases with adiabatic indices $\gamma = 5/3$ and $\gamma = 4/3$.

The gas is assumed to approach the shock with supersonic velocity, $\mathcal{M}_1 > 1$. Then, (3.376) tells us that

$$r = \frac{\gamma + 1}{\gamma - 1 + 2\mathcal{M}_1^{-2}} = \frac{\gamma + 1}{\gamma + 1 + 2(\mathcal{M}_1^{-2} - 1)} > 1 \; . \tag{3.379}$$

The density is higher downstream of the shock, the velocity must correspondingly be lower. According to (3.377), the pressure also increases because

$$q = \frac{2\gamma\mathcal{M}_1^2 - \gamma + 1}{\gamma + 1} > \frac{2\gamma - \gamma + 1}{\gamma + 1} = 1 \; . \tag{3.380}$$

In the ultrasonic limit, $\mathcal{M}_1 \gg 1$, the density jump approaches

$$r = \frac{\gamma + 1}{\gamma - 1} \; , \tag{3.381}$$

or $r = 4$ for a monatomic, ideal, non-relativistic gas. The closer γ gets towards unity, the larger r will become (Figure 3.11). For a relativistic gas, $\gamma = 4/3$ and $r = 7$. The pressure and temperature jumps across such strong shocks can become arbitrarily large. Both can rise strongly, showing that the gas will be hot downstream the shock.

Eliminating the Mach number between (3.376) and (3.377) gives either of the two equations

$$r = \frac{(\gamma + 1)q + (\gamma - 1)}{(\gamma - 1)q + (\gamma + 1)} \; , \quad q = \frac{(\gamma + 1)r - (\gamma - 1)}{(\gamma + 1) - (\gamma - 1)r} \tag{3.382}$$

relating the pressure and the density jumps to each other.

Caution The adiabatic index for a gas composed of molecules with f degrees of freedom is

$$\gamma = \frac{f + 2}{f}$$

for a non-relativistic and

$$\gamma = \frac{f + 1}{f}$$

for a relativistic gas. For $f = 3$, i.e. if the molecules have only the three translational but no internal (rotational or vibrational) degrees of freedom, $\gamma = 5/3$ in the non-relativistic and $\gamma = 4/3$ in the relativistic case. ◀

3.5.4 Shock velocity

Having seen how conservation laws alone predict the discontinuities in the density, the velocity and the pressure of a fluid through the Rankine-Hugoniot conditions, we now turn to the question how fast the shock itself will propagate through the fluid. To this end, consider a long pipe filled with gas and closed with a piston at its left end. At time $t = 0$, we imagine that the piston is instantaneously accelerated to some high and constant velocity u.

The sudden acceleration will drive a shock into the unperturbed gas ahead of the piston. Downstream of the shock, the gas will be at rest, $v_1 = 0$, while it will have the velocity of the piston, $v_2 = u$, upstream. In between, the shock will move with a yet unknown velocity v_s.

To analyse this situation, we transform into the rest frame of the shock and mark all velocities in the rest frame of the shock with primes. In that frame, by construction, $v'_s = 0$, further $v'_1 = v_1 - v_s = -v_s$ and $v'_2 = v_2 - v_s = u - v_s$, while the velocity difference between down- and upstream remains of course unchanged, $v_2 - v_1 = u = v'_2 - v'_1$.

Solving the identity

$$u = v'_2 - v'_1 = v'_1 \left(\frac{1}{r} - 1 \right) \tag{3.383}$$

for $v'_1 = -v_s$ immediately gives the shock velocity

$$v_s = \frac{ru}{r-1} \tag{3.384}$$

in terms of the velocity u of the piston. A strong shock, which has $r = (\gamma + 1)/(\gamma - 1)$ as we have seen before, thus moves with the velocity

$$v_s = \frac{\gamma + 1}{2} u \ . \tag{3.385}$$

?

Test several limiting cases of the result (3.388) for the shock velocity and interpret the results.

In an ideal, monatomic, nonrelativistic gas, for example, the shock velocity exceeds the velocity of the piston by $4/3 - 1 \approx 33\,\%$. We now know the shock speed only as a function of the velocity u of the piston. Sometimes this is unknown, sometimes it is irrelevant because we want to know the shock speed in terms of the intrinsic properties r and q of the shock. To achieve this, we simply write

$$v_s = -v'_1 = |\mathcal{M}_1| c_{s1} \ , \tag{3.386}$$

solve either one of the Rankine-Hugoniot shock jump conditions (3.376) or (3.377) for $\mathcal{M}_1^2$,

$$\mathcal{M}_1^2 = \frac{2r}{(\gamma + 1) - (\gamma - 1)r} = \frac{(\gamma + 1)q + (\gamma - 1)}{2\gamma} \ , \tag{3.387}$$

and insert the result into (3.386) to find the shock velocity as a function of either the density jump r or the pressure jump q,

$$v_s = c_{s1} \sqrt{\frac{2r}{(\gamma + 1) - (\gamma - 1)r}} = c_{s1} \sqrt{\frac{(\gamma + 1)q + (\gamma - 1)}{2\gamma}} \ . \tag{3.388}$$

3.5.5 The Sedov solution

An impressive example for a shock wave is given by an explosion, i.e. by an event in which in very short time energy is being released within a very small volume (Figure 3.12). We now consider such an event under the following simplifying assumptions: (1) The shock is very strong, meaning that the pressure of the surrounding medium can be neglected, $P_1 \ll P_2$. (2) The explosion energy E is released instantaneously and the energy of the surrounding material is negligible compared to E, i.e. the explosion energy dominates that of the surroundings. (3) The gas be polytropic with an adiabatic index γ.

Under these conditions, our shock jump condition for the density is

$$r = \frac{\rho_2}{\rho_1} \approx \frac{\gamma + 1}{\gamma - 1} , \tag{3.389}$$

as it has to be for a very strong shock. The densities ρ_1 and ρ_2 are completely determined by each other, which implies that the behaviour of the shock must be entirely determined by the explosion energy E and the surrounding matter density ρ_1.

Let us now consider the shock at a time t after the explosion, when it has already reached an unknown radius $R(t)$. The only quantity with the dimension of a length that can be formed from E, t and ρ_1 is

$$\left(\frac{Et^2}{\rho_1}\right)^{1/5} , \tag{3.390}$$

which suggests the ansatz

$$R(t) = R_0 \left(\frac{Et^2}{\rho_1}\right)^{1/5} \tag{3.391}$$

with a dimension-less constant R_0 which remains to be determined. The shock velocity is obviously the time derivative of $R(t)$,

$$v_\mathrm{s} = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2}{5}\frac{R}{t} = \frac{2R_0}{5}\left(\frac{E}{\rho_1}\right)^{1/5} t^{-3/5} , \tag{3.392}$$

but it also has to obey the relation (3.388) found above. Solving the latter for the pressure jump q and inserting $c_\mathrm{s}^2 = \gamma P_1/\rho_1$ gives

$$q = \frac{P_2}{P_1} = \frac{1}{\gamma + 1}\left[\frac{2v_\mathrm{s}^2\rho_1}{P_1} - (\gamma - 1)\right] . \tag{3.393}$$

Under the assumption that $P_1 \ll P_2$, we can neglect the second term on the right-hand side and approximate the pressure inside the shock by

$$P_2 = \frac{2v_\mathrm{s}^2\rho_1}{\gamma + 1} . \tag{3.394}$$

The jump condition (3.389) for the density ratio shows that the density inside the shock must remains constant in time because ρ_1 is constant. Given the shock velocity (3.392), the pressure inside the shock must be

$$P_2 = \frac{8R_0^2}{25(\gamma + 1)} E^{2/5}\rho_1^{3/5} t^{-6/5} . \tag{3.395}$$

