

Chapter 2

Radiation Processes

This chapter deals with radiation processes. These are defined as processes by which electromagnetic radiation is either scattered, emitted or absorbed by matter. In the first five sections, radiation will be treated as a classical electromagnetic wave. We shall begin with the very illustrative case of Thomson scattering, then give a general description of spectra, proceed to synchrotron radiation and bremsstrahlung and finally consider the drag that a charged particle experiences as it moves through a radiation field. Up to that point, our main theoretical instrument will be Larmor's formula, either in its fully relativistic form (1.138) or in its non-relativistic approximation (1.142), which quantifies the radiation power of a charge moving with a velocity $\vec{\beta}$ and accelerated by $\dot{\vec{\beta}}$. Then, we shall leave the classical picture of electromagnetic waves and consider quantum properties of radiation. The theory of Compton scattering treats electromagnetic radiation as a stream of photons. Emission of radiation by quantum systems will be discussed next, treating their interaction with electromagnetic radiation at a semi-classical, perturbative level, i.e. without quantisation of the electromagnetic field. This will lead us to the calculation of radiative transition probabilities and finally to the shape of spectral lines.

2.1 Thomson scattering

Thomson scattering describes perhaps the simplest case of interaction between an electromagnetic wave and a point charge: The wave accelerates the charge transversally to its propagation direction. Due to its accelerated motion, the charge radiates according to the non-relativistic Larmor formula. The emitted radiation power, divided by the flux density of the incoming radiation, is the Thomson cross section. Its differential, polarisation-dependent or polarisation-averaged forms (2.13) and (2.14) as well as the total Thomson cross section (2.15) are the main results of this section.

Let us begin with a monochromatic, polarised, plane electromagnetic wave hitting an electron at rest. By the Lorentz force, it will accelerate the electron to move harmonically. Because of this accelerated motion, the electron will radiate according to Larmor's formula, as we have seen in Sect. 1.3.5. We ask

now how the energy radiated by the electron relates to the energy transported by the incoming wave.

For definiteness, we introduce a coordinate frame such that the infalling electromagnetic wave propagates into the \hat{e}_z direction. The \vec{E} and \vec{B} vectors must then fall into the x - y plane because electromagnetic waves in vacuum are transversal. The polarisation angle will be fixed below. We place the electron at rest into the origin of the coordinate frame.

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Why are electromagnetic waves in vacuum transversal? Can you construct situations in which longitudinal electromagnetic waves occur?

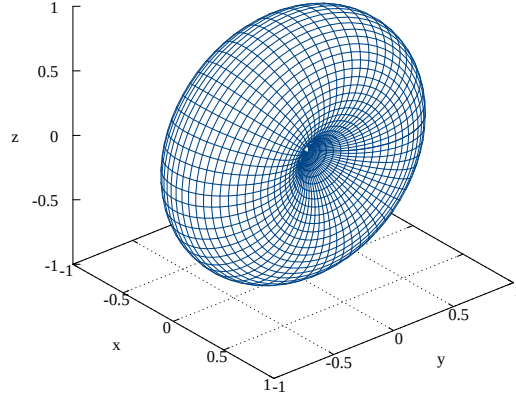


Figure 2.1 The spatial radiation pattern of a non-relativistic charge accelerated along the x axis is shown here. (The x axis points horizontally towards the bottom right).

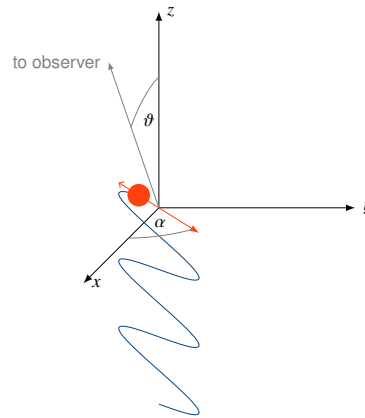


Figure 2.2 Choice of the coordinate system for the treatment of Thomson scattering in the text.

The electron experiences the Lorentz force

$$m_e \ddot{\vec{x}} = -e\vec{E} - \frac{e}{c} \vec{v} \times \vec{B} = -e(\vec{E} + \vec{\beta} \times \vec{B}) . \quad (2.1)$$

For the incoming wave, $|\vec{B}| = |\vec{E}|$. If the electron moves non-relativistically, $|\vec{v}| \ll c$, the magnetic contribution to the Lorentz force can be neglected since

the \vec{E} and \vec{B} fields of an electromagnetic wave in vacuum have equal magnitude. The equation of motion for the electron then reduces to

$$\ddot{\vec{x}} = c\dot{\vec{\beta}} = -\frac{e}{m_e}\vec{E}. \quad (2.2)$$

The non-relativistic limit of the Larmor formula (1.138) is

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c} \left| \hat{e} \times \dot{\vec{\beta}} \right|^2. \quad (2.3)$$

It gives the energy radiated per unit time into the solid-angle element $d\Omega$ around the vector \hat{e} pointing from the charge to the observer (Figure 2.1). Since the electron's motion is non-relativistic, retardation effects can be neglected. Inserting the acceleration by the Lorentz force (2.2) with (2.3) gives

$$\frac{dP}{d\Omega} = \frac{e^4}{4\pi m_e^2 c^3} \left| \hat{e} \times \vec{E} \right|^2. \quad (2.4)$$

We rotate the coordinate frame (Figure 2.2) such that the observer lies in the x - z plane,

$$\hat{e} = \begin{pmatrix} \sin \theta \\ 0 \\ \cos \theta \end{pmatrix}, \quad (2.5)$$

and introduce the polarisation angle α of the incoming \vec{E} field as the angle enclosed by the \vec{E} vector with the \hat{e}_x axis,

$$\vec{E} = E \begin{pmatrix} \cos \alpha \\ \sin \alpha \\ 0 \end{pmatrix}. \quad (2.6)$$

With this choice, we find

$$\hat{e} \times \vec{E} = E \begin{pmatrix} -\sin \alpha \cos \theta \\ \cos \alpha \cos \theta \\ \sin \alpha \sin \theta \end{pmatrix}, \quad (2.7)$$

and the radiated power per solid angle given by (2.4) turns into

$$\frac{dP}{d\Omega} = \frac{e^4 E^2}{4\pi m_e^2 c^3} (1 - \sin^2 \theta \cos^2 \alpha). \quad (2.8)$$

The infalling energy current density is quantified by the Poynting vector of the incoming wave,

$$\vec{S} = \frac{c}{4\pi} \left| \vec{E} \right|^2 \hat{e}_z. \quad (2.9)$$

This is the energy per unit area and unit time impinging on the electron. The ratio between the energy radiated per unit time and unit solid angle and the energy current density,

$$\frac{1}{|\vec{S}|} \frac{dP}{d\Omega} = \frac{d\sigma}{d\Omega} = \frac{e^4}{m_e^2 c^4} (1 - \sin^2 \theta \cos^2 \alpha), \quad (2.10)$$

has the dimension of an area. It is the *differential Thomson cross section* for polarised light (Figure 2.3a).

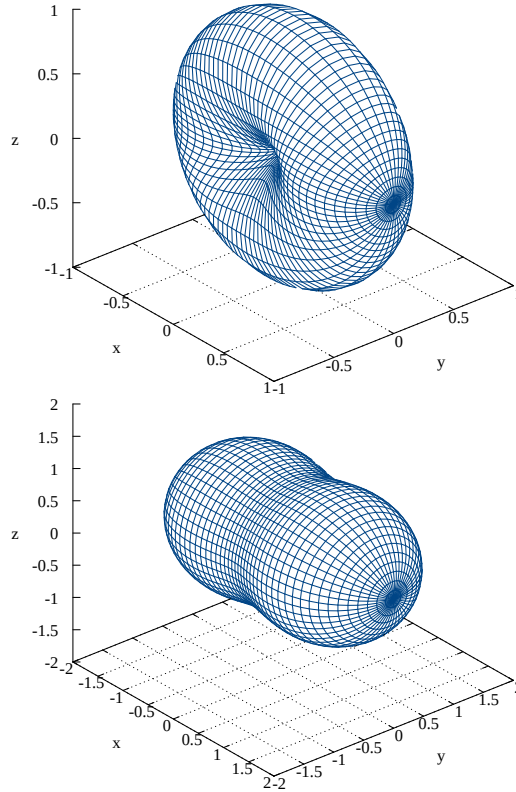


Figure 2.3 These bodies illustrate the polarised and the unpolarised Thomson cross sections for electromagnetic waves propagating along the positive x direction. *Top panel:* The directional dependence of the polarised Thomson cross section on the scattering angle is shown here, with the polarisation angle being the angle enclosed with the z axis (i.e. the polar angle). *Bottom panel:* The unpolarised Thomson cross section is forward-backward symmetric.

The prefactor $e^4/m_e^2c^4$ has an interesting and intuitive meaning. Suppose we want to explain the entire rest-energy of the electron by the electrostatic energy of the charge e distributed over a sphere of radius r_e . We would then require

$$m_e c^2 = \frac{e^2}{r_e} \quad (2.11)$$

and find the classical electron radius

$$r_e = \frac{e^2}{m_e c^2} \approx 2.81 \cdot 10^{-13} \text{ cm} . \quad (2.12)$$

For ions composed of N nucleons and having a charge number Z , this classical radius is at least approximately $Z^2/(1800 N)$ times smaller because of their much higher mass. The Thomson cross section of ions is therefore generally negligibly small compared to that of the electrons. Electromagnetic radiation flowing through, say, a hydrogen plasma is scattered by the electrons, which then interact mainly by Coulomb collisions with the ions.

The classical electron radius brings the differential, polarised Thomson cross section (2.10) into the simple, intuitive form

$$\frac{d\sigma}{d\Omega} = r_e^2 (1 - \sin^2 \theta \cos^2 \alpha) . \quad (2.13)$$

For unpolarised light, we need to average (2.13) over all polarisation angles α . This average leads to the unpolarised, differential Thomson cross section (Figure 2.3b)

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle_\alpha = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\sigma}{d\Omega} d\alpha = \frac{r_e^2}{2} (2 - \sin^2 \theta) = \frac{r_e^2}{2} (1 + \cos^2 \theta) . \quad (2.14)$$

If we finally integrate over all directions into which the radiation is scattered, we find the *total Thomson cross section*

$$\begin{aligned} \sigma_T &= \int d\Omega \left\langle \frac{d\sigma}{d\Omega} \right\rangle_\alpha = \pi r_e^2 \int_{-1}^1 d(\cos \theta) (1 + \cos^2 \theta) \\ &= \frac{8\pi}{3} r_e^2 \approx 6.64 \cdot 10^{-25} \text{ cm}^2 . \end{aligned} \quad (2.15)$$

This can be interpreted as the area that a single, non-relativistic electron puts in the way of incoming, unpolarised radiation.

Problems

1. Work out the mean molecular mass for a mixture of neutral, atomic hydrogen and helium as a function of the hydrogen mass fraction X .
2. Consider an electron at the origin of the coordinate system, illuminated by two unpolarised electromagnetic wave bundles propagating along the $-y$ and $-z$ axes with different energy current densities S_y and S_z .
 - (a) Find the radiation power radiated into the x direction.
 - (b) Is the scattered radiation polarised?

2.2 Spectra

This brief section discusses how electromagnetic spectra of accelerated charges can be computed. The starting point is Larmor's equation in its relativistic or non-relativistic forms, giving the radiation power. The total energy radiated away is the time integral over the power which, by Parseval's equation for Fourier-conjugate functions, can be converted to a frequency integral. Its integrand is the energy per unit frequency, i.e. the spectrum. This allows us to derive the fully relativistic expression (2.36) for the spectrum. The substantially simplified versions (2.39) and (2.42) for non-relativistic charges can be directly derived from the non-relativistic Larmor equation.

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Before moving on, verify (2.13), then average over polarisation angles and integrate over the solid angle.

Example: Eddington Luminosity

Let us immediately apply the Thomson scattering cross section to the following situation. Suppose ionised gas surrounds a hot, spherically-symmetric, radiating body of mass M . The radiation carries the momentum current density

$$\frac{\vec{S}}{c} = \frac{1}{4\pi} |\vec{E}|^2 \hat{e}, \quad (2.16)$$

given by the components of Maxwell's stress-energy tensor. Since this is the momentum flowing per unit time through unit area, it corresponds to a force per unit area, or a pressure exerted on an ideally absorbing wall.

The total energy emitted by the star per unit time is its luminosity L . Exploiting the spherical symmetry, we have

$$L = \int \vec{S} \cdot d\vec{a} = 4\pi R^2 \cdot \frac{c}{4\pi} |\vec{E}(R)|^2, \quad (2.17)$$

where $\vec{E}(R)$ is the electric field strength at radius R . According to (2.16), the radiation pressure there is expressed by L after eliminating the electric field \vec{E} ,

$$\frac{\vec{S}}{c} = \frac{L}{4\pi c R^2} \hat{e}. \quad (2.18)$$

Each electron in the surrounding plasma has a Thomson-scattering cross section of σ_T and thus experiences the force

$$\vec{F}_R = \frac{\vec{S}}{c} \cdot \sigma_T = \frac{L}{4\pi R^2 c} \sigma_T \hat{e} \quad (2.19)$$

by the radiation pressure. Recall that the force on the ions in the plasma is lower by a factor of $\approx Z^2/(1800N)$ if the ions have the charge Ze and are composed of N nucleons. This radiation-pressure force acting radially outward is counter-acted by the gravitational force of the mass M of the central body,

$$\vec{F}_G = -\frac{GMm}{R^2} \hat{e}. \quad (2.20)$$

Both forces compensate each other if the luminosity L satisfies

$$\frac{L}{4\pi R^2 c} \sigma_T = \frac{GMm}{R^2}, \quad (2.21)$$

i.e. if the luminosity reaches the *Eddington limit*

$$L = L_{\text{Edd}} = \frac{4\pi GMm}{\sigma_T} c. \quad (2.22)$$

Inserting a solar mass for M and a proton mass for m here results in

$$L_{\text{Edd}} = 1.26 \cdot 10^{38} \frac{\text{erg}}{\text{s}} = 3.28 \cdot 10^4 L_{\odot} \quad (2.23)$$

(see Tabs. 1.3 and 1.4). ◀

Example: Eddington Luminosity (continued)

Note that we have deliberately not specified the particle mass m in (2.20) and the following equations to be the electron mass. Consider a hydrogen plasma consisting of an equal mixture of electrons and protons. By (2.12), the Thomson cross section of a proton is about $1800^2 \approx 3.2 \cdot 10^6$ times smaller than that of an electron. However, while essentially only the electrons feel the radiation pressure, they are tightly coupled by Coulomb interactions to the protons. The radiation-pressure force thus needs to compensate the gravitational force felt by the electrons and the protons together. The particle mass m inserted in (2.22) should therefore be the total mass per electron rather than the electron mass alone. For a fully ionised hydrogen plasma, we can approximate m by the proton mass m_p . ◀

The energy received by an observer from a radiating electron, flowing into the solid angle $d\Omega$, is

$$\frac{dE}{d\Omega} = \int dt \frac{dP}{d\Omega}, \quad (2.24)$$

where $dP/d\Omega$ is given by the Larmor formula (1.138). Often, we are interested in the radiation spectrum, i.e. in the distribution of the energy over frequency rather than time. Realising that the time t and the frequency ω are Fourier conjugates, this is most easily found using Plancherel's theorem,

$$\int_{-\infty}^{\infty} dt |f(t)|^2 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\hat{f}(\omega)|^2, \quad (2.25)$$

which specialises Parseval's equation for continuous Fourier transforms. It relates the integral over a function to that over its Fourier transform. The negative frequencies ω in (2.25) may appear strange here. Nonetheless, they obtain a well-defined meaning because we require that $f(t)$ be real. Then, its Fourier transform $\hat{f}(\omega)$ must satisfy the relation $\hat{f}(-\omega) = \hat{f}^*(\omega)$.

Applying Plancherel's theorem to (2.24) and inserting the Larmor formula (1.138), we find

$$\begin{aligned} \frac{dE}{d\Omega} &= \frac{e^2}{4\pi c} \int_{-\infty}^{\infty} dt \left| \frac{\hat{\mathbf{e}} \times \left[(\hat{\mathbf{e}} - \vec{\beta}) \times \dot{\vec{\beta}} \right]}{(1 - \hat{\mathbf{e}} \cdot \vec{\beta})^3} \right|^2 = \frac{e^2}{4\pi c} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\hat{f}(\omega)|^2 \\ &= \int_{-\infty}^{\infty} d\omega \frac{d^2E}{d\Omega d\omega}, \end{aligned} \quad (2.26)$$

where $\hat{f}(\omega)$ is now specified to be the Fourier transform of the function

$$f(t) = \frac{\hat{\mathbf{e}} \times \left[(\hat{\mathbf{e}} - \vec{\beta}) \times \dot{\vec{\beta}} \right]}{(1 - \hat{\mathbf{e}} \cdot \vec{\beta})^3} \quad (2.27)$$

that can directly be read off the Larmor formula (1.138). The spectrum is then given by the absolute square of $\hat{f}(\omega)$,

$$\frac{d^2E}{d\Omega d\omega} = \frac{e^2}{8\pi^2 c} |\hat{f}(\omega)|^2. \quad (2.28)$$

The Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dt \frac{\hat{e} \times \left[(\hat{e} - \vec{\beta}) \times \dot{\vec{\beta}} \right]}{(1 - \hat{e} \cdot \vec{\beta})^3} e^{-i\omega t} \quad (2.29)$$

simplifies considerably realising that the integrand needs to be evaluated at the retarded time $t' = t - R/c$, where R is the distance from the observer to the electron at the retarded time. Taking into account that the differential dt' of the retarded time is related to dt by (1.136), we can first cancel one factor $(1 - \hat{e} \cdot \vec{\beta})$ from the denominator and write

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dt' \frac{\hat{e} \times \left[(\hat{e} - \vec{\beta}) \times \dot{\vec{\beta}} \right]}{(1 - \hat{e} \cdot \vec{\beta})^2} e^{-i\omega(t'+R/c)}. \quad (2.30)$$

Furthermore, a short calculation shows that the integrand can be written as a total derivative with respect to the retarded time t' ,

$$\frac{\hat{e} \times \left[(\hat{e} - \vec{\beta}) \times \dot{\vec{\beta}} \right]}{(1 - \hat{e} \cdot \vec{\beta})^2} = \frac{d}{dt'} \left[\frac{\hat{e} \times (\hat{e} \times \vec{\beta})}{(1 - \hat{e} \cdot \vec{\beta})} \right]. \quad (2.31)$$

Verify (2.31) by your own calculation.

This leaves us with

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dt' \frac{d}{dt'} \left[\frac{\hat{e} \times (\hat{e} \times \vec{\beta})}{(1 - \hat{e} \cdot \vec{\beta})} \right] e^{-i\omega(t'+R/c)}, \quad (2.32)$$

which calls for partial integration. Before we get to that, however, we decompose the distance vector \vec{R} from the radiating electron to the observer into the distance vector \vec{x} from the center of the orbit to the electron and the distance vector \vec{r} from the center of the orbit to the observer,

$$\vec{R} = \vec{r} - \vec{x}. \quad (2.33)$$

The idea behind this decomposition is that the motion of the radiating charge is confined to a distant source, and thus to a volume which is far away and small compared to its distance from the observer. The retarded distance R is then $\hat{e} \cdot \vec{R}$, and its derivative with respect to the retarded time is

$$\frac{d}{dt'} \frac{R}{c} = -\hat{e} \cdot \dot{\vec{\beta}}; \quad (2.34)$$

compare (1.126). Assuming that the emission in the distant past and in the far future can be neglected, we can ignore the boundary terms appearing in the partial integration of (2.32). Taking (2.34) into account, the partial integration gives

$$\begin{aligned} \hat{f}(\omega) &= - \int_{-\infty}^{\infty} dt' \left[\frac{\hat{e} \times (\hat{e} \times \vec{\beta})}{(1 - \hat{e} \cdot \vec{\beta})} \right] \frac{d}{dt'} e^{-i\omega(t'+R/c)} \\ &= i\omega \int_{-\infty}^{\infty} dt' \left[\hat{e} \times (\hat{e} \times \vec{\beta}) \right] e^{-i\omega(t' - \hat{e} \cdot \vec{x}/c)}, \end{aligned} \quad (2.35)$$

where we have ignored the constant phase factor $e^{i\omega\hat{e}\cdot\vec{r}}$. It is irrelevant because we later need to take the absolute value of $\hat{f}(\omega)$ anyway.

It is worth noting again that we have made a single approximation in the preceding calculation which is perfectly legitimate in typical astrophysical situations: We have assumed that the radiating electron is confined to a distant volume that is small compared to its distance from the observer. This has allowed us to derive a general prescription for calculating radiation spectra, expressed by (2.29) with $\hat{f}(\omega)$ given by the Fourier transform (2.35),

$$\frac{d^2 E}{d\Omega d\omega} = \frac{e^2}{8\pi^2 c} |\hat{f}(\omega)|^2 = \frac{e^2 \omega^2}{8\pi^2 c} \left| \int_{-\infty}^{\infty} dt' \left[\hat{e} \times (\hat{e} \times \vec{\beta}) \right] e^{-i\omega(t' - \hat{e}\cdot\vec{x}(t')/c)} \right|^2, \quad (2.36)$$

understanding that the integrand has to be evaluated at the retarded time t' and that $\vec{x} = \vec{x}(t')$ describes the electron's orbit about a fixed reference point within the volume it is confined to. We can now apply this general result to different circumstances relevant in astrophysics.

The calculation simplifies considerably for non-relativistically moving charges. Then, relativistic beaming is irrelevant, retardation effects can be ignored, and terms of higher than linear order in β and β^2 can be neglected. We can then begin with the direction-integrated, non-relativistic Larmor formula following from (1.141) by setting $\gamma = 1$ and dropping the fourth-order term in β . Then,

$$E = \int_{-\infty}^{\infty} P dt = \frac{2e^2}{3c^3} \int_{-\infty}^{\infty} |\ddot{\vec{a}}(t)|^2 dt, \quad (2.37)$$

where $\ddot{\vec{a}} = \ddot{\vec{x}}$ is the acceleration experienced by the charge. Employing Plancherel's theorem (2.25) once more, we can continue writing (2.37) as

$$E = \frac{2e^2}{3c^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\hat{\ddot{\vec{a}}}(\omega)|^2 = \int_{-\infty}^{\infty} d\omega \frac{dE}{d\omega}, \quad (2.38)$$

which yields the non-relativistic, direction-integrated spectrum

$$\frac{dE}{d\omega} = \frac{e^2}{3\pi c^3} |\hat{\ddot{\vec{a}}}(\omega)|^2. \quad (2.39)$$

The Fourier transform $\hat{\ddot{\vec{a}}}$ of the acceleration can easily be expressed by the Fourier transform of the orbit itself. Since

$$\ddot{\vec{a}} = \ddot{\vec{x}} = \frac{d^2}{dt^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{\vec{x}}(\omega) e^{-i\omega t} = - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega^2 \hat{\vec{x}}(\omega) e^{-i\omega t}, \quad (2.40)$$

the Fourier transform of $\hat{\ddot{\vec{a}}}$ is

$$\hat{\ddot{\vec{a}}} = -\omega^2 \hat{\vec{x}}(\omega), \quad (2.41)$$

which allows us to calculate the spectrum directly from

$$\frac{dE}{d\omega} = \frac{e^2 \omega^4}{3\pi c^3} |\hat{\vec{x}}(\omega)|^2. \quad (2.42)$$

Example: Electron on a circular orbit

A quite simple example is an electron orbiting on a circle of radius r with an angular frequency ω_0 . Since its orbit is given by

$$\vec{x}(t) = r \begin{pmatrix} \cos \omega_0 t \\ \sin \omega_0 t \\ 0 \end{pmatrix}, \quad x_1(t) + ix_2(t) = r e^{i\omega_0 t}, \quad (2.43)$$

the Fourier transform of x_1 and x_2 together is

$$\hat{x}_1(\omega) + i\hat{x}_2(\omega) = r \int_{-\infty}^{\infty} e^{i(\omega_0 - \omega)t} dt = 2\pi r \delta_D(\omega_0 - \omega). \quad (2.44)$$

Its spectrum is thus a single, sharp line emitting the energy

$$\int_{-\infty}^{\infty} d\omega \frac{dE}{d\omega} = \frac{4\pi}{3} \frac{e^2 r^2 \omega_0^4}{c^3}. \quad (2.45)$$

Example: Electron under constant acceleration

For another illustrative example, suppose an electron is accelerated with constant acceleration \vec{d} during a finite time interval $-\tau/2 \leq t \leq \tau/2$. The Fourier transform of this acceleration is

$$\hat{\vec{d}}(\omega) = \vec{d} \int_{-\tau/2}^{\tau/2} e^{-i\omega t} dt = -\frac{i\vec{d}}{\omega} (e^{i\omega\tau/2} - e^{-i\omega\tau/2}) = \frac{2\vec{d}}{\omega} \sin \frac{\omega\tau}{2}, \quad (2.46)$$

which we can insert directly into (2.39) to find the spectrum

$$\frac{dE}{d\omega} = \frac{4e^2 \vec{d}^2}{3\pi c^3 \omega^2} \sin^2 \frac{\omega\tau}{2}. \quad (2.47)$$

Problems

1. Verify equation (2.31).
2. Let the Fourier transform of a function $f(x)$ and the inverse transform of its Fourier conjugate $\hat{f}(k)$ be defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx f(x) e^{ikx}, \quad f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{-ikx}. \quad (2.48)$$

Prove the following identities:

$$(a) \quad \widehat{f^*}(-k) = \hat{f}(k) \quad (2.49)$$

for real functions, $f(x) \in \mathbb{R}$.

$$(b) \quad \widehat{f * g} = \hat{f} \hat{g} \quad (2.50)$$

if

$$(f * g)(x) := \int_{-\infty}^{\infty} dy f(x-y)g(y) \quad (2.51)$$

is the convolution of the two functions f and g .

$$(c) \quad \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) \hat{g}^*(k) = \int_{-\infty}^{\infty} dx f(x) g^*(x) \quad (2.52)$$

(Parseval's equation).

$$(d) \quad \int_{-\infty}^{\infty} \frac{dk}{2\pi} |\hat{f}(k)|^2 = \int_{-\infty}^{\infty} dx |f(x)|^2. \quad (2.53)$$

3. Consider an electron whose one-dimensional trajectory $x(t)$ satisfies the differential equation of a damped harmonic oscillator,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0. \quad (2.54)$$

- (a) What is the oscillator frequency ω if ω_0 is the system's eigenfrequency? *Hint:* Try the ansatz $x(t) \propto e^{\pm i\omega t}$. What does a complex frequency mean physically?
- (b) Show that the solution of the differential equation is given by

$$x(t) = \frac{v_0}{\bar{\omega}} e^{-\gamma t} \sin \bar{\omega} t \quad \text{with} \quad \bar{\omega} = \sqrt{\omega_0^2 - \gamma^2}. \quad (2.55)$$

if $\omega_0 > \gamma$ and the initial conditions are $x(t=0) = 0$ and $\dot{x}(t=0) = v_0$.

- (c) Calculate the Fourier transform $\hat{x}(\omega)$. Assume that $x(t) = 0$ for $t < 0$.
- (d) Calculate the spectrum $dE/d\omega$ of the moving electron.
- (e) What does the spectrum look like if both $\omega \gg \omega_0$ and $\omega \gg \gamma$?

2.3 Synchrotron radiation

In this section, the power and the spectrum radiated by a relativistic charge gyrating in a magnetic field are calculated. This is entirely an application of Larmor's equation from classical electrodynamics and the general formulae for calculating spectra derived in the preceding section. We shall first consider the trajectory of the electron, then discuss relativistic beaming and its effects, and proceed directly to the synchrotron power in (2.68) and the synchrotron spectrum in (2.86). The main assumptions are that the emitting charge is confined to a volume whose dimensions are small compared to its distance from the observer and that the source is ultra-relativistic. Besides the shape of the synchrotron spectrum, an important result of the discussion is that relativistic beaming allows the observer to see the signal only during a very short time per orbit, which substantially broadens the spectrum since frequency and time are Fourier conjugates.

