Appendix B

Summary of Differential Geometry

B.1 Manifold

An n-dimensional manifold M is a suitably well-behaved space that is locally homeomorphic to $\mathbb{R}^n$, i.e. that locally "looks like" $\mathbb{R}^n$.

A chart h, or a coordinate system, is a homeomorphism from $D \subset M$ to $U \subset \mathbb{R}^n$,

$$h : D \to U \ , \quad p \mapsto h(p) = (x^1, \ldots, x^n) \ , \tag{B.1}$$

i.e. it assigns an n-tupel of coordinates $\{x^i\}$ to a point $p \in D$.

An atlas is a collection of charts whose domains cover the entire manifold. If all coordinate changes between charts of the atlas with overlapping domains are differentiable, the manifold and the atlas themselves are called differentiable.

B.2 Tangent and dual spaces

The tangent space T_pM at a point $p \in M$ is the vector space of all derivations. A derivation v is a map from the space $\mathcal{F}_p$ of C^∞ functions in p into the real numbers,

$$v : \mathcal{F}_p \to \mathbb{R} \ , \quad f \mapsto v(f) \ . \tag{B.2}$$

A derivation is a linear map which satisfies the Leibniz rule,

$$v(\lambda f + \mu g) = \lambda v(f) + \mu v(g) \ , \quad v(fg) = v(f)g + fv(g) \ . \tag{B.3}$$

Tangent vectors generalise directional derivatives of functions.

A coordinate basis of the tangent vector space is given by the partial derivatives $\{\partial_i\}$. Tangent vectors can then be expanded in this basis,

$$v = v^i \partial_i \;, \quad v(f) = v^i \partial_i f \;. \tag{B.4}$$

A dual vector w is a linear map assigning a real number to a vector,

$$w : TM \to \mathbb{R} \;, \quad v \mapsto w(v) \;. \tag{B.5}$$

The space of dual vectors to a tangent vector space TM is the dual space T^*M.

Specifically, the differential of a function $f \in \mathcal{F}$ is a dual vector defined by

$$\mathrm{d}f : TM \to \mathbb{R} \;, \quad v \mapsto \mathrm{d}f(v) = v(f) \;. \tag{B.6}$$

Accordingly, the differentials of the coordinate functions x^i form a basis $\{\mathrm{d}x^i\}$ of the dual space which is orthonormal to the coordinate basis $\{\partial_i\}$ of the tangent space,

$$\mathrm{d}x^i(\partial_j) = \partial_j(x^i) = \delta^i_j \;. \tag{B.7}$$

B.3 Tensors

A tensor $t \in \mathcal{T}^r_s$ of rank (r, s) is a multilinear mapping of r dual vectors and s vectors into the real numbers. For example, a tensor of rank $(0, 2)$ is a bilinear mapping of 2 vectors into the real numbers,

$$t : TM \times TM \to \mathbb{R} \;, \quad (x, y) \mapsto t(x, y) \;. \tag{B.8}$$

The tensor product is defined component-wise. For example, two dual vectors $w_{1,2}$ can be multiplied to form a rank-$(0, 2)$ tensor $w_1 \otimes w_2$

$$(v_1, v_2) \mapsto (w_1 \otimes w_2)(v_1, v_2) = w_1(v_1)\, w_2(v_2) \;. \tag{B.9}$$

A basis for tensors of arbitrary rank is obtained by the tensor product of suitably many elements of the bases $\{\partial_i\}$ of the tangent space and $\{\mathrm{d}x^j\}$ of the dual space. For example, a tensor $t \in \mathcal{T}^0_2$ can be expanded as

$$t = t_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j \;. \tag{B.10}$$

If applied to two vectors $x = x^k \partial_k$ and $y = y^l \partial_l$, the result is

$$t(x, y) = t_{ij}\, \mathrm{d}x^i(x^k \partial_k)\, \mathrm{d}x^j(y^l \partial_l) = t_{ij} x^k y^l \delta^i_k \delta^j_l = t_{ij} x^i y^j \;. \tag{B.11}$$

The contraction of a tensor $t \in \mathcal{T}^r_s$ is defined by

$$C : \mathcal{T}^r_s \to \mathcal{T}^{r-1}_{s-1} \;, \quad t \mapsto Ct \tag{B.12}$$

such that one of the dual vector arguments and one of the vector arguments are filled with pairs of basis elements and summed over all pairs. For example, the contraction of a tensor $t \in \mathcal{T}_1^1$ is

$$Ct = t(\mathrm{d}x^k, \partial_k) = (t^i_j\,\partial_i \otimes \mathrm{d}x^j)(\mathrm{d}x^k, \partial_k) = t^i_j \delta^k_i \delta^j_k = t^k_k \;. \tag{B.13}$$