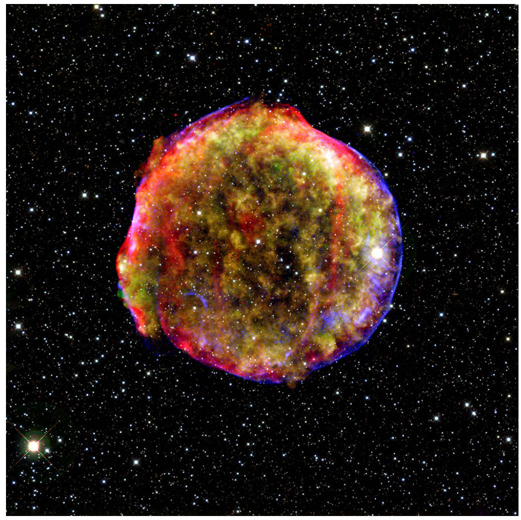

Figure 3.12 This image of the remnant of Tycho's supernova combines X-ray and infrared data from the Chandra and Spitzer space telescopes, respectively. Blue-red colours show the energetic X-ray emission at the shock, while yellow-green colours show the infrared emission inside the shock front.

According to (3.385), the velocity of the gas behind the shock then falls off in the same way,

$$u = \frac{2v_s}{\gamma + 1} = \frac{4R_0}{5(\gamma + 1)} \left(\frac{E}{\rho_1}\right)^{1/5} t^{-3/5} \,. \tag{3.396}$$

We can interpret these relations as follows: A shock wave driven by the sudden release of a large amount of energy E, which propagates outward with the time-dependent radius $R(t)$, sweeps up surrounding material with the mass

$$M = \frac{4\pi}{3}\rho_1 R^3 \,, \tag{3.397}$$

which is accelerated from rest to a velocity $\approx R/t$. Thus, the kinetic energy

$$E_{\text{kin}} \approx \frac{4\pi}{3}\rho_1 \frac{R^5}{t^2} \tag{3.398}$$

must be put into the swept-up material. Equating this to the explosion energy E, we immediately find

$$R = \left(\frac{3Et^2}{4\pi\rho_1}\right)^{1/5} \,, \tag{3.399}$$

i.e. the scaling relation (3.391) simply expresses energy conservation.

Without solving any of the hydrodynamical equations, we now know how the radius of the explosion shock, its velocity as well as the pressure and the density at its inside. They are completely determined by the released amount of energy E and the density ρ_1 of the surrounding material.

Problems

1. In astrophysics, the hydrodynamical equations often simplify a lot due to spherical symmetry, e.g. when a young star of mass M accretes material. In this case, $\vec{v} = v(r)\hat{e}_r$, and the continuity and the Euler equations simplify to

$$\partial_t\rho + \frac{1}{r^2}\partial_r\left(r^2\rho v\right) = 0\ , \quad \partial_t v + v\partial_r v + \frac{c_s^2}{\rho}\partial_r\rho = -\partial_r\Phi = \frac{GM}{r^2}\ , \tag{3.400}$$

respectively, where we have used $\vec{\nabla}P = c_s^2\vec{\nabla}\rho$. Let us for simplicity further assume that the sound speed c_s is constant. In order to use the method of characteristics, the equations have to be brought into the form

$$T_{ij}\partial_t u^j + R_{ij}\partial_r u^j = Z_i\ , \tag{3.401}$$

where summing over j is implied. The matrix elements R_{ij} and T_{ij} are given by the coefficients in front of the partial derivatives, while the Z_i are given by the inhomogeneities.

   (a) Bring the continuity and the Euler equations into the form (3.401) and identify T, R, $\vec{u}$ and $\vec{Z}$.

   (b) Determine a relation between the differentials $\mathrm{d}r$ and $\mathrm{d}t$ from the condition $\det(T\mathrm{d}r - R\mathrm{d}t) = 0$. What does the result mean physically?

   (c) The goal is to find the differentials $\mathrm{d}s^\top = (\mathrm{d}s_1, \mathrm{d}s_2)$ and the vector $\vec{L}^\top = (L_1, L_2)$ such that

$$\mathrm{d}\vec{s}^\top T = \vec{L}^\top \mathrm{d}t\ , \quad \mathrm{d}\vec{s}^\top T = \vec{L}^\top \mathrm{d}r \tag{3.402}$$

   are satisfied. Determine $\mathrm{d}s_1$ and $\mathrm{d}s_2$ from $\mathrm{d}\vec{s}^\top(T\mathrm{d}r - R\mathrm{d}t) = 0$ and L_1 from (3.402), arbitrarily setting $L_2 = 1$.

   (d) Multiplying (3.401) by $\mathrm{d}\vec{s}^\top$ from the left and using (3.402) leads to $\vec{L}^\top \cdot \mathrm{d}\vec{u} = \mathrm{d}\vec{s}^\top \cdot \vec{Z}$. Set up the latter equation which defines the Riemann invariants and carry the necessary integration out as far as possible.

2. Assume that the coordinate system is chosen such that the yz-plane is parallel to a shock front and the x-direction perpendicular to it and the gas flows from the side 1 to the side 2. With the energy-momentum tensor

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)u^\mu u^\nu - \eta^{\mu\nu}P \tag{3.403}$$

and the four-velocity $(u^\mu) = \gamma(c, \vec{v}^\top)$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor, the relativistic generalisations of the continuity conditions for the densities of the particle current, the momentum current and the energy current are given by

$$n_1 u_1^x = n_1 u_1^x\ , \quad T_1^{xx} = T_2^{xx}\ , \quad cT_1^{0x} = cT_2^{0x}\ , \tag{3.404}$$

respectively.

(a) Express the continuity conditions as functions of the velocities $\beta_i \equiv \beta_i^x$ in units of the light speed, the enthalpies per volume $h_i = \varepsilon_i + P_i$ with $\varepsilon_i = \rho c^2$, and the specific volumes per particle $V_i \equiv n_i^{-1}$ (here and in the following, $i = 1, 2$).

(b) Determine the velocities on both sides of the discontinuity as a function of P_i and ε_i. *Hint*: It is helpful to introduce the rapidity $\theta \equiv \operatorname{artanh}(v/c)$.

(c) What is the relative velocity v_{12} of the gases on both sides of the discontinuity?

(d) In the ultrarelativistic case, $P = \varepsilon/3$. Determine the velocities on both sides of the discontinuity in the case of a very strong shock front ($\varepsilon_2 \to \infty$).

3.6 Instabilities

This section concludes the chapter on hydrodynamics with a discussion of fluid instabilities. The method of analysis is common to most of them: The governing equations are linearised by a perturbation ansatz. Decomposing the perturbations into plane waves turns these linearised differential equations into systems of linear algebraic equations which the dispersion relations can directly be derived from requiring non-trivial solutions. We begin with surface waves on a fluid in a gravitational field which are shown to satisfy the non-linear dispersion relation (3.418). The Rayleigh-Taylor or buoyancy instability follows, whose dispersion relation (3.426) shows that a specifically lighter fluid placed below a specifically heavier fluid tends to develop an unstable boundary. The Kelvin-Helmholtz or shear instability arises at the boundary between two fluids one of which flows with respect to the other. Its dispersion relation is given in (3.436). Thermal instability sets in if and when heating a gas leads to less efficient cooling, or cooling leads to less efficient heating. The dispersion relation of this instability is shown in (3.456). After a brief intermezzo on heat conduction, we consider heat transport by convection and show in (3.486) that convection sets in if the temperature gradient is steep. Finally, we briefly discuss turbulence and derive the Kolmogorov spectrum (3.500) for the energy distribution over scales of turbulent eddies.