2.3.1 Larmor frequency and relativistic focussing

Consider now an electron moving relativistically in a homogeneous magnetic field. Without electric field, $\vec{E} = 0$, the Lorentz force (1.146) causes the acceleration

$$\frac{d(\gamma\vec{v})}{dt} = -\frac{e}{mc}\vec{v} \times \vec{B}. \quad (2.56)$$

Since this purely magnetic Lorentz force is perpendicular to the velocity, it cannot change the electron's energy, thus $\gamma = \text{const}$. Let us rotate the coordinate frame such that \vec{B} is aligned with the z axis, hence $\vec{B} = B\hat{e}_z$. Then,

$$\frac{d(\gamma v_z)}{dt} = \gamma \dot{v}_z = 0 \quad (2.57)$$

while

$$\dot{v}_x = -\frac{eB}{\gamma mc}v_y, \quad \dot{v}_y = \frac{eB}{\gamma mc}v_x. \quad (2.58)$$

Taking a second time derivative of either of the two equations (2.58) and combining it with the respective other equation gives

$$\ddot{v}_i + \left(\frac{eB}{\gamma mc}\right)^2 v_i = 0, \quad i = x, y. \quad (2.59)$$

This is the equation of a harmonic oscillator with the Larmor frequency

$$\omega_L = \frac{eB}{\gamma mc} = 17.6 \text{ Hz } \gamma^{-1} \left(\frac{B}{\mu\text{G}}\right) \left(\frac{m_e}{m}\right). \quad (2.60)$$

In a constant magnetic field, the electron therefore describes a circular orbit with cyclic frequency ω_L in the plane perpendicular to \vec{B} , while it moves with constant velocity along \vec{B} (Figure 2.4). If it has $v_z \neq 0$ initially, it orbits on a helix with constant radius and pitch angle.

Let us now assume for simplicity that $v_z = 0$ so that the electron moves on a circle in the plane perpendicular to \vec{B} (Figure 2.5). Alternatively, we can transform into a reference frame co-moving with the mean motion of the

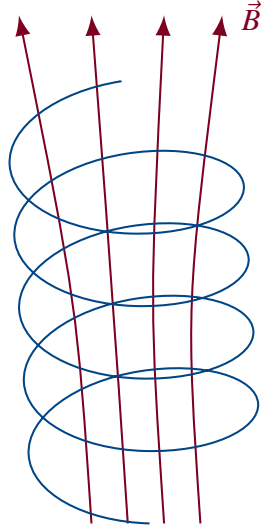


Figure 2.4 The trajectory of a charge in a locally constant magnetic field \vec{B} is a helix.

electron. On a circular orbit, the acceleration is perpendicular to the velocity, $\vec{\beta} \perp \dot{\vec{\beta}}$ or $\vec{\beta} \cdot \dot{\vec{\beta}} = 0$. Since the electron is supposed to move relativistically, $\beta \approx 1$, and we can approximate

$$1 - \beta = \frac{1 - \beta^2}{1 + \beta} \approx \frac{1}{2\gamma^2}, \quad \beta \approx 1 - \frac{1}{2\gamma^2}. \tag{2.61}$$

Introducing the angle θ between \hat{e} and $\vec{\beta}$ by $\beta \cos \theta = \hat{e} \cdot \vec{\beta}$, we see that the factor $(1 - \hat{e} \cdot \vec{\beta})^{-1} = (1 - \beta \cos \theta)^{-1}$ in the Larmor formula (1.138) is very large. In the direction of the motion, $\theta = 0$,

$$(1 - \hat{e} \cdot \vec{\beta})^{-1} = (1 - \beta)^{-1} \approx 2\gamma^2, \tag{2.62}$$

and the factor $(1 - \beta \cos \theta)^{-1}$ drops to half its maximum within a narrow angle. Requiring

$$\frac{1}{1 - \hat{e} \cdot \vec{\beta}} \gtrsim \frac{1}{2(1 - \beta)}, \tag{2.63}$$

we find the condition

$$\cos \theta \approx 1 - \frac{\theta^2}{2} \gtrsim 2 - \frac{1}{\beta}, \tag{2.64}$$

from which we can read off

$$\theta \lesssim \sqrt{2\left(\frac{1}{\beta} - 1\right)} \approx \sqrt{2(1 - \beta)} \approx \frac{1}{\gamma}. \tag{2.65}$$

The energy radiated by the electron is thus confined to a very narrow beam with opening angle $\lesssim \gamma^{-1}$. This will allow us to introduce several well-justified approximations as we go along.

?

Why is it appropriate and consistent to approximate $1 + \beta \approx$ in (2.61)?

2.3.2 Synchrotron power

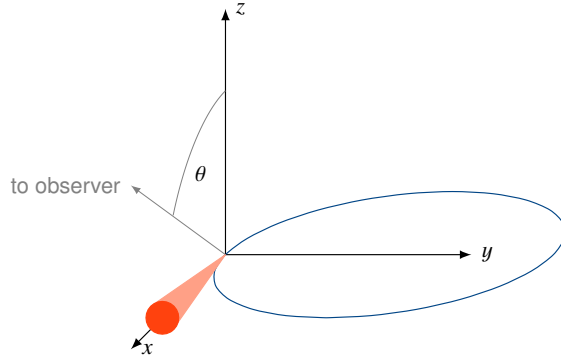


Figure 2.5 Illustration of how the coordinate frame is chosen for the calculation of the synchrotron power and the synchrotron spectrum carried out in the text.

Let us first introduce a coordinate frame oriented such that the electron's orbit falls into the x - y plane, while the observer is in the x - z plane. Furthermore, we shift the coordinate origin into the centre of the circular orbit and choose the zero point of the retarded time t' such that the electron moves into the \hat{e}_x direction at $t' = 0$. Then, we can write

$$\hat{e} = \begin{pmatrix} \sin \theta \\ 0 \\ \cos \theta \end{pmatrix}, \quad \vec{x} = x \begin{pmatrix} \sin \varphi \\ \cos \varphi \\ 0 \end{pmatrix},$$

$$\vec{\beta} = \beta \begin{pmatrix} \cos \varphi \\ -\sin \varphi \\ 0 \end{pmatrix}, \quad \dot{\vec{\beta}} = \beta \dot{\varphi} \begin{pmatrix} -\sin \varphi \\ -\cos \varphi \\ 0 \end{pmatrix}, \quad (2.66)$$

where x is the radius of the orbit and the dimension-less velocity is $\beta = x\dot{\varphi}/c = x\omega_L/c$.

The total synchrotron power follows directly from the integrated Larmor formula (1.141). Since $\vec{\beta} \perp \dot{\vec{\beta}}$ in the case of synchrotron radiation, we first obtain

$$P = \frac{2e^2}{3c} \gamma^6 \left[\dot{\beta}^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2 \right] = \frac{2e^2}{3c} \gamma^6 \dot{\beta}^2 (1 - \beta^2) = \frac{2e^2}{3c} \gamma^4 \dot{\beta}^2. \quad (2.67)$$

Since $\dot{\beta} = \beta \dot{\varphi} \approx \omega_L$, we can further simplify

$$P = \frac{2e^2}{3c} \gamma^4 \omega_L^2 = \frac{2e^2}{3c} \gamma^4 \left(\frac{eB}{\gamma mc} \right)^2 = \frac{8\pi}{3} r_c^2 c \gamma^2 \frac{B^2}{4\pi} = c \gamma^2 \sigma_T U_B, \quad (2.68)$$

where $U_B = B^2/4\pi$ is the energy density in the magnetic field that can be read off Maxwell's energy-momentum tensor, see (1.112), and σ_T is the Thomson cross section, derived in the non-relativistic regime. As we shall see later, this is a very intuitive expression for the total synchrotron power.

2.3.3 Synchrotron spectrum

We now turn to the evaluation of the spectrum (2.36) under the given circumstances. Expanding first the double vector product in (2.36), we find

$$\hat{e} \times (\hat{e} \times \vec{\beta}) = (\hat{e} \cdot \vec{\beta}) \hat{e} - \vec{\beta} = \beta \begin{pmatrix} -\cos \varphi \cos^2 \theta \\ \sin \varphi \\ \cos \varphi \sin \theta \cos \theta \end{pmatrix}. \quad (2.69)$$

This vector must be perpendicular to the line-of-sight, whose direction is given by \hat{e} . We can thus expand it into two basis vectors perpendicular to \hat{e} , which we choose to be \hat{e}_y and

$$\hat{e}_\perp = \hat{e} \times \hat{e}_y = \begin{pmatrix} -\cos \theta \\ 0 \\ \sin \theta \end{pmatrix}. \quad (2.70)$$

In this basis,

$$\hat{e} \times (\hat{e} \times \vec{\beta}) = \beta \cos \varphi \cos \theta \hat{e}_\perp + \beta \sin \varphi \hat{e}_y. \quad (2.71)$$

The phase ψ of the exponential in (2.36) is

$$\omega \left(t' - \frac{\hat{e} \cdot \vec{x}}{c} \right) =: \psi = \omega \left(t' - \frac{x \sin \theta \sin \varphi}{c} \right). \quad (2.72)$$

We can now make use of the fact that the radiation of the relativistically moving electron is strongly focussed into its forward direction, since the opening angle of the radiation cone is approximately confined to $[-\gamma^{-1}, \gamma^{-1}]$, as we have discussed before. This implies that our observer will see the radiation only when $|\varphi| \lesssim \gamma^{-1}$ and $|\theta - \pi/2| \lesssim \gamma^{-1}$. Since $\gamma \gg 1$, the angle θ is close to $\pi/2$. We introduce its complement $\theta \equiv \pi/2 - \theta \ll 1$ and approximate

$$\sin \theta = \sin \left(\frac{\pi}{2} - \theta \right) = \cos \theta \approx 1 + \frac{\theta^2}{2}, \quad \cos \theta = \cos \left(\frac{\pi}{2} - \theta \right) = \sin \theta \approx \theta. \quad (2.73)$$

The expansion in φ is effectively an expansion in t' , for $\varphi = \omega_L t'$. We shall see later that we need to carry it to order t'^3 , hence

$$\sin \varphi \approx \omega_L t' \left(1 - \frac{\omega_L^2 t'^2}{6} \right), \quad \cos \varphi \approx 1 - \frac{\omega_L^2 t'^2}{2}. \quad (2.74)$$

We thus have

$$\hat{e} \times (\hat{e} \times \vec{\beta}) \approx \beta \theta \hat{e}_\perp + \beta \omega_L t' \hat{e}_y \approx \theta \hat{e}_\perp + \omega_L t' \hat{e}_y, \quad (2.75)$$

and the Fourier phase becomes

$$\psi \approx \omega t' \left[1 - \beta \left(1 - \frac{\omega_L^2 t'^2}{6} \right) \left(1 - \frac{\theta^2}{2} \right) \right] \approx \frac{\omega t'}{2} \left(\frac{1}{\gamma^2} + \theta^2 + \frac{\omega_L^2 t'^2}{3} \right) \quad (2.76)$$

where we have used the relations

$$\beta = \frac{x \omega_L}{c}, \quad 1 - \beta = \frac{1}{2\gamma^2}, \quad \beta \approx 1. \quad (2.77)$$

To further simplify the expression for the Fourier phase, we pull the factor $(\gamma^{-2} + \theta^2)$ out of the parenthesis in (2.76) to find first

$$\psi \approx \frac{\omega t'}{2} \left(\frac{1}{\gamma^2} + \theta^2 \right) \left(1 + \frac{\tau^2}{3} \right) \quad (2.78)$$

with the new dimension-less time variable

$$\tau := \frac{\omega_L t'}{\sqrt{\gamma^{-2} + \theta^2}}. \quad (2.79)$$

Defining further the dimension-less frequency

$$\xi := \frac{\omega}{3\omega_L} \left(\frac{1}{\gamma^2} + \theta^2 \right)^{3/2}, \quad (2.80)$$

we can write the phase as

$$\psi = \frac{3\xi\tau}{2} \left(1 + \frac{\tau^2}{3} \right). \quad (2.81)$$

?

Verify the approximate expression (2.81) for the phase function ψ .

Combining expression (2.75) for the double vector product, inserting the Fourier phase ψ from (2.78) and transforming the integration variable from t' to τ as defined in (2.79), we find that we can split the function $\hat{f}(\omega)$ introduced in (2.35) as

$$\hat{f}(\omega) = \hat{f}_\perp(\omega)\hat{e}_\perp + \hat{f}_\parallel(\omega)\hat{e}_y, \quad (2.82)$$

where the perpendicular and parallel Fourier amplitudes are

$$\begin{aligned} \hat{f}_\perp(\omega) &= -i \frac{\omega}{\omega_L} \theta \left(\frac{1}{\gamma^2} + \theta^2 \right)^{1/2} \int_{-\infty}^{\infty} d\tau e^{-i\psi}, \\ \hat{f}_\parallel(\omega) &= -i \frac{\omega}{\omega_L} \left(\frac{1}{\gamma^2} + \theta^2 \right) \int_{-\infty}^{\infty} \tau d\tau e^{-i\psi}. \end{aligned} \quad (2.83)$$

The remaining integrals are Bessel functions of fractional order,

$$\int_{-\infty}^{\infty} d\tau e^{-i\psi} = \frac{2}{\sqrt{3}} K_{1/3}(\xi), \quad \int_{-\infty}^{\infty} \tau d\tau e^{-i\psi} = -\frac{2i}{\sqrt{3}} K_{2/3}(\xi). \quad (2.84)$$

Putting these results together, we can express (2.36) as

$$\frac{d^2 E}{d\Omega d\omega} = \frac{e^2}{4\pi c} \left[|\hat{f}_\perp(\omega)|^2 + |\hat{f}_\parallel(\omega)|^2 \right], \quad (2.85)$$

or, introducing the preceding results for the functions f_\perp and f_\parallel ,

$$\frac{d^2 E}{d\Omega d\omega} = \frac{e^2 \omega^2}{3\pi c \omega_L^2} \left(\frac{1}{\gamma^2} + \theta^2 \right)^2 \left[\frac{\theta^2}{\gamma^{-2} + \theta^2} K_{1/3}^2(\xi) + K_{2/3}^2(\xi) \right]. \quad (2.86)$$

This is the synchrotron spectrum (Figure 2.6).

To obtain further insight into the shape of the spectrum, let us shift the observer into the orbital plane of the electron. Since the radiation is focussed into a narrow cone with $|\theta| \lesssim \gamma^{-1}$, this is not a strong simplification. We first realise

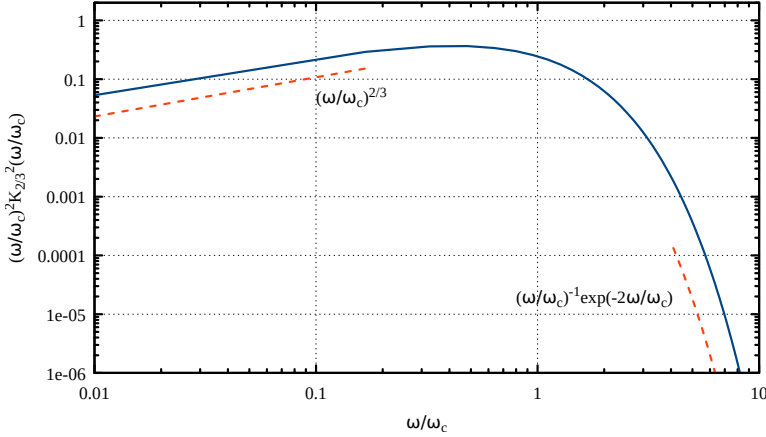


Figure 2.6 The shape of the synchrotron spectrum is shown in arbitrary units as a function of the scaled frequency $\xi = \omega/\omega_c$. The polar angle was set to $\theta = \pi/2$, i.e. this is the spectral shape in the orbital plane of the electron. The Lorentz factor is irrelevant here since it only affects the amplitude, not the shape of the spectrum.

that the intensity of the radiation component polarised perpendicular to the electron's orbit vanishes since $f_{\perp}(\omega) = 0$. In the orbital plane, synchrotron radiation is thus completely linearly polarised in the orbital plane, or perpendicular to the guiding magnetic field. Then, with $\theta = 0$, (2.86) simplifies to

$$\frac{d^2E}{d\Omega d\omega} = \frac{3}{\pi} \frac{e^2 \gamma^2}{c} \left(\frac{\omega}{\omega_c}\right)^2 K_{2/3}^2\left(\frac{\omega}{\omega_c}\right), \quad \omega_c = 3\omega_L \gamma^3 \quad (2.87)$$

where we have introduced the cutoff frequency

$$\omega_c = 3\omega_L \gamma^3 = \frac{3\gamma^2 eB}{mc}. \quad (2.88)$$

The Bessel function $K_{2/3}(\xi)$ follows a falling power law for $\xi \ll 1$ and drops approximately exponentially for $\xi \gg 1$,

$$K_{2/3}(\xi) \approx \begin{cases} \frac{1}{2} \Gamma\left(\frac{2}{3}\right) \left(\frac{\xi}{2}\right)^{-2/3} & (\xi \ll 1) \\ \frac{\sqrt{\pi}}{\sqrt{2\xi}} e^{-\xi} & (\xi \gg 1) \end{cases}. \quad (2.89)$$

For small ξ , the synchrotron spectrum is thus a power law in frequency,

$$\frac{d^2E}{d\Omega d\omega} \approx \frac{3 \cdot 2^{4/3}}{4\pi} \Gamma^2\left(\frac{2}{3}\right) \frac{e^2 \gamma^2}{c} \left(\frac{\omega}{\omega_c}\right)^{2/3} \quad (2.90)$$

Since $\gamma \gg 1$, the frequency range covered by this power-law behaviour is very wide. Only far above the Larmor frequency, the spectrum is cut off exponentially near the cutoff frequency ω_c . This is a direct consequence of the narrow radiation cone: During each orbit of the electron, its radiation is only received by the observer in a very short time interval. The Fourier transform of this time interval, however, corresponds to a wide frequency range, similar to

the uncertainty principle in quantum mechanics. For frequencies near or above the cutoff frequency, the spectrum is approximated by

$$\frac{d^2E}{d\Omega d\omega} \approx \frac{3}{2} \frac{e^2 \gamma^2}{c} \left(\frac{\omega}{\omega_c} \right) \exp\left(-2 \frac{\omega}{\omega_c}\right). \quad (2.91)$$

Problems

1. Due to the Lorentz force, a non-relativistic electron moving with a velocity v through the magnetic field \vec{B} experiences the acceleration

$$\ddot{x} = -\frac{e}{mc} (\vec{v} \times \vec{B}). \quad (2.92)$$

- (a) What is the average amount of energy per unit time and volume, $d^2E/(dt dV)$, radiated away by an isotropic electron distribution with number density n_e ?
- (b) Assume now further that the electrons are in thermal equilibrium. In this case, the probability for an electron to have the velocity $v = |\vec{v}|$ is given by the Maxwell-Boltzmann distribution

$$p(v)dv = \sqrt{\frac{2}{\pi}} \left(\frac{m_e}{k_B T} \right)^{3/2} v^2 \exp\left(-\frac{m_e v^2}{2k_B T}\right), \quad (2.93)$$

where T is the temperature of the electron gas, k_B is Boltzmann's constant and m_e the electron mass. Calculate $d^2E/(dt dV)$ as a function of the electron temperature T and the magnetic field \vec{B} .

Hint: You can use that

$$\int_0^\infty dx x^4 e^{-ax^2} = \frac{3\sqrt{\pi}}{8} a^{-5/2}. \quad (2.94)$$

2. The synchrotron spectrum in the orbital plane of a single electron with Larmor frequency ω_L is

$$\frac{d^2E}{d\omega d\Omega} = \frac{3e^2 \gamma^2}{\pi c} \left(\frac{\omega}{\omega_c} \right)^2 K_{2/3}^2 \left(\frac{\omega}{\omega_c} \right), \quad (2.95)$$

where $\omega_c = 3\omega_L \gamma^3$ and $K_{2/3}(x)$ is the modified Bessel function of order $2/3$ of the second kind.

- (a) In stochastic particle-acceleration processes, the accelerated electrons typically follow an energy distribution of the power-law form

$$\frac{dN}{dE} dE = A E^{-\alpha} dE, \quad (2.96)$$

where A is a normalisation constant. Calculate the spectrum for such a population of electrons. *Hint:* Express the energy E by γ and use

$$\int_0^\infty dx x^a K_{2/3}^2(bx^2) = b^{-(a+1)/2} \frac{\sqrt{\pi} \Gamma\left(\frac{3a-5}{12}\right) \Gamma\left(\frac{3a+11}{12}\right) \Gamma\left(\frac{a+1}{4}\right)}{8 \Gamma\left(\frac{a+3}{4}\right)}, \quad (2.97)$$

valid for $a > 5/3$.

- (b) Draw the expected spectrum schematically in a double-logarithmic plot.

2.4 Bremsstrahlung

This section is concerned with a conceptually simple, but mathematically involved problem: Electrons scattering off ions follow hyperbolic orbits, are accelerated accordingly and emit free-free radiation or bremsstrahlung. A thermal ensemble of such electrons emits a spectrum characterised by an exponential cut-off, reflecting the Boltzmann factor of their energy distribution. The mathematical difficulty arises because, as we have seen in our general derivation of electromagnetic spectra, the hyperbolic electron orbits appears in the phase of a Fourier transform. This gives rise to Hankel functions of continuous order, which are difficult to handle. The main results of this section are the bremsstrahlung spectrum (2.118) of a single electron, the mean bremsstrahlung spectrum (2.122) after integrating over electron impact parameters, and the bremsstrahlung emissivity (2.131) obtained after integrating over a thermal electron population.

2.4.1 Orbit of an electron scattering off an ion

As we have seen before in (2.42), the spectrum of a non-relativistically moving charge is determined by the Fourier transform of its orbit $\vec{x}(t)$. Classically, an electron coming from infinity, scattering off an ion with charge Ze and leaving to infinity describes a hyperbolic orbit, much like a comet in the Solar System (Figure 2.7). We borrow the description of the orbit from the treatment of Kepler's problem in classical mechanics. By angular-momentum conservation, the orbit will be confined to a plane, in which we introduce plane polar coordinates (r, φ) .

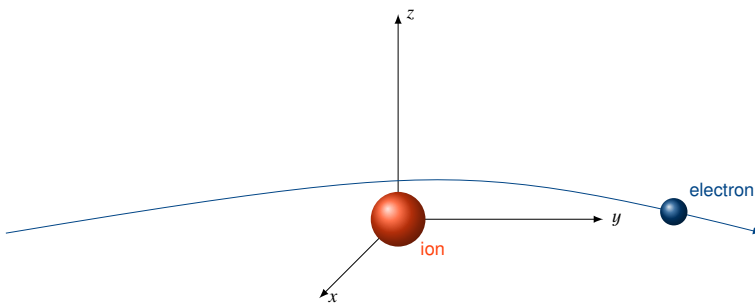


Figure 2.7 On the origin of bremsstrahlung: An electron is accelerated by the Coulomb force of an ion. It performs a hyperbolic orbit around the ion.

The (positive) energy of the electron is

$$E = \frac{m}{2} \dot{r}^2 + \frac{l^2}{2mr^2} - \frac{Ze^2}{r}, \quad (2.98)$$

where l is the conserved angular momentum. The solution of Kepler's problem tells us that this equation is solved by the conical sections (Figure 2.8), described in polar coordinates by

$$r(\varphi) = \frac{p}{1 + \varepsilon \cos \varphi}, \quad (2.99)$$

with the orbital parameter p and the numerical eccentricity ε expressing the angular momentum and the energy,

$$p = \frac{l^2}{Ze^2m}, \quad \varepsilon^2 = 1 + \frac{2Ep}{Ze^2}. \quad (2.100)$$

Since e^2 must have the dimension erg cm in the Gaussian cgs system, it is quite easy to convince oneself that p is a length and ε is dimension-less. We further introduce the length scale a by

$$p = a(\varepsilon^2 - 1). \quad (2.101)$$

For a bound elliptical orbit, a is the semi-major axis. Combining (2.101) with the second equation (2.100), we can express the energy by the orbital parameter a as

$$E = \frac{Ze^2}{2a}. \quad (2.102)$$

?

If needed, recapitulate the derivation of equation (2.99) for Kepler orbits, and the conditions for it to be valid.

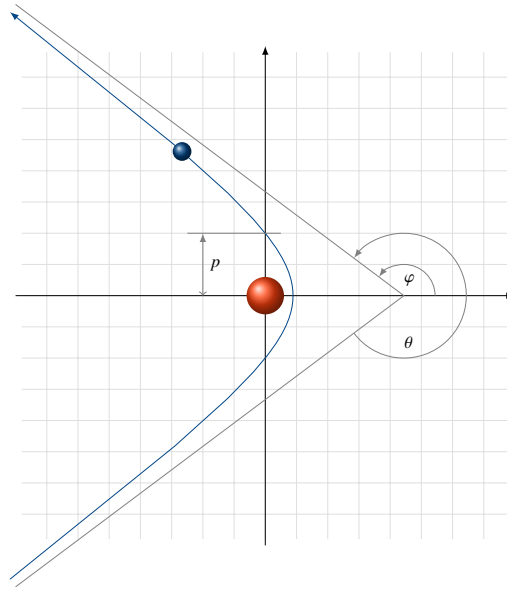


Figure 2.8 Hyperbolic orbit of an unbound particle in an attractive field of force.

We now replace the polar angle φ by the so-called *eccentric anomaly* ψ . For an unbound orbit, ψ is implicitly defined by

$$r(\psi) = a(\varepsilon \cosh \psi - 1), \quad (2.103)$$

which, together with (2.99) and (2.101) implies

$$\cos \varphi = \frac{\varepsilon - \cosh \psi}{\varepsilon \cosh \psi - 1}. \quad (2.104)$$

Now, we eliminate the squared angular momentum l^2 between (2.100) and (2.98), insert the expression (2.102) for the energy into the resulting equation and solve it for \dot{r}^2 ,

$$\dot{r}^2 = \frac{2}{m} \left[\frac{Ze^2}{2a} + \frac{Ze^2}{r} - \frac{Ze^2 a(\varepsilon^2 - 1)}{2r^2} \right] = \frac{2Ze^2}{mr^2} \left[\frac{r^2}{2a} + r - \frac{a(\varepsilon^2 - 1)}{2} \right]. \quad (2.105)$$

Next, we use (2.103) to introduce the eccentric anomaly into the following terms,

$$\dot{r} = a\varepsilon \sinh \psi \dot{\psi}, \quad \frac{r^2}{2a} + r - \frac{a(\varepsilon^2 - 1)}{2} = \frac{a\varepsilon^2}{2} \sinh^2 \psi. \quad (2.106)$$

These finally allow us to bring (2.105) into the form

$$\dot{\psi}^2 = \frac{Ze^2}{ma^3(\varepsilon \cosh \psi - 1)^2}. \quad (2.107)$$

Separating the variables ψ and t , we can express the time needed by the particle to get from $\psi = 0$ to ψ as

$$t = \int_0^t dt' = \sqrt{\frac{ma^3}{Ze^2}} \int_0^\psi d\psi' (\varepsilon \cosh \psi' - 1) = \tau(\varepsilon \sinh \psi - \psi), \quad (2.108)$$

where the time scale τ was introduced. Equation (2.108) is *Kepler's equation* for a hyperbolic orbit.