The metric $g \in \mathcal{T}_2^0$ is a symmetric, non-degenerate tensor field of rank $(0, 2)$, i.e. it satisfies

$$g(x, y) = g(y, x)\;, \quad g(x, y) = 0\ \forall\ y \quad \Rightarrow \quad x = 0\;. \tag{B.14}$$

The metric defines the scalar product between two vectors,

$$\langle x, y\rangle = g(x, y)\;. \tag{B.15}$$

B.4 Covariant derivative

The covariant derivative or a connection linearly maps a pair of vectors to a vector,

$$\nabla : TM \times TM \to TM\;, \quad (x, y) \mapsto \nabla_x y \tag{B.16}$$

such that for a function $f \in \mathcal{F}$

$$\nabla_{fx} y = f \nabla_x y\;, \quad \nabla_x(fy) = f\nabla_x y + x(f)y\;. \tag{B.17}$$

The covariant derivative of a function f is its differential,

$$\nabla_v f = vf = \mathrm{d}f(v)\;. \tag{B.18}$$

Due to the linearity, it is completely specified by the covariant derivatives of the basis vectors,

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k\;. \tag{B.19}$$

The functions Γ^k_{ij} are called connection coefficients or Christoffel symbols. They are not tensors.

By means of the exponential map, so-called normal coordinates can always be introduced locally in which the Christoffel symbols all vanish.

The covariant derivative ∇y of a vector y is a rank-$(1, 1)$ tensor field defined by

$$\nabla y : T^*M \times TM \to \mathbb{R}\;, \quad \nabla y(w, v) = w(\nabla_v y)\;. \tag{B.20}$$

In components,

$$(\nabla y)^i_j = \nabla y(\mathrm{d}x^i, \partial_j) = \mathrm{d}x^i(\nabla_{\partial_j} y^k \partial_k) = \partial_j y^i + \Gamma^i_{jk} y^k\;. \tag{B.21}$$

The covariant derivative of a tensor field is defined to obey the Leibniz rule and to commute with contractions. Specifically, the covariant derivative of a dual vector field $w \in T^*M$ is a tensor of rank $(0, 2)$ with components

$$(\nabla w)_{ij} = \partial_j w_i - \Gamma^k_{ij} w_k\;. \tag{B.22}$$

B.5 Parallel transport and geodesics

A curve γ is defined as a map from some interval $I \subset \mathbb{R}$ to the manifold,

$$\gamma : I \to M \;, \quad t \mapsto \gamma(t) \;. \tag{B.23}$$

Its tangent vector is $\dot{\gamma}(t)$.

A vector v is said to be parallel transported along γ if

$$\nabla_{\dot{\gamma}} v = 0 \;. \tag{B.24}$$

A geodesic curve is defined as a curve whose tangent vector is parallel transported along γ,

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \;. \tag{B.25}$$

In coordinates, let $u = \dot{\gamma}$ be the tangent vector to γ and with components $\dot{x}^i = u^i$, then

$$\ddot{x}^k + \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0 \;. \tag{B.26}$$

B.6 Torsion and curvature

The torsion of a connection is defined by

$$T : TM \times TM \to TM \;, \quad (x, y) \mapsto T(x, y) = \nabla_x y - \nabla_y x - [x, y] \;. \tag{B.27}$$

It vanishes if and only if the connection is symmetric.

On a manifold M with a metric g, a symmetric connection can always be uniquely defined by requiring that $\nabla g = 0$. This is the Levi-Civita connection, whose Christoffel symbols are

$$\Gamma^i_{jk} = \frac{1}{2} g^{ia} \left(\partial_j g_{ak} + \partial_k g_{ja} - \partial_a g_{jk} \right) \;. \tag{B.28}$$

From now on, we shall assume that we are working with the Levi-Civita connection whose torsion vanishes.