We shall now proceed to examine hydrodynamical instabilities, i.e. the evolution of situations in which an equilibrium configuration is slightly perturbed. Such investigations follow standard procedures. The equilibrium configuration is taken as given. Small perturbations are applied and the relevant equations are linearised in these perturbations. The linear equations resulting therefrom are then decomposed into Fourier modes whose dispersion relation is derived. Instable situations are characterised by complex or imaginary frequencies, which signal exponential growth of the perturbations. For simplicity, we shall assume that the fluids are inviscid and incompressible, hence $\vec{\nabla} \cdot \vec{v} = 0$.

We transform into the rest frame of one of the unperturbed solutions, in such a way that the velocity field there is given by the velocity perturbation only. Then, only terms linear in the velocity need to be retained. The vorticity equation (3.272) then tells us that

$$\partial_t \vec{\Omega} = \partial_t \left(\vec{\nabla} \times \vec{v} \right) = 0 \; . \tag{3.405}$$

Vorticity cannot build up, and we can assume that $\vec{\nabla} \times \vec{v} = 0$. Then, there exists a velocity potential ψ such that $\vec{v} = \vec{\nabla}\psi$. Since the fluid is also incompressible, the velocity potential must satisfy Laplace's equation

$$\vec{\nabla}^2 \psi = 0 \; . \tag{3.406}$$

In addition, Bernoulli's law in the form (3.315) must hold with the term quadratic in the velocity neglected,

$$\partial_t \psi + \tilde{h} + \Phi = 0 \; . \tag{3.407}$$

Moreover, for an incompressible fluid, the enthalpy per unit mass is determined by

$$\mathrm{d}\tilde{h} = \frac{\mathrm{d}P}{\rho} = \mathrm{d}\left(\frac{P}{\rho} \right) \tag{3.408}$$

since $\mathrm{d}\rho = 0$, hence $\tilde{h} = P/\rho$, and Bernoulli's law turns into

$$P = -\rho \left(\partial_t \psi + \Phi \right) \; . \tag{3.409}$$

?

Interpret the physical meaning of (3.409). To do so, taking the gradient is helpful.

We begin with two situations in which two fluids are separated by a surface. Our fundamental set of equations for these investigations will be (3.406) and (3.409).

3.6.1 Gravity waves

Consider a fluid which rests under local gravity such that its surface is a plane. Above the surface, we imagine a gas which is much less dense than the fluid and sets the pressure P_2 = const. at the surface. We introduce a coordinate frame such that the surface of the fluid coincides with the x-y plane. We wish to find out how perturbations in the fluid surface propagate.

To this end, we introduce a function $\zeta(x, y, t)$ describing the perturbed fluid surface (Figure 3.13). The velocity in z direction is the change of ζ with time, hence

$$v_z = \partial_t \zeta + \vec{v} \cdot \vec{\nabla} \zeta \approx \partial_t \zeta \; , \tag{3.410}$$

where the last step was possible since $\vec{v}$ and ζ are both small quantities. Moreover, the velocity in z direction is the z derivative of the velocity potential ψ,

$$v_z = \partial_z \psi \; . \tag{3.411}$$

Given the external, e.g. atmospheric pressure P_0, equation (3.409) demands

$$P_0 = -\rho \left(\partial_t \psi + g\zeta \right) \; , \tag{3.412}$$

where $\Phi = g\zeta$ is the local gravitational potential due to the gravitational acceleration g, evaluated at the surface where $z = \zeta$. Since only the spatial

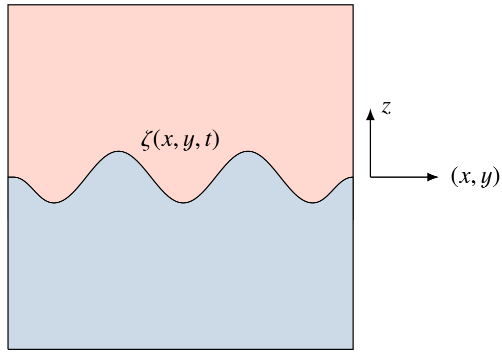

Figure 3.13 Illustration of the perturbed boundary between two fluids, described by a function $z = \zeta(x, y, t)$.

gradient of ψ is relevant for the velocity $\vec{v}$, we can absorb the constant pressure term into ψ by the gauge choice

$$\psi \to \psi + \frac{P_0 t}{\rho} \ . \tag{3.413}$$

The equation we have to solve is then simply

$$\partial_t \psi = -g\zeta \ . \tag{3.414}$$

Taking another time derivative leads to the equation

$$\partial_t \zeta = v_z = \partial_z \psi = -\frac{1}{g}\partial_t^2 \psi \ , \tag{3.415}$$

which has to be evaluated at $z = \zeta$. The velocity potential thus has to satisfy the Laplace equation (3.406) and Bernoulli's equation (3.415).

We begin our solution with the *ansatz*

$$\psi = f(z)\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{3.416}$$

for a wave-like solution propagating in x direction. The Laplace equation, applied to this *ansatz*, constrains $f(z)$ to satisfy

$$f''(z) - k^2 f(z) = 0 \ , \tag{3.417}$$

which is solved by $f(z) = f_0 \exp(\pm kz)$. Since this velocity potential is confined to $z < 0$, we need the branch $f(z) = f_0 \exp(kz)$ here.

?

The choice $f(z) = f_0 \mathrm{e}^{kz}$ hides the selection of a specific boundary condition for the solution of the Laplace equation (3.417). Which is it? How could different boundary conditions be set?

Inserting this result together with (3.416) into Bernoulli's equation (3.415) now immediately gives the dispersion relation for gravity waves,

$$\omega^2 = kg \ . \tag{3.418}$$

Interestingly, the group velocity $v_\mathrm{g} = \partial_k \omega$ of such waves depends on the wavelength,

$$v_\mathrm{g} = \partial_k \sqrt{kg} = \frac{1}{2}\sqrt{\frac{g}{k}} \ : \tag{3.419}$$

Longer gravity waves travel faster.

3.6.2 The Rayleigh-Taylor instability

We now consider a somewhat more involved situation. Imagine two different fluids meeting at a common, unperturbed surface perpendicular to the local gravitational acceleration g. The fluid above has density ρ_1 and height h_1, the fluid below has density ρ_2 and depth h_2. Both fluids are initially at rest. We choose the coordinate system such that the unperturbed surface coincides with the x-y plane. If this is perturbed as described by a function $\zeta(x, y, t)$, how do the perturbations develop?