Since the energy E must be the kinetic energy of the electron at infinite distance from the ion, we can eliminate a from (2.102),

$$\frac{m}{2} v_\infty^2 = E = \frac{Ze^2}{2a} \Rightarrow a = \frac{Ze^2}{mv_\infty^2}. \quad (2.109)$$

In terms of v_∞ , the time scale τ is thus given by

$$\tau = \sqrt{\frac{ma^3}{Ze^2}} = \frac{Ze^2}{mv_\infty^3} = \frac{a}{v_\infty}. \quad (2.110)$$

2.4.2 Fourier transform of the orbit

By means of Kepler's equation (2.108), we can now substitute the time t by the eccentric anomaly ψ in the Fourier transform of the electron's orbit. First, we combine (2.103) and (2.104) to write the Cartesian coordinates

$$\begin{aligned} x(\psi) &= r \cos \phi = a(\varepsilon - \cosh \psi), \\ y(\psi) &= \sqrt{r^2 - x^2} = a \sqrt{\varepsilon^2 - 1} \sinh \psi. \end{aligned} \quad (2.111)$$

Moreover, we have from (2.108)

$$dt = \tau(\varepsilon \cosh \psi - 1) d\psi, \quad e^{i\omega t} = e^{i\omega\tau(\varepsilon \sinh \psi - \psi)}. \quad (2.112)$$

It is now convenient to compute the Fourier transform of the velocity, $\hat{\vec{v}} = -i\omega\hat{\vec{x}}$, instead of the Fourier transform of the orbit, $\hat{\vec{x}}$. We thus write

$$\begin{aligned} \hat{x}(\omega) &= -\frac{\hat{x}}{i\omega} \int_{-\infty}^{\infty} dt \dot{x} e^{-i\omega t} = \frac{i}{\omega} \int_{-\infty}^{\infty} dt \frac{dx}{d\psi} \dot{\psi} e^{-i\omega t} \\ &= \frac{i}{\omega} \int_{-\infty}^{\infty} d\psi \frac{dx}{d\psi} e^{-i\omega\tau(\psi)}, \end{aligned} \quad (2.113)$$

Caution Note that Kepler's equation is transcendental and can thus only be solved numerically. ◀

and likewise for $\hat{y}(\omega)$. With (2.111), this gives

$$\begin{aligned}\hat{x}(\omega) &= -\frac{ia}{\omega} \int_{-\infty}^{\infty} d\psi \sinh \psi e^{-i\omega\tau(\varepsilon \sinh \psi - \psi)}, \\ \hat{y}(\omega) &= \frac{ia\sqrt{\varepsilon^2 - 1}}{\omega} \int_{-\infty}^{\infty} d\psi \cosh \psi e^{-i\omega\tau(\varepsilon \sinh \psi - \psi)}.\end{aligned}\quad (2.114)$$

These integrals can be expressed by the Hankel function of the first kind of order ν , $H_\nu^{(1)}(x)$, and its derivative, $H_\nu^{(1)\prime}(x)$. In terms of these, we have

$$\hat{x}(\omega) = \frac{\pi a}{\omega} H_{iv}^{(1)\prime}(iv\varepsilon), \quad \hat{y}(\omega) = -\frac{\pi a\sqrt{\varepsilon^2 - 1}}{\omega\varepsilon} H_{iv}^{(1)}(iv\varepsilon), \quad (2.115)$$

where the order $\nu = \omega\tau$. With (2.42), we thus find the bremsstrahlung spectrum

$$\frac{dE}{d\omega} = \frac{2\pi^2 a^2 e^2 \omega^2}{3c^3} \left\{ [H_{iv}^{(1)\prime}(iv\varepsilon)]^2 - \left(1 - \frac{1}{\varepsilon^2}\right) [H_{iv}^{(1)}(iv\varepsilon)]^2 \right\} \quad (2.116)$$

for a single electron moving on a hyperbolic orbit with eccentricity ε . The sign in front of the second term in brackets is negative because $H_{iv}^{(1)}(iv\varepsilon)$ is purely imaginary, while its derivative $H_{iv}^{(1)\prime}(iv\varepsilon)$ is real. Before we can continue, we need to integrate (2.116) over a realistic distribution of the eccentricity ε .

The following relation between Bessel functions and their derivatives comes to help, which also applies to the Hankel functions,

$$z \left[Z_p^{\prime 2}(z) - \left(1 - \frac{p^2}{z^2}\right) Z_p^2(z) \right] = \frac{d}{dz} (z Z_p(z) Z_p'(z)). \quad (2.117)$$

Setting $z = iv\varepsilon$ and $p = iv$, this allows us to write (2.116) as

$$\frac{dE}{d\omega} = -i \frac{2\pi^2 a^2 e^2 \omega}{3\tau\varepsilon c^3} \frac{d}{d\varepsilon} \left[\varepsilon H_{iv}^{(1)}(iv\varepsilon) H_{iv}^{(1)\prime}(iv\varepsilon) \right], \quad (2.118)$$

where we have used that the order $\nu = \omega\tau$. The prefactor $-i$ is necessary because the Hankel function $H_{iv}^{(1)}(iv\varepsilon)$ is imaginary.

2.4.3 Integration over impact parameters

The numerical eccentricity ε of an particle's orbit is determined by its angular momentum l , which is in turn controlled by the orbit's impact parameter b . This is defined as the closest distance of the scattering centre from the straight line which would be the electron's unperturbed trajectory. Combining the two equations (2.100) with $E = mv_\infty^2/2$, we find

$$\varepsilon^2 = 1 + \frac{v_\infty^2 l^2}{Z^2 e^4} \quad (2.119)$$

for the squared numerical eccentricity. Then, using the expression $l = bmv_\infty$ for the angular momentum and replacing the constants occurring by means of (2.109), the simple result is

$$\varepsilon^2 = 1 + \frac{b^2}{a^2}. \quad (2.120)$$

Caution The Hankel function of the first kind is the complex linear combination

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$$

of the Bessel functions J_ν and Y_ν of the first and second kinds. Both solve Bessel's differential equation

$$x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + (x^2 - \nu^2) f = 0.$$

We now take the spectrum (2.118) produced by a single electron and multiply it with the number of scattering events between electrons and ions per unit time and unit volume. Let n_i and n_e be the number densities of ions and electrons, respectively, and v_∞ the velocity of the electrons relative to the ions. Consider a single ion and surround it by a cylindrical shell of radius b , width db , and height $v_\infty dt$. Then, all

$$n_e \cdot (2\pi b db) \cdot (v_\infty dt) = 2\pi n_e v_\infty a^2 \varepsilon d\varepsilon dt \quad (2.121)$$

electrons contained in this shell will scatter off the ion within the time interval dt . Multiplying this number with n_i , we find the total number of scatterings between ions and electrons with relative velocity v_∞ and impact parameter within $[b, b + db]$ per unit time and unit volume. Further multiplying this number with the spectrum (2.118), and integrating over all impact parameters b or eccentricities ε , then gives the spectrum emitted by such electrons per unit time and volume,

$$\frac{d^3 E}{d\omega dt dV} = i \frac{4\pi^3 Z^2 e^6 n_i n_e}{3m^2 c^3 v_\infty} \left(\frac{Ze^2 \omega}{mv_\infty^3} \right) H_{iv}^{(1)}(iv) H_{iv}^{(1)\prime}(iv) . \quad (2.122)$$

For arriving at this expression, we have used (2.109) and (2.110) to substitute a and τ and regrouped terms for later convenience.

The Hankel functions and their derivatives need to be numerically evaluated, but we can insert their asymptotic forms for small and large arguments. These are

$$\begin{aligned} v \ll 1 : H_{iv}^{(1)}(iv) &\approx \frac{2}{i\pi} \ln\left(\frac{2}{\gamma v}\right), & H_{iv}^{(1)\prime}(iv) &\approx \frac{2}{\pi v} \\ v \gg 1 : H_{iv}^{(1)}(iv) &\approx -\frac{i}{\pi \sqrt{3}} \left(\frac{6}{v}\right)^{1/3} \Gamma(1/3), & H_{iv}^{(1)\prime}(iv) &\approx \frac{1}{\pi \sqrt{3}} \left(\frac{6}{v}\right)^{2/3} \Gamma(2/3). \end{aligned} \quad (2.123)$$

Now, with the further help of

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \quad \Gamma(1/3)\Gamma(2/3) = \frac{\pi}{\sin(\pi/3)} = \frac{2\pi}{\sqrt{3}}, \quad (2.124)$$

we find the low- and high-frequency approximations

$$\frac{d^3 E}{d\omega dt dV} = \frac{16\pi Z^2 e^6 n_i n_e}{3m^2 c^3 v_\infty} \begin{cases} \ln\left(\frac{2}{\gamma} \frac{mv_\infty^3}{Ze^2 \omega}\right) & \omega \ll \tau^{-1} \\ \frac{\pi}{\sqrt{3}} & \omega \gg \tau^{-1} \end{cases} . \quad (2.125)$$

Recall from (2.110) that $\tau = av_\infty^{-1}$

2.4.4 Average over electron velocities, thermal bremsstrahlung

The dependence of the spectrum on ω is mild for low ω , and absent for high ω , which is a very interesting result: The energy emitted per unit frequency

is (almost) independent of the frequency. These asymptotic results motivate writing the complete spectrum of non-relativistic bremsstrahlung in the form

$$\frac{d^3E}{d\omega dV} = j(\omega) = \frac{16\pi^2 Z^2 e^6 n_i n_e}{3\sqrt{3}m^2 c^3} \frac{g_{\text{ff}}(v_\infty, \omega)}{v_\infty}, \quad (2.126)$$

introducing the so-called Gaunt factor $g_{\text{ff}}(v_\infty, \omega)$. In the high-frequency limit, g_{ff} tends to unity, as (2.125) shows, and depends generally only weakly on v_∞ and ω . It is thus reasonable to introduce a velocity-averaged Gaunt factor by

$$\left\langle \frac{g_{\text{ff}}(v_\infty, \omega)}{v_\infty} \right\rangle = \bar{g}_{\text{ff}}(\omega) \left\langle \frac{1}{v_\infty} \right\rangle \quad (2.127)$$

and average the reciprocal velocity over some velocity distribution.

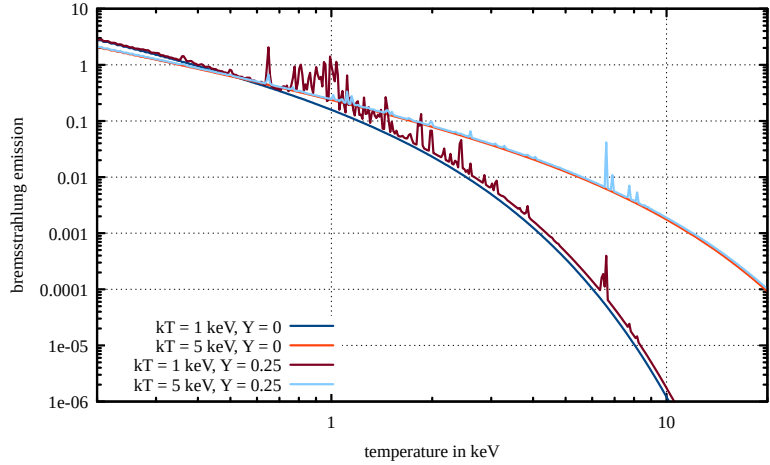


Figure 2.9 Thermal bremsstrahlung without and with line emission, for plasma temperatures of 1 keV and 5 keV. The spectra were produced with the `xspec` software package using a Raymond-Smith plasma model.

If the electrons scattering off the ions form a thermal population, their velocity distribution is Maxwellian,

$$p(v_\infty)dv_\infty = 4\pi \left(\frac{m}{2\pi k_B T} \right)^{3/2} v_\infty^2 \exp\left(-\frac{mv_\infty^2}{2k_B T}\right) dv_\infty. \quad (2.128)$$

For emitting at least a single photon of frequency ω or energy $\hbar\omega$, an electron has to satisfy

$$\frac{mv_\infty^2}{2} \geq \hbar\omega \quad \Rightarrow \quad v_\infty \geq v_{\text{min}} = \sqrt{\frac{2\hbar\omega}{m}}. \quad (2.129)$$

The average of v_∞^{-1} then turns out to be

$$\begin{aligned} \left\langle \frac{1}{v_\infty} \right\rangle &= 4\pi \left(\frac{m}{2\pi k_B T} \right)^{3/2} \int_{v_{\text{min}}}^{\infty} v_\infty dv_\infty \exp\left(-\frac{mv_\infty^2}{2k_B T}\right) \\ &= \sqrt{\frac{2m}{\pi k_B T}} \exp\left(-\frac{\hbar\omega}{k_B T}\right). \end{aligned} \quad (2.130)$$

Combined with (2.126), this finally gives the emissivity of non-relativistic, thermal bremsstrahlung (Figure 2.9)

$$j(\omega) = \frac{16\pi^2 Z^2 e^6 n_i n_e}{3\sqrt{3} m^2 c^3} \bar{g}_{\text{ff}}(\omega) \sqrt{\frac{2m}{\pi k_B T}} \exp\left(-\frac{\hbar\omega}{k_B T}\right). \quad (2.131)$$

The Gaunt factor is typically tabulated, but for many astrophysical applications, $\bar{g}_{\text{ff}}(\omega) \approx 1$ is a sufficient approximation.

Problems

1. A simplified derivation of the bremsstrahlung emissivity begins with Born's approximation, asserting that the electron's acceleration can be evaluated along a straight, undeflected orbit.
 - (a) Evaluate the electron's acceleration by an ion along a straight line.
 - (b) Fourier transform the acceleration and calculate the approximate bremsstrahlung spectrum.
 - (c) Carry out the integration over impact parameters. Which problem occurs?

2.5 Radiation damping

Remarkably, electrodynamics is incomplete in the following sense: Consider an electron moving in a homogeneous magnetic field in the absence of electric fields. The Lorentz force then causes the electron to move on a spiral orbit without changing the electron's energy. At that level, the prediction of electrodynamics would be that the electron keeps moving in this way forever. However, the motion along the spiral is an accelerated motion, which implies that the electron loses energy by radiation. This loss of energy is not contained in the equation of motion for the electron. The back-reaction of the radiation emitted by an accelerated charge on the motion of that same charge has to be described separately. This is a fundamental limit of electrodynamics: As a linear theory, it cannot encompass this kind of back-reaction. In this section, the backreaction of the radiation on the radiating charge itself is derived. The loss of energy by the charge due to the radiation can be described by an effective force, the radiation-damping force, the expression (2.139) for which will be the first main result. As an important application, the energy transfer from a charge moving through a sea of radiation to that radiation field itself is developed next, which leads to the very intuitive result (2.163) for the transferred power.

2.5.1 Damping force

The loss of energy by radiation can be described as the action of an effective damping force \vec{F}_{rad} acting on the electron. The energy radiated away within a certain time interval $-\tau/2 \leq t \leq \tau/2$,

$$E = \int_{-\tau/2}^{\tau/2} dt P(t), \quad (2.132)$$

P being the radiative power, must then equal the work exerted by this radiation-damping force on the electron during the same time,

$$\int_{-\tau/2}^{\tau/2} dt P(t) = - \int \vec{F}_{\text{rad}} \cdot d\vec{s}. \quad (2.133)$$

The solid-angle integrated Larmor formula (1.141) shows that the power radiated by an accelerated electron is homogeneous of degree $k = 2$ in the acceleration $\dot{\vec{\beta}}$, that is, if the acceleration is scaled by a dimension-less factor a , the power changes by a factor a^2 . Generally, a function $f(x)$ is called homogeneous of degree k if $f(ax) = a^k f(x)$ for $a \in \mathbb{R}$. The Larmor power thus satisfies Euler's theorem for homogeneous functions: If $f(\vec{x})$ is a homogeneous function of degree k in \vec{x} , then its derivative satisfies

$$\vec{x} \cdot \frac{df(\vec{x})}{d\vec{x}} = kf(\vec{x}). \quad (2.134)$$

?

Can you prove Euler's theorem (2.134) for homogeneous functions? Otherwise, look it up.

When applied to the radiation power, Euler's theorem thus says

$$\dot{\vec{\beta}} \cdot \frac{\partial}{\partial \dot{\vec{\beta}}} P(\dot{\vec{\beta}}) = 2P(\dot{\vec{\beta}}). \quad (2.135)$$

We use this statement to express the power in (2.133) by its derivative and obtain

$$- \int \vec{F}_{\text{rad}} \cdot d\vec{s} = \frac{1}{2} \int dt \dot{\vec{\beta}} \cdot \frac{\partial}{\partial \dot{\vec{\beta}}} P(\dot{\vec{\beta}}) = -\frac{1}{2} \int dt \dot{\vec{\beta}} \cdot \frac{d}{dt} \frac{\partial}{\partial \dot{\vec{\beta}}} P(\dot{\vec{\beta}}) \quad (2.136)$$

by partial integration, omitting the boundary terms. This is generally no substantial restriction because we can typically choose the integration boundaries wide enough for the radiation power to vanish at both of them. Now, since $\dot{\vec{\beta}} dt = d\vec{s}/c$, we can identify the expression

$$\vec{F}_{\text{rad}} = \frac{1}{2c} \frac{d}{dt} \frac{\partial}{\partial \dot{\vec{\beta}}} P(\dot{\vec{\beta}}) \quad (2.137)$$

with the radiation-damping force. In the non-relativistic limit (1.142),

$$P = \frac{2e^2}{3c} \dot{\vec{\beta}}^2, \quad \frac{\partial}{\partial \dot{\vec{\beta}}} P(\dot{\vec{\beta}}) = \frac{4e^2}{3c} \dot{\vec{\beta}}, \quad (2.138)$$

whence the radiation-damping force turns out to be

$$\vec{F}_{\text{rad}} = \frac{2e^2}{3c^2} \ddot{\vec{\beta}} \quad (2.139)$$

Example: Scattering off bound electrons

We will now directly apply this result to an electron on a bound harmonic orbit with an angular frequency ω_0 . Let the electron be externally driven by an incoming electromagnetic wave with frequency ω . This wave exerts the electric Lorentz force

$$\vec{F}_L = -\frac{e}{c} \vec{E}_0 e^{i\omega t} \quad (2.140)$$

on the electron. We assume that the electron moves non-relativistically such that we can ignore the magnetic part of the Lorentz force. Including radiation damping with a damping constant γ to be determined shortly, the equation of motion

$$\ddot{\vec{x}} + \gamma \dot{\vec{x}} + \omega_0^2 \vec{x} = -\frac{e}{m} \vec{E}_0 e^{i\omega t} \quad (2.141)$$

describes a harmonically driven and damped harmonic oscillator. Its particular solution is immediately found to read

$$\vec{x} = -\frac{e}{m} \frac{\vec{E}_0 e^{i\omega t}}{\omega_0^2 - \omega^2 - i\omega\gamma} \quad (2.142)$$

after an initial settling phase during which a possible oscillation with the eigenfrequency ω_0 of the bound orbit decays exponentially. We thus have

$$\ddot{\vec{\beta}} = -\omega^2 \vec{\beta}, \quad (2.143)$$

allowing us to write the radiation-damping force as

$$\vec{F}_{\text{rad}} = -\frac{2e^2\omega^2}{3c^2} \vec{\beta} \quad (2.144)$$

and to identify the damping constant

$$\gamma = \gamma_0 \omega^2 \quad \text{with} \quad \gamma_0 = \frac{2}{3} \frac{e^2}{mc^3} = \frac{2}{3} \frac{r_e}{c}, \quad (2.145)$$

where r_e is the classical electron radius introduced in (2.12). According to (2.142), the electron's acceleration is

$$\dot{\vec{\beta}} = -\omega^2 \frac{\vec{x}}{c} = \frac{e}{mc} \frac{\vec{E}_0 e^{i\omega t} \omega^2}{\omega_0^2 - \omega^2 - i\gamma_0 \omega^3}, \quad (2.146)$$

which we can now insert into the non-relativistic, integrated Larmor equation (1.142) to find

$$P = \frac{2e^2}{3c} \left| \dot{\vec{\beta}} \right|^2 = \frac{2e^4}{3m^2c^3} \vec{E}_0^2 \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma_0^2 \omega^6}. \quad (2.147)$$

The incoming energy current density is given by the amplitude of the Poynting vector $|\vec{S}| = c\vec{E}_0^2/(4\pi)$, and thus the cross section for scattering off a harmonically bound charge becomes

$$\sigma = \frac{P}{|\vec{S}|} = \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma_0^2 \omega^6} \quad (2.148)$$

with the typical resonance behaviour near $\omega = \omega_0$ (Figure 2.10). ◀

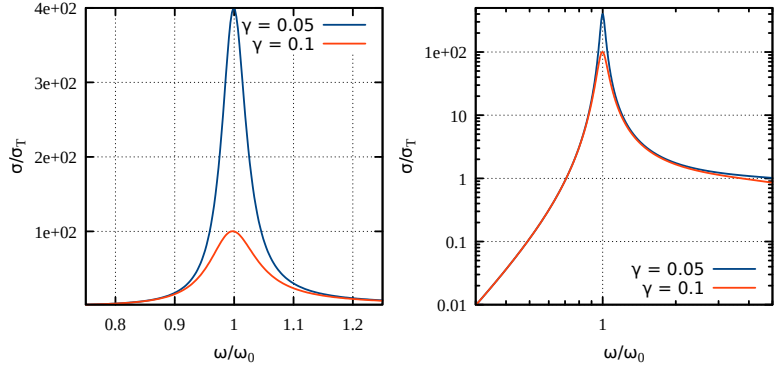


Figure 2.10 Illustration of the cross section for scattering of electromagnetic radiation off a harmonically bound electron. *Left*: The cross section (2.148) is shown (in units of the Thomson cross section σ_T) for two values of the damping constant. *Right*: The same curves, now plotted double-logarithmically, reveal the ω^4 scaling for low frequencies, i.e. the regime of Rayleigh scattering.

in this limit. Since $\vec{\beta} = \dot{\vec{x}}/c$, this involves a third time derivative of the electron's orbit. This is one of the rare cases of a third-order time derivative in physics.

Some limiting cases of the general cross section (2.148) for scattering off bound electrons are of particular interest. First, in the high-frequency limit $\omega \gg \omega_0$ and $\omega \gg \gamma_0^{-1}$, the driving force oscillates so fast that radiation damping is strong. The cross section (2.148) then falls off like ω^{-2} ,

$$\sigma \approx \frac{\sigma_T}{\gamma_0^2 \omega^2}. \quad (2.149)$$

Notice, however, that the electron will be unbound if the incoming radiation has too high frequency, and then its cross section will turn into the Thomson cross section, $\sigma \approx \sigma_T$.

In the opposite limit, when $\omega \ll \omega_0$ and $\omega \ll \gamma_0^{-1}$, we find the limit of Rayleigh scattering,

$$\sigma \approx \sigma_T \left(\frac{\omega}{\omega_0} \right)^4, \quad (2.150)$$

with the scattering cross section depending on the fourth power of the frequency. For $\omega \approx \omega_0$ and weak damping, $\omega_0 \ll \gamma_0^{-1}$, we approximate

$$\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) \approx 2\omega_0(\omega - \omega_0) \quad (2.151)$$

in (2.148) and find

$$\sigma \approx \frac{\pi \sigma_T}{2 \gamma_0} \phi_\Gamma(\omega - \omega_0), \quad \Gamma := \gamma_0 \omega_0^2, \quad (2.152)$$

where the function $\phi_\Gamma(\omega - \omega_0)$ is the so-called *Lorentz profile*,

$$\phi_\Gamma(\omega - \omega_0) = \frac{1}{\pi} \frac{\Gamma/2}{(\omega - \omega_0)^2 + (\Gamma/2)^2} \quad (2.153)$$

shown in Fig. 2.11. The Lorentz profile will recur several times in later Sections. It is normalised to unity.

?

Verify the combined results (2.152) and (2.153) and confirm that the Lorentz profile is normalised to unity.

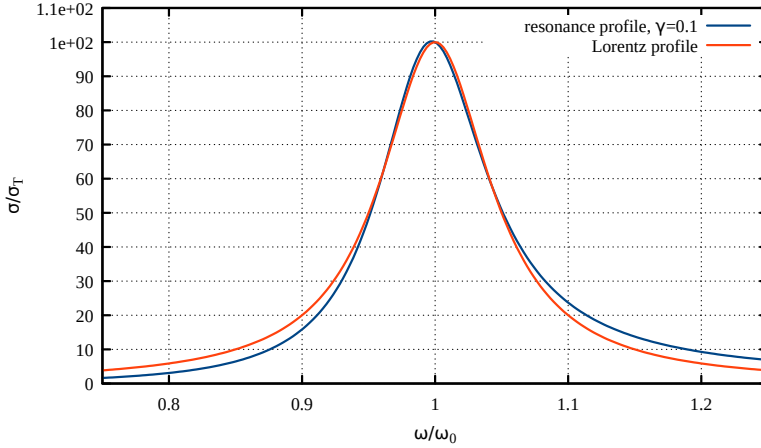


Figure 2.11 Near the resonance, the scattering cross section is reasonably approximated by the Lorentz profile.

2.5.2 Transfer of energy from a moving charge to a radiation field

Consider now an electron moving with possibly relativistic speed $\vec{\beta}$ through an isotropic radiation field, whose electric and magnetic field components satisfy

$$\langle \vec{E} \rangle = 0 = \langle \vec{B} \rangle, \tag{2.154}$$

where the average is taken over time intervals long compared to typical oscillation period $2\pi\omega^{-1}$ of the radiation field. Now we transform to the rest frame of the electron. The electron experiences the field components \vec{E}' , \vec{B}' given by the Lorentz transform (1.87). They accelerate the electron through the electric Lorentz force

$$\ddot{\vec{x}}' = \frac{1}{m} \vec{F}'_L = \frac{e}{m} \vec{E}' \tag{2.155}$$

since the magnetic part of the Lorentz force vanishes in the electron's rest frame, where $\vec{v}' = 0$. We can now calculate the power radiated by the accelerated electron with the non-relativistic Larmor formula, for which we need to evaluate

$$\langle |\ddot{\vec{x}}'|^2 \rangle = \frac{e^2}{m^2} \langle |\vec{E}'|^2 \rangle \tag{2.156}$$

in the electron's rest frame. Here, we can directly insert the Lorentz transform of the fields from (1.87) and carry out the average. Doing so, we have to take into account that the electromagnetic field in its rest frame is randomly oriented and has an energy density U . This allows us to use

$$\langle E_i^2 \rangle = \frac{\langle \vec{E}^2 \rangle}{3} = \frac{4\pi}{3} U = \langle B_j^2 \rangle \tag{2.157}$$

for the squares of the electric and magnetic field components and

$$\langle E_i B_j \rangle = 0 \tag{2.158}$$

for any combination of i and j . These relations enable us to write

$$\langle |\ddot{\vec{x}}'|^2 \rangle = 4\pi\gamma^2 U \frac{e^2}{m^2} \left(\frac{2}{3} + \frac{1}{3\gamma^2} + \frac{2}{3}\beta^2 \right) = 4\pi\gamma^2 U \frac{e^2}{m^2} \left(1 + \frac{\beta^2}{3} \right) \tag{2.159}$$

and thus

$$P_{\text{em}} = \gamma^2 U \frac{8\pi e^4}{3m^2 c^3} \left(1 + \frac{\beta^2}{3}\right) = cU\sigma_T \gamma^2 \left(1 + \frac{\beta^2}{3}\right) \quad (2.160)$$

for the power radiated by the electron in its rest frame. However, since the power is

$$P = \frac{dE}{dt} \quad (2.161)$$

and both the energy E and the time t transform like the zero components of four-vectors, the power is invariant under Lorentz transforms. Therefore, the result (2.160) also holds in the rest frame of the radiation field. On the other hand, the power absorbed by the electron is given by the Poynting vector times the cross section,

$$P_{\text{abs}} = \left| \vec{S} \right| \sigma_T = \frac{c}{4\pi} \vec{E}^2 \sigma_T = cU\sigma_T. \quad (2.162)$$

The net power transferred by the electron to the radiation field is thus

$$P = P_{\text{em}} - P_{\text{abs}} = cU\sigma_T \left[\gamma^2 \left(1 + \frac{\beta^2}{3}\right) - 1 \right] = \frac{4}{3} \beta^2 \gamma^2 cU\sigma_T. \quad (2.163)$$

We can now proceed to calculate the back-reaction on the electron by its transfer of energy to the radiation field. Clearly, the loss of kinetic energy of the electron must equal the negative radiation power (2.163),

$$\frac{dE}{dt} = mc^2 \frac{d\gamma}{dt} = -\frac{4}{3} \beta^2 \gamma^2 cU\sigma_T = -\frac{4}{3} (\gamma^2 - 1) cU\sigma_T. \quad (2.164)$$

Separating the variables γ and t and integrating over time gives

$$\int_{\gamma}^0 \frac{dx}{x^2 - 1} = -\frac{t}{\tau} \quad \text{with} \quad \tau := \frac{3mc}{4U\sigma_T}. \quad (2.165)$$

Noticing that

$$\frac{1}{x^2 - 1} = \frac{1}{2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right), \quad (2.166)$$

we can readily carry this integral out, finding

$$\frac{1}{2} \ln \frac{\gamma - 1}{\gamma + 1} = -\frac{t}{\tau}. \quad (2.167)$$

This equation can now easily be solved for γ or β , giving the essentially exponential decrease

$$\beta(t) = \frac{2 \exp(-t/\tau)}{1 + \exp(-2t/\tau)} \quad (2.168)$$

of the electron's velocity with time. This result shows that relativistic electrons, or charges in general, lose energy on a characteristic time scale

$$\tau = \frac{3mc}{4U\sigma_T} \quad (2.169)$$

when interacting with a radiation field. As the expression shows, the time scale is given by the rest-mass energy of the electron, divided by the energy of the radiation field flowing per unit time through the Thomson cross section. Similarly, the characteristic path length for a relativistic electron to lose its energy in a radiation field is

$$\lambda = c\tau = \frac{3mc^2}{4U\sigma_T}. \quad (2.170)$$

?