The curvature is defined by

$$\begin{aligned} &\bar{R} : TM \times TM \times TM \to TM \;, \\ &(x, y, z) \mapsto \bar{R}(x, y)z = \left(\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]} \right) z \;. \end{aligned} \tag{B.29}$$

The curvature or Riemann tensor $\bar{R} \in \mathcal{T}^1_3$ is given by

$$\bar{R} : T^*M \times TM \times TM \times TM \to \mathbb{R} \;, \quad (w, x, y, z) \mapsto w[\bar{R}(x, y)z] \;. \tag{B.30}$$

Its components are

$$\bar{R}^i_{jkl} = \mathrm{d}x^i [\bar{R}(\partial_k, \partial_l)\partial_j] = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^a_{jl} \Gamma^i_{ak} - \Gamma^a_{jk} \Gamma^i_{al} \;. \tag{B.31}$$

The Riemann tensor obeys three important symmetries,

$$\bar{R}_{ijkl} = -\bar{R}_{jikl} = \bar{R}_{jilk} \;, \quad \bar{R}_{ijkl} = \bar{R}_{klij} \;, \tag{B.32}$$

which reduce its $4^4 = 256$ components in four dimensions to 21.

In addition, the Bianchi identities hold,

$$\sum_{(x,y,z)} \bar{R}(x,y)z = 0 \;, \quad \sum_{(x,y,z)} \nabla_x \bar{R}(y,z) = 0 \;, \tag{B.33}$$

where the sums extend over all cyclic permutations of x, y, z. The first Bianchi identity reduces the number of independent components of the Riemann tensor to 20. In components, the second Bianchi identity can be written

$$\bar{R}^i_{\;j[kl;m]} = 0 \;, \tag{B.34}$$

where the indices in brackets need to be antisymmetrised.

The Ricci tensor is the contraction of the Riemann tensor over its first and third indices, thus its components are

$$R_{jl} = \bar{R}^i_{\;jil} = R_{lj} \;. \tag{B.35}$$

A further contraction yields the Ricci scalar,

$$\mathcal{R} = R^i_i \;. \tag{B.36}$$

The Einstein tensor is the combination

$$G = R - \frac{\mathcal{R}}{2} g \;. \tag{B.37}$$

Contracting the second Bianchi identity, we find the contracted Bianchi identity,

$$\nabla \cdot G = 0 \;. \tag{B.38}$$

B.7 Pull-back, Lie derivative and Killing vector fields

A differentiable curve $\gamma_t(p)$ defined at every point $p \in M$ defines a diffeomorphic map $\phi_t : M \to M$. If $\dot{\gamma}_t = v$ for a vector field $v \in TM$, ϕ_t is called the flow of v.

The pull-back of a function f defined on the target manifold at $\phi_t(p)$ is given by

$$(\phi_t^* f)(p) = (f \circ \phi_t)(p) \;. \tag{B.39}$$

This allows vectors defined at p to be pushed forward to $\phi_t(p)$ by

$$(\phi_{t*} v)(f) = v(\phi_t^* f) = v(f \circ \phi_t) \;. \tag{B.40}$$

Dual vectors w can then be pulled back by

$$(\phi_t^* w)(v) = w(\phi_{t*} v) \ . \tag{B.41}$$

For diffeomorphisms ϕ_t, the pull-back and the push-forward are inverse, $\phi_t^* = \phi_{t*}^{-1}$.

As for vectors and dual vectors, the pull-back and the push-forward can also be defined for tensors of arbitrary rank.

The Lie derivative of a tensor field T into direction v is given by the limit

$$\mathcal{L}_v T = \lim_{t\to 0} \frac{\phi_t^* T - T}{t} \ , \tag{B.42}$$

where ϕ_t is the flow of v. The Lie derivative quantifies how a tensor changes as the manifold is transformed by the flow of a vector field.

The Lie derivative is linear and obeys the Leibniz rule,

$$\mathcal{L}_x(y+z) = \mathcal{L}_x y + \mathcal{L}_x z \ , \quad \mathcal{L}_x(y\otimes z) = \mathcal{L}_x y \otimes z + y \otimes \mathcal{L}_x z \ . \tag{B.43}$$

It commutes with the contraction. Further important properties are

$$\mathcal{L}_{x+y} = \mathcal{L}_x + \mathcal{L}_y \ , \quad \mathcal{L}_{\lambda x} = \lambda \mathcal{L}_x \ , \quad \mathcal{L}_{[x,y]} = [\mathcal{L}_x, \mathcal{L}_y] \ . \tag{B.44}$$

The Lie derivative of a function f is the ordinary differential

$$\mathcal{L}_v f = v(f) = \mathrm{d}f(v) \ . \tag{B.45}$$

The Lie derivative and the differential commute,

$$\mathcal{L}_v \mathrm{d}f = \mathrm{d}\mathcal{L}_v f \ . \tag{B.46}$$

The Lie derivative of a vector x is the commutator

$$\mathcal{L}_v x = [v, x] \ . \tag{B.47}$$

By its commutation with contractions and the Leibniz rule, the Lie derivative of a dual vector w turns out to be

$$(\mathcal{L}_x w)(v) = x[w(v)] - w([x, v]) \ . \tag{B.48}$$

Lie derivatives of arbitrary tensors can be similarly derived. For example, if $g \in \mathcal{T}_2^0$, we find

$$(\mathcal{L}_x g)(v_1, v_2) = x[g(v_1, v_2)] - g([x, v_1], v_2) - g(v_1, [x, v_2]) \tag{B.49}$$

with $v_{1,2} \in TM$.

