Appendix C

Penrose-Carter diagrams

For studying in particular the causal structure of spacetimes, a compactification has proven useful whose illustration has become known as Penrose-Carter diagram. We first demonstrate its construction at the example of Minkowski spacetime. It proceeds in three steps:

1. We assume a spherically-symmetric spacetime and consider the time t and the radial coordinate r only. We transform (ct, r) to null coordinates $(\tilde{u}, \tilde{v})$ by
$$(ct, r) \mapsto (\tilde{u}, \tilde{v}) \; , \quad \tilde{u} := ct - r, \; \tilde{v} := ct + r \; . \tag{C.1}$$
With $t \in (-\infty, \infty)$ and $r \in [0, \infty)$, we have $\tilde{u}, \tilde{v} \in (-\infty, \infty)$ and $\tilde{u} \leq \tilde{v}$. The line element of the metric then transforms to
$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 = -\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} + \frac{1}{4}(\tilde{v} - \tilde{u})^2\,\mathrm{d}\Omega^2 \; . \tag{C.2}$$

2. Next, we map the null coordinates to compact intervals by the transform
$$(\tilde{u}, \tilde{v}) \mapsto (U, V) \; , \quad U := \arctan \tilde{u} \; , \quad V := \arctan \tilde{v} \; . \tag{C.3}$$
With
$$\mathrm{d}\tilde{u} = \frac{\mathrm{d}U}{\cos^2 U} \; , \quad \mathrm{d}\tilde{v} = \frac{\mathrm{d}V}{\cos^2 V} \; , \quad \tilde{v} - \tilde{u} = \frac{\sin(U - V)}{\cos U \cos V} \; , \tag{C.4}$$
the line elements turns into
$$\mathrm{d}s^2 = -\frac{1}{\cos^2 U \cos^2 V}\left[\mathrm{d}U\mathrm{d}V + \frac{1}{4}\sin^2(V - U)\,\mathrm{d}\Omega^2\right] \; . \tag{C.5}$$

3. Finally, we return to time- and space-like coordinates (T, R) defined by
$$T := V + U \; , \quad R := V - U \; . \tag{C.6}$$

Using

$$U = \frac{1}{2}(T - R) , \quad V = \frac{1}{2}(T + R) , \tag{C.7}$$

we find

$$\mathrm{d}U\mathrm{d}V = \frac{1}{4}\left(\mathrm{d}T^2 - \mathrm{d}R^2\right) \tag{C.8}$$

and thus the line element

$$\mathrm{d}s^2 = \omega^{-2}(T, R)\left(-\mathrm{d}T^2 + \mathrm{d}R^2 + \sin^2 R\,\mathrm{d}\Omega^2\right) \tag{C.9}$$

with the conformal factor

$$\omega(T, R) = 2\cos U \cos V = \cos T + \cos R . \tag{C.10}$$

With $\tilde{u}, \tilde{v} \in \mathbb{R}$ and $\tilde{u} \leq \tilde{v}$, the compactified coordinates (U, V) obey

$$U \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) , \quad V \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) , \quad U \leq V , \tag{C.11}$$

which implies

$$R \in [0, \pi) , \quad |T| + R \in [0, \pi) . \tag{C.12}$$

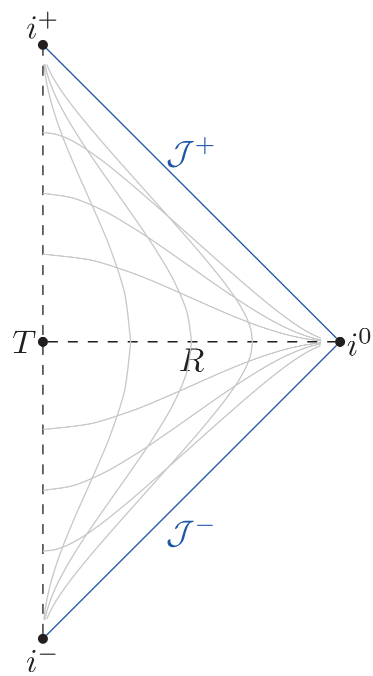

Figure C.1 Penrose-Carter diagram of Minkowski spacetime. The light-gray curves are lines of constant radius (running from i^- to i^+) and lines of constant time (emerging from i^0).

The following points and lines are particularly important for the causal structure of the spacetime:

- i^+: Future time-like infinity, i.e. $(T, R) = (\pi, 0)$;
- i^0: Spatial infinity, i.e. $(T, R) = (0, \pi)$;
- i^-: Past time-like infinity, i.e. $(T, R) = (-\pi, 0)$;
- $\mathcal{J}^+$: Future null infinity, i.e. $T = \pi - R$, $0 < R < \pi$;
- $\mathcal{J}^-$: Past null infinity, i.e. $T = R - \pi$, $0 < R < \pi$;

Lines of constant radius emerge from i^- and end in i^+, while lines of constant time all end in i^0.

The Penrose-Carter diagram of Minkowski spacetime is shown in Fig. C.1. In addition, Fig. C.2 shows the Penrose-Carter diagram for the Kruskal extension of the Schwarzschild spacetime. There, the four domains of this extension are marked with Roman numerals, as discussed in Chap. 10 and indicated in Fig. 10.3.

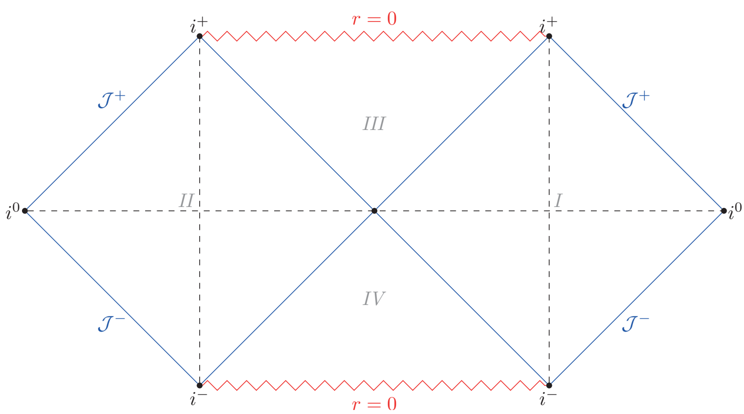

Figure C.2 Penrose-Carter diagram for the Kruskal extension of the Schwarzschild spacetime. The four different domains of the Kruskal extension are indicated by Roman numerals as in Fig. 10.3.