Carrying out the description following (2.156), verify the expressions (2.159) and (2.160) by your own calculation.

?

Explain the integral boundaries on the left-hand side of (2.165).

Problems

1. Derive the solution (2.145) of the equation of motion (2.143).
2. Calculate the time scale (2.169) for an electron travelling through the Cosmic Microwave Background.
3. The velocity of an electron in a homogeneous magnetic field changes due to the Lorentz force according to

$$\frac{d(\gamma\vec{v})}{dt} = \frac{e}{m_e c} (\vec{v} \times \vec{B}) \quad (2.171)$$

- (a) Set up the equations of motion for the individual components of \vec{x} in the field $\vec{B} = B\hat{e}_z$.
- (b) How does the equation of motion change if the radiation damping force

$$\vec{F}_{\text{rad}} = \frac{2e^2}{3c^3} \ddot{\vec{x}} \quad (2.172)$$

is also taken into account? Assume that the energy loss per orbit is small, i.e. the damping force can be evaluated using the undamped solution from (a). Under which circumstances is the former assumption valid?

- (c) Solve the differential equations for the components x_i with the boundary conditions $\vec{x}(t=0) = (x_0, 0, 0)^T$ and $\vec{v}(t=0) = (0, v_0, 0)^T$. Draw the solution schematically.

2.6 Compton scattering

This section introduces the photon picture for electromagnetic radiation. So far, incoming electromagnetic waves could only accelerate charges perpendicular to their direction of motion, which implies that they could not transfer momentum to the charges. With the discussion of radiation damping in the preceding section, we have seen how charges experience an effective force against their direction of motion due to the radiation they emit. In the discussion of Compton scattering, the incoming radiation is described as a stream of photons transferring both energy and momentum to the charges they scatter off from. The main result derived here is the mean energy loss per photon per collision (2.183). We then proceed to calculating the energy gained by a moving charge from a sea of radiation by Compton scattering and combine it with the loss due to radiation damping to find the total rate (2.193) of energy transfer between the charge and the photons. Compton scattering is then combined with the Fokker-Planck approach to work out photon diffusion in phase space due to scattering with electrons. The main result there is the approximation (2.220) to the so-called Kompaneets equation which neglects effects from quantum statistics, but is nonetheless appropriate for many astrophysical circumstances.

2.6.1 Energy change in the scattering process

So far, we have studied how charges radiate when they are accelerated under several kinds of circumstances. We have seen in the last section how a charge can transfer energy to a radiation field by radiation damping.

Recall the physical situation we had in mind: A charge, say an electron, moving through an isotropic sea of radiation keeps being accelerated by the randomly oriented electromagnetic fields of the radiation sea. Due to this acceleration, the charge radiates away part of its kinetic energy and thus transfers energy to the radiation field.

Let us now consider the reverse question: Suppose we have an electron at rest and a radiation field streaming past it. Does the radiation field transfer any energy to the charge? In the classical picture of radiation being composed of electromagnetic waves, the charge is accelerated by the Lorentz force of the randomly superposed electromagnetic waves constituting the radiation field. The magnetic part of the Lorentz force can never change the charge's energy since it acts perpendicular to the charge's velocity.

Since electromagnetic waves in vacuum are transversal, the electric part of the Lorentz force cannot act in the streaming direction of the radiation in the charge's rest frame. Driven by the electric Lorentz force of the radiation, the charge will thus oscillate perpendicular to the streaming direction. If the radiation is unpolarised, the electric field experienced by the charge will be randomly superposed of waves with arbitrary orientations and random phases. Does this mean that there is no net energy transfer from the radiation field to the charge?

At this point, it is necessary to change to the photon picture and describe radiation as a stream of particles, each carrying a four-momentum

$$k^\mu = \frac{\omega}{c} \begin{pmatrix} 1 \\ \hat{e} \end{pmatrix}, \quad (2.173)$$

where \hat{e} is the direction of motion. The total energy-momentum four-vector of the electron, p^μ , and the photon $\hbar k^\mu$ is conserved, and thus (Figure 2.12)

$$p^\mu + \hbar k^\mu = p'^\mu + \hbar k'^\mu, \quad (2.174)$$

where primes denote quantities after scattering. Recall the result (1.63) from relativistic dynamics, showing that the four-momentum of the electron has the components

$$p^\mu = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = \gamma m \begin{pmatrix} c \\ \vec{v} \end{pmatrix} \quad (2.175)$$

and the Minkowski square given by (1.65), $\langle p, p \rangle = -m^2 c^2$, which implies the relativistic energy-momentum relation (1.66),

$$E^2 = c^2 \vec{p}^2 + m^2 c^4. \quad (2.176)$$

We first leave the electron momentum \vec{p} arbitrary and later transform into the frame in which the electron is initially at rest. The $\mu = 0$ component of (2.174) gives

$$E + \hbar\omega = E' + \hbar\omega', \quad (2.177)$$

while its spatial components give

$$c\vec{p} + \hbar\omega\hat{e} = c\vec{p}' + \hbar\omega'\hat{e}' . \quad (2.178)$$

Squaring (2.178), using the relativistic energy-momentum relation (2.176) and eliminating \vec{p}' through (2.178) yields

$$E'^2 = E^2 + 2\hbar c\vec{p} \cdot (\omega\hat{e} - \omega'\hat{e}') + \hbar^2 (\omega\hat{e} - \omega'\hat{e}')^2 . \quad (2.179)$$

Next, we use (2.177) to eliminate E' and find after brief rearranging

$$E(\omega - \omega') = \hbar\omega\omega'(1 - \cos\theta) + c\vec{p} \cdot (\omega\hat{e} - \omega'\hat{e}') , \quad (2.180)$$

where the scattering angle θ of the photon was introduced by $\cos\theta = \hat{e} \cdot \hat{e}'$.

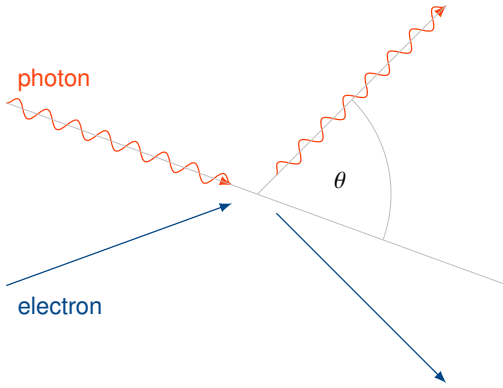


Figure 2.12 Sketch of the kinematics of a Compton-scattering event. The total incoming four-momentum is conserved.

Let us now transform into the rest frame of the electron before the scattering event. There, we can set $\vec{p} = 0$ and $E = mc^2$ in (2.180). The remaining equation is quickly solved for the frequency of the photon after scattering,

$$\frac{\omega'}{\omega} = \frac{1}{1 + \varepsilon(1 - \cos\theta)} , \quad (2.181)$$

where $\varepsilon = \hbar\omega/E_0$ is the energy ratio between the photon energy and the electron's rest-energy. Averaging this last result over angles, taking the unpolarised Thomson cross section (2.14) into account, we find the mean relative frequency or energy change per photon,

$$\begin{aligned} \frac{\langle \Delta E_\gamma \rangle}{E_\gamma} &= \frac{\langle \omega' \rangle}{\omega} - 1 = \frac{1}{\sigma_T} \frac{r_e^2}{2} \int \frac{(1 + \cos^2\theta) \sin\theta d\theta d\phi}{1 + \varepsilon(1 - \cos\theta)} - 1 \\ &= \frac{\pi r_e^2}{\sigma_T} \int_{-1}^1 \frac{(1 + \mu^2) d\mu}{1 + \varepsilon(1 - \mu)} - 1 \\ &= \frac{\pi r_e^2}{\sigma_T} \frac{\ln(1 + 2\varepsilon)(2\varepsilon^2 + 2\varepsilon + 1) - 2\varepsilon(1 + \varepsilon)}{\varepsilon^3} - 1 . \end{aligned} \quad (2.182)$$

Notice that no approximation has so far been made in the rest frame of the electron prior to scattering. Now, we introduce the often appropriate limiting

?

Convince yourself of the result (2.180) by your own calculation.

case of photons whose energy is much below the rest energy of the electron, $\varepsilon \ll 1$. Then, by Taylor-expanding the ε -dependent first term in (2.182) to second order, we find the energy change per photon per Compton-scattering event

$$\frac{\langle \Delta E_\gamma \rangle}{E_\gamma} \approx \frac{8\pi r_e^2}{3\sigma_T} (1 - \varepsilon) - 1 = -\varepsilon = -\frac{\hbar\omega}{mc^2} . \quad (2.183)$$

This is our first important result: The relative energy loss of a photon scattering off an electron is given by the ratio of the photon energy and the rest energy of the electron.

2.6.2 Net energy transfer

We now have two competing effects. An electron moving through a sea of radiation is accelerated by the Lorentz force of the electromagnetic radiation field, hence it radiates and transfers the power given by (2.163) to the radiation field. At the same time, photons transfer part of their energy through Compton collisions back to the electrons. For comparing both effects, we first transform our previous result (2.163) from an energy loss per electron per unit time to an energy increase per photon per unit time.

When we studied the energy transfer from a moving charge to an isotropic radiation field, we saw that the power transferred from the electron is proportional to the energy density U of the radiation field. Let now U_ω be the specific energy density of the radiation field contributed by photons with frequency ω . We must then satisfy the normalisation condition

$$U = \int U_\omega d\omega . \quad (2.184)$$

According to (2.163), a single electron increases the energy in such photons by the amount

$$\frac{dE_\omega^+}{dt} = \frac{4}{3} \beta^2 \gamma^2 c U_\omega \sigma_T \quad (2.185)$$

per unit time. Let the number density of electrons with velocity β be $n_e(\beta)$, and the total number density of all electrons be

$$n_e = \int_0^\infty d\beta n_e(\beta) . \quad (2.186)$$

Then, the electrons contained in a unit of volume increase the energy density in photons with frequency ω by the amount

$$\frac{dU_\omega^+}{dt} = \frac{4}{3} n_e(\beta) \beta^2 \gamma^2 c U_\omega \sigma_T \quad (2.187)$$

per unit time, irrespective of the photon frequency. Since the spatial number density of photons of frequency ω is

$$n_\gamma(\omega) = \frac{U_\omega}{\hbar\omega} , \quad (2.188)$$

the sought energy gained per photon per unit time from the electrons with velocity β is

$$\frac{dE_\gamma^+}{dt} = \frac{dU_\omega^+}{dt} n_\gamma^{-1}(\omega) = \frac{4}{3} \beta^2 \gamma^2 n_e(\beta) c \hbar \omega \sigma_T . \quad (2.189)$$

?

Show that the result (2.182) is correct and that second-order Taylor approximation in ε leads to (2.183).

To find the total energy gain of the photons due to the complete electron population with number density n_e , we need to integrate over the velocity β , see (2.186). Define the average of $\beta^2\gamma^2$ by

$$\langle \beta^2\gamma^2 \rangle = n_e^{-1} \int_0^\infty d\beta \beta^2\gamma^2 n_e(\beta), \quad (2.190)$$

then the energy gain per photon and unit time due to electrons of all velocities is

$$\frac{dE_\gamma^+}{dt} = \frac{4}{3} \langle \beta^2\gamma^2 \rangle n_e c \hbar \omega \sigma_T. \quad (2.191)$$

Similarly, the number of Compton collisions that a photon experiences with electrons of total number density n_e is $cn_e\sigma_T$. According to (2.183), the energy change per photon per unit time is

$$\frac{dE_\gamma^-}{dt} = -cn_e\sigma_T \frac{(\hbar\omega)^2}{mc^2}. \quad (2.192)$$

Now we can compare the energy gained per photon per unit time, expressed by (2.191), with the energy loss (2.192) per photon per unit time. The total energy change per photon per unit time is the sum of gain and loss,

$$\frac{dE_\gamma}{dt} = \frac{dE_\gamma^+}{dt} + \frac{dE_\gamma^-}{dt} = cn_e\sigma_T \hbar\omega \left(\frac{4}{3} \langle \beta^2\gamma^2 \rangle - \frac{\hbar\omega}{mc^2} \right). \quad (2.193)$$

2.6.3 The Kompaneets equation

An illustrative combination of the Fokker-Planck approach and Compton scattering leads to an evolution equation for the phase-space density of photons passing through a hot electron gas. This is most useful in the context of the Cosmic Microwave Background (CMB). The CMB decouples from the quickly recombining cosmic plasma when its temperature falls to ≈ 3000 K, corresponding to a thermal energy of ≈ 0.3 eV. After that, the CMB photons are redshifted by a factor of $\approx 100 \dots 1000$ before they propagate through plasma inside galaxies or galaxy clusters. They have thus typical thermal energies in the meV range or further below. The electron energies even in relatively cool plasmas are typically higher by factors $\gtrsim 10^6$, but still well non-relativistic. In such circumstances, it is appropriate to study Compton scattering under the approximations

$$\frac{\hbar\omega}{c} \ll p_e \ll mc, \quad (2.201)$$

where p_e is the electron momentum.

Let us return with these approximations to the exact equation (2.180) for the frequency change of the scattered photon and stay in the laboratory frame, thus leave $\vec{p}_e \neq 0$. Due to our approximations, we can then neglect the first term on the right-hand side of (2.180) and write

$$\omega - \omega' \approx \frac{c\vec{p}_e \cdot (\omega\hat{e} - \omega'\hat{e}')}{mc^2}. \quad (2.202)$$

Example: Thermal equilibrium between electrons and photons

For a specific example, suppose now that the electrons have a thermal velocity distribution with a temperature T_e such that $kT_e \ll mc^2$. The electrons are then non-relativistic, allowing us to set $\gamma \approx 1$. By the equipartition theorem, for systems in thermal equilibrium, their mean-squared velocity must be

$$\langle \beta^2 \rangle = \frac{3kT_e}{mc^2} \approx \langle \beta^2 \gamma^2 \rangle. \quad (2.194)$$

For averaging the energy gain (2.191) over all photon frequencies, we need to adopt a photon spectrum and calculate the mean energy $\langle \hbar\omega \rangle$ as well as the mean squared energy $\langle (\hbar\omega)^2 \rangle$. Suppose that the photons have a Planck spectrum with temperature T_γ . In terms of the dimension-less energy parameter

$$x := \frac{\hbar\omega}{kT_\gamma}, \quad (2.195)$$

the number of photon states in an infinitesimally thin spherical shell with radius x and width dx is

$$n_x(T_\gamma)dx = \frac{1}{\pi^2} \left(\frac{kT_\gamma}{\hbar c} \right)^3 \frac{x^2 dx}{\exp(x) - 1} \quad (2.196)$$

according to the Bose-Einstein occupation number in (2.392). By means of the integral

$$\int_0^\infty \frac{x^n dx}{\exp(x) - 1} = n! \zeta(n+1), \quad (2.197)$$

the moments of the photon-energy distribution can be calculated to be

$$\langle \hbar\omega \rangle = kT_\gamma \frac{3\zeta(4)}{\zeta(3)}, \quad \langle (\hbar\omega)^2 \rangle = (kT_\gamma)^2 \frac{12\zeta(5)}{\zeta(3)}. \quad (2.198)$$

When inserted into (2.193) together with the mean-squared velocity (2.194) of the electrons, they give the mean energy gain per photon per unit time due to thermal electrons,

$$\left\langle \frac{dE_\gamma}{dt} \right\rangle = \frac{12\zeta(4)}{\zeta(3)} c n_e \sigma_T \frac{(kT_\gamma)(kT_e)}{mc^2} \left(1 - \frac{\zeta(5)T_\gamma}{\zeta(4)T_e} \right). \quad (2.199)$$

This is a highly intriguing result: The energy transfer between thermal populations of electrons and photons should vanish if the temperature of the electrons was slightly *higher* than that of the photons,

$$\frac{T_e}{T_\gamma} = \frac{\zeta(5)}{\zeta(4)}, \quad (2.200)$$

even if the ratio between the temperatures is near unity? This could imply one of two conclusions: Either, finite energy transfer from the photons to the electrons would remain in thermal equilibrium between the two species, *defined* to occur at equal temperatures, or the net energy transfer would cease if the two species were slightly *out of* thermal equilibrium? ◀

Example: Thermal equilibrium between electrons and photons (continued)

Needless to say, a perpetuum mobile could be constructed if either one of these conclusions would be correct, but a perpetuum mobile is forbidden by the second law of thermodynamics. Therefore, the result (2.199) cannot be quite right. The error sneaked in when, in (2.196), we assumed a Bose-Einstein distribution for the photons with vanishing chemical potential, $\mu = 0$. The conclusion from (2.199), combined with the second law of thermodynamics, is therefore much more interesting: If a photon and an electron population coexist in thermal equilibrium, the photons must acquire a finite chemical potential. Then, they cannot maintain their Planck spectrum any longer, but must obtain a spectrum that is slightly deformed by the finite chemical potential. ◀

The energy change of the photon will thus also be small, and we can proceed to approximate

$$\vec{p}_e \cdot (\omega \hat{e} - \omega' \hat{e}') \approx \omega \vec{p}_e \cdot (\hat{e} - \hat{e}') = \omega p_e |\hat{e} - \hat{e}'| \cos \theta, \quad (2.203)$$

where we have introduced the angle θ between the electron momentum \vec{p}_e and the vector $(\hat{e} - \hat{e}')$. Since the modulus of the difference vector $(\hat{e} - \hat{e}')$ is

$$|\hat{e} - \hat{e}'| = \sqrt{2 - 2 \cos \theta}, \quad (2.204)$$

we can write (2.202) as

$$\delta\omega \approx -\frac{\omega p_e}{mc} \cos \theta \sqrt{2 - 2 \cos \theta}. \quad (2.205)$$

This is a typical case suggesting a treatment with the Fokker-Planck approach. The change of the phase-space density $f(\omega)$ of the photons with time is then described by the radial Fokker-Planck equation (1.175)

$$\frac{\partial f}{\partial t} + \frac{1}{p^2} \frac{\partial (j_p p^2)}{\partial p} = 0, \quad (2.206)$$

where p is the photon momentum. The current density of the radial photon momentum is given by (1.177),

$$j_p = D_2 f \frac{\partial}{\partial p} (\ln f - \ln \bar{f}). \quad (2.207)$$

To be specific, the distributions f and \bar{f} are the actual and the equilibrium phase-space distributions of the photons. In thermal equilibrium with the electrons, the photons would attain a Bose-Einstein distribution with the appropriate chemical potential and the temperature of the electrons, T_e , which is many orders of magnitude larger than the actual photon temperature. For this reason, the term involving \bar{f} in (2.207) can be neglected altogether in our application, allowing us to approximate simply

$$j_p \approx D_2 \frac{\partial f}{\partial p}. \quad (2.208)$$

This leaves the Fokker-Planck equation (2.206) in the simple form

$$\frac{\partial f}{\partial t} + \frac{1}{p^2} \frac{\partial}{\partial p} \left(D_2 p^2 \frac{\partial f}{\partial p} \right) = 0. \quad (2.209)$$

Next, we need to work out the diffusion coefficient D_2 . As we have emphasised in Sect. 1.4.4, its physical meaning is one half of the mean-squared momentum change per unit time of the population of scattered particles, i.e. of the photons in the present case. From the frequency change per scattering (2.205), we find the mean-squared momentum change

$$\begin{aligned} D_2 &= \frac{1}{2} \langle \delta p^2 \rangle = \frac{1}{2} \frac{\hbar^2}{c^2} \langle \delta \omega^2 \rangle \\ &= \left(\frac{\hbar \omega}{mc^2} \right)^2 \langle p_e^2 \cos^2 \theta \rangle n_e c \int d\Omega \frac{d\sigma}{d\Omega} (1 - \cos \theta). \end{aligned} \quad (2.210)$$

By the equipartition theorem, an electron population in thermal equilibrium must have the mean-squared momentum

$$\langle p_e^2 \rangle = 2m \frac{3}{2} kT_e = 3mkT_e, \quad (2.211)$$

while the mean-squared $\cos \theta$ gives a factor of 1/3. For the differential cross section, we use the unpolarised Thomson cross section (2.14),

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2} (1 + \cos^2 \theta). \quad (2.212)$$

The solid-angle integral in (2.210) then simply gives the total Thomson cross section

$$\sigma_T = \frac{8\pi}{3} r_e^2. \quad (2.213)$$

Taking all factors together, we obtain

$$D_2 = \left(\frac{\hbar \omega}{mc^2} \right)^2 m c n_e \sigma_T kT_e = \frac{p^2}{mc} n_e \sigma_T kT_e. \quad (2.214)$$

Putting this result back into the Fokker-Planck equation (2.209), we find

$$\frac{\partial f}{\partial t} + c n_e \sigma_T \frac{kT_e}{mc^2} \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^4 \frac{\partial f}{\partial p} \right) = 0. \quad (2.215)$$

Let us finally replace the time by the so-called Compton parameter y , defined by

$$dy = \frac{kT_e}{mc^2} c n_e \sigma_T dt. \quad (2.216)$$

This has an intuitive physical meaning: The first factor is the relative energy change of a photon with energy kT by Compton scattering; cf. (2.183). The second factor is the probability for a photon experiencing a Compton-scattering event within the time interval dt . Thus, the differential Compton- y parameter quantifies the mean relative energy change of a photon within the time interval dt . It allows us to bring the Fokker-Planck-equation into the form

$$\frac{\partial f}{\partial y} + \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^4 \frac{\partial f}{\partial p} \right) = 0. \quad (2.217)$$

This is *not* exactly the so-called Kompaneets equation, which is often derived and used in this context. However, it is the appropriate limit of the Kompaneets equation, which reveals its origin from the much more general approach of Fokker-Planck theory.

Let us now insert the Bose-Einstein distribution for the photons with vanishing chemical potential,

$$f = \frac{1}{e^x - 1} \quad \text{with} \quad x := \frac{cp}{kT} \quad (2.218)$$

into the Kompaneets equation (2.217). Since p appears to fourth order in the numerator as well as the denominator in the second term of (2.217), we can replace p by x directly. Further, we use

$$f' = -f^2 e^x \quad \text{and} \quad f'' = -f e^x (f + 2f') . \quad (2.219)$$

After brief rearrangement, this turns the Kompaneets equation into

$$\frac{\partial f}{\partial y} = \frac{x e^x}{(e^x - 1)^2} \left(x \frac{e^x + 1}{e^x - 1} - 4 \right) . \quad (2.220)$$

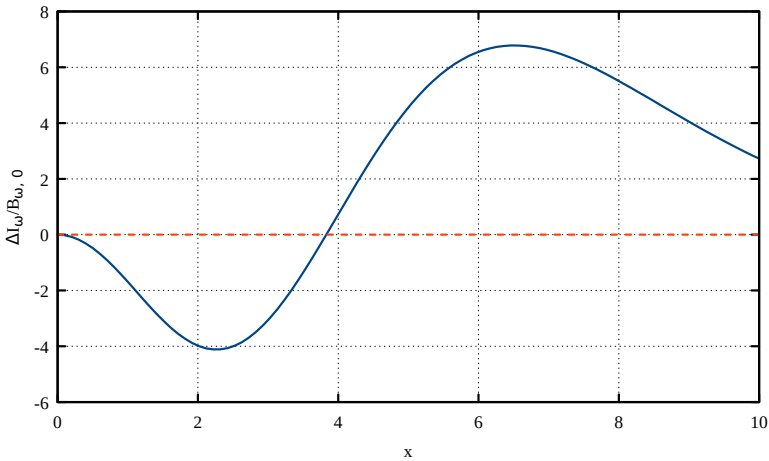


Figure 2.13 The relative change $\Delta I_\omega/B_{\omega,0}$ of the intensity of a black-body spectrum due to Compton scattering is shown as a function of the dimension-less frequency $x = \hbar\omega/(k_B T)$. The intensity is lowered at frequencies below $x = 3.83$ and increased above.

Aiming at astrophysical applications, we are not quite done yet. Notice that (2.220) describes the change of the phase-space density (or the occupation number) of the photons with the Compton- y parameter as they propagate through a plasma. As we shall show below, the intensity is related to the occupation number by (2.396). To obtain the change of intensity with the Compton- y parameter instead, we need to multiply the Kompaneets equation (2.220) by a factor $B_{\omega,0} x^3$, with amplitude $B_{\omega,0}$ of the Planck spectrum defined in (2.401). Thus, we find after integration over y

$$\Delta I_\omega = B_{\omega,0} \frac{x^4 e^x}{(e^x - 1)^2} \left(x \frac{e^x + 1}{e^x - 1} - 4 \right) y =: B_{\omega,0} g(x) y . \quad (2.221)$$

?

The complete Kompaneets equation reads

$$\frac{\partial f}{\partial y} + \frac{1}{p^2} \frac{\partial}{\partial p} \left[p^4 \left(\frac{\partial f}{\partial p} + f + f^2 \right) \right] = 0 .$$

Where could the additional terms arise from, and why is (2.217) an appropriate limit for our purposes?

This intensity change (Figure 2.13) has an intuitive origin: By Compton scattering, photons are neither created nor destroyed, but only re-distributed in frequency. Based on our initial assumption (2.201), we have studied the effect of high-energy electrons scattering low-energy photons. By the inverse Compton effect, the electrons scatter way more photons from low to high energy rather than the other way. The net effect is thus a depletion of photons relative to the Planck spectrum at low frequencies, and an enhancement at high frequencies. The division between low and high frequencies is set by the root of the function $g(x)$ defined in (2.221), which is numerically found to be at $x_0 = 3.83$. For the Planck spectrum of the CMB, we shall see in (2.414) that the frequency characteristic for its temperature is

$$\nu_{\text{CMB}} = \frac{k_{\text{B}}T_{\text{CMB}}}{h} = 56.8 \text{ GHz} , \quad (2.222)$$

which allows to convert x_0 to the frequency

$$\nu_0 = x_0 \frac{k_{\text{B}}T_{\text{CMB}}}{h} = 217.5 \text{ GHz} . \quad (2.223)$$

Any hot plasma between us and the CMB will therefore reduce the specific CMB intensity below 217.5 GHz, and enhance it above.

Perhaps the most prominent example of huge bodies of hot plasma on the way between the CMB and us are galaxy clusters whose plasma has temperatures of $1 \text{ keV} \lesssim k_{\text{B}}T \lesssim 10 \text{ keV}$ and radii of order $R \approx 1 \text{ Mpc} \approx 3.1 \cdot 10^{24} \text{ cm}$. Their electron number densities are typically $n_e \approx 10^{-2} \text{ cm}^{-3}$. A crude estimate for their Compton- y parameter is

$$y \approx \frac{k_{\text{B}}T}{m_e c^2} \sigma_{\text{T}} n_e R \approx 10^{-4} . \quad (2.224)$$

Galaxy clusters thus have a very specific spectral signature against the CMB: They cast shadows on the CMB below 217.5 GHz and appear as sources above. The amplitude of the shadows and the sources are of order a milli-Kelvin. This thermal Sunyaev-Zel'dovich effect has turned into an important means for discovering and probing galaxy clusters.

Problems

1. Carry out the steps leading from (2.178) to (2.180).
2. Electrons passing through a plasma lose energy also by Coulomb scattering, i.e. by their interaction with ions through the Coulomb force. A detailed treatment of the Coulomb scattering process shows that the relative energy loss in a single Coulomb-scattering event is

$$\frac{\Delta E}{E} = 4 \frac{m_e}{m_i} , \quad (2.225)$$

irrespective of the impact parameter.

- (a) Derive the ratio between the remaining energy of an electron and its initial energy after n Coulomb collisions.

- (b) Approximating $m_e \ll m_i$, how many collisions are needed for the electron to lose half its initial energy?
3. The differential cross section for photons with energy $\hbar\omega$ that are scattered off free electrons is given by the Klein-Nishina formula

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2} F^2(\omega, \theta) \left[F(\omega, \theta) + \frac{1}{F(\omega, \theta)} - 1 + \cos^2 \theta \right], \quad (2.226)$$

where r_e is the classical electron radius and

$$F(\omega, \theta) = \left[1 + \frac{\hbar\omega}{m_e c^2} (1 - \cos \theta) \right]^{-1}. \quad (2.227)$$

- (a) What is the ratio $\hbar\omega/m_e c^2$ for visible light? How does the Klein-Nishina formula simplify in this case? Is the solution familiar to you?
- (b) Assume that an electron is hit by a γ photon with $\hbar\omega = m_e c^2$. Calculate the total cross section

$$\sigma_{\text{KN}} = \int d\Omega \frac{d\sigma}{d\Omega} \quad (2.228)$$

and compare it to the classical Thomson cross section σ_T .