Killing vector fields K define isometries of the metric, i.e. the metric does not change under the flow of K. This implies the Killing equation

$$\mathcal{L}_K g = 0 \quad \Rightarrow \quad \nabla_i K_j + \nabla_j K_i = 0 \ . \tag{B.50}$$

B.8 Differential forms

Differential p-forms $\omega \in \bigwedge^p$ are totally antisymmetric tensor fields of rank $(0, p)$. Their components satisfy

$$\omega_{i_1 \dots i_p} = \omega_{[i_1 \dots i_p]} \ . \tag{B.51}$$

The exterior product $\wedge$ is defined by

$$\wedge : \bigwedge^p \times \bigwedge^q \to \bigwedge^{p+q} \ , \quad (\omega, \eta) \mapsto \omega \wedge \eta = \frac{(p+q)!}{p!q!} \mathcal{A}(\omega \otimes \eta) \ , \tag{B.52}$$

where $\mathcal{A}$ is the alternation operator

$$(\mathcal{A}t)(v_1, \dots, v_p) = \frac{1}{p!} \sum_\pi \operatorname{sgn}(\pi) t(v_{\pi(1)}, \dots, v_{\pi(p)}) \ . \tag{B.53}$$

On the vector space $\bigwedge$ of differential forms, the wedge product defines an associative, skew-commutative Grassmann algebra,

$$\omega \wedge \eta = (-1)^{pq} \, \eta \wedge \omega \ , \tag{B.54}$$

with $\omega \in \bigwedge^p$ and $\eta \in \bigwedge^q$.

A basis for the p-forms is

$$\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p} \ , \tag{B.55}$$

which shows that the dimension of $\bigwedge^p$ is

$$\dim \bigwedge^p = \begin{pmatrix} n \\ p \end{pmatrix} \ . \tag{B.56}$$

The interior product i_v is defined by

$$i : TM \times \bigwedge^p \to \bigwedge^{p-1} \ , \quad (v, \omega) \mapsto i_v(\omega) = \omega(v, \dots) \ . \tag{B.57}$$

In components, the interior product is given by

$$(i_v \omega)_{i_2 \dots i_p} = v^j \omega_{j i_2 \dots i_p} \ . \tag{B.58}$$

The exterior derivative turns p-forms ω into $(p+1)$-forms $\mathrm{d}\omega$,

$$\mathrm{d} : \bigwedge^p \to \bigwedge^{p+1} \ , \quad \omega \mapsto \mathrm{d}\omega = \sum_{i_1 < \dots < i_p} \mathrm{d}\omega_{i_1 \dots i_p} \wedge \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p} \ . \tag{B.59}$$

Accordingly, the components of the exterior derivative are given by partial derivatives,

$$(\mathrm{d}\omega)_{i_1, \dots, i_{p+1}} = (p+1) \, \partial_{[i_1} \omega_{i_2 \dots i_{p+1}]} \tag{B.60}$$

A differential form α is called exact if a differential form β exists such that $\alpha = \mathrm{d}\beta$. It is called closed if $\mathrm{d}\alpha = 0$.

B.9 Cartan's structure equations

Let $\{e_i\}$ be an arbitrary basis and $\{\theta^i\}$ its dual basis such that

$$\langle \theta^i, \mathrm{e}_j \rangle = \delta^i_j \ . \tag{B.61}$$

The connection forms $\omega^i_j \in \bigwedge^1$ are defined by

$$\nabla_v e_i = \omega^j_i(v)\, e_j \ . \tag{B.62}$$

In terms of Christoffel symbols, they can be expressed as

$$\omega^i_j = \Gamma^i{}_{kj}\ \theta^k \ . \tag{B.63}$$

They satisfy the antisymmetry relation

$$\mathrm{d}g_{ij} = \omega_{ij} + \omega_{ji} \ . \tag{B.64}$$

The covariant derivative of a dual basis vector is

$$\nabla_v \theta^i = -\omega^i_j(v)\, \theta^j \ . \tag{B.65}$$

Covariant derivatives of arbitrary vectors x and dual vectors α are then given by

$$\nabla_v x = \langle \mathrm{d}x^i + x^j \omega^i_j, v \rangle\, e_i \ , \quad \nabla_v \alpha = \langle \mathrm{d}\alpha_i - \alpha_j \omega^j_i, v \rangle\, \theta^i \tag{B.66}$$

or

$$\nabla x = e_i \otimes (\mathrm{d}x^i + x^j \omega^j_i) \ , \quad \nabla \alpha = \theta^i \otimes (\mathrm{d}\alpha_i - \alpha_j \omega^j_i) \ . \tag{B.67}$$