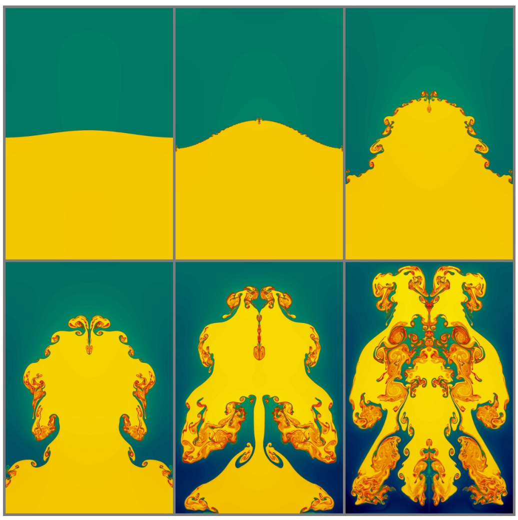

Figure 3.14 This sequence of images shows the onset of the Rayleigh-Taylor instability in a simulation. The boundaries of the rising "Rayleigh-Taylor finger" additionally become Kelvin-Helmholtz unstable. (courtesy of Volker Springel)

Now, we have to find two velocity potentials, ψ_1 and ψ_2, subject to the following conditions: Both have to satisfy the Laplace equation (3.406), the Bernoulli equation (3.415) at the surface and the boundary conditions that the pressure and the velocity at the surface must be continuous and that the velocities at $z = h_1$ and $z = -h_2$ must both vanish,

$$\begin{aligned} &(P_1 - P_2)|_{z=\zeta} = 0\ , \quad (\partial_z\psi_1 - \partial_z\psi_2)|_{z=\zeta} = 0\ , \\ &(\partial_z\psi_1)|_{z=h_1} = 0 = (\partial_z\psi_2)|_{z=-h_2}\ . \end{aligned} \tag{3.420}$$

Laplace's equation, together with the third of these conditions suggests the *ansatz*

$$\begin{aligned} \psi_1 &= A_1 \cosh[k(z - h_1)]\mathrm{e}^{\mathrm{i}(kx-\omega t)}\ , \\ \psi_2 &= A_2 \cosh[k(z + h_2)]\mathrm{e}^{\mathrm{i}(kx-\omega t)}\ . \end{aligned} \tag{3.421}$$

With Bernoulli's equation (3.412), the first of the boundary conditions (3.420) requires

$$\zeta = \frac{\rho_2\partial_t\psi_2 - \rho_1\partial_t\psi_1}{g(\rho_1 - \rho_2)}\ , \tag{3.422}$$

where we can now evaluate the right-hand side at $z = 0$ rather than at $z = \zeta$, since ζ is supposed to be a small perturbation. Another time derivative of (3.422) gives

$$\partial_t \zeta = v_z = \partial_z \psi_1 = \frac{\rho_2 \partial_t^2 \psi_2 - \rho_1 \partial_t^2 \psi_1}{g(\rho_1 - \rho_2)} \ . \tag{3.423}$$

Inserting the ansatz (3.421) here, we first find

$$-A_1 k \sinh(kh_1) = \frac{\omega^2}{g(\rho_1 - \rho_2)} \left[A_2 \rho_2 \cosh(kh_2) - A_1 \rho_1 \cosh(kh_1)\right] \ , \tag{3.424}$$

and the second boundary condition (3.420) finally requires

$$-A_1 \sinh(kh_1) = A_2 \sinh(kh_2) \ . \tag{3.425}$$

We now eliminate A_2 between the latter two equations and obtain

$$\omega^2 = \frac{kg(\rho_2 - \rho_1)}{\rho_2 \coth(kh_2) + \rho_1 \coth(kh_1)} \ . \tag{3.426}$$

This dispersion relation shows the highly interesting result that the frequency becomes imaginary if the specifically lighter fluid is placed beneath the specifically heavier one. This is the Rayleigh-Taylor or buoyancy instability (Figure 3.14): In such a configuration, small perturbations of the surface between the two fluids cause the fluids to begin exchanging their stratification.

3.6.3 The Kelvin-Helmholtz instability

We now come to another hydrodynamical instability, caused by a tangential velocity perturbation at the boundary between two fluids. Again, the boundary is described by a surface $\zeta(x, y, t)$ which, as long as the surfaces remain unperturbed, coincides with the plane $z = 0$. We proceed as follows. We imagine an unperturbed situation in which the upper fluid is streaming with velocity $\vec{v} = v\hat{e}_x$ into the x direction. We anticipate that this shear flow will excite wave-like perturbations $\delta\vec{v}$ in the velocity, δP in the pressure and $\delta z = \zeta$ in the boundary, and express this anticipation by adopting

$$\begin{aligned} \delta\vec{v} &= \delta\vec{v}_0 \, \mathrm{e}^{\mathrm{i}(kx - \omega t)} \ , \quad \delta P = \delta P_0 \, f(z) \, \mathrm{e}^{\mathrm{i}(kx - \omega t)} \ , \\ \zeta &= \zeta_0 \, \mathrm{e}^{\mathrm{i}(kx - \omega t)} \ . \end{aligned} \tag{3.427}$$

The equations we need to solve are the linearised Euler equation

$$\partial_t \delta\vec{v} + v \partial_x \delta\vec{v} = -\frac{\vec{\nabla} \delta P}{\rho} \tag{3.428}$$

for an incompressible fluid, $\vec{\nabla} \cdot \delta\vec{v} = 0$. Applying the divergence to (3.428) shows that the pressure perturbation δP now needs to satisfy the Laplace equation

$$\vec{\nabla}^2 \delta P = 0 \ , \tag{3.429}$$

which immediately implies the equation $f''(z) - k^2 f(z) = 0$ for $f(z)$ or

$$f(z) = f_0 \mathrm{e}^{-kz} \tag{3.430}$$

above the surface since the solution is then confined to $z > 0$. Now, the z component of Euler's equation shows that the velocity perturbation in z direction above the surface needs to satisfy

$$\delta v_z = \frac{k\delta P_1}{\mathrm{i}\rho_1(kv - \omega)} \ . \tag{3.431}$$

> **?** Notice once more the choice of a boundary condition for the Laplace equation (3.429) implied by the ansatz (3.430).

At the same time, δv_z must be given by the derivatives of the surface ζ,

$$\delta v_z = \partial_t\zeta + v\partial_x\zeta = \mathrm{i}(kv - \omega)\zeta \ . \tag{3.432}$$

Equating both expressions for δv_z, we find the relation

$$\delta P_1 = -\frac{\rho_1\zeta}{k}(kv - \omega)^2 \tag{3.433}$$

for the pressure fluctuation above the boundary. Below the boundary, $v = 0$ and the minus sign is changed to a plus sign since $f(z) = f_0 \exp(kz)$ there. Thus,

$$\delta P_2 = \frac{\rho_2\zeta}{k}\omega^2 \ . \tag{3.434}$$

Since the pressure fluctuation needs to be continuous at the boundary, $\delta P_1 = \delta P_2$ at $z = \zeta$,

$$\rho_2\omega^2 + \rho_1(kv - \omega)^2 = 0 \ . \tag{3.435}$$

This quadratic equation for ω leads to the dispersion relation

$$\omega_\pm = \frac{kv}{\rho_1 + \rho_2}\left(\rho_1 \pm \mathrm{i}\sqrt{\rho_1\rho_2}\right) \ . \tag{3.436}$$

Unless $\rho_1 = 0$, this frequency is complex and thus necessarily implies an instability, the so-called *Kelvin-Helmholtz* instability (Figures 3.15 and 3.15).

Figure 3.15 Examples for the Kelvin-Helmholtz instability in the atmospheres of the Earth (left panel) and Jupiter (right panel), in both cases indicated by clouds. (Wikipedia)

3.6.4 Thermal Instability

Let us now consider a physical system that gains energy by heating processes and loses energy by cooling mechanisms. Both heating and cooling can occur in many ways. Frequent heating mechanisms are heating by compression, by the injection of hot particles or by radiation from nearby sources. Cooling may occur by expansion or through radiation losses, but also for example by the emission of energetic particles. The net effect of the heating and cooling processes taken together is described by the cooling function

$$\mathcal{L}(\rho, T) , \tag{3.437}$$

which describes the total energy loss per unit mass and unit time. It is typically a function of the density ρ and the temperature T, but may of course depend on other parameters, such as the chemical composition of the system under consideration.

If the system is in thermal equilibrium, we require that the net cooling function vanish, $\mathcal{L}(\rho, T) = 0$: thermal equilibrium requires that the rates of energy gain by heating and loss by cooling exactly balance each other. This condition implicitly defines a relation between ρ and T.