4. Consider a photon with frequency ω scattered by a resting electron under the angle θ . By the scattering process, its frequency changes to $\omega' < \omega$. One can transform into the barycentre system, defined by $\vec{p}_{\text{tot}} = 0$ before and after the scattering, by applying a proper Lorentz boost

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \quad (2.229)$$

to the four-momentum $(p^\mu) = (E/c, \vec{p})^T$, assuming that the incoming photon moves along the negative z -direction.

- (a) Calculate the energies and momenta of both the electron and the photon in the barycentre system as a function of β .
- (b) Determine the velocity β as a function of ω and the electron mass m_e .
- (c) Express the scattering angle θ^* in the barycentre system as a function of the scattering angle θ in the rest frame of the electron, ω and m_e .

2.7 Radiative Quantum Transitions

This section deals with the interaction of electromagnetic radiation with quantum systems such as atoms or ions. We first derive the interaction

Hamiltonian (2.262) by a semi-classical approach treating the electromagnetic field as a classical rather than a quantum field. Next, we relate the amplitude of the interaction Hamiltonian to the intensity of the incoming radiation, enabling us to express the quantum-mechanical transition probability (2.270) by the intensity and the transition matrix element between the initial and the final state. We then introduce the dipole approximation and simplify the transition probability (2.291) accordingly. Cross sections for quantum transitions are defined, and expressions for bound-bound and bound-free transitions are given in (2.291) and (2.291).

2.7.1 Transition probability

Up to this point, we have treated electromagnetic radiation either as composed of classical electromagnetic waves, as in Thomson scattering and our treatment of continuous emission spectra, or as a stream of photons, as in Compton scattering. From both points of view, the particles interacting with the radiation had no internal structure. Effects of radiation on their internal structure, or radiative processes caused by transitions between internal configurations, were neglected so far.

We now proceed to see how electromagnetic radiation can cause transitions between quantum states, e.g. in atoms, but also between bound and free electron states. We begin by recalling a result from time-dependent perturbation theory in quantum mechanics.

Suppose the Hamiltonian \hat{H} of a quantum-mechanical system can be decomposed into a time-independent part $\hat{H}^{(0)}$ and a time-dependent perturbation $\hat{H}^{(1)}(t)$,

$$\hat{H}(t) = \hat{H}^{(0)} + \hat{H}^{(1)}(t). \quad (2.230)$$

Let the time-dependent eigenstates of the unperturbed Hamiltonian $\hat{H}^{(0)}$ with eigenvalue E_n be

$$|n(t)\rangle = |n\rangle e^{-iE_n t/\hbar}, \quad (2.231)$$

where the state vector $|n\rangle$ does not depend on time. We expand the eigenstates $|\psi_n(t)\rangle$ of the complete Hamiltonian $\hat{H}(t)$ into eigenstates of the unperturbed Hamiltonian,

$$|\psi_n(t)\rangle = \sum_k c_{nk} |n(t)\rangle, \quad (2.232)$$

and demand that they solve Schrödinger's equation,

$$i\hbar |\dot{\psi}_n(t)\rangle = [\hat{H}^{(0)} + \hat{H}^{(1)}(t)] |\psi_n(t)\rangle. \quad (2.233)$$

In a first step, this leads to

$$i\hbar \left(\dot{c}_{nk} - c_{nk} \frac{iE_k}{\hbar} \right) |n(t)\rangle = \sum_k c_{nk} [E_k + \hat{H}^{(1)}(t)] |n(t)\rangle \quad (2.234)$$

since the $|n(t)\rangle$ are eigenstates of the unperturbed Hamiltonian $\hat{H}^{(0)}$. Now, we multiply by $\langle m|$ and use the orthonormality of the unperturbed eigenstates to arrive at

$$\dot{c}_{nm} = -\frac{i}{\hbar} \sum_k c_{nk} \langle m | \hat{H}^{(1)}(t) | k \rangle e^{i\omega_{nm}t}, \quad (2.235)$$

Carry out all steps leading from Schrödinger's equation (2.233) to the evolution equation (2.235) yourself.

where

$$\omega_{mn} = \frac{E_m - E_n}{\hbar} \quad (2.236)$$

is the frequency associated with the difference between the energy eigenvalues of the unperturbed states $|n\rangle$ and $|m\rangle$.

The evolution equation (2.235) for the expansion coefficients c_{nm} is exact, but in general difficult to solve. To proceed, we assume that the system is in the eigenstate $|n\rangle$ of the unperturbed Hamiltonian when the perturbation sets in at $t = 0$, thus $c_{nk} = \delta_{nk}$, and that the coefficients c_{nk} with $k \neq n$ remain small even while the perturbation is acting. Then, (2.235) simplifies to

$$\dot{c}_{nm} = -\frac{i}{\hbar} \langle m | \hat{H}^{(1)}(t) | n \rangle e^{i\omega_{mn}t} \quad (2.237)$$

and can immediately be integrated once the time dependence of the perturbation Hamiltonian $\hat{H}^{(1)}(t)$ is given.

In our context, perturbations by electromagnetic radiation are most important. We can decompose them into monochromatic waves with frequency ω and thus write the perturbation Hamiltonian as

$$\hat{H}^{(1)}(t) = \hat{V} e^{i\omega t} \theta(t) \quad (2.238)$$

with an operator \hat{V} representing the constant amplitude of the wave. The step function $\theta(t)$ expresses that the perturbation is supposed to begin at $t = 0$. Inserting expression (2.238) into (2.237) and integrating leads us to

$$\begin{aligned} c_{nm} &= -\frac{i}{\hbar} \langle m | \hat{V} | n \rangle \int_0^t dt' e^{i(\omega_{mn}-\omega)t'} \\ &= -\frac{V_{mn}}{\hbar(\omega_{mn}-\omega)} \left[e^{i(\omega_{mn}-\omega)t} - 1 \right] \end{aligned} \quad (2.239)$$

with the transition-matrix element

$$V_{mn} := \langle m | \hat{V} | n \rangle \quad (2.240)$$

of the amplitude \hat{V} of the perturbation Hamiltonian.

The absolute square of c_{nm} is the transition probability into state $|m\rangle$. Dividing this probability by t gives the transition rate Γ . Using

$$1 - \cos x = 2 \sin^2 \frac{x}{2}, \quad (2.241)$$

we find directly from (2.239) the transition rate

$$\Gamma = \frac{|V_{nm}|^2 t}{\hbar^2} \left[\frac{\sin(\omega_{mn}-\omega)t/2}{(\omega_{mn}-\omega)t/2} \right]^2. \quad (2.242)$$

If we can furthermore take the limit $t \rightarrow \infty$, i.e. if the perturbation acts for a time long compared to the time the system takes for the transition from the state $|n\rangle$ to the state $|m\rangle$, we can use

$$\lim_{a \rightarrow \infty} a \left(\frac{\sin ax}{ax} \right)^2 = \pi \delta_D(x) \quad (2.243)$$

with $a = t/2$ to bring the transition rate into the form

$$\Gamma = \frac{2\pi |V_{nm}|^2}{\hbar^2} \delta_D(\omega_{mn}-\omega). \quad (2.244)$$

?

Can you confirm the expression (2.242) for the transition rate Γ ? How could you prove (2.243)?

2.7.2 Perturbing Hamiltonian

Since we are interested in radiative transitions, we need to know the perturbation Hamiltonian belonging to an incident electromagnetic wave. We have seen in (1.59) that the Lagrange function of a free relativistic particle is

$$L = -mc^2 \sqrt{1 - \beta^2}. \quad (2.245)$$

If the particle has an electromagnetic charge q , it couples to an electromagnetic field. With the four-potential A^μ of the field and the four-velocity u^μ of the particle, the Lagrange function is extended by a coupling term

$$L = \left(-mc^2 + \frac{q}{c} A_\mu u^\mu \right) \sqrt{1 - \beta^2}. \quad (2.246)$$

Since the four-potential and the four-velocity have the components

$$A^\mu = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix}, \quad u^\mu = \gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix}, \quad (2.247)$$

this Lagrange function can be written as

$$L = -mc^2 \sqrt{1 - \beta^2} - q\Phi + \frac{q}{c} \vec{A} \cdot \vec{v}. \quad (2.248)$$

The momentum conjugate to the velocity \vec{v} is

$$\frac{\partial L}{\partial \vec{v}} = \vec{P} = \gamma m \vec{v} + \frac{q}{c} \vec{A} = \vec{p} + \frac{q}{c} \vec{A}, \quad (2.249)$$

where $\vec{p} = \gamma m \vec{v}$ is the momentum of the free particle. The Legendre transform

$$H = \vec{P} \cdot \vec{v} - L \quad (2.250)$$

then turns the Lagrange- into the Hamilton function of a charged particle in an electromagnetic field,

$$H = \frac{1}{2m} \left(\vec{P} - \frac{q}{c} \vec{A} \right)^2 + q\Phi + mc^2. \quad (2.251)$$

According to the correspondence principle, we shall interpret this Hamilton function as a Hamilton operator. In particular, this implies that \vec{P} will have to be replaced by the momentum operator \hat{P} ,

$$\vec{P} \rightarrow \hat{P} = -i\hbar \vec{\nabla}. \quad (2.252)$$

Remaining in quantum mechanics, avoiding the step into quantum electrodynamics, the electromagnetic field components A^μ will be treated as classical fields rather than field operators. Yet, they depend on spatial coordinates \vec{x} . These need to be interpreted as position operators, which do not commute with the momentum operator \hat{P} . Thus, we also write the vector potential as an operator \hat{A} , understanding that this merely reflects that spatial coordinates x_i in the vector potential need to be replaced by position operators \hat{x}_i . Expanding the square in (2.251), we thus need to distinguish between

$$\hat{P} \cdot \hat{A} \quad \text{and} \quad \hat{A} \cdot \hat{P}. \quad (2.253)$$

?

Determine the equations of motion from the Lagrange function (2.246). Which force term do you expect?

However, we have not employed the gauge freedom of electrodynamics yet. Choosing the Coulomb gauge,

$$\vec{\nabla} \cdot \vec{A} = 0, \quad (2.254)$$

we can pull the momentum operator \hat{P} past the vector-potential operator \hat{A} . In addition to and without conflict with the Coulomb gauge, we can further gauge Φ away, $\hat{\Phi} = 0$, and obtain the Hamilton operator

$$\hat{H} = \frac{\hat{P}^2}{2m} + mc^2 - \frac{e}{mc} \hat{A} \cdot \hat{P} + \frac{e^2}{2mc^2} \hat{A}^2. \quad (2.255)$$

The first two terms reproduce the Hamiltonian $\hat{H}^{(0)}$ of an unperturbed, free particle, if \hat{P} is interpreted as the momentum operator in absence of the electromagnetic field.

Let us now compare the two final terms in (2.255) containing the vector potential. Their ratio η can be estimated by

$$\eta \approx \frac{e A}{2c P}, \quad (2.256)$$

with typical values A and P of the vector potential and the momentum. In Coulomb gauge with $\Phi = 0$, the electric field is

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}. \quad (2.257)$$

If we decompose \vec{A} into plane waves and consider a single mode with frequency ω ,

$$\vec{E} = -\frac{i\omega}{c} \vec{A} = -ik \vec{A} = -\frac{2\pi i}{\lambda} \vec{A}, \quad (2.258)$$

where we have used the dispersion relation $k = \omega/c$ for electromagnetic waves in vacuum. Thus,

$$A \approx \frac{\lambda E}{2\pi}. \quad (2.259)$$

The momentum of the electron in a hydrogen atom is

$$P \approx \alpha mc, \quad \text{where} \quad \alpha = \frac{e^2}{\hbar c} = \frac{1}{137.04} \quad (2.260)$$

is the fine-structure constant. Combining all terms, we estimate

$$\eta \approx \frac{1}{4\pi\alpha} \frac{\lambda e E}{mc^2}. \quad (2.261)$$

The numerator of the second factor is the work done on the electron by a single wave of the incident electromagnetic field. This is compared to the electron's rest energy! Unless the electromagnetic field is so intense that it can deposit a sizeable fraction of the electron's rest energy on the electron by a single wave, we can safely ignore the term quadratic in \vec{A} in (2.255). Our perturbing Hamiltonian is thus

$$\hat{H}^{(1)}(t) = \frac{e}{mc} \hat{A} \cdot \hat{P} = -i \frac{\hbar e}{mc} \hat{A} \cdot \vec{\nabla}. \quad (2.262)$$

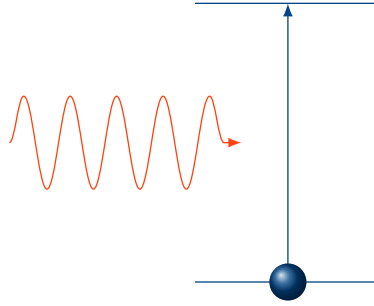


Figure 2.14 Illustration of an incoming electromagnetic wave causing a transition between two quantum states.

2.7.3 Decomposition of the electromagnetic field

Let us now decompose the incident electromagnetic field (cf. Figure 2.14) into plane waves,

$$\vec{A}(\vec{x}, t) = A_0 \hat{e} e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \quad (2.263)$$

where A_0 is a scalar, time-independent amplitude and \hat{e} is the polarisation direction. Coulomb gauge immediately implies transversality, $\hat{e} \cdot \vec{k} = 0$. We know that this decomposition into plane waves is possible because the vector potential of electromagnetic waves in vacuum must satisfy the d'Alembert equation $\square \vec{A} = 0$, what plane waves do if only they obey the dispersion relation $k = \omega/c$.

With (2.244), these plane electromagnetic waves in the perturbing Hamiltonian (2.262) give the transition rate

$$\Gamma = \frac{e^2}{m^2 c^2} |A_0|^2 \left| \langle m | e^{i\vec{k} \cdot \vec{x}} \hat{e} \cdot \vec{\nabla} | n \rangle \right|^2 \delta_D(\omega_{mn} - \omega). \quad (2.264)$$

We can now relate the absolute square $|A_0|^2$ of the vector-potential amplitude to the intensity of the incoming light. The energy flux density carried by the electromagnetic wave is expressed by its Poynting vector,

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{c}{4\pi} \vec{E}^2 \hat{e}_k, \quad (2.265)$$

where \hat{e}_k is a unit vector pointing into the direction of the wave vector \vec{k} . The mean energy flowing past the quantum-mechanical system per unit area and unit time is thus

$$\langle |\vec{S}| \rangle = \frac{1}{T} \int_{-T/2}^{T/2} dt |\vec{S}| = \frac{c}{4\pi T} \int_{-T/2}^{T/2} dt \vec{E}^2. \quad (2.266)$$

In the limit of very long times, the time integral can be transformed to a frequency integral by Plancherel's theorem (2.25), which brings (2.266) into the form

$$\langle |\vec{S}| \rangle = \frac{c}{4\pi T} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\hat{E}|^2. \quad (2.267)$$

The specific intensity, i.e. the energy per unit area, time and frequency, is thus

$$I_\omega = \frac{c}{8\pi^2 T} \left| \hat{\vec{E}} \right|^2. \quad (2.268)$$

We can now use (2.258) to continue writing

$$I_\omega = \frac{c}{8\pi^2 T} \frac{\omega^2}{c^2} |A_0|^2 \quad (2.269)$$

and return to the transition rate (2.264) with this result. This gives

$$\Gamma = \frac{8\pi^2 e^2}{m^2 c} \frac{I_\omega T}{\omega^2} \left| \langle m | e^{i\vec{k} \cdot \vec{x}} \hat{\vec{e}} \cdot \vec{\nabla} | n \rangle \right|^2 \delta_D(\omega_{mn} - \omega) \quad (2.270)$$

for the transition rate between the states $|n\rangle$ and $|m\rangle$, given the specific intensity I_ω acting on the system for time T .

2.7.4 Dipole approximation

Before we evaluate the transition matrix element occurring in (2.270), we can apply a further approximation. Expand the phase factor $\exp(i\vec{k} \cdot \vec{x})$ into a Taylor series,

$$e^{i\vec{k} \cdot \vec{x}} \approx 1 + i\vec{k} \cdot \vec{x} - \frac{1}{2} (\vec{k} \cdot \vec{x})^2 + \dots \quad (2.271)$$

Already the first-order term, $\vec{k} \cdot \vec{x}$, is very much smaller than unity, as the following estimate shows. By the dispersion relation, the wave number k of the electromagnetic wave must be

$$k = \frac{\omega_{mn}}{c} = \frac{E_m - E_n}{\hbar c}, \quad (2.272)$$

while $|\vec{x}| = x$ must be of the order of the Bohr radius a_0 ,

$$x \approx a_0 = \frac{\hbar^2}{me^2} = 5.2918 \cdot 10^{-9} \text{ cm}. \quad (2.273)$$

Thus, their product can be estimated to be

$$\vec{k} \cdot \vec{x} \approx kx \approx \frac{\hbar}{me^2 c} (E_m - E_n) = \frac{E_m - E_n}{\alpha mc^2}, \quad (2.274)$$

where we have identified the fine-structure constant α , see (2.260). As long as the energy difference between the transitions is very small compared to the rest-energy of the electron, it is thus very well justified to replace the phase factor by unity. Consider transitions in the hydrogen atom as an example. The ionisation energy of hydrogen is 13.6 eV, while $\alpha mc^2 \approx (511/137) \text{ keV} \approx 3.7 \cdot 10^3 \text{ eV}$. In this case, $kx \approx 3.7 \cdot 10^{-3}$. We are then left to evaluate the matrix element

$$\langle m | \hat{\vec{e}} \cdot \vec{\nabla} | n \rangle = \frac{i}{\hbar} \langle m | \hat{\vec{e}} \cdot \hat{\vec{p}} | n \rangle \quad (2.275)$$

Since $|n\rangle$ and $|m\rangle$ are eigenstates of the *unperturbed* Hamiltonian, it is most useful to replace the momentum operator $\hat{\vec{p}}$ by means of the following commutation relation,

$$[\hat{x}, \hat{p}^2] = \hat{p}_x [\hat{x}, \hat{p}_x] + [\hat{x}, \hat{p}_x] \hat{p}_x = 2i\hbar \hat{p}_x. \quad (2.276)$$

Caution Notice that the Bohr radius can be expressed by the classical electron radius (2.12) and the fine-structure constant α as

$$a_0 = \frac{\hbar^2}{me^2} = \frac{e^2 \hbar^2 c^2}{mc^2 e^4} = \frac{r_e}{\alpha^2}.$$

We shall use this relation in (2.300) below. ◀

It allows us to write

$$[\hat{x}, \hat{H}] = \frac{i\hbar}{m} \hat{p}, \quad (2.277)$$

which turns the transition matrix element (2.275) into

$$\langle m | \hat{e} \cdot \vec{\nabla} | n \rangle = -\frac{m\omega_{mn}}{\hbar} \langle m | \hat{e} \cdot \hat{x} | n \rangle = -\frac{m\omega_{mn}}{e\hbar} \langle m | \hat{e} \cdot \hat{d} | n \rangle, \quad (2.278)$$

where the dipole operator $\hat{d} = e\hat{x}$ was introduced. For this reason, the approximation $\exp(i\vec{k} \cdot \vec{x}) \approx 1$ is called the *dipole approximation*.

If the dipole matrix element $\langle m | \hat{e} \cdot \hat{d} | n \rangle$ vanishes, we need to proceed to the next order in the Taylor expansion of the phase factor, arriving at the level of the so-called quadrupole transitions. The rate (2.270) for dipole transitions has now assumed the form

$$\Gamma = \frac{8\pi^2 I_\omega T}{c\hbar^2} |\langle m | \hat{e} \cdot \hat{d} | n \rangle|^2 \delta_D(\omega_{mn} - \omega). \quad (2.279)$$

Finally, for the frequent case of unpolarised radiation, the mean-squared projection of \hat{d} on \hat{e} gives a factor of 1/3, and we arrive at

$$\Gamma = \frac{8\pi^2 I_\omega T}{3c\hbar^2} |\vec{d}_{mn}|^2 \delta_D(\omega_{mn} - \omega), \quad (2.280)$$

where the dipole matrix element $\vec{d}_{mn} = \langle m | \hat{d} | n \rangle$ was defined.

2.7.5 Cross sections

We would like to convert the expression (2.280) for the rate of transitions between the states $|n\rangle$ and $|m\rangle$ into an expression for the transition cross section. We shall consider two cases; transitions between two bound states and transitions between a bound and a free state.

Let us begin with transitions between two bound states, which we assume for simplicity to be non-degenerate. Thus, the initial and the final states can be occupied by a single electron each. The two states differ by the discrete energy $E_m - E_n$, which has to be supplied or carried away by a photon with energy $\hbar\omega_{mn} = |E_m - E_n|$. To be specific, we choose to consider the absorption of photons, thus $E_m > E_n$. Of the incoming specific intensity I_ω , only those photons can be absorbed whose frequency precisely equals ω_{mn} . This is expressed by the product $I_\omega \delta_D(\omega_{mn} - \omega)$ in the transition probability (2.280). Notice that the frequency integral over the Dirac delta function must be dimension-less, so the delta function must have the dimension [frequency]⁻¹. The number of incoming photons at the frequency ω during the time T per area is

$$\frac{I_\omega T}{\hbar\omega}. \quad (2.281)$$

Dividing (2.280) by this number gives the desired absorption cross section

$$\sigma_{mn} = \frac{8\pi^2}{3c\hbar} \omega_{mn} |\vec{d}_{mn}|^2 \delta_D(\omega_{mn} - \omega), \quad (2.282)$$

?

Verify the relations (2.276) and (2.277).

with the Dirac delta function expressing that the transition is assumed for now to be needle-sharp in frequency. Conventionally, the dimension-less quantity

$$f_{mn} := \frac{2m\omega_{mn}}{3\hbar e^2} \left| \vec{d}_{mn} \right|^2 \quad (2.283)$$

is called the *oscillator strength* of the transition from the state $|n\rangle$ to the state $|m\rangle$. Identifying it in (2.282) allows us to write the cross section in the simple form

$$\sigma_{mn} = \frac{4\pi^2 e^2}{mc} f_{mn} \delta_D(\omega_{mn} - \omega) = 4\pi^2 r_e c f_{mn} \delta_D(\omega_{mn} - \omega) , \quad (2.284)$$

where the classical electron radius $r_e = 2.81 \cdot 10^{-13}$ cm was introduced from (2.12).

In realistic situations, as we shall see below, the absorption cross section does not have the needle-sharp delta profile adopted here, but a broader one. If this profile is described by a function $\phi(\omega_{mn} - \omega)$ which is normalised to unity, the cross section reads

$$\sigma_{mn} = \frac{4\pi^2 e^2}{mc} f_{mn} \phi(\omega_{mn} - \omega) = 4\pi^2 r_e c f_{mn} \phi(\omega_{mn} - \omega) . \quad (2.285)$$

As we shall see shortly, there is a characteristic line profile function, called the Voigt profile.

For bound-free transitions, we can proceed in an analogous way as for bound-bound transitions, except that we have to take the number of available free electron states into account. We arrive at the bound-free absorption cross section σ_{bf} if we multiply the transition rate (2.270) by the number of final electron states and divide, as before, by the number of photons incoming per unit area per unit time. The number of final electron states in an infinitesimally thin momentum shell in phase space is

$$\frac{4\pi p_f^2 dp_f}{(2\pi\hbar)^3} V = \frac{p_f^2 dp_f}{2\pi^2 \hbar^3} V = \frac{k_f^2 dk_f}{2\pi^2} V \quad (2.286)$$

if the shell has the width $dp_f = \hbar dk_f$ in the final electron momentum. Energy conservation implies that the energy of an incoming photon, $\hbar\omega$, must come up for the binding energy E_1 of the electron plus the energy of the free electron after ionisation,

$$\hbar\omega = \frac{p_f^2}{2m} + E_1 = \frac{\hbar^2 k_f^2}{2m} + E_1 . \quad (2.287)$$

This allows us to relate the width dk_f of the shell of electron momenta to the width $d\omega$ in photon frequency,

$$\hbar d\omega = \frac{\hbar^2 k_f dk_f}{m} \Rightarrow k_f dk_f = \frac{m d\omega}{\hbar} . \quad (2.288)$$

The number of final electron states can thus be expressed by

$$\frac{k_f^2 dk_f}{2\pi^2} V = \frac{k_f m d\omega}{2\pi^2 \hbar} V , \quad (2.289)$$

?

Is the oscillator strength (2.283) really dimension-less, as claimed?

and the number of photons with frequency within $[\omega, \omega + d\omega]$ incoming during the time T per unit area is given by

$$\frac{I_\omega T}{\hbar\omega} d\omega. \quad (2.290)$$

Multiplying the transition rate (2.270) with the number of electron states (2.289) and dividing by the number (2.290) of incoming photons gives the cross section

$$\sigma_{\text{bf}} = \frac{4e^2 k_f V}{mc \omega} \left| \langle f | e^{i\vec{k}\cdot\vec{x}} \hat{\epsilon} \cdot \vec{\nabla} | b \rangle \right|^2 \quad (2.291)$$

between the bound state $|b\rangle$ and the free state $|f\rangle$, where the transition matrix elements still needs to be worked out.

2.7.6 Photoionisation cross section

To give one specific and simple example for the calculation of a bound-free cross section, we consider the photoionisation of the hydrogen atom from its ground state. In the position representation, the bound and free electron states are given by the wave functions

$$\begin{aligned} \psi_b(\vec{x}) &= \langle x | b \rangle = (\pi a_0^3)^{-1/2} e^{-r/a_0}, \\ \psi_f(\vec{x}) &= \langle x | f \rangle = V^{-1/2} e^{i\vec{k}_f \cdot \vec{x}}, \end{aligned} \quad (2.292)$$

where the final electron state is assumed to be confined to the volume V .

If the energy difference between the final and initial electron states is small compared to the rest-energy of the electron, i.e. as long as the electron remains non-relativistic, we can evaluate the transition matrix element in dipole approximation. We thus set $\exp(i\vec{k} \cdot \vec{x}) \approx 1$ in (2.291) and use the Hermitian property of the momentum operator to exchange the final and the initial states,

$$\left| \langle f | \hat{\epsilon} \cdot \vec{\nabla} | b \rangle \right|^2 = \left| \langle b | \hat{\epsilon} \cdot \vec{\nabla} | f \rangle^* \right|^2 = \left| \langle b | \hat{\epsilon} \cdot \vec{\nabla} | f \rangle \right|^2. \quad (2.293)$$

Inserting the initial and final wave functions, the matrix element is now easily evaluated,

$$\begin{aligned} \langle b | \hat{\epsilon} \cdot \vec{\nabla} | f \rangle &= (\pi a_0^3 V)^{-1/2} \int d^3x e^{-r/a_0} \hat{\epsilon} \cdot \vec{\nabla} e^{i\vec{k}_f \cdot \vec{x}} \\ &= (\pi a_0^3 V)^{-1/2} i \hat{\epsilon} \cdot \vec{k}_f \int d^3x e^{-r/a_0 + i\vec{k}_f \cdot \vec{x}}. \end{aligned} \quad (2.294)$$

The remaining integral is quickly worked out in polar coordinates,

$$\begin{aligned} \int d^3x e^{-r/a_0 + i\vec{k}_f \cdot \vec{x}} &= 2\pi \int_0^\infty r^2 dr e^{-r/a_0} \int_{-1}^1 d \cos \theta e^{ik_f r \cos \theta} \\ &= 4\pi \int_0^\infty r^2 dr e^{-r/a_0} \frac{\sin(k_f r)}{k_f r} \\ &= \frac{8\pi a_0^3}{(1 + k_f^2 a_0^2)^2} \approx \frac{8\pi}{k_f^4 a_0^4}, \end{aligned} \quad (2.295)$$

?

Can you confirm that the wave functions (2.292) correctly represent the states of an electron bound in the ground state of a hydrogen atom, and a free electron, respectively? Are they properly normalised?

where the final approximation is allowed if the energy of the final state is much larger than that of the initial state.

