Appendix A

Electrodynamics

A.1 Electromagnetic field tensor

Electric and magnetic fields are components of the antisymmetric field tensor

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \tag{A.1}$$

formed from the four-potential

$$A^{\mu} = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix} . \tag{A.2}$$

The field tensor can be conveniently summarised as

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & \vec{E}^{\top} \\ -\vec{E} & \mathcal{B} \end{pmatrix} \tag{A.3}$$

with

$$\mathcal{B}_{ij} = \epsilon_{ija} B^{a} . \tag{A.4}$$

Given the signature $(-, +, +, +)$ of the Minkowski metric, its associated rank-$(0, 2)$ tensor has the components

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -\vec{E}^{\top} \\ \vec{E} & \mathcal{B} \end{pmatrix} . \tag{A.5}$$

A.2 Maxwell's equations

The homogeneous Maxwell equations read

$$\partial_{[\alpha} F_{\beta\gamma]} = 0 . \tag{A.6}$$

For $\alpha = 0$, $(\beta, \gamma) = (1, 2)$, $(1, 3)$ and $(2, 3)$, this gives

$$\dot{\vec{B}} + c\vec{\nabla} \times \vec{E} = 0 , \tag{A.7}$$

and for $\alpha = 1$, $(\beta, \gamma) = (2, 3)$, we find

$$\vec{\nabla} \cdot \vec{B} = 0 \ . \tag{A.8}$$

The inhomogeneous Maxwell equations are

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu \ , \tag{A.9}$$

where

$$j^\mu = \begin{pmatrix} \rho c \\ \vec{j} \end{pmatrix} \tag{A.10}$$

is the four-current density. For $\mu = 0$ and $\mu = i$, (A.9) gives

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \ , \quad c\vec{\nabla} \times \vec{B} - \dot{\vec{E}} = 4\pi\vec{j} \ , \tag{A.11}$$

respectively.

With the definition (A.1) and the Lorenz gauge condition $\partial_\mu A^\mu = 0$, the inhomogeneous equations (A.9) can be written as

$$\Box A^\mu = -\frac{4\pi}{c} j^\mu \ , \tag{A.12}$$

where $\Box = -\partial_0^2 + \vec{\nabla}^2$ is the d'Alembert operator. The particular solution of the homogeneous equation is given by the convolution of the source with the retarded Greens function

$$G(t, t', \vec{x}, \vec{x}') = \frac{1}{\left|\vec{x} - \vec{x}'\right|} \delta\left(t - t' - \frac{\left|\vec{x} - \vec{x}'\right|}{c}\right) \ , \tag{A.13}$$

i.e. by

$$A^\mu(t, \vec{x}) = \frac{1}{c} \int \mathrm{d}^3x' \int \mathrm{d}t' G(t, t', \vec{x}, \vec{x}') \, j^\mu(t, \vec{x}') \ . \tag{A.14}$$

A.3 Lagrange density and energy-momentum tensor

The Lagrange density of the electromagnetic field coupled to matter is

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} A_\mu j^\mu \ , \tag{A.15}$$

from which Maxwell's equations follow by the Euler-Lagrange equations,

$$\partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} - \frac{\partial \mathcal{L}}{\partial A^\mu} = 0 \ . \tag{A.16}$$

From the Lagrange density of the free electromagnetic field,

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \ , \tag{A.17}$$

we find the energy-momentum tensor

$$T^{\mu\nu} = -2\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} + g^{\mu\nu}\mathcal{L} = \frac{1}{4\pi}\left(F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) . \tag{A.18}$$

From (A.3), we find first

$$F^{\alpha\beta} F_{\alpha\beta} = -2(\vec{E}^2 - \vec{B}^2) \ , \tag{A.19}$$

and the energy-momentum tensor can be written as

$$T^{\mu\nu} = \frac{1}{8\pi} \begin{pmatrix} \vec{E}^2 + \vec{B}^2 & 2\vec{E}^\top \mathcal{B}^\top \\ 2\mathcal{B}\vec{E} & -\vec{E}^2 - \vec{B}^2 + \mathcal{B}\mathcal{B}^\top \end{pmatrix} . \tag{A.20}$$

This yields the energy density

$$T^{00} = \frac{\vec{E}^2 + \vec{B}^2}{8\pi} \tag{A.21}$$

of the electromagnetic field and the Poynting vector

$$cT^{0i} = \frac{c}{4\pi} \vec{E} \times \vec{B} \ . \tag{A.22}$$