Example: Thermal bremsstrahlung

For thermal bremsstrahlung, for example, the cooling function is proportional to the squared density times the square root of the temperature,

$$\mathcal{L}(\rho, T) = C\,\rho\,\sqrt{T} - (\text{heating terms}) . \tag{3.438}$$

The density ρ appears linear here rather than quadratic because we refer the cooling function to unit mass rather than unit volume. More realistic cooling functions contain terms caused by so-called line cooling, i.e. cooling by the emission of energy via spectral lines. ◀

The cooling function $\mathcal{L}(\rho, T)$ can adopt various forms, in particular because cooling processes are often related to thermal occupation numbers of quantum states and the quantum-mechanical transition probabilities between atomic or molecular excitations. Since the Boltzmann factor decreases exponentially with the energy of states scaled by the thermal energy $k_\mathrm{B}T$, sometimes small temperature changes can give rise to large changes in occupation numbers. If quantum transitions contribute to the cooling processes, the discrete atomic or molecular energy levels involved introduce discrete thresholds. A typical curve in the ρ-T plane characterised by the equilibrium condition $\mathcal{L}(\rho, T) = 0$ may thus contain flat plateaus and steep steps.

As is common in thermodynamics, there may be several equilibria for the system to attain, such as mechanical, thermal or phase equilibrium. Mechanical equilibrium could be established by the system adapting to some external pressure P. In such a case, the pressure at the system's boundary is set externally. Then, the equation of state, i.e. the relation between pressure, temperature and density $P = P(\rho, T)$, may define an additional curve in the ρ-T plane. For an ideal gas, for example, we must satisfy

$$P = \frac{\rho k_\mathrm{B} T}{m_\mathrm{particle}} . \tag{3.439}$$

Maintaining mechanical equilibrium with a constant external pressure P_ext then defines a hyperbola in the ρ-T plane, or a straight line in the $\log\rho$-$\log T$ plane.

If thermal and mechanical equilibrium need to be maintained at the same time, the system may occupy only such points in the ρ-T plane where the curves defined by the conditions $\mathcal{L}(\rho, T) = 0$ and $P(\rho, T) = P_{\rm ext}$ intersect. If the condition $\mathcal{L}(\rho, T) = 0$ contains plateaus and steps, this may occur in several points (Figure 3.16).

Mechanical equilibrium is usually established much faster than thermal or phase equilibria. Thus, if the external, mechanical conditions change, the system will first rearrange itself to maintain mechanical equilibrium. Doing so, it moves to another point in the ρ-T plane on the appropriate curve defined by $P(\rho, T) = P_{\rm ext}$. This will bring it out of thermal equilibrium.

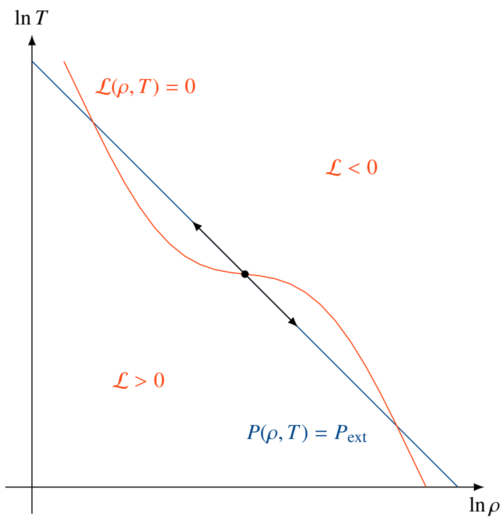

Figure 3.16 Illustration of the discussion on the onset of thermal instability in mechanical equilibrium with constant external pressure.

Example: Unstable or stable evolution

To construct a specific example, suppose that the system is heated by extra energy at constant external pressure. It will typically reestablish mechanical equilibrium faster than thermal equilibrium. To sufficient approximation, it will thus first expand and move along its isobaric curve towards lower density and higher temperature. Suppose this drives the system to a place where the cooling function is negative, $\mathcal{L} < 0$. Now, the energy gain will be larger than the energy loss. Then, the temperature will increase further, the density will decrease by further expansion, and the system will move even further away from thermal equilibrium. It will then be thermally unstable. Suppose, on the contrary, that the heating drives the system to a point where the cooling function is positive, $\mathcal{L} > 0$. It can then cool from its new position and move back to thermal equilibrium. ◀

Let us now consider a simple model for the thermal instability. Besides the continuity and Euler equations, taken in the form

$$\partial_t\rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \quad \text{and} \quad \partial_t\vec{v} + \frac{\vec{\nabla}v^2}{2} - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) = -\frac{\vec{\nabla}P}{\rho}\,, \qquad (3.440)$$

we write the energy-conservation equation

$$T\left[\partial_t s + \left(\vec{v}\cdot\vec{\nabla}\right)s\right] = -\mathcal{L}(\rho, T) \tag{3.441}$$

in terms of the specific entropy. The cooling function $\mathcal{L}$ gives the net energy loss per unit mass.

Let now the thermal equilibrium state be characterised by the density $\rho = \rho_0$ and the temperature $T = T_0$. By definition, the cooling function vanishes there, $\mathcal{L}(\rho_0, T_0) = 0$. We shall further assume that the unperturbed fluid velocity $\vec{v}_0 = 0$, which can be achieved by transforming to the comoving frame of the unperturbed flow. We perturb this state by small deviations $\delta\rho$, δT and $\delta\vec{v}$ to the density, the temperature and the velocity, respectively, and linearise in these perturbations. The linearised continuity and Euler equations read

$$\partial_t\delta\rho + \vec{\nabla}\cdot(\rho_0\delta\vec{v}) = 0\ , \quad \partial_t\delta\vec{v} = -\frac{\vec{\nabla}\delta P}{\rho_0}\ . \tag{3.442}$$

As usual, we eliminate the divergence of the velocity perturbation by combining the time derivative of the continuity equation with the divergence of the Euler equation. This enables us to write

$$\partial_t^2\delta\rho = \vec{\nabla}^2\delta P\ . \tag{3.443}$$

We first allow perturbations with $\delta P \neq 0$ and ask later for the conditions for instability under constant pressure.

We continue by linearising the entropy equation. We expand the specific entropy near its equilibrium value s_0 as

$$s = s_0 + \frac{\partial s}{\partial P}\delta P + \frac{\partial s}{\partial\rho}\delta\rho = s_0 + c_\mathrm{v}\frac{\delta P}{P_0} - c_\mathrm{p}\frac{\delta\rho}{\rho_0}\ , \tag{3.444}$$

where the earlier result (3.250) was used in the second step. Likewise, we expand the cooling function on the right-hand side of (3.441),

$$\mathcal{L}(\rho, T) = \mathcal{L}_0 + \left(\frac{\partial\mathcal{L}}{\partial T}\right)_\rho\delta T_\rho + \left(\frac{\partial\mathcal{L}}{\partial T}\right)_P\delta T_P\ , \tag{3.445}$$

distinguishing temperature changes at constant density, δT_ρ, and at constant pressure, δT_P. For later convenience, we introduce the abbreviations

$$\mathcal{L}_P := \frac{1}{c_\mathrm{p}}\left(\frac{\partial\mathcal{L}}{\partial T}\right)_P \quad \text{and} \quad \mathcal{L}_V := \frac{1}{c_\mathrm{v}}\left(\frac{\partial\mathcal{L}}{\partial T}\right)_\rho \tag{3.446}$$

for the derivatives of the cooling function with respect to temperature, taken at constant pressure or constant density. Since the specific entropy s_0 at equilibrium must satisfy the unperturbed entropy equation (3.441), we can insert the expansions (3.444) and (3.445) into (3.441), eliminate the equilibrium expressions and linearise in the perturbations to obtain

$$T_0\partial_t\left(c_\mathrm{v}\frac{\delta P}{P_0} - c_\mathrm{p}\frac{\delta\rho}{\rho_0}\right) = -c_\mathrm{v}\mathcal{L}_V\delta T_\rho - c_\mathrm{p}\mathcal{L}_P\delta T_P\ . \tag{3.447}$$