Putting the last results back into the bound-free cross section (2.291), we obtain

$$\sigma_{\text{bf}} = \frac{256\pi}{3} \frac{e^2}{m c \omega} \frac{1}{(a_0 k_f)^5} = \frac{256\pi}{3} \frac{\alpha \hbar}{m \omega} \frac{1}{(a_0 k_f)^5}, \quad (2.296)$$

where we have averaged over all polarisation directions to replace

$$\langle \hat{e} \cdot \vec{k}_f \rangle^2 = \frac{1}{3} k_f^2. \quad (2.297)$$

Our previous approximation that the energy of the final state largely exceeds that of the initial state allows us to ignore the binding energy E_1 in (2.287) and to substitute

$$k_f = \sqrt{\frac{2m\omega}{\hbar}} \quad (2.298)$$

and bring the photoionisation cross section into the form

$$\sigma_{\text{bf}} = \frac{64\pi}{3} \frac{\alpha}{\sqrt{2}} \frac{1}{a_0^5} \left(\frac{\hbar}{m\omega} \right)^{7/2}. \quad (2.299)$$

Rearranging the constants, inserting the Bohr radius (2.273) in the form

$$a_0 = \frac{r_e}{\alpha^2} \quad (2.300)$$

with the classical electron radius (2.12) as well as the Rydberg energy

$$\text{Ry} = \frac{m e^4}{2\hbar^2} = \frac{\alpha^2 m c^2}{2} = 13.6 \text{ eV}, \quad (2.301)$$

we can bring the expression for the bound-free cross section into the more intuitive form

$$\sigma_{\text{bf}} = \left(\frac{4}{\alpha} \right)^3 \sigma_{\text{T}} \left(\frac{\text{Ry}}{\hbar\omega} \right)^{7/2} = 1.09 \cdot 10^{-16} \text{ cm}^2 \left(\frac{\text{Ry}}{\hbar\omega} \right)^{7/2} \quad (2.302)$$

containing the Thomson cross section (2.15). It should be kept in mind, however, that this equation is only valid for photon energies much larger than the Rydberg energy, $\hbar\omega \gg \text{Ry}$.

Problems

1. The cross section for a transition between an initial state $|n\rangle$ and a final state $|m\rangle$ was derived as

$$\sigma_{mn} = \frac{4\pi}{3c\hbar} \omega_{mn} \left| \vec{d}_{mn} \right|^2 \delta_{\text{D}}(\omega_{mn} - \omega), \quad (2.303)$$

where $\vec{d}_{mn} = \langle m | e \hat{x} | n \rangle$ is the dipole matrix element and $\omega_{mn} = (E_m - E_n)/\hbar$ is the frequency corresponding to the energy difference between the states $|m\rangle$ and $|n\rangle$. The delta distribution assures that only those photons contribute to the cross section that have the correct frequency.

?

Can we really integrate the radius to infinity in (2.295)? What is the crucial approximation behind doing so? Carry out the final radial integral in (2.295) yourself.

- (a) Consider the one-dimensional harmonic oscillator with energy levels $E_n = \hbar\omega(n + 1/2)$ and corresponding wave functions

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \quad (2.304)$$

with the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \quad (2.305)$$

Calculate the cross section σ_{10} for the transition from the ground state ($n = 0$) to the first excited state ($n = 1$). *Hint:* It may be helpful to use

$$\int_{-\infty}^{\infty} dx x^2 e^{-\alpha x^2} = - \int_{-\infty}^{\infty} dx \frac{\partial}{\partial \alpha} e^{-\alpha x^2}. \quad (2.306)$$

- (b) Consider now an infinitely deep potential well of length L with energy levels

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \quad (2.307)$$

with $n \in \mathbb{N}$ and wave functions

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi}{L}x\right) & \text{if } n \text{ is odd} \\ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) & \text{if } n \text{ is even} \end{cases}, \quad (2.308)$$

with $x \in [-L/2, L/2]$. What is the cross section σ_{21} for the transition from the ground state ($n = 1$) to the first excited state ($n = 2$)? Compare the factor in front of the delta distribution with that for the harmonic oscillator.

2.8 Shapes of Spectral Lines

In this section, three different statements on spectral lines are derived and applied. First, it is shown that spontaneous transitions between quantum states broaden spectral lines emitted by electromagnetic transitions between these states from the needle-sharp profile expected for ideally sharp transitions to a Lorentz profile whose width is determined by the spontaneous transition rate. The first main result is the Lorentzian profile function (2.318). Collisions between emitting quantum systems are shown to have the same effect, with the spontaneous transition rate replaced by the collision rate. Second, the Doppler broadening by the motion of the emitting quantum systems leads to the Gaussian line profile (2.330) if the motion is thermal. Third, the combined effects of spontaneous or collisional transitions and Doppler broadening are shown to create the Voigt line profile (2.337). This combined line profile is then used to determine how the equivalent widths of spectral lines change with the number of absorbers, leading to the curve-of-growth described by (2.352).

2.8.1 Natural line width

Consider now two states of a quantum-mechanical system, for simplicity called $|m\rangle$ and $|n\rangle$, which are separated by the energy difference $E_n - E_m > 0$. If the system is in the upper state $|n\rangle$, it has a finite probability to decay spontaneously to the lower state $|m\rangle$. The state $|n\rangle$ thus has a finite lifetime, which causes an uncertainty in its energy E_n . The transition energy between the two states $|m\rangle$ and $|n\rangle$ will thus be distributed around its precise value $E_n - E_m$. We shall now work out the shape of this distribution.

We begin with the evolution equation (2.235) for the expansion coefficients c_{nm} perturbed state $|\psi(t)\rangle$ in terms of the eigenstates $|k\rangle$ of an unperturbed Hamiltonian,

$$\dot{c}_{nm} = -\frac{i}{\hbar} \sum_k c_{nk} \langle m | \hat{H}^{(1)}(t) | k \rangle e^{i\omega_{mk}t} \quad (2.309)$$

and assume a radiative perturbation Hamiltonian $\hat{H}^{(1)}(t)$ with periodic time dependence as in (2.238),

$$\hat{H}^{(1)}(t) = \hat{V} e^{-i\omega t} \theta(t), \quad (2.310)$$

with a time-independent operator \hat{V} .

Let us now restrict our attention to a radiative transition between any two states $|n\rangle$ and $|m\rangle$. Their energies E_m and E_n are supposed to satisfy $E_m > E_n$, respectively. Initially, we assume the system to be in the state $|n\rangle$, which could be its ground state, and thus begin the evolution with $c_{nm} = 1$ and $c_{nk} = 0$ for $k \neq n$. Restricting our general result (2.309) to this simplified two-state system, the coefficient c_{nm} evolves in time according to

$$\dot{c}_{nm} = -\frac{i}{\hbar} \langle m | \hat{V} | n \rangle e^{i(\omega_{nm} - \omega)t}. \quad (2.311)$$

Strictly speaking, c_{nm} is also time dependent, so we would have to solve a system of coupled differential equations for c_{nm} and c_{nm} . In a first step of what could turn into an iterative approach, we now assume that the ground state remains populated as the transitions are going on, hence $c_{nm} = 1$ for all times. This is justified if the transition probability from $|n\rangle$ to $|m\rangle$ is small. Should this be unreasonable in the situation considered, a first solution for $c_{nm}(t)$ can then be inserted into the evolution equation for c_{nm} to determine a correction, and so forth.

Taking, however, $c_{nm} = 1$ for now, we can immediately solve (2.311) by direct integration, enforcing the initial condition $c_{nm} = 0$ at $t = 0$. This gives

$$c_{nm}(t) = -\frac{\langle m | \hat{V} | n \rangle}{\hbar(\omega_{nm} - \omega)} \left[e^{i(\omega_{nm} - \omega)t} - 1 \right], \quad (2.312)$$

as in (2.239).

However, this result has the problem that it was derived ignoring that the excited state $|m\rangle$ can decay spontaneously. Very much like radioactive decay, we can phenomenologically model such a spontaneous decay by the introducing a contribution

$$\dot{c}_{nm} \rightarrow \dot{c}_{nm} - \frac{\Gamma}{2} c_{nm} \quad (2.313)$$

?

Why would an excited state spontaneously decay into a less energetic state?

into the differential equation (2.311), where the spontaneous decay rate $\Gamma/2$ was inserted with a factor of 1/2 that will be convenient later. After this ad-hoc modification, c_{nm} is supposed to evolve according to

$$\dot{c}_{nm} = -\frac{i}{\hbar} \langle m | \hat{V} | n \rangle e^{i(\omega_{mn}-\omega)t} - \frac{\Gamma}{2} c_{nm} . \quad (2.314)$$

After bringing this additional term $\Gamma c_{nm}/2$ to the left-hand side and multiplying the equation with $e^{\Gamma t/2}$, we see that we can write

$$\partial_t (c_{nm} e^{\Gamma t/2}) = -\frac{i}{\hbar} \langle m | \hat{V} | n \rangle e^{[i(\omega_{mn}-\omega)+\Gamma/2]t} . \quad (2.315)$$

Again, we can directly integrate this equation with the same initial condition as before, $c_{nm} = 0$ at $t = 0$. This gives

$$c_{nm}(t) = \frac{\langle m | \hat{V} | n \rangle}{\hbar} \frac{e^{-\Gamma t/2} - e^{i(\omega_{mn}-\omega)t}}{(\omega - \omega_{mn}) + i\Gamma/2} . \quad (2.316)$$

After a sufficiently long initial time $t \gg \Gamma^{-1}$, the exponential term in the numerator dies off. Then, the absolute square of c_{nm} , which gives the probability for finding the system in state $|m\rangle$, becomes time-independent and reads

$$|c_{nm}|^2 = \frac{|\langle m | \hat{V} | n \rangle|^2}{\hbar^2} \frac{1}{(\omega - \omega_{mn})^2 + \Gamma^2/4} . \quad (2.317)$$

The dependence of the transition probability on frequency is thus described by the Lorentz profile function

$$\phi_{\Gamma}(\omega - \omega_{12}) = \frac{1}{\pi} \frac{\Gamma/2}{(\omega - \omega_{12})^2 + \Gamma^2/4} \quad (2.318)$$

first encountered in (2.153). Recall that the prefactor in (2.318) is chosen such that ϕ_{Γ} integrates to unity.

2.8.2 Collisional broadening

When a quantum-mechanical system interacts with another in a collision, its phase is randomly changed, or reset. We model this process by assuming that there is a random phase shift $\delta\phi$ in each collision, which we choose to be drawn from the interval $[-\pi, \pi]$. The probability distribution of $\delta\phi$ within this interval is supposed to be flat such that all phase shifts within $[-\pi, \pi]$ are equally likely.

We cannot know the phase shift after a single collision. However, the average phase factor after a single collision must vanish,

$$\langle e^{i\delta\phi} \rangle = 0 , \quad (2.319)$$

because of the flat distribution of $\delta\phi \in [-\pi, \pi]$. If more than one collision occurs, the mean phase factor will still vanish: The phase shift after N collisions will have a flat distribution over the interval $[-N\pi, N\pi]$, hence the average phase factor will vanish also if an arbitrary number of collisions has occurred.

The number of collisions within a given time t can be modelled as a Poisson process. Let Γ_c be the collision rate. Then the expected number of collisions

within the time t is $\Gamma_c t$, and the probability for k collisions to occur during that time is given by the Poisson distribution,

$$p_k = \frac{(\Gamma_c t)^k}{k!} e^{-\Gamma_c t}. \quad (2.320)$$

Since the phase factor after time t is zero if *any* collision has happened, the mean phase factor will be

$$\langle e^{i\delta\phi} \rangle = p_0 = e^{-\Gamma_c t}. \quad (2.321)$$

Since the probability for the system to be in state $|2\rangle$ is given by $|c_{nm}|^2$, this corresponds to modifying the evolution equation for c_{nm} by a term

$$\dot{c}_{nm} = -\frac{\Gamma_c}{2} c_{nm}. \quad (2.322)$$

A comparison to the treatment of the natural line width above, see (2.314) and (2.315), shows that the only change to the previous solution (2.316) for the transition probability $|c_{nm}|^2$ is that the decay rate Γ for spontaneous transitions is replaced by the sum of the spontaneous and the collisional decay rates,

$$\Gamma \rightarrow \Gamma + \Gamma_c. \quad (2.323)$$

The shape of the line profile function (2.318) will thus remain unchanged, only its width will be enhanced by an amount determined by the sum of the spontaneous and the collisional decay rates.

2.8.3 Doppler broadening of spectral lines

A further broadening mechanism is caused by the Doppler effect. If the emitting quantum-mechanical systems, e.g. atoms or molecules, move along the line-of-sight with the velocity v_{\parallel} , we observe the frequency

$$\omega = \omega_0 \left(1 + \frac{v_{\parallel}}{c} \right) \quad (2.324)$$

instead of the emitted frequency ω_0 . This is the non-relativistic approximation to the Doppler effect, which we can safely use for atoms or molecules moving thermally. In the thermal case, the emitters can further be expected to have a Maxwellian velocity distribution with a width σ_v determined by their temperature. The equipartition theorem demands

$$\frac{m}{2} \sigma_v^2 = \frac{kT}{2} \Rightarrow \sigma_v^2 = \frac{kT}{m}. \quad (2.325)$$

The single velocity component v_{\parallel} then has a Gaussian distribution, and the observed line profile is then given by

$$\int_{-\infty}^{\infty} \frac{dv_{\parallel}}{\sqrt{2\pi\sigma_v^2}} \delta_D \left[\omega - \omega_0 \left(1 + \frac{v_{\parallel}}{c} \right) \right] \exp \left[-\frac{(v_{\parallel} - \bar{v})^2}{2\sigma_v^2} \right], \quad (2.326)$$

where \bar{v} is the mean velocity of the emitting system. Using the identity

$$\delta_D(ax) = \frac{1}{a} \delta_D(x), \quad (2.327)$$

?

If the velocity distribution of the emitting atoms and molecules would follow a power law, what would the line profile look like?

for the Dirac delta function, the Gaussian line profile proportional to

$$\exp\left[-\frac{1}{2\sigma_v^2}\left(\frac{\omega - \omega_0}{\omega_0}c - \bar{v}\right)^2\right] = \exp\left[-\frac{c^2}{2\sigma_v^2}\left(\frac{\omega - \bar{\omega}}{\omega_0}\right)^2\right] \quad (2.328)$$

emerges. Here, we have defined the centre frequency

$$\bar{\omega} \equiv \omega_0 \left(1 + \frac{\bar{v}}{c}\right), \quad (2.329)$$

i.e. the average frequency of the line emission, shifted by the Doppler effect due to the mean motion of the emitting or absorbing medium. The line-profile function $\phi(\omega - \omega_0)$ for thermally moving atoms is thus the (normalised) Gaussian

$$\phi(\omega - \omega_0) = \frac{c}{\omega_0 \sqrt{2\pi\sigma_v^2}} \exp\left[-\frac{c^2}{2\sigma_v^2}\left(\frac{\omega - \bar{\omega}}{\omega_0}\right)^2\right]. \quad (2.330)$$

2.8.4 The Voigt profile

In presence of all three line-broadening effects, i.e. spontaneous, collisional and Doppler broadening, the line profile is a convolution of the Lorentz profile for the line broadened by spontaneous and collisional decays with the Gaussian velocity distribution taking account of the Doppler effect. The combined line profile is thus determined by the integral

$$\frac{1}{\sqrt{2\pi}\sigma_v} \int_{-\infty}^{\infty} dv_{\parallel} \phi\left[\omega - \omega_{12}\left(1 + \frac{v_{\parallel}}{c}\right)\right] \exp\left(-\frac{v_{\parallel}^2}{2\sigma_v^2}\right), \quad (2.331)$$

which can be brought into a standard form by a sequence of substitutions. First, we introduce a velocity scale v_0 and a dimension-less velocity q by

$$v_0 \equiv \sqrt{2}\sigma_v \quad \text{and} \quad q \equiv \frac{v_{\parallel}}{\sqrt{2}\sigma_v} = \frac{v_{\parallel}}{v_0} \quad (2.332)$$

to bring the Gaussian factor in (2.331) into the form

$$\frac{dv_{\parallel}}{\sqrt{2\pi}\sigma_v} \exp\left(-\frac{v_{\parallel}^2}{2\sigma_v^2}\right) = \frac{dq}{\sqrt{\pi}} e^{-q^2}. \quad (2.333)$$

The Lorentz profile (2.318), with the centre frequency shifted by the Doppler effect, reads

$$\phi_{\Gamma}\left[\omega - \omega_{12}\left(1 + \frac{v_{\parallel}}{c}\right)\right] = \frac{1}{\pi} \frac{\Gamma/2}{\left[\omega - \omega_{12}\left(1 + \frac{v_{\parallel}}{c}\right)\right]^2 + \Gamma^2/4}. \quad (2.334)$$

The further substitutions of a centred, normalised frequency u and a normalised collision rate a , defined by

$$u \equiv \frac{\omega - \omega_{12}}{\omega_{12}} \frac{c}{v_0} \quad \text{and} \quad a \equiv \frac{\Gamma}{2\omega_{12}} \frac{c}{v_0}, \quad (2.335)$$

bring this profile into the form

$$\phi(u) = \frac{c}{\pi\omega_{12}v_0} \frac{a}{(u - q)^2 + a^2}. \quad (2.336)$$

The result of the convolution (2.331) thus reads

$$\phi(u) = \frac{ac}{\pi \sqrt{\pi} \omega_{12} \nu_0} \int_{-\infty}^{\infty} dq \frac{e^{-q^2}}{(u-q)^2 + a^2}, \quad (2.337)$$

which is the so-called the Voigt profile (Figure 2.15). Near its centre, this line profile has a Gaussian shape, while its wings retain the Lorentzian shape.

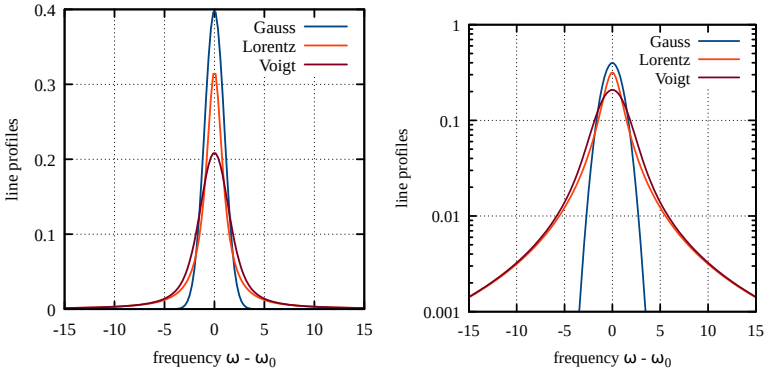


Figure 2.15 The Gauss, the Lorentz and the Voigt profiles are shown for $\sigma = 1$ and $\Gamma = 1$ in arbitrary units. The left and right panels are distinguished only by the linear and logarithmic scaling of the ordinate. The right panel illustrates in particular the broad, Lorentzian wings of the Voigt profile.

2.8.5 Equivalent widths and curves-of-growth

Two concepts have been found useful describing the information contained in observed spectral lines, namely their equivalent width and their curve-of-growth. The equivalent width quantifies the area under a spectral line. If I_0 is the local specific intensity of the spectral continuum, that is the continuum intensity in the vicinity of the line, the equivalent width is defined as

$$W \equiv \int \frac{I_0 - I(\omega)}{I_0} d\omega, \quad (2.338)$$

where $I(\omega)$ is the specific intensity within the line. Thus, the equivalent width of an absorption line is a measure for the total intensity removed from the spectrum. An analogous definition can be given for the equivalent width of emission lines, which then quantifies the total intensity added to the spectrum. The optical depth within the line is given by the number density of absorbers n , the geometrical extent L of the absorbing medium and the frequency-dependent cross section $\sigma(\omega)$,

$$\tau = n L \sigma(\omega), \quad (2.339)$$

where the specific dependence of τ on the frequency may be given by (2.279) in dipole approximation. The specific intensity within the line is then lowered compared to the specific continuum intensity I_0 by

$$I(\omega) = I_0 e^{-\tau(\omega)}, \quad (2.340)$$

and thus the equivalent width is the integral

$$W = \int d\omega [1 - e^{-\tau(\omega)}] \quad (2.341)$$

across the line.

Since the cross section is proportional to the profile function $\phi(\omega)$, (2.341) can equally well be written as

$$W = \int d\omega [1 - e^{-C\phi(\omega)}] . \quad (2.342)$$

As shown in (2.280), the frequency-independent constant C inserted here is $C = 2\pi n L r_e c f_{12}$ for a dipole transition between levels 2 and 1 with oscillator strength f_{12} . For low optical depth, $\tau \ll 1$, the exponential function in (2.341) or (2.342) can be replaced by its first-order Taylor expansion. This results in

$$W = \int d\omega n L \sigma(\omega) = 2\pi n L r_e c f_{12} \quad (2.343)$$

because the profile function is defined to be normalised such that its integral over frequency ω gives unity. Thus, for low optical depth, we introduce the column density $N = nL$ and have

$$W \propto N , \quad (2.344)$$

i.e. the equivalent width is simply growing linearly with the column density of absorbers along the line-of-sight from the observer.

In the opposite, optically thick case $\tau \gg 1$, the line profile can be approximated by a sudden drop from the continuum level I_0 to zero intensity within a frequency range of width 2Δ , and a sudden rise back to the continuum level. The spectral line is thus simply described as a rectangular stripe cut from the spectrum. By definition of the equivalent width,

$$W \approx 2\Delta \quad (2.345)$$

in this case. We now need to distinguish whether $\tau \approx 1$ is reached only in the core or already in the wings of the spectral-line profile. We first consider the case of an optically thick core, but optically thin wings. If the line is Doppler-broadened, as most lines are, the line profile has a Gaussian core, and we can approximate the optical depth as

$$\tau = NL\sigma(\omega) = 2\pi NL \frac{r_e c^2 f_{12}}{\sqrt{2\pi}\sigma_v} \exp\left(-\frac{c^2(\omega - \omega_{12})^2}{2\sigma_v^2}\right) \quad (2.346)$$

with a thermal velocity dispersion σ_v given by (2.325). We now determine the half-width $\Delta = \omega - \omega_{12}$ by setting $\tau = 1$ in (2.346) and solving for Δ ,

$$\exp\left(-\frac{c^2\Delta^2}{2\sigma_v^2}\right) \stackrel{!}{=} \frac{\sigma_v}{\pi \sqrt{2\pi} N L r_e c^2 f_{12}} . \quad (2.347)$$

Thus, the width Δ and therefore also the equivalent width scale with N like

$$\Delta \propto \sqrt{\ln N} , \quad W \propto \sqrt{\ln N} . \quad (2.348)$$

?

Why would the specific intensity fall off exponentially with the optical depth, as in (2.340)?

i.e. they depend only very weakly on the number N of absorbers.

If, however, $\tau \approx 1$ is reached already in the Lorentzian wings of the line, we adopt the Lorentz profile (2.318) instead of the Voigt profile and further simplify the Lorentz profile by assuming a damping rate Γ small compared to the frequency difference to the line centre, $\Gamma \ll \omega - \omega_{12}$. Then,

$$\phi_{\Gamma}(\omega) \approx \frac{\Gamma}{2\pi(\omega - \omega_{12})^2}, \quad (2.349)$$

and the optical depth becomes

$$\tau = NL\sigma(\omega) \approx \pi NLr_e c f_{12} \frac{\Gamma}{(\omega - \omega_{12})^2}. \quad (2.350)$$

As above, we find the width Δ from this equation by setting $\tau = 1$ in (2.350) and solving for $\Delta = \omega - \omega_{12}$. This reveals that in this case of very high optical depth, Δ and the equivalent width W scale with the number N of absorbers like

$$\Delta \propto \sqrt{N}, \quad W \propto \sqrt{N}. \quad (2.351)$$

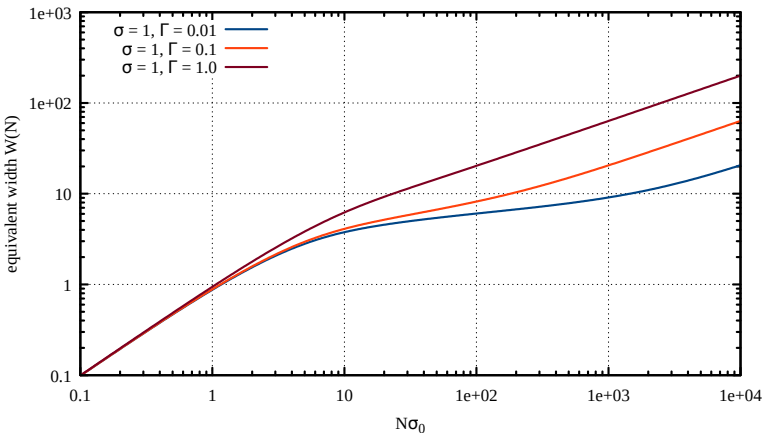


Figure 2.16 Curves-of-growth as a function of $N\sigma_0$ for three different line profiles with the same Gaussian broadening, $\sigma = 1$, but different values for the damping Γ .

Summarising, the curve-of-growth $W(N)$ behaves as

$$W(N) \propto \begin{cases} N & \text{small } N \\ \sqrt{\ln N} & \text{intermediate } N \\ \sqrt{N} & \text{large } N \end{cases}. \quad (2.352)$$

For determining the number N of absorbers, lines with different oscillator strengths f are required because then the spectral lines fall into different sections of the curve-of-growth $W(N)$ for the same N . This may prove difficult when some lines fall into the flat section of $W(N)$ where $W(N) \propto \sqrt{\ln N}$.

Problems

1. Besides their natural line width, emission lines with transition frequency ω_0 are broadened due to collisions of the emitting atoms and their thermal velocities. The collisional broadening leads to a line shape that is described by a Lorentz profile

$$\phi_{\Gamma_c}(\omega - \omega_{12}) = \frac{1}{\pi} \frac{\Gamma_c/2}{(\omega - \omega_{12})^2 + \Gamma_c^2/4}, \quad (2.353)$$

where $\Gamma_c = \sigma \langle nv \rangle$ is the collision rate, σ is the cross section for collisions, n is the number density of atoms, v their velocity and $\langle \cdot \rangle$ indicates the thermal average. The Doppler broadening leads to the Gaussian profile function

$$\phi_D = \frac{c}{\sqrt{2\pi}\omega_0\sigma_v} \exp\left[-\frac{c^2}{2\sigma_v^2} \left(\frac{\omega - \omega_0}{\omega_0}\right)^2\right], \quad (2.354)$$

where σ_v is the velocity dispersion.

- (a) Estimate the line width for Doppler broadening from the full width at half maximum (FWHM) $\Delta\omega_D$, defined by $\phi_D(\omega_0 \pm \Delta\omega_D/2) = \phi_D(\omega_0)/2$, as a function of temperature T .
- (b) Estimate the line width $\Delta\omega_c$ due to collisions from the FWHM of $\phi_c(\omega)$ as a function of T . Assume that σ is set by the Bohr radius a_0 and that the density does not depend on temperature.
- (c) How can the results from (a) and (b) be combined to determine the density of an emitting medium?
- (d) Calculate the ratio $\Delta\omega_c/\Delta\omega_D$ for the H α line (6563 Å) emitted from a cloud of atomic hydrogen with $n = 16 \text{ cm}^{-3}$.

2.9 Radiation Quantities

In our treatment of radiation processes, we began with the classical picture of electromagnetic waves and their emission by accelerated charges. We added the photon picture when it became necessary for the treatment of momentum exchange between electromagnetic radiation and charges, and discussed quantum transitions caused by radiation. We shall proceed to discuss in this section the propagation and the transport of radiation, treating radiation in close analogy to a fluid. The main results are the definition of the specific intensity I_ω in (2.360), the angular moments (2.370) of the intensity and the demonstration that the quantity I_ω/ω^3 is relativistically invariant.

2.9.1 Specific Intensity

Let us therefore consider radiation again as a stream of particles which carry energy and momentum. In order to characterise the flow of radiation, we imagine setting up a small screen of differential area $d\vec{A}$ and arbitrary orientation. Our first question is: What amount of energy is streaming per time interval dt

into a direction enclosing the angle θ with the normal to the screen into the solid angle element $d\Omega$ and within the frequency interval $d\omega$?