Torsion and curvature are expressed by the torsion 2-form $\Theta^i \in \bigwedge^2$ and the curvature 2-form $\Omega^i_j \in \bigwedge^2$ as

$$T(x,y) = \Theta^i(x,y)\, e_i \ , \quad \bar{R}(x,y) e_i = \Omega^j_i(x,y)\, e_j \ . \tag{B.68}$$

The torsion and curvature forms are related to the connection forms and the dual basis vectors by Cartan's structure equations

$$\Theta^i = \mathrm{d}\theta^i + \omega^i_k \wedge \theta^k \ , \quad \Omega^i_j = \mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j \ . \tag{B.69}$$

The components of the torsion and curvature tensors are determined by

$$\Theta^i = T^i_{jk}\, \theta^j \wedge \theta^k \ , \quad \Omega^i_j = \bar{R}^i{}_{jkl}\ \theta^k \wedge \theta^l \ . \tag{B.70}$$

B.10 Differential operators and integration

The Hodge star operator turns a p-form into an $(n-p)$-form,

$$* : \bigwedge^p \to \bigwedge^{n-p} , \quad \omega \mapsto *\omega . \tag{B.71}$$

If $\{e^i\}$ is an orthonormal basis of the dual space, the Hodge star operator is uniquely defined by

$$* \left(e^1 \wedge \ldots \wedge e^{i_p}\right) = e^{i_{p+1}} \wedge \ldots \wedge e^{i_n} , \tag{B.72}$$

where the indices $i_1 \ldots i_n$ appear in their natural order or a cyclic permutation thereof. For example, the coordinate differentials $\{\mathrm{d}x^i\}$ are an orthonormal dual basis in $\mathbb{R}^3$, and

$$*\mathrm{d}x^1 = \mathrm{d}x^2 \wedge \mathrm{d}x^3 , \quad *\mathrm{d}x^2 = \mathrm{d}x^3 \wedge \mathrm{d}x^1 , \quad *\mathrm{d}x^3 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 . \tag{B.73}$$

The codifferential is a differentiation lowering the order of a p-form by one,

$$\delta : \bigwedge^p \to \bigwedge^{p-1} , \quad \omega \mapsto \delta\omega , \tag{B.74}$$

which is defined by

$$\delta\omega = \mathrm{sgn}(g)(-1)^{n(p+1)} \,(*\mathrm{d}*)\, \omega . \tag{B.75}$$

It generalises the divergence of a vector field and thus has the components

$$(\delta\omega)_{i_2 \ldots i_p} = \frac{1}{\sqrt{|g|}} \partial_{i_1} \left(\sqrt{|g|}\, \omega^{i_1 i_2 \ldots i_p} \right) . \tag{B.76}$$

The Laplace-de Rham operator

$$\mathrm{d} \circ \delta + \delta \circ \mathrm{d} \tag{B.77}$$

generalises the Laplace operator.

The canonical volume form is an n-form given by

$$\eta = \sqrt{|g|}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n . \tag{B.78}$$

The integration of n-forms $\omega = f(x_1, \ldots, x_n)\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ over domains $D \subset M$ is defined by

$$\int_D \omega = \int_D f(x_1, \ldots, x_n)\, \mathrm{d}x^1 \ldots \mathrm{d}x^n . \tag{B.79}$$

Functions f are integrated by means of the canonical volume form,

$$\int_D f\eta = \int_D f \sqrt{|g|}\, \mathrm{d}x^1 \ldots \mathrm{d}x^n . \tag{B.80}$$

The theorems of Stokes and Gauss can be expressed as

$$\int_D \mathrm{d}\alpha = \int_{\partial D} \alpha , \quad \int_D \delta v^\flat \eta = \int_{\partial D} *v^\flat , \tag{B.81}$$

where $\alpha \in \bigwedge^{n-1}$ is an $(n-1)$-form and $v \in TM$ is a vector field.