Now, for an ideal gas, the perturbed equation of state is

$$\delta P = \frac{\rho_0 k_\mathrm{B} T_0}{\bar{m}} \left(\frac{\delta T}{T_0} + \frac{\delta\rho}{\rho_0} \right) = P_0 \left(\frac{\delta T}{T_0} + \frac{\delta\rho}{\rho_0} \right) , \tag{3.448}$$

implying that the temperature changes at constant density and at constant pressure are

$$\delta T_\rho = T_0 \frac{\delta P}{P_0} \quad \text{and} \quad \delta T_P = -T_0 \frac{\delta\rho}{\rho_0} . \tag{3.449}$$

These expressions allow us to bring (3.447) into the form

$$\partial_t \left(c_\mathrm{v} \frac{\delta P}{P_0} - c_\mathrm{p} \frac{\delta\rho}{\rho_0} \right) = c_\mathrm{p} \mathcal{L}_P \frac{\delta\rho}{\rho_0} - c_\mathrm{v} \mathcal{L}_V \frac{\delta P}{P_0} . \tag{3.450}$$

After multiplying with P_0, replacing

$$\frac{P_0}{\rho_0} = \frac{c_\mathrm{s}^2}{\gamma} \tag{3.451}$$

according to (3.240) and recalling $c_\mathrm{p} = \gamma c_\mathrm{v}$, we arrive at

$$\partial_t \left(\delta P - c_\mathrm{s}^2 \delta\rho \right) = c_\mathrm{s}^2 \mathcal{L}_P \delta\rho - \mathcal{L}_V \delta P . \tag{3.452}$$

At this point, we take the Laplacian and insert the result (3.443) obtained previously from the linearly perturbed continuity and Euler equations. This leads to the equation

$$\partial_t \left(\partial_t^2 \delta\rho - c_\mathrm{s}^2 \vec{\nabla}^2 \delta\rho \right) = c_\mathrm{s}^2 \mathcal{L}_P \vec{\nabla}^2 \delta\rho - \mathcal{L}_V \partial_t^2 \delta\rho \tag{3.453}$$

for the density perturbations $\delta\rho$.

As usual in linear stability analysis, we evaluate this equation for a plane wave

$$\delta\rho = \delta\hat{\rho} \mathrm{e}^{\mathrm{i}(kx - \omega t)} , \tag{3.454}$$

for which (3.453) turns into

$$\partial_t \left[\left(c_\mathrm{s}^2 k^2 - \omega^2 \right) \delta\rho \right] = \left(\mathcal{L}_V \omega^2 - \mathcal{L}_P c_\mathrm{s}^2 k^2 \right) \delta\rho , \tag{3.455}$$

which yields the cubic dispersion relation

$$-\mathrm{i}\omega \left(c_\mathrm{s}^2 k^2 - \omega^2 \right) = \mathcal{L}_V \omega^2 - \mathcal{L}_P c_\mathrm{s}^2 k^2 . \tag{3.456}$$

In general, this equation is difficult to solve. In the limiting case of small wave lengths, $c_\mathrm{s}^2 k^2 \gg \omega^2$, we can approximate

$$\mathrm{i}\omega \approx \mathcal{L}_P \quad \text{or} \quad \omega \approx -\mathrm{i}\mathcal{L}_P . \tag{3.457}$$

Then, the density perturbation depends on time as

$$\delta\rho = \delta\hat{\rho}\, \mathrm{e}^{-\mathcal{L}_P t} , \tag{3.458}$$

hence it grows exponentially if $\mathcal{L}_P < 0$. Thermal instability thus sets in on small scales if

$$\mathcal{L}_P = \frac{1}{c_\mathrm{p}} \left(\frac{\partial \mathcal{L}}{\partial T} \right)_P < 0 , \tag{3.459}$$

i.e. if the cooling function decreases upon temperature increases at constant pressure. In the opposite limiting case of very large wave length, $c_s^2 k^2 \ll \omega^2$, the dispersion relation (3.456) demands that

$$\omega \approx -\mathrm{i}\mathcal{L}_V \; . \tag{3.460}$$

Thermal intability then sets in if

$$\mathcal{L}_V = \frac{1}{c_\mathrm{v}}\left(\frac{\partial \mathcal{L}}{\partial T}\right)_\rho < 0 \; . \tag{3.461}$$

The conditions (3.459) and (3.461) show that thermal instability must be expected if the cooling function decreases upon temperature increases, or in other words, if higher temperature leads to reduced cooling or conversely if lower temperature implies enhanced cooling. These conditions are of course quite intuitive: If a system can cool more efficiently the cooler it gets, or if it can cool less efficiently the hotter it gets, any small temperature fluctuation can be expected to grow.

3.6.5 Heat conduction

Since mechanical equilibrium can typically be established much faster than thermal equilibrium, systems can be in mechanical equilibrium, but out of thermal equilibrium. Perhaps the most straightforward example is a star which is being kept in mechanical equilibrium by the balance between gravity and the pressure gradient, but nonetheless continuously radiates energy. In this case, a temperature gradient is maintained between the core and the surface by the central energy production, and the entropy equation reads

$$\rho T \frac{\mathrm{d}s}{\mathrm{d}t} = \vec{\nabla} \cdot \left(\kappa \vec{\nabla} T\right) + \sigma_{ij} \frac{\partial v^i}{\partial x_j} \; . \tag{3.462}$$

If the velocity gradient on the right-hand side can be considered too small to drive any matter currents, the second term on the right-hand side can be neglected. At constant pressure, we can write the change δq of heat per unit mass as

$$\delta q = c_\mathrm{p} \mathrm{d}T = T \mathrm{d}s \; , \tag{3.463}$$

which we can solve for the differential

$$\mathrm{d}s = c_\mathrm{p} \mathrm{d} \ln T \tag{3.464}$$

of the specific entropy. This allows us to reduce the entropy equation (3.462) to read

$$\rho c_\mathrm{p} \frac{\mathrm{d}T}{\mathrm{d}t} = \kappa \vec{\nabla}^2 T \; . \tag{3.465}$$

Introducing the transport coefficient $\chi \equiv \kappa/(\rho c_\mathrm{p})$ for the temperature, we can re-write the entropy equation as a diffusion equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \chi \vec{\nabla}^2 T \tag{3.466}$$

for the temperature T with the diffusion coefficient χ.

In close analogy to radiative energy transport, in particular the result (2.484) for the radiative energy-current density, we now define a conductive opacity $\kappa_{\rm cond}$ through the conductive energy current

$$\vec{F}_{\rm cond} = -\frac{c}{3\rho\kappa_{\rm cond}}\vec{\nabla}\left(aT^4\right) = -\frac{4acT^3}{3\rho\kappa_{\rm cond}}\vec{\nabla}T \equiv -\kappa\vec{\nabla}T \;, \tag{3.467}$$

from which we obtain the relation

$$\kappa = \frac{4caT^3}{3\rho\kappa_{\rm cond}} \tag{3.468}$$

between the heat conductivity κ and the conductive opacity $\kappa_{\rm cond}$. If both radiative and conductive energy transport are present, an effective opacity $\kappa_{\rm eff}$ can thus usefully be defined by

$$\frac{1}{\kappa_{\rm eff}} = \frac{1}{\kappa_{\rm rad}} + \frac{1}{\kappa_{\rm cond}} \quad \Rightarrow \quad \kappa_{\rm eff} = \frac{\kappa_{\rm rad}\kappa_{\rm cond}}{\kappa_{\rm rad} + \kappa_{\rm cond}} \;. \tag{3.469}$$

?