We begin with the occupation number of photon states. Let $n_{\alpha\vec{p}}$ be the spatial number density of photons with momentum \vec{p} and the polarisation state α ($\alpha = 1, 2$). The energy-momentum four-vector of a photon with energy $\hbar\omega$ is

$$p^\mu = \hbar k^\mu = \frac{\hbar\omega}{c} \begin{pmatrix} 1 \\ \hat{e} \end{pmatrix}, \tag{2.355}$$

with the unit vector \hat{e} pointing into the direction of light propagation. Since photons are massless particles, the wave four-vector and hence also the four-momentum are null vectors, $\langle k, k \rangle = 0 = \langle p, p \rangle$. Therefore,

$$E = cp \quad \text{with} \quad \vec{p} = \frac{\hbar\omega}{c} \hat{e}, \quad p = \frac{\hbar\omega}{c}. \tag{2.356}$$

A volume element $d\Gamma = d^3x d^3p$ of phase space is divided in cells of size $(2\pi\hbar)^3$ to account for Heisenberg's uncertainty principle: If the position of a particle is confined to dx in one spatial direction, its momentum in the same direction cannot be confined to better than $dx dp = 2\pi\hbar$. The number of cells per phase-space volume element $d\Gamma$ is thus

$$\frac{d^3x d^3p}{(2\pi\hbar)^3} = dV \frac{p^2 dp d\Omega}{(2\pi\hbar)^3} = dV \frac{\omega^2 d\omega d\Omega}{(2\pi c)^3}, \tag{2.357}$$

where we have expressed the momentum by the frequency ω in the last step.

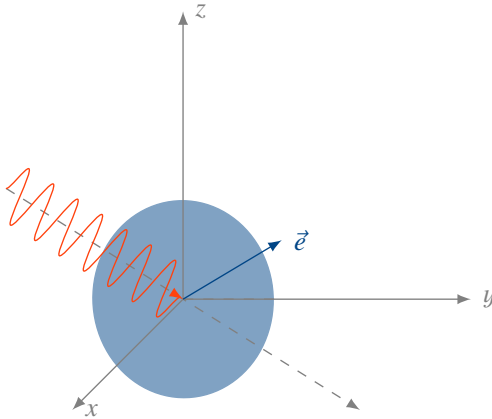


Figure 2.17 Illustration of photons streaming through an inclined area element.

The amount of energy carried by photons with momentum \vec{p} through the infinitesimal screen $d\vec{A}$ (Figure 2.17) is now given by the number of available phase space cells from (2.357), times the number of photons per phase-space cell with polarisation state α and momentum \vec{p} , times the energy $E = cp = \hbar\omega$ per photon, times the volume $dV = c dt d\vec{A} \cdot \hat{e}$ covered by the screen relative to the stream of photons. Thus, we find

$$dE = \frac{\omega^2 d\omega d\Omega}{(2\pi c)^3} \sum_{\alpha=1}^2 n_{\alpha\vec{p}} \hbar\omega dA \cos \theta c dt, \tag{2.358}$$

?

Compare (2.355) with the dispersion relation for electromagnetic waves, and (2.356) with the relativistic energy-momentum relation.

where θ is the angle between $d\vec{A}$ and \vec{p} . The energy flowing through the screen per unit screen area dA , unit time dt and unit frequency $d\omega$ into the unit solid angle $d\Omega$ defines the specific intensity I_ω of the radiation by the assignment

$$\frac{dE}{dt d\omega dA d\Omega} = \sum_{\alpha=1}^2 n_{\alpha\vec{p}} \frac{\hbar\omega^3}{(2\pi)^3 c^2} \cos\theta \equiv I_\omega \cos\theta. \quad (2.359)$$

For unpolarised light, $n_{1\vec{p}} = n_{2\vec{p}}$, so the sum in (2.359) merely gives a factor of two. Then, the specific intensity is related to the occupation number and the frequency by

$$I_\omega = \frac{2\hbar\omega^3}{(2\pi)^3 c^2} n_{\alpha\vec{p}} = \frac{\hbar\omega^3}{4\pi^3 c^2} n_{\alpha\vec{p}}. \quad (2.360)$$

Two powers of ω in the numerator are due to the volume element in phase space, the additional factor $\hbar\omega$ is the photon energy.

2.9.2 Moments of the intensity

Let us approach the intensity from a different point of view. For an electromagnetic wave in vacuum, the Poynting vector is

$$\vec{S} = \frac{c}{4\pi} \vec{E}^2 \hat{k} = cU\hat{k}, \quad (2.361)$$

where U is the energy density. It is the energy current density in electromagnetic radiation, i.e. the electromagnetic energy flowing per unit time through unit area. Dividing by the solid angle 4π of the sphere, we find the intensity

$$I = \frac{cU}{4\pi} = \frac{1}{4\pi} |\vec{S}| \quad (2.362)$$

and its relation to the absolute magnitude of the Poynting vector. The first equation (2.362) shows that the integral of I/c over the solid angle is the energy density,

$$\int d\Omega \frac{I}{c} = U. \quad (2.363)$$

According to the second equation (2.362), we can write the Poynting vector as $\vec{S} = 4\pi I \hat{k}$. Its integral over a sphere with arbitrary (small) radius R ,

$$\int \vec{S} \cdot d\vec{A} = 4\pi \int d\Omega I \hat{k} \cdot \hat{e}_r R^2 = 4\pi R^2 \int d\Omega I \cos\theta \quad (2.364)$$

must be the energy flowing per unit time through the sphere. Dividing by the surface area of the sphere, we find the total energy current density

$$F = \int d\Omega I \cos\theta \quad (2.365)$$

averaged over the complete solid angle.

Maxwell's stress-energy tensor \vec{T} , whose components are given in (1.111), express the momentum current density. Multiplied with an oriented area element $d\vec{A}$, we find the force $d\vec{F} = \vec{T} d\vec{A}$ exerted per area dA by the momentum current density since the momentum current density times an area is the momentum per

unit time, hence the force. Without loss of generality, we choose $d\vec{A} = dA\hat{e}_z$ for an area element in the x - y plane. Since the magnetic contribution to \vec{T} equals the electric contribution, we write

$$d\vec{F} = \vec{T} d\vec{A} = \frac{1}{2\pi} \left(\frac{\vec{E}^2}{2} \mathbb{1}_3 - \vec{E} \otimes \vec{E} \right) dA \hat{e}_z = \frac{1}{2\pi} \left(\frac{\vec{E}^2}{2} \hat{e}_z - E_3 \vec{E} \right) dA. \quad (2.366)$$

It is easily seen that $dF_{1,2} = 0$: There is no net force on the area element in its own plane, as expected. For dF_3 , we rather have the force per unit area, or the radiation pressure,

$$P_{\text{rad}} = \frac{dF_3}{dA} = \frac{1}{2\pi} \left(\frac{\vec{E}^2}{2} - E_3^2 \right). \quad (2.367)$$

Averaging over the solid angle, using $\langle \vec{E}^2 \rangle = 3\langle E_3^2 \rangle$ for a locally isotropic radiation field, writing $E_3 = E \cos \theta$, and replacing $\vec{E}^2 = 4\pi U$, we can conclude

$$P_{\text{rad}} = \frac{1}{4\pi} \int d\Omega U \cos^2 \theta = \frac{1}{4\pi} \int d\Omega U \cos^2 \theta = \frac{U}{3}. \quad (2.368)$$

On the one hand, this confirms the well-known result valid for all relativistic boson gases that their pressure equals a third of their energy density. On the other hand, we can substitute the intensity I from (2.362) in the first equation (2.368) to see that the radiation pressure is the second angular moment of I/c ,

$$P_{\text{rad}} = \int d\Omega \frac{I}{c} \cos^2 \theta. \quad (2.369)$$

We have thus established the relations

$$\int d\Omega \frac{I}{c} = U, \quad F = \int d\Omega I \cos \theta, \quad P_{\text{rad}} = \int d\Omega \frac{I}{c} \cos^2 \theta \quad (2.370)$$

between the energy density U , the integrated energy current density F and the radiation pressure P_{rad} with the three lowest-order angular moments of the intensity. They will turn out to be important shortly in our discussion of radiation transport.

2.9.3 Relativistic invariance of I_ω/ω^3

Suppose now that the screen $d\vec{A}$ is fixed at the origin of an unprimed coordinate frame such that it points into the \hat{e}_z direction. Let it be observed from another, primed, frame moving with velocity v into the common \hat{e}_z direction of the two frames. For simplicity, clocks are supposed to be synchronised such that $t = 0 = t'$ when the two frames coincide. An experimentalist resting in the unprimed frame finds by counting that

$$dN = d\Gamma n_{\vec{p}} = 2 \frac{p^2 dp d\Omega}{(2\pi\hbar)^3} n_{\vec{p}} dA c \cos \theta dt \quad (2.371)$$

photons have passed the screen after a time interval dt . A fellow experimentalist resting in the primed frame counts

$$dN' = 2 \frac{p'^2 dp' d\Omega'}{(2\pi\hbar)^3} n'_{\vec{p}'} dA' (c \cos \theta' - v) dt' \quad (2.372)$$

photons in the time interval dt' measured on his clock. If the two experimentalists agree to synchronise the durations of their measurements by

$$dt' = \gamma dt \quad (2.373)$$

to account for the relativistic time dilation, they must count the same number of photons, $dN = dN'$. In order to see what this implies for the occupation numbers $n_{\vec{p}}$ in the unprimed and $n'_{\vec{p}'}$ in the primed frame, we need to Lorentz transform the absolute value p of the photon momentum, the solid-angle element $d\Omega$ and the direction cosine $\cos \theta$. The area elements are unchanged, $dA = dA'$, for they are perpendicular to the direction of the relative motion of the two frames.

We have seen in (1.45) and (1.47) that angles and solid angles change like

$$\cos \theta' = \frac{\beta + \cos \theta}{1 + \beta \cos \theta} \quad \text{and} \quad d\Omega' = \frac{d\Omega}{\gamma^2 (1 + \beta \cos \theta)^2} \quad (2.374)$$

under Lorentz transforms. The absolute value of the momentum is $p = E/c$, as shown in (2.356), and thus transforms like the zero component of a four-vector,

$$p' = p^0 = \gamma(p^0 + \beta p^3) = \gamma(p + \beta \cos \theta p) = \gamma p (1 + \beta \cos \theta). \quad (2.375)$$

We now insert the primed quantities into (2.372) to find

$$\begin{aligned} dN' &= 2 \frac{p^2 dp d\Omega}{(2\pi\hbar)^3} \frac{[\gamma(1 + \beta \cos \theta)]^3}{\gamma^2 (1 + \beta \cos \theta)^2} n'_{p'} dA c \left(\frac{\beta + \cos \theta}{1 + \beta \cos \theta} - \beta \right) \gamma dt \\ &= 2 \frac{p^2 dp d\Omega}{(2\pi\hbar)^3} \gamma^2 (1 + \beta \cos \theta) n'_{p'} dA c \cos \theta \frac{1 - \beta^2}{1 + \beta \cos \theta} dt \\ &= 2 \frac{p^2 dp d\Omega}{(2\pi\hbar)^3} n'_{p'} dA c \cos \theta dt \end{aligned} \quad (2.376)$$

for the number of photons counted by the experimentalist resting in the primed frame. This agrees with dN from (2.371) if, and only if, the occupation numbers transform as $n_{\vec{p}} = n'_{\vec{p}'}$. With (2.360), this implies the important result that the specific intensity divided by ω^3 is invariant,

$$\frac{I_\omega}{\omega^3} = \frac{I'_\omega}{\omega'^3}. \quad (2.377)$$

2.10 The Planck spectrum and Einstein coefficients

In this section, the Planck spectrum is derived from first principles, i.e. from the grand-canonical partition sum of a photon gas in thermal equilibrium with a heat bath of given temperature. The first main result is the specific intensity (2.396) as a function of frequency at given temperature of thermal (black-body) radiation. Then, the Einstein coefficients for absorption, stimulated and spontaneous transition are introduced. The relations (2.425) between them required by the Planck spectrum are derived, showing that spontaneous transitions are necessary and that the rates of stimulated emission and absorption must be equal.

Example: The dipole of the Cosmic Microwave Background

To give an example, let us study an instructive consequence of the relativistic invariance of I_ω/ω^3 . In its rest frame, the Cosmic Microwave Background (CMB) is an isotropic radiation field with a Planck spectrum. The occupation number is thus given by

$$n_p = \left[\exp\left(\frac{\hbar\omega}{kT_{\text{CMB}}}\right) - 1 \right]^{-1}, \quad (2.378)$$

where T_{CMB} is the CMB temperature. The energy of a photon measured by an observer moving with a four-velocity u with respect to the rest frame of the radiation is the (negative) projection of the photon's four-momentum on the four-velocity,

$$E = -\langle p, u \rangle. \quad (2.379)$$

This is quickly verified for an observer at rest in the rest frame of the radiation, who has $u^\mu = (c, 0)^T$ there. With p^μ from (2.355), the projection (2.379) is

$$-\langle p, u \rangle = -p^\mu u_\mu = \hbar\omega \quad (2.380)$$

for this observer, as it should be. An observer moving instead with $u' = \gamma(c, \vec{v})^T = \gamma c(1, \vec{\beta})^T$ relative to the rest frame of the radiation, however, measures the photon energy

$$E' = -\langle p, u' \rangle = \hbar\omega\gamma(1 - \vec{\beta} \cdot \hat{e}) = E\gamma(1 - \vec{\beta} \cdot \hat{e}). \quad (2.381)$$

This is the relativistic Doppler shift: The moving observer measures a relative energy change of

$$\frac{E' - E}{E} = \gamma(1 - \beta \cos \theta) \quad (2.382)$$

compared to the observer at rest. For $\theta = 0$, this result simplifies to

$$\frac{E' - E}{E} = \frac{1 - \beta}{\sqrt{1 - \beta^2}} = \sqrt{\frac{1 - \beta}{1 + \beta}} \approx 1 - \beta, \quad (2.383)$$

where the approximation in the final step is valid for $\beta \ll 1$.

Returning to the CMB, the moving observer sees the occupation number

$$n'_{p'} = \left[\exp\left(\frac{\hbar\omega'}{kT'_{\text{CMB}}}\right) - 1 \right]^{-1} = \left[\exp\left(\frac{\hbar\omega\gamma(1 - \beta \cos \theta)}{kT'_{\text{CMB}}}\right) - 1 \right]^{-1}, \quad (2.384)$$

which must be the same as n_p from (2.378). This can only be achieved if the moving observer sees a direction-dependent temperature

$$T'_{\text{CMB}} = T_{\text{CMB}}\gamma(1 - \beta \cos \theta) \approx T_{\text{CMB}}\left(1 - \frac{v}{c} \cos \theta\right), \quad (2.385)$$

where the approximation is again valid for non-relativistic motion, $v \ll c$. The motion of the Earth relative to the rest frame of the CMB thus imprints a dipolar pattern on the measured CMB temperature (Figure 2.18). With $\beta \approx 10^{-3}$ and $T_{\text{CMB}} \approx 3$ K, the amplitude of this temperature dipole is of milli-Kelvin order. ◀

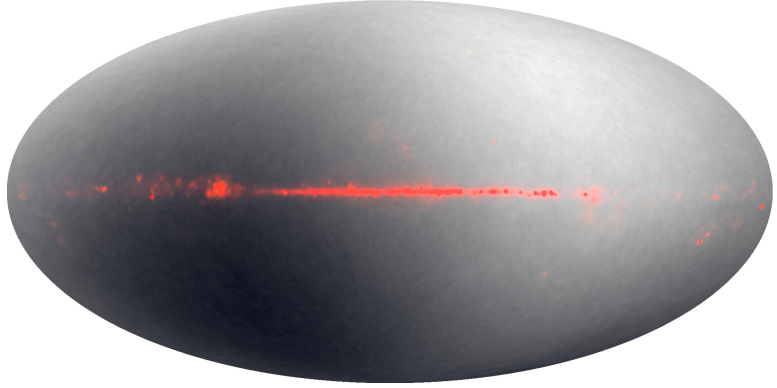


Figure 2.18 The gray-scale image shows the dipole of the Cosmic Microwave Background, measured by the Wilkinson Microwave Anisotropy Probe. In red, the emission from the Galactic disk is shown. (Provided by the WMAP Science Team)

2.10.1 The Planck spectrum

We begin by recalling some general results from statistical physics. Suppose we have an ensemble of quantum states whose occupation is in equilibrium with a heat bath of temperature T . For convenience, we shall express the temperature by the inverse thermal energy β below, $\beta = (k_B T)^{-1}$. Let these states be labelled by an abstract index α which may be composed of various quantum numbers, as appropriate for the system at hand. The energies of these quantum states are called ε_α , and the quantum states are occupied n_α times. If the total number

$$N = \sum_{\alpha} n_{\alpha} \quad (2.386)$$

of occupied states is unspecified, the ensemble has the grand-canonical partition sum

$$Z_{GC} = \sum_{N=0}^{\infty} e^{\beta \mu N} \sum_{\{n_{\alpha}\}} \exp\left(-\beta \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha}\right), \quad (2.387)$$

where the summation over $\{n_{\alpha}\}$ is meant to indicate that the set $\{n_{\alpha}\}$ must obey condition (2.386). The chemical potential μ is the energy required to change the occupation number by unity. With (2.386), the partition sum (2.387) can be written

$$Z_{GC} = \sum_{N=0}^{\infty} \sum_{\{n_{\alpha}\}} \exp\left[-\beta \sum_{\alpha} (\varepsilon_{\alpha} - \mu) n_{\alpha}\right] = \sum_{n_{\alpha}} \exp\left[-\beta \sum_{\alpha} (\varepsilon_{\alpha} - \mu) n_{\alpha}\right], \quad (2.388)$$

where the decisive last step was possible because the sum over n_{α} , constrained by the fixed total occupation number N and followed by a sum over all possible values of N , amounts to an unconstrained sum over n_{α} . The sum in the exponential translates to a product, and we find

$$Z_{GC} = \prod_{\alpha} Z_{\alpha}, \quad Z_{\alpha} = \sum_{n_{\alpha}} e^{-\beta(\varepsilon_{\alpha} - \mu)n_{\alpha}}. \quad (2.389)$$

For Fermi-Dirac systems, $n_\alpha \in \{0, 1\}$, while $n_\alpha \in [0, \infty)$ for Bose-Einstein systems. Thus,

$$Z_\alpha^{\text{FD}} = 1 + e^{-\beta(\varepsilon_\alpha - \mu)}, \quad Z_\alpha^{\text{BE}} = \frac{1}{1 - e^{-\beta(\varepsilon_\alpha - \mu)}}, \quad (2.390)$$

where we have carried out a geometrical series for the Bose-Einstein case. The mean occupation numbers are

$$\bar{n}_\alpha = \frac{1}{Z_\alpha} \sum_\alpha n_\alpha e^{-\beta(\varepsilon_\alpha - \mu)n_\alpha} = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln Z_\alpha. \quad (2.391)$$

Applying this to (2.390), we find

$$\bar{n}_\alpha^{\text{FD}} = \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)} + 1}, \quad \bar{n}_\alpha^{\text{BE}} = \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)} - 1}. \quad (2.392)$$

For a free photon gas, $\mu = 0$ because photons can spontaneously be created or destroyed. If we label photon energies ε_α by their momentum, $\varepsilon_\alpha = cp$. The energy density contained per unit photon momentum in the photon gas is then

$$dU_p = 2 \cdot \frac{4\pi p^2 dp}{(2\pi\hbar)^3} \cdot cp \cdot \bar{n}_p^{\text{BE}} = \frac{c}{\pi^2 \hbar^3} \frac{p^3 dp}{e^{\beta cp} - 1}, \quad (2.393)$$

where the factor of two accounts for the two polarisation states of each photon. Substituting the momentum p by the frequency ω through

$$E = \hbar\omega = cp, \quad (2.394)$$

we find the spectral energy density

$$\frac{dU_\omega}{d\omega} = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}. \quad (2.395)$$

Multiplying with $c/(4\pi)$ according to the definition of the specific intensity in (2.360), we find the Planck spectrum

$$I_\omega = \frac{c}{4\pi} \frac{dU_\omega}{d\omega} =: B_\omega(T) = \frac{\hbar}{4\pi^3 c^2} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}. \quad (2.396)$$

This is often expressed in terms of the frequency $\nu = \omega/(2\pi)$, for which we obtain

$$B_\nu(T) = \frac{c}{4\pi} \frac{dU_\nu}{d\nu} = \frac{2h}{c^2} \frac{\nu^3}{e^{\beta h\nu} - 1}. \quad (2.397)$$

The Planck spectrum has the characteristic frequency

$$\omega_0 = \frac{k_B T}{\hbar}, \quad \nu_0 = \frac{k_B T}{h} = \frac{\omega_0}{2\pi}, \quad (2.398)$$

which can conveniently be used to introduce the dimension-less frequency

$$x := \frac{\omega}{\omega_0} = \frac{\nu}{\nu_0}, \quad (2.399)$$

in terms of which the Planck spectrum becomes (Figure 2.19)

$$B_\omega = \frac{(k_B T)^3}{4\pi^3 (\hbar c)^2} \frac{x^3}{e^x - 1}, \quad B_\nu = \frac{2(k_B T)^3}{(hc)^2} \frac{x^3}{e^x - 1}. \quad (2.400)$$

The prefactors of B_ω and B_ν in (2.400) evaluate to

$$B_{\omega,0} := \frac{(k_B T)^3}{4\pi^3 (\hbar c)^2} = 2.12 \cdot 10^{-17} \frac{\text{erg}}{\text{cm}^2 \text{ s Hz sr}} \left(\frac{T}{\text{K}}\right)^3, \quad B_{\nu,0} = 2\pi B_{\omega,0}. \quad (2.401)$$

The corresponding spectral energy density is

$$\frac{4\pi}{c} B_{\omega,0} = 8.88 \cdot 10^{-27} \frac{\text{erg}}{\text{cm}^3 \text{ Hz}} \left(\frac{T}{\text{K}}\right)^3. \quad (2.402)$$

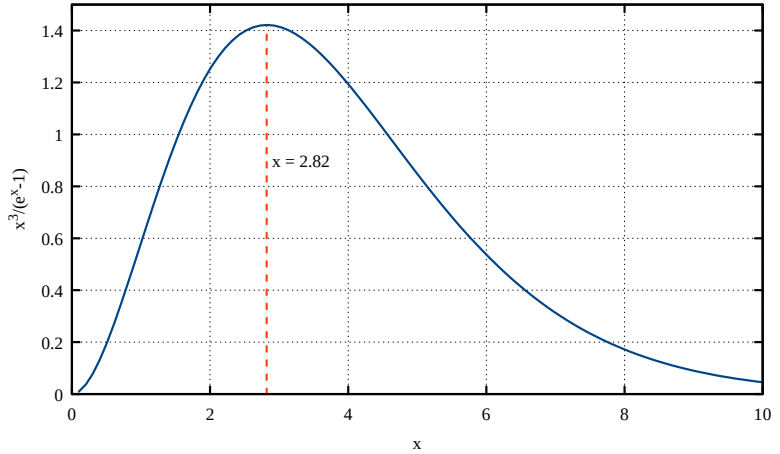


Figure 2.19 This figure shows the function $x^3/(e^x - 1)$ describing the frequency dependence of the Planck spectrum. The vertical line marks the location of the maximum at $x_{\max} \approx 2.82$.

The maximum of the Planck spectrum is located where

$$\frac{d}{dx} \frac{x^3}{e^x - 1} = 0 \quad \Rightarrow \quad (3 - x)e^x = 3, \quad (2.403)$$

which is a transcendental equation solved by $x_{\max} \approx 2.82$. With the help of (2.403), we have

$$\frac{x_{\max}^3}{e^{x_{\max}} - 1} = x_{\max}^2 (3 - x_{\max}) \approx 1.43 \quad (2.404)$$

there, hence the maximum amplitude of the Planck spectrum is approximately $1.43 B_{\omega,0}$.

For high frequencies, $x \gg 1$, the exponential in the denominator of (2.400) dominates, and the Planck spectrum can be approximated by Wien's law,

$$B_\omega \approx B_{\omega,0} x^3 e^{-x}, \quad (2.405)$$

while it turns into the Rayleigh-Jeans law for low frequencies, $x \ll 1$. Then, $e^x - 1 \approx x$, which allows the approximation

$$B_\omega \approx B_{\omega,0} x^2 \quad (2.406)$$

of the spectrum. The Rayleigh-Jeans law is often used to define a radiation temperature T_{rad} by requiring

$$\frac{2\nu^2}{c^2} k_B T_{\text{rad}} \stackrel{!}{=} I_\nu. \quad (2.407)$$

?

How would you solve an equation like (2.403)? Remind yourself of the Newton-Raphson method. Test how quickly the method converges starting with $x_0 = 3$ or $x_0 = 5$. What happens if you start with $x_0 \leq 2$?

Obviously, this agrees well with the thermodynamic temperature if $x \ll 1$ or $h\nu \ll 2.82 k_B T$ and $I_\nu = B_\nu$, but the deviation may become considerable for higher frequencies.

The total energy density contained in the photon ensemble is

$$U(T) = \int_0^\infty \frac{dU_\omega}{d\omega} d\omega = \frac{(k_B T)^4}{\pi^2 (\hbar c)^3} \int_0^\infty \frac{x^3 dx}{e^x - 1}. \quad (2.408)$$

The remaining integral is best carried out after expanding the integrand into a geometrical series,

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \int_0^\infty \frac{e^{-x} x^3 dx}{1 - e^{-x}} = \int_0^\infty x^3 dx e^{-x} \sum_{j=0}^\infty e^{-jx} = \sum_{j=1}^\infty \int_0^\infty x^3 dx e^{-jx}. \quad (2.409)$$

Each individual integral in (2.409) gives

$$\int_0^\infty x^3 dx e^{-jx} = \frac{\Gamma(4)}{j^4}. \quad (2.410)$$

Returning with this result to the energy density in (2.408), we find

$$U(T) = \frac{\Gamma(4) (k_B T)^4}{\pi^2 (\hbar c)^3} \sum_{j=1}^\infty \frac{1}{j^4} = \frac{\Gamma(4) \zeta(4) (k_B T)^4}{\pi^2 (\hbar c)^3} = \frac{\pi^2 (k_B T)^4}{15 (\hbar c)^3} =: a T^4, \quad (2.411)$$

where $\zeta(4) = \pi^4/90$ and $\Gamma(4) = 3! = 6$ were used in the step next to the last. Finally, the derived constant

$$a := \frac{\pi^2 k_B^4}{15 (\hbar c)^3} = 7.57 \cdot 10^{-15} \frac{\text{erg}^4}{\text{cm}^3 \text{K}^4} \quad (2.412)$$

was introduced, which is sometimes called the Stefan-Boltzmann constant. Using the same approach, we find that the number density of the photons is given by

$$n_\gamma(T) = \int_0^\infty d\omega \frac{dU_\omega}{d\omega} \frac{1}{\hbar\omega} = \frac{(k_B T)^3}{\pi^2 (\hbar c)^3} \int_0^\infty \frac{x^2 dx}{e^x - 1} = \frac{2\zeta(3) (k_B T)^3}{\pi^2 (\hbar c)^3}, \quad (2.413)$$

with $\zeta(3) \approx 1.202$.

?

Carry out the integration (2.410) yourself, and confirm the result (2.413).

2.10.2 Transition Balance and the Einstein coefficients

Suppose now that we have an ensemble of simplified atoms with just two energy levels E_1 and $E_2 > E_1$ which are supposed to be in equilibrium with an ambient radiation field characterised by a temperature T . We consider the mean transition rates in an emission- and absorption process between the photons of the radiation field and transitions between the two energy levels.

Besides absorption and spontaneous emission, we will have to take stimulated emission into account, which is a consequence of the Bose character of the photons. If a quantum state is already occupied by photons, an increase in the occupation number is more likely.

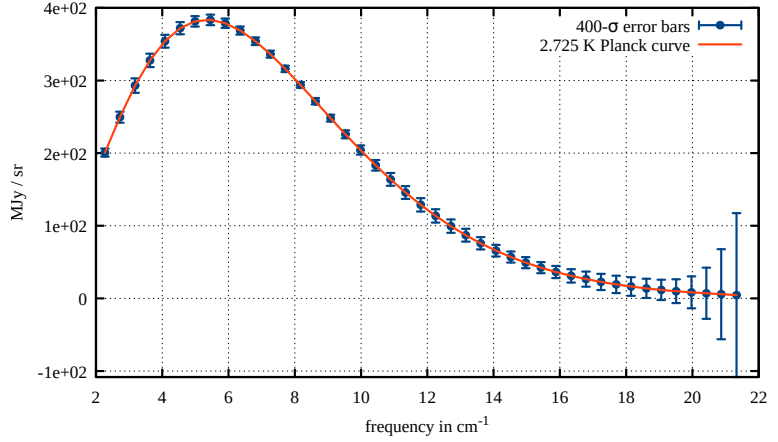


Figure 2.20 Spectrum of the Cosmic Microwave Background measured by the FIRAS instrument on-board the COBE satellite [5].