Why is it useful to define the inverse of the effective opacity as the sum of the inverse radiative and conductive opacities as in (3.469)?

3.6.6 Convection

We have just seen that temperature gradients drive energy currents of either by electromagnetic radiation or heat conduction. If the temperature gradient is too large, convection sets in. Then, warm, rising bubbles cannot cool sufficiently and adapt to their environment as they rise. Instead, they remain warmer than the surrounding medium and continue to rise. We now investigate this situation, considering a volume $V(P, s)$ of gas characterised by the pressure P and the specific entropy s as it rises against the gravitational force (Figure 3.17).

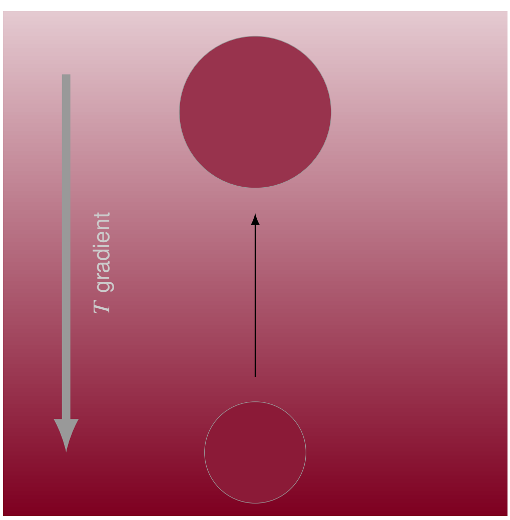

Figure 3.17 Convection sets in if rising bubbles retain higher entropy than their environment.

We ignore again thermal compared to mechanical adaptation processes because they are typically slower. We watch the bubble with volume $V(P, s)$ as it rises by

Example: Heat conductivity due to electrons

When we discussed the heat conductivity as a phenomenon due to particle transport, we saw in (3.135) that it can be written as

$$\kappa = \frac{nv}{\sqrt{3}} c_{\mathrm{v}} \lambda \tag{3.470}$$

in terms of the mean-free path λ of the fluid particles. If we consider electrons whose mean free path is determined by scattering off the ions,

$$\lambda = \frac{1}{n_{\mathrm{i}}\sigma} , \tag{3.471}$$

where n_{i} and σ are the number density of the ions and their scattering cross section with the electrons.

Typically, an electron will approach an ion up to a distance r_{i} where the kinetic and potential energies equal,

$$\frac{mv^2}{2} \approx \frac{Ze^2}{r_{\mathrm{i}}} \quad \Rightarrow \quad r_{\mathrm{i}} \approx \frac{2Ze^2}{mv^2} . \tag{3.472}$$

The cross section for electron-ion scattering can then be crudely approximated by

$$\sigma \approx \pi r_{\mathrm{i}}^2 , \tag{3.473}$$

and we obtain the expression

$$\kappa = \frac{n_{\mathrm{e}} v_{\mathrm{e}}}{\sqrt{3}} c_{\mathrm{v}} \frac{m_{\mathrm{e}}^2 v_{\mathrm{e}}^4}{4\pi n_{\mathrm{i}} Z^2 e^4} = \frac{1}{\sqrt{3}} \left(\frac{m_{\mathrm{e}}^2}{4\pi Z^2 e^4} \right) \left(\frac{n_{\mathrm{e}}}{n_{\mathrm{i}}} \right) c_{\mathrm{v}} v_{\mathrm{e}}^5 \tag{3.474}$$

for the heat conductivity contributed by electrons scattered by ions.

In a thermal electron gas, the heat capacity at constant volume is $c_{\mathrm{v}} = 3k_{\mathrm{B}}/2$ per particle and the thermal electron velocity is

$$v_{\mathrm{e}}^2 = \frac{3k_{\mathrm{B}} T_{\mathrm{e}}}{m_{\mathrm{e}}} . \tag{3.475}$$

Inserted into (3.474), these results give the heat conductivity

$$\kappa = \frac{\sqrt{3} k_{\mathrm{B}}}{2} \left(\frac{m_{\mathrm{e}}^2}{4\pi Z^2 e^4} \right) \left(\frac{n_{\mathrm{e}}}{n_{\mathrm{i}}} \right) \left(\frac{3k_{\mathrm{B}} T_{\mathrm{e}}}{m_{\mathrm{e}}} \right)^{5/2} \tag{3.476}$$

for classical (non-degenerate) electrons. ◀

Example: Heat conductivity due to electrons (continued)

If we identify the Thomson cross section σ_T here, we can alternatively write

$$\kappa = \frac{9k_\mathrm{B}c}{Z^2\sigma_\mathrm{T}}\left(\frac{n_\mathrm{e}}{n_\mathrm{i}}\right)\left(\frac{k_\mathrm{B}T_\mathrm{e}}{m_\mathrm{e}}\right)^{5/2} , \tag{3.477}$$

which obviously has the required dimension

$$[\kappa] = \frac{\mathrm{erg}}{\mathrm{cm\,s\,K}} . \tag{3.478}$$

Numerically, we find

$$\kappa \approx 9.5 \cdot 10^{12}\, Z^{-2}\left(\frac{n_\mathrm{e}}{n_\mathrm{i}}\right)\left(\frac{kT_\mathrm{e}}{1\,\mathrm{keV}}\right)^{5/2} . \tag{3.479}$$

◂

an amount Δz, where its volume after the essentially instantaneous mechanical adaptation is $V' = V(P', s)$. Having risen, the bubble experiences a buoyancy force determined by the volume $V'' = V(P', s')$ which the bubble *would* adopt if it had the specific entropy s' of its new environment. This situation is stable if the actual bubble volume $V' = V(P', s)$ is smaller than the adapted volume $V'' = V(P', s')$, because then gravity will dominate the buoyancy force, and the bubble will then sink down again. We thus have the condition

$$V(P', s') = V'' > V' = V(P', s) \tag{3.480}$$

for convective stability.

The entropy at the increased height $z + \Delta z$ is

$$s' = s + \left.\frac{\mathrm{d}s}{\mathrm{d}z}\right|_z \Delta z , \tag{3.481}$$

and the volume change of the bubble with specific entropy at constant pressure is

$$\mathrm{d}V = \left(\frac{\partial V}{\partial s}\right)_P \mathrm{d}s = c_\mathrm{p}\left(\frac{\partial V}{\partial s}\right)_P \frac{\mathrm{d}T}{T} . \tag{3.482}$$

In its new environment at increased height $z + \Delta z$ with its specific entropy s', the bubble thus attains the new volume

$$V'' = V' + \left(\frac{\partial V}{\partial s}\right)_P \Delta s = V' + \left(\frac{\partial V}{\partial s}\right)_P \left.\frac{\mathrm{d}s}{\mathrm{d}z}\right|_z \Delta z . \tag{3.483}$$

The stability condition (3.480) is thus satisfied if the specific entropy increases with the height z,

$$\left.\frac{\mathrm{d}s}{\mathrm{d}z}\right|_z > 0 . \tag{3.484}$$

The derivative of the specific entropy with respect to the height z can be expressed by

$$\frac{\mathrm{d}s}{\mathrm{d}z} = \left(\frac{\partial s}{\partial T}\right)_P \frac{\mathrm{d}T}{\mathrm{d}z} + \left(\frac{\partial s}{\partial P}\right)_T \frac{\mathrm{d}P}{\mathrm{d}z} = c_\mathrm{v}\left[\gamma\frac{\mathrm{d}\ln T}{\mathrm{d}z} - (\gamma - 1)\frac{\mathrm{d}\ln P}{\mathrm{d}z}\right] , \tag{3.485}$$

where we have used the partial derivatives (3.246) of the entropy found earlier. The stability condition (3.484) then shows that the temperature gradient must satisfy

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln z} > \frac{\gamma - 1}{\gamma}\frac{\mathrm{d}\ln P}{\mathrm{d}\ln z} \tag{3.486}$$

for the gas stratification to be *unstable* against convection. The quantity

$$\frac{\gamma - 1}{\gamma} \equiv \nabla_{\mathrm{ad}} \tag{3.487}$$

is often called the adiabatic temperature gradient ("nabla adiabatic"). Using this, we stability condition is written in the compact form

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} \equiv \nabla < \nabla_{\mathrm{ad}} \ . \tag{3.488}$$

Once convection sets in, it is a very efficient means of transporting heat, but viscosity can hinder the convective energy transport.

?

Why is ∇_{ad} from (3.487) called "adiabatic" temperature gradient? To see this, work out

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}$$

for an adiabatically stratified gas.

3.6.7 Turbulence

Turbulence is a very rich, important and fascinating field of its own. By no means can we treat turbulence here in any depth. We confine the discussion here to one physically and methodically aspect, i.e. the derivation of the Kolmogorov power spectrum for the scale dependence of energy in subsonic turbulence.

Hydrodynamical flows with large Reynolds numbers turn out to be highly unstable. For high viscosity (low Reynolds number), stable solutions of the Navier-Stokes equation exist which develop instabilities above a critical Reynolds number

$$\mathcal{R} = \frac{uL}{\nu} \gtrsim \mathcal{R}_{\mathrm{cr}} \ . \tag{3.489}$$

A full analysis of such instabilities is very difficult and in general an unsoved problem. Turbulence sets in, in the course of which energy is being transported from large to small scales until it is dissipated by the production of viscous heat on sufficiently small scales. Turbulence can be seen as the transitional regime between macroscopic, ordered as opposed to microscopic, unordered or thermal motion. On the macroscopic scale, turbulence is driven by some stirring mechanism acting on some linear scale. Eddies form on that scale which feed a cascade of eddies of decreasing size. This proceeds until the smallest eddies reach a size comparable to the mean-free path of the fluid particles. The energy fed into the turbulent cascade on the driving scale propagates through the cascade of eddies and is finally dissipated into heat by dissipation, i.e. by the viscosity of the fluid.

Let λ be the size of an eddy within the turbulent cascade of the fluid flow, and v_λ the linear velocity that the eddy rotates with. The characteristic time scale of one turn-over of the eddy is $\tau_\lambda = \lambda / v_\lambda$. Let further ε_λ be the specific energy, i.e.

the energy per unit mass characteristic for the material in such an eddy. Then, the flow of the specific energy through an eddy of the size λ is

$$\dot{\varepsilon} \approx \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} \approx \overbrace{\left(\frac{v_\lambda^2}{2}\right)}^{\text{specific energy}} \underbrace{\left(\frac{\lambda}{v_\lambda}\right)^{-1}}_{\text{inverse time scale}} \approx \frac{v_\lambda^3}{\lambda} . \tag{3.490}$$

Let L be the macroscopic length scale on which the energy is being fed into the turbulent cascade, and u be the typical stirring velocity. From there, the energy cascades through the turbulent eddies to progressively smaller scales until it is finally viscously dissipated on a scale λ_{visc}. In between, i.e. on scales λ satisfying the scale hierarchy

$$\lambda_{\text{visc}} < \lambda < L , \tag{3.491}$$

the energy flow $\dot{\varepsilon}$ must be independent of scale because the energy cannot be accumulated at any scale in the cascade. Therefore, we conclude from (3.490) that the typical eddy velocity must change with the eddy scale λ as

$$v_\lambda \propto \lambda^{1/3} . \tag{3.492}$$

Together with the boundary condition that the velocity be u on the driving scale L, we thus expect

$$v_\lambda \approx u\left(\frac{\lambda}{L}\right)^{1/3} . \tag{3.493}$$

The largest eddies thus rotate with the highest velocities, but the smallest have the highest vorticity,

$$\Omega \approx \frac{v_\lambda}{\lambda} \approx \frac{u}{\lambda}\left(\frac{\lambda}{L}\right)^{1/3} \approx \frac{u}{(\lambda^2 L)^{1/3}} . \tag{3.494}$$

To estimate the viscous scale λ_{visc}, we compare the viscous dissipation with the specific energy flow $\dot{\varepsilon}$. The viscous heating rate h_{visc} can be estimated by the viscosity times the squared velocity gradient, as can be seen from the dissipation term on the right-hand side of the Navier-Stokes equation (3.148). Thus,

$$h_{\text{visc}} \approx \eta\left(\frac{v_\lambda}{\lambda}\right)^2 \approx \eta\left(\frac{v_\lambda^3}{\lambda}\right)^{2/3} \lambda^{-4/3} = \eta\dot{\varepsilon}^{2/3}\lambda^{-4/3} . \tag{3.495}$$

Therefore, h_{visc} is negligibly small on large scales, but if the heating rate becomes of the order of the energy flow rate,

$$h_{\text{visc}} \approx \rho\dot{\varepsilon} , \tag{3.496}$$

viscous dissipation begins dominating. According to (3.495), this happens on a length scale λ_{visc} given by

$$\eta\dot{\varepsilon}^{2/3}\lambda_{\text{visc}}^{-4/3} \approx \rho\dot{\varepsilon} . \tag{3.497}$$

Solving for λ_{visc} and inserting $\dot{\varepsilon} = u^3/L$ gives

$$\lambda_{\text{visc}} = \left(\frac{\eta L^{1/3}}{\rho u}\right)^{3/4} = L\left(\frac{\nu}{uL}\right)^{3/4} = \frac{L}{\mathcal{R}^{3/4}} , \tag{3.498}$$

where $\mathcal{R}$ is the Reynolds number on the scale L.

Finally, we consider how the specific energy is distributed over scales. Doing so, we assess how the specific energy scales with the wave number k. Since the squared velocity scales like $\lambda^{2/3}$, its Fourier transform scales like

$$\widehat{\left(v_\lambda^2\right)} \propto \lambda^3 \, \lambda^{2/3} \propto \lambda^{11/3} \propto k^{-11/3} \ . \tag{3.499}$$

The number of Fourier modes in shells of width dk is $\propto k^2$, hence the energy spectrum as a function of wave number is

$$E(k) \propto k^2 \, k^{-11/3} = k^{-5/3} \ . \tag{3.500}$$

This is the famous energy spectrum derived by Kolmogorov in 1941, showing how the energy in a turbulent cascade is distributed over scales identified by their wave number k.

Problems

1. Studying gravity waves, we have solved (3.417) with the implicitly assumed boundary condition $f(z) = 0$ for $z \to -\infty$. The dispersion relation (3.418) is thus valid for infinitely deep water.

    (a) Derive the solution of (3.417) satisfying the boundary condition $f(z) = 0$ at finite depth, $z = -h$.

    (b) Which dispersion relation do you find for water with finite depth?

Suggested further reading: [10, 11, 12, 13, 14, 15, 16, 17]