Example: The spectrum of the Cosmic Microwave Background

The best measured Planck spectrum that we know of is the spectrum of the cosmic microwave background (Figure 2.20). The CMB temperature of $T_{\text{CMB}} = 2.726$ K sets the characteristic frequency

$$\omega_{0,\text{CMB}} = \frac{k_{\text{B}} T_{\text{CMB}}}{\hbar} = 356.88 \cdot 10^9 \text{ s}^{-1}, \quad \nu_{0,\text{CMB}} = \frac{\omega_{0,\text{CMB}}}{2\pi} = 56.80 \text{ GHz} \quad (2.414)$$

and the frequency of the maximum specific intensity is

$$\nu_{\text{max,CMB}} = \frac{\omega_{\text{max,CMB}}}{2\pi} = 160.18 \text{ GHz}. \quad (2.415)$$

There, the specific intensity and the spectral energy density are

$$\begin{aligned} B_{\nu_{\text{max,CMB}}} &= 1.90 \cdot 10^{-16} \frac{\text{erg}}{\text{cm}^2 \text{ s Hz sr}}, \\ U_{\nu_{\text{max,CMB}}} &= 7.98 \cdot 10^{-26} \frac{\text{erg}}{\text{cm}^3 \text{ Hz}}. \end{aligned} \quad (2.416)$$

The total energy density in the CMB is

$$U = 4.17 \cdot 10^{-13} \frac{\text{erg}}{\text{cm}^3}, \quad (2.417)$$

which is contributed by

$$n_{\gamma} \approx 410 \text{ cm}^{-3} \quad (2.418)$$

photons per cubic centimetre. ◀

The rates of absorption and of stimulated emission, B_{12} and B_{21} , respectively, will be proportional to the specific intensity I_{ω} ,

$$(\text{absorption rate}) \propto I_{\omega} B_{12} \quad \text{and} \quad (\text{stimulated emission rate}) \propto I_{\omega} B_{21}, \quad (2.419)$$

while the rate of spontaneous emission, A_{21} , will not depend on I_ω ,

$$(\text{spontaneous emission rate}) \propto A_{21} . \quad (2.420)$$

The rate coefficients A and B are called *Einstein coefficients*.

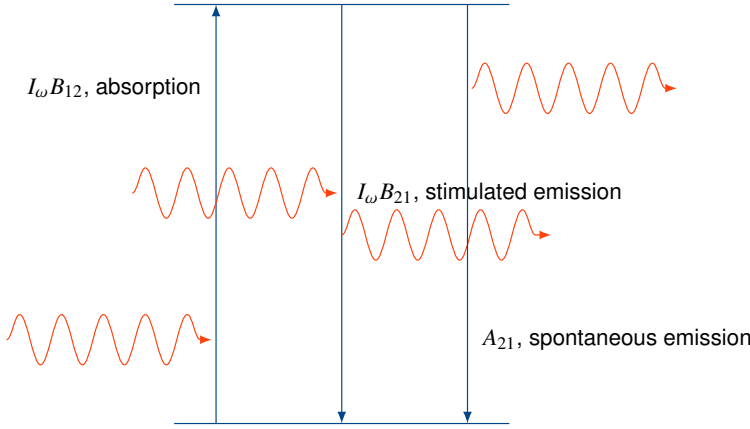


Figure 2.21 Illustration of radiative transitions between two quantum levels and the Einstein coefficients.

Now, let N_1 and N_2 be the mean occupation numbers of states with the energies E_1 and E_2 . Equilibrium between transitions (Figure 2.21) will require as many transitions per unit time from E_1 to E_2 as from E_2 to E_1 ,

$$N_1 I_\omega B_{12} = N_2 [A_{21} + I_\omega B_{21}] . \quad (2.421)$$

Solving for I_ω , we see that this can be satisfied if the specific intensity is

$$I_\omega = \frac{N_2 A_{21}}{N_1 B_{12} - N_2 B_{21}} . \quad (2.422)$$

Since we assume thermal equilibrium, the occupation numbers N_2 and N_1 must also be related by a Boltzmann factor,

$$\frac{N_2}{N_1} = e^{-\beta(E_2 - E_1)} = e^{-\beta \hbar \omega} . \quad (2.423)$$

Inserting this into (2.422), we find

$$I_\omega = \frac{A_{21} e^{-\beta \hbar \omega}}{B_{12} - B_{21} e^{-\beta \hbar \omega}} = \frac{A_{21}}{B_{12} e^{\beta \hbar \omega} - B_{21}} . \quad (2.424)$$

We can bring this into agreement with Planck's spectrum derived from quantum statistics (2.396) if, and only if, the rate coefficients satisfy Einstein's relations,

$$B_{12} = B_{21} \quad \text{and} \quad A_{21} = \frac{\hbar \omega^3}{4\pi^3 c^2} B_{21} . \quad (2.425)$$

This is a very interesting result, obtained by Einstein long before quantum statistics was established. It shows that without stimulated emission $B_{21} = 0$, the Planck spectrum cannot be obtained, and the microscopic rates of absorption and stimulated emission, B_{12} and B_{21} must be equal.

Problems

1. Consider an ensemble of Hydrogen atoms of temperature T . Besides the ground state, a fraction of atoms is thermally excited so that their electrons occupy higher energy levels.
 - (a) Calculate the fraction n_j/n_1 of Hydrogen atoms in the excited state j relative to the ground state for $j = 2, 3$.
 - (b) Calculate the relative intensity of the Lyman- β ($3 \rightarrow 1$) and Lyman- α ($2 \rightarrow 1$) lines for a cloud of atomic hydrogen with $T = 100$ K. The oscillator strengths and the wavelengths are $f_\beta = 0.0791$, $f_\alpha = 0.4162$ and $\lambda_\alpha = 1216 \text{ \AA}$, $\lambda_\beta = 1026 \text{ \AA}$, respectively.
2. For an ensemble of atoms with temperature T in thermal equilibrium with a radiation field, the rates for spontaneous emission A_{21} , induced emission B_{21} and absorption B_{12} between the energy levels 1 and 2 satisfy

$$N_1 I_\omega B_{12} = N_2 (A_{21} + I_\omega B_{21}), \quad (2.426)$$

where I_ω is the specific intensity of the radiation field and $N_{1,2}$ are the numbers of atoms in the first and the second energy levels, respectively. We can use the former equation to deduce the Lyman- α cross section σ_α .

- (a) Show that the rate equation can be written as

$$3A_{21}n(\omega_\alpha) = \int d\omega \sigma_\alpha(\omega) \frac{\omega^2}{\pi^2 c^2} n(\omega), \quad (2.427)$$

where $n(\omega) = [\exp(\omega/k_B T) - 1]^{-1}$ is the occupation number and ω_α the circular frequency of the Lyman- α transition. The transition rate B_{12} is written in terms of the cross section σ_α .

- (b) The cross section can be written as $\sigma_\alpha = C\phi(\omega - \omega_\alpha)$ with the line profile function ϕ and a constant C . Determine C . Use $A_{21} = 6.25 \cdot 10^8 \text{ s}^{-1}$ and λ_α given before. You may assume that the profile function is very narrow, i.e. it can be approximated by a Dirac delta function.

2.11 Absorption and Emission

This section begins with the definition of macroscopic coefficients for the spontaneous emission and absorption of radiation, leading to the net absorption coefficient, the opacity and the emissivity. The derivation of Kirchhoff's law (2.436) follows, which relates these quantities to the specific intensity. We then set up the radiation-transport equation and solve it under simplifying assumptions, leading to the solution (2.445). The section concludes with a discussion of continuous rather than discrete transitions.

2.11.1 Absorption coefficients and emissivity

We now want to describe how the energy transported by light is changed as the light propagates through an absorbing and emitting medium. The *absorption coefficient* α_ω is defined in terms of the energy absorbed per unit volume, time and frequency from the solid angle $d^2\Omega$,

$$\alpha_\omega I_\omega = \left(\frac{dE}{dV dt d\omega d^2\Omega} \right)_{\text{abs}} . \quad (2.428)$$

Since the stimulated emission is also proportional to I_ω , an analogous definition applies for what is called the *induced emission coefficient*,

$$\alpha_\omega^{\text{ind}} I_\omega = \left(\frac{dE}{dV dt d\omega d^2\Omega} \right)_{\text{ind}} . \quad (2.429)$$

To further account for the spontaneous emission, we define the *emissivity*

$$j_\omega = \left(\frac{dE}{dV dt d\omega d^2\Omega} \right)_{\text{spn}} , \quad (2.430)$$

which is the energy emitted spontaneously per unit volume, time and frequency into the solid-angle element $d^2\Omega$. Effectively, the net absorption is the difference between absorption and stimulated emission,

$$\alpha_\omega^{\text{net}} = \alpha_\omega - \alpha_\omega^{\text{ind}} . \quad (2.431)$$

Since the dimension of the specific intensity I_ω is

$$\frac{\text{energy}}{\text{time} \cdot \text{area} \cdot \text{frequency} \cdot \text{solid angle}} , \quad (2.432)$$

α_ω must obviously have the dimension $(\text{length})^{-1}$. The inverse absorption coefficient α_ω^{-1} thus characterises a length, which can be identified with the mean free path for a photon of frequency ω .

Let now σ_ω be the cross section of an atom, molecule or other particle for the absorption of light of frequency ω . The number density of such absorbing particles be n , and their mass density be ρ . Then, the absorption must be due to the combined cross sections of these particles,

$$\alpha_\omega = n\sigma_\omega =: \rho\kappa . \quad (2.433)$$

The quantity κ introduced in the last step, characterising the absorption by unit mass of the medium, is called *opacity*. Its physical dimension must be an absorption cross section per unit mass, thus an area per unit mass,

$$[\kappa] = \frac{\text{cm}^2}{\text{g}} . \quad (2.434)$$

If the absorbing and emitting material is in equilibrium with the radiation field passing through it, the emitted and absorbed amounts of energy must equal, hence

$$j_\omega + \alpha_\omega^{\text{ind}} I_\omega = \alpha_\omega I_\omega \quad (2.435)$$

or, by the definition (2.431) of the net absorption coefficient, the ratio between emissivity and net absorption coefficient must equal the specific intensity I_ω ,

$$I_\omega = \frac{j_\omega}{\alpha_\omega^{\text{net}}}, \quad (2.436)$$

which is *Kirchhoff's law*. At the same time, we have the relation (2.421) between the specific intensity and the Einstein coefficients, which must themselves be related by Einstein's relations (2.425). Combining these results, we find the relation

$$\frac{j_\omega}{\alpha_\omega^{\text{net}}} = \frac{\hbar\omega^3}{4\pi^3c^2} \left(\frac{N_1}{N_2} - 1 \right)^{-1} \quad (2.437)$$

between the occupation numbers N_1 and N_2 of two energy levels contributing to the radiation balance on one side, and the emissivity and the net absorption coefficient on the other. Thus, if the occupation numbers are known, the emission and absorption properties in equilibrium can be calculated, and vice versa. In particular, in thermal equilibrium between radiation and matter, the specific intensity must be given by the Planck spectrum, $I_\omega = B_\omega$, hence

$$\alpha_\omega^{\text{net}} = \frac{j_\omega}{B_\omega}. \quad (2.438)$$

2.11.2 Radiation Transport in a Simple Case

Let us now consider an emitting and absorbing medium in which scattering can be ignored. The medium be characterised by its emissivity j_ω and a net absorption coefficient $\alpha_\omega^{\text{net}}$. A light bundle passing through it has its intensity changed per unit path length by an amount

$$dI_\omega = \underbrace{j_\omega dl}_{\text{emission}} - \underbrace{\alpha_\omega^{\text{net}} I_\omega dl}_{\text{absorption}}, \quad (2.439)$$

from which we obtain the equation of radiation transport in its simplest case,

$$\frac{dI_\omega}{dl} = j_\omega - \alpha_\omega^{\text{net}} I_\omega. \quad (2.440)$$

The homogeneous equation (2.440) is readily solved. Setting $j_\omega = 0$ for the moment,

$$\frac{dI_\omega}{dl} = -\alpha_\omega^{\text{net}} I_\omega \quad \Rightarrow \quad d \ln I_\omega = -\alpha_\omega^{\text{net}} dl, \quad (2.441)$$

thus

$$I_\omega(l) = I_{\omega,0} \exp\left(-\int_0^l \alpha_\omega^{\text{net}}(l') dl'\right), \quad (2.442)$$

with an integration constant $I_{\omega,0}$ set by the incoming specific intensity.

The inhomogeneous equation (2.440) can now be solved by a standard technique called the variation of constants. We extend the definition of the former integration constant $I_{\omega,0}$ to allow its dependence on the light path, $I_{\omega,0} = I_{\omega,0}(l)$, and find

$$j_\omega - \alpha_\omega^{\text{net}} I_\omega \stackrel{!}{=} \frac{dI_\omega}{dl} = \left[I'_{\omega,0}(l) - I_{\omega,0}(l) \alpha_\omega^{\text{net}} \right] \exp\left(-\int_0^l \alpha_\omega^{\text{net}}(l') dl'\right), \quad (2.443)$$

Why does scattering have to be ignored for (2.440) to hold?

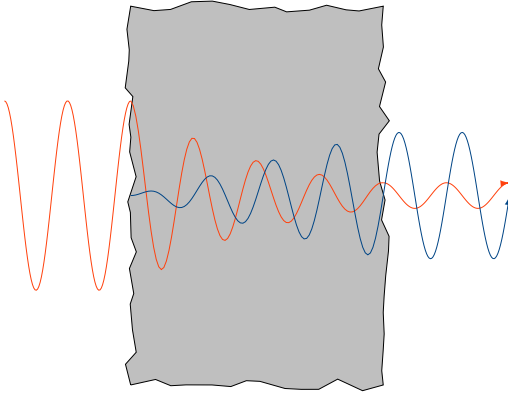


Figure 2.22 Illustration of radiation transport. While the incoming radiation is damped by the absorption, the spontaneous and the stimulated emission of the medium increase the intensity.

which, with (2.442), implies

$$j_{\omega}(l) = I'_{\omega,0}(l) \exp\left(-\int_0^l \alpha_{\omega}^{\text{net}}(l') dl'\right). \quad (2.444)$$

By separation of variables, this differential equation for $I_{\omega,0}(l)$ has the solution

$$I_{\omega,0}(l) = \int dl \left[j_{\omega}(l) \exp\left(\int_0^l \alpha_{\omega}^{\text{net}}(l') dl'\right) \right] + C \quad (2.445)$$

with another integration constant C set by boundary conditions.

Example: Constant emission and absorption

If $\alpha_{\omega}^{\text{net}}$ and j_{ω} are constant along the light path, the inner integral in (2.445) is simply

$$\int_0^l \alpha_{\omega}^{\text{net}} dl = \alpha_{\omega}^{\text{net}} l, \quad (2.446)$$

while the outer integration gives

$$I_{\omega,0}(l) = \frac{j_{\omega}}{\alpha_{\omega}^{\text{net}}} e^{\alpha_{\omega}^{\text{net}} l} + C. \quad (2.447)$$

By (2.442), the specific intensity then develops according to (Figure 2.22)

$$I_{\omega}(l) = \frac{j_{\omega}}{\alpha_{\omega}^{\text{net}}} - C e^{-\alpha_{\omega}^{\text{net}} l} \quad (2.448)$$

along the path length of the light bundle. If, for example, the specific intensity satisfies the boundary condition $I_{\omega} = 0$ at $l = 0$, it changes as a function of path length like

$$I_{\omega}(l) = \frac{j_{\omega}}{\alpha_{\omega}^{\text{net}}} \left(1 - e^{-\alpha_{\omega}^{\text{net}} l}\right). \quad (2.449)$$

Example: Radiation transport in the limiting cases of optically thick media

Interesting limiting cases of radiation transport are those of optically thin or thick media. Optically thin means that the mean free path of photons is large compared to the overall length L of the light path through the medium, $\alpha_\omega^{\text{net}} L \ll 1$, while optically thick means the opposite, $\alpha_\omega^{\text{net}} L \gg 1$. In the optically thin case, we can expand $1 - e^{-x} \approx x$ to first order and approximate

$$I_\omega(L) \approx \frac{j_\omega}{\alpha_\omega^{\text{net}}} \alpha_\omega^{\text{net}} L \approx j_\omega L . \quad (2.450)$$

The specific intensity is then simply the emissivity times the total path length. In the optically thick case, the exponential in (2.449) tends to zero, and

$$I_\omega(L) \approx \frac{j_\omega}{\alpha_\omega^{\text{net}}} . \quad (2.451)$$

This closes the loop: If the radiation is in thermal equilibrium with the optically thick medium through which it propagates, we can complete (2.451) with (2.438) to find

$$I_\omega \approx B_\omega . \quad (2.452)$$

This shows that radiation in thermal equilibrium with an optically thick medium leaves the medium with a Planck spectrum. ◀

Example: Radiation transport in the limiting case of optically thin media

As a further illustrative example, let us now consider optically thin, thermal emission of radio waves. As we have seen, an optically thin medium satisfies $\alpha_\omega^{\text{net}} L \ll 1$ and $I_\omega = j_\omega L$, while thermal equilibrium requires $I_\omega \approx B_\omega$. Combining these conditions, we find

$$B_\omega \approx I_\omega \approx j_\omega L = \alpha_\omega^{\text{net}} L B_\omega \ll B_\omega . \quad (2.453)$$

This evidently contradictory conclusion demonstrates that the two assumptions, thermal equilibrium *and* optically-thin radiation, are in manifest conflict with each other: Radiation cannot attain thermal equilibrium with an optically thin medium. ◀

Example: Planck spectrum shining through gas

For yet another instructive example, consider gas at temperature T_1 in thermal equilibrium with radiation having a Planck spectrum with temperature T_0 before it enters the gas. In the gas, Kirchhoff's law (2.436) demands

$$j_\omega = \alpha_\omega^{\text{net}} B_\omega(T_1) \tag{2.454}$$

because of the thermal equilibrium. For simplicity, we assume that T_1 and $\alpha_\omega^{\text{net}}$ are constant. In the present situation, (2.445) implies

$$\begin{aligned} I_{\omega,0}(l) &= B_\omega(T_0) + B_\omega(T_1)\alpha_\omega^{\text{net}} \int_0^l dl' e^{\alpha_\omega^{\text{net}} l'} \\ &= B_\omega(T_0) + B_\omega(T_1) \left(e^{\alpha_\omega^{\text{net}} l} - 1 \right) . \end{aligned} \tag{2.455}$$

Then, the specific intensity (2.442) is given by

$$I_\omega(l) = B_\omega(T_0)e^{-\alpha_\omega^{\text{net}} l} + B_\omega(T_1) \left(1 - e^{-\alpha_\omega^{\text{net}} l} \right) , \tag{2.456}$$

which is a weighed average between the two Planck spectra for temperatures T_0 and T_1 . As the radiation propagates into the gas, its original Planck spectrum is gradually being replaced by the Planck spectrum determined by the gas temperature. ◀

2.11.3 Emission and Absorption in the Continuum Case

In the case of transitions between discrete energy levels, the emitted energy is determined by the number of transitions times the energy released per transition,

$$\underbrace{N_2 A_{21}}_{\text{(transition number)}} \cdot \underbrace{\hbar\omega_{12}}_{\text{(energy per transition)}} = \delta E . \tag{2.457}$$

The emissivity, defined as the energy emitted per unit time and unit volume into a unit solid angle, is thus related to the transition number by

$$j_\omega = \frac{N_2 A_{21} \hbar\omega_{12}}{4\pi} \rightarrow \frac{N_2 A_{21} \hbar\omega}{4\pi} \delta_D(\omega - \omega_{12}) , \tag{2.458}$$

if N_2 is taken to be the occupation number of quantum states per unit volume. The Dirac delta function is introduced here for modeling a needle-sharp line transition. We generalise this last expression by replacing it with a more detailed or realistic line profile function $\phi(\omega)$,

$$j_\omega = \frac{N_2 A_{21} \hbar\omega}{4\pi} \phi(\omega) , \tag{2.459}$$

which quantifies the transition probability as a function of frequency. By a completely analogous procedure for the absorption coefficient, we find

$$\alpha_\omega = \frac{N_1 B_{12}}{4\pi} \hbar\omega \phi(\omega) . \tag{2.460}$$

Now, we consider an electron of energy E which emits the energy

$$\frac{dE}{d\omega dt} \equiv P(\omega, E) \tag{2.461}$$

?

Beginning with the definition of the Einstein coefficients, deduce (2.460) yourself.

per unit time and unit frequency. Let further $f(\vec{p})$ be the momentum distribution of the electrons, then the number of electrons with energies between E and $E + dE$ is

$$n(E) dE = f(\vec{p}) \frac{d^3p}{dE} dE = 4\pi p^2 \frac{dp}{dE} f(\vec{p}) dE \quad (2.462)$$

if we assume the electron distribution to be isotropic in momentum space. Since each electron emits the energy

$$dE = P(\omega, E) d\omega dt, \quad (2.463)$$

we obtain the emissivity

$$j_\omega = \frac{1}{4\pi} \int_0^\infty n(E) P(\omega, E) dE = \int_0^\infty p^2 f(p) \frac{dp}{dE} P(\omega, E) dE. \quad (2.464)$$

By the relation (2.458) between the emissivity and the Einstein coefficient A_{21} , we have for a single transition described by the continuous line profile function $\phi(\omega)$

$$P(\omega, E_2) = \hbar\omega \int_0^{E_2} A_{21} \phi(\omega) dE_1, \quad (2.465)$$

since electrons with the energy E_2 can emit through transitions to all possible states with $E_1 < E_2$. Using now Einstein's relation (2.424) between A_{21} and B_{21} , we find

$$P(\omega, E_2) = \hbar\omega \frac{\hbar\omega^3}{4\pi^3 c^2} \int_0^{E_2} B_{21} \phi(\omega) dE_1. \quad (2.466)$$

Similarly, the net absorption coefficient is

$$\alpha_\omega = \frac{\hbar\omega}{4\pi} \int dE_1 \int dE_2 \left[\underbrace{n(E_1) B_{12}}_{(\text{absorption})} - \underbrace{n(E_2) B_{21}}_{(\text{stimulated emission})} \right] \phi(\omega). \quad (2.467)$$

Exchanging the order of integrations and inserting (2.466) into the second term, that term can be rewritten as

$$\frac{\hbar\omega}{4\pi} \int dE_2 n(E_2) \int dE_1 B_{21} \phi(\omega) = \frac{\pi^2 c^2}{\hbar\omega^3} \int dE_2 n(E_2) P(\omega, E_2). \quad (2.468)$$

By the same procedure and using $E_2 = E_1 + \hbar\omega$, the first term can be transformed into

$$\frac{\hbar\omega}{4\pi} \int dE_2 n(E_1 - \hbar\omega) \int dE_1 B_{12} \phi(\omega) = \frac{\pi^2 c^2}{\hbar\omega^3} \int dE_2 n(E_2 - \hbar\omega) P(\omega, E_2). \quad (2.469)$$

We thus obtain the absorption coefficient

$$\alpha_\omega = \frac{\pi^2 c^2}{\hbar\omega^3} \int dE [n(E - \hbar\omega) - n(E)] P(\omega, E). \quad (2.470)$$

In thermal equilibrium with a heat bath of temperature T and far from degeneracy, the electron number density must be proportional to a Boltzmann factor,

$$n(E) \propto \exp\left(-\frac{E}{k_B T}\right), \quad (2.471)$$

thus the difference of the electron number densities at different energies is

$$n(E - \hbar\omega) - n(E) = n(E) \left[\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1 \right]. \quad (2.472)$$

Inserting to (2.470) with this expression, we find

$$\alpha_\omega = \frac{\pi^2 c^2}{\hbar\omega^3} \left(e^{\hbar\omega/k_B T} - 1 \right) \int dE n(E) P(\nu, E). \quad (2.473)$$

The remaining integral is $4\pi j_\omega$, as (2.464) shows, allowing us to write

$$\alpha_\omega = j_\omega \frac{4\pi^3 c^2}{\hbar\omega^3} \left(e^{\hbar\omega/k_B T} - 1 \right). \quad (2.474)$$

A glance at (2.396) finally reveals that the factor multiplying the emissivity is the inverse Planck spectrum $B_\omega(T)$. We can thus reduce (2.474) to the relation

$$\alpha_\omega = \frac{j_\omega}{B_\omega(T)} \quad (2.475)$$

between absorption and emission, just as in the discrete case.

2.11.4 Energy transport through absorbing media

It is useful to re-write the transport equation (2.440) for radiation in spherical polar coordinates. To do so, we write the total differential dI_ω of the specific intensity as

$$dI_\omega = \partial_r I_\omega dr + \partial_\theta I_\omega d\theta \quad (2.476)$$

and use the relations

$$dr = \cos\theta dl, \quad d\theta = -\frac{\sin\theta}{r} dl \quad (2.477)$$

between the coordinate differentials $dr, d\theta$ and the path length dl . The radiation-transport equation then reads

$$\partial_r I_\omega \cos\theta - \partial_\theta I_\omega \frac{\sin\theta}{r} = -\alpha_\omega^{\text{net}} I_\omega + j_\omega. \quad (2.478)$$

We now integrate over frequencies ω , introduce the averaged net absorption coefficient $\bar{\alpha}^{\text{net}}$ defined by

$$\int_0^\infty d\omega \alpha_\omega^{\text{net}} I_\omega = \bar{\alpha}^{\text{net}} I \quad (2.479)$$

and find

$$\partial_r I \cos\theta - \partial_\theta I \frac{\sin\theta}{r} = -\bar{\alpha}^{\text{net}} I + j. \quad (2.480)$$

Next, we multiply this equation by $\cos\theta/c$ and integrate over the complete solid angle $d\Omega = \sin\theta d\theta d\varphi$. Due to the isotropy of the emissivity j , the second term on the right-hand side then vanishes altogether. The second term on the left-hand side is partially integrated to shift the derivative with respect to θ away from the intensity I . This results in

$$\partial_r \int \frac{I}{c} \cos^2\theta d\Omega + \frac{1}{r} \int \frac{I}{c} \partial_\theta (\sin^2\theta \cos\theta) d\theta d\varphi = -\bar{\alpha}^{\text{net}} \int \frac{I}{c} \cos\theta d\Omega. \quad (2.481)$$

We have seen earlier in (2.370) that the angular moments of the intensity are related to the energy density U , the energy current density F and the radiation pressure P_{rad} . Furthermore, the integral in the second term on the left-hand side of (2.481) is

$$\int \frac{I}{c} \partial_\theta (\sin^2 \theta \cos \theta) d\theta d\varphi = \int \frac{I}{c} (3 \cos^2 \theta - 1) d\Omega = 3P_{\text{rad}} - U = 0, \quad (2.482)$$

leaving (2.481) in the simple form $c\partial_r P = -\bar{\alpha}^{\text{net}} F$ or, with the opacity κ defined in (2.433),

$$F = -\frac{c}{\rho\kappa} \partial_r P. \quad (2.483)$$

The energy current density F is determined by the gradient of the radiation pressure. Since the radiation pressure P is a third of the energy density U , which is in turn given by $U = aT^4$ according to (2.411), we can write the result (2.483) in the very intuitive form

$$F = -\frac{4acT^3}{3\rho\kappa} \partial_r T, \quad (2.484)$$

which clearly says that the radiative energy current density through an absorbing medium is driven by the temperature gradient, and inhibited by the opacity κ .

Problems

1. The change of the specific intensity I_ω in matter per unit length is given by the radiation transport equation

$$\frac{dI_\omega}{dl} = j_\omega - \alpha_\omega^{\text{net}} I_\omega, \quad (2.485)$$

where j_ω is the emissivity and $\alpha_\omega^{\text{net}}$ is the net absorption coefficient. Assume that radio waves travel through a medium which has a temperature profile $T(l) = T_0 \exp(-l/\lambda)$, where T_0 is the temperature at the surface and λ is a typical length scale for the temperature gradient.

- (a) Let $\alpha_\omega^{\text{net}}$ be constant throughout the medium, and the radiation be in local thermal equilibrium with the medium. Solve the radiation transport equation under the condition that the incoming specific intensity at $l = 0$ is $I_{\omega,\text{in}}$ and $\hbar\omega \ll k_B T$.
- (b) Assume that the incoming spectrum is given by a power law, $I_{\omega,\text{in}} = I_0 (\omega/\omega_0)^{-\gamma}$, which can be seen in many astrophysical phenomena. Determine the spectrum of the radiation once it has travelled by a distance $L \sim \lambda$ with $(\alpha_\omega^{\text{net}})^{-1} \ll L$. What happens to the shape of the spectrum?

Suggested further reading: [2, 6, 7, 8, 9]