Chapter 13

Two Examples of Relativistic Astrophysics

13.1 Light bundles

13.1.1 Geodesic deviation

We had seen in (6.16) that the separation vector n between two geodesics out of a congruence evolves in a way determined by the equation of geodesic deviation or *Jacobi equation*

$$\nabla_u^2 n = \bar{R}(u, n)u \; , \tag{13.1}$$

where u is the tangent vector to the geodesics and $\bar{R}(u, n)u$ is the curvature as defined in (3.51).

Caution Recall that, as introduced in Chapter 6, a congruence is a bundle of world lines in this context. ◂

We apply this now to a light bundle, i.e. a congruence of light rays or null geodesics propagating from a source to an observer moving with a four-velocity u_o. Let k be the wave vector of the light rays, then the frequency of the light at the observer is

$$\omega_\mathrm{o} = \langle k, u_\mathrm{o} \rangle \; , \tag{13.2}$$

and we introduce the normalised wave vector $\tilde{k} = k/\omega_\mathrm{o}$ which satisfies $\langle \tilde{k}, u_\mathrm{o} \rangle = 1$. Since k is a null vector, so is $\tilde{k}$.

Next, we introduce a *screen* perpendicular to k and to u_o. It thus falls into the local three-space of the observer, where it is perpendicular to the light rays. Since it is two-dimensional, it can be spanned by two orthonormal vectors $E_{1,2}$, which are parallel-transported along the light bundle such that

$$\nabla_k E_i = 0 = \nabla_{\tilde{k}} E_i \quad (i = 1, 2) \; . \tag{13.3}$$

Notice that the parallel transport along a null geodesic implies that the E_i remain perpendicular to $\tilde{k}$,

$$\nabla_{\tilde{k}}\langle\tilde{k}, E_i\rangle = \langle\nabla_{\tilde{k}}\tilde{k}, E_i\rangle + \langle\tilde{k}, \nabla_{\tilde{k}}E_i\rangle = 0\ . \tag{13.4}$$

In a coordinate basis $\{e_\alpha\}$ and its conjugate dual basis $\{\theta^i\}$, they can be written as

$$E_i = E_i^\alpha e_\alpha \quad \text{with} \quad E_i^\alpha = \theta^\alpha(E_i)\ . \tag{13.5}$$

The separation vector n between rays of the bundle can now be expanded into the basis $E_{1,2}$,

$$n = n^\alpha e_\alpha = N^i E_i\ , \tag{13.6}$$

showing that its components n^α in the basis $\{e_\alpha\}$ are

$$n^\alpha = \theta^\alpha(n) = \theta^\alpha(N^i E_i) = N^i E_i^\alpha\ . \tag{13.7}$$

Substituting the normalised wave vector $\tilde{k}$ for the four-velocity u in the Jacobi equation (13.1), we first have

$$\nabla^2_{\tilde{k}} n = \bar{R}(\tilde{k}, n)\tilde{k}\ . \tag{13.8}$$

Writing $n = N^i E_i$ and using (13.3), we find

$$\nabla_{\tilde{k}} n = E_i \nabla_{\tilde{k}} N^i\ , \quad \nabla^2_{\tilde{k}} n = E_i \nabla^2_{\tilde{k}} N^i\ , \tag{13.9}$$

and thus

$$E_i \nabla^2_{\tilde{k}} N^i = \bar{R}(\tilde{k}, E_j)\tilde{k}\, N^j\ . \tag{13.10}$$

Finally, we multiply equation (13.10) with E^i from the left and use the orthonormality of the vectors $E_{1,2}$,

$$\left\langle E^i, E_j \right\rangle = \delta^i_j\ , \tag{13.11}$$

to find the equation

$$\nabla^2_{\tilde{k}} N^i = \left\langle E^i, \bar{R}\left(\tilde{k}, E_j\right)\tilde{k}\right\rangle N^j\ , \tag{13.12}$$

describing how the perpendicular cross section of a light bundle changes along the bundle.

13.1.2 Ricci and Weyl contributions

It will turn out convenient to introduce the *Weyl tensor* $\bar{C}$, whose components are determined by

$$\bar{R}_{\alpha\beta\gamma\delta} = \bar{C}_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha} - \frac{\mathcal{R}}{3} g_{\alpha[\gamma}g_{\delta]\beta}\ . \tag{13.13}$$

In contrast to the Riemann tensor, the Weyl tensor (representing the Weyl curvature) is trace-free, $\bar{C}^{\alpha}{}_{\beta\alpha\delta} = 0$, but otherwise has the same symmetries,

$$\bar{C}_{\alpha\beta\gamma\delta} = -\bar{C}_{\beta\alpha\gamma\delta} = -\bar{C}_{\alpha\beta\delta\gamma} = \bar{C}_{\gamma\delta\alpha\beta} \; . \tag{13.14}$$

Inserting the second, third, and fourth terms on the right-hand side of (13.12) into (13.12), we see that

$$\begin{aligned} g_{\alpha[\gamma}R_{\delta]\beta}\, E^{i\alpha}\tilde{k}^{\beta}\tilde{k}^{\gamma}E^{\delta}_{j} &= \frac{1}{2}\left(\left\langle E^{i},\tilde{k}\right\rangle R\left(E_{j},\tilde{k}\right) - \left\langle E^{i},E_{j}\right\rangle R\left(\tilde{k},\tilde{k}\right)\right) \; , \\ g_{\beta[\gamma}R_{\delta]\alpha}\, E^{i\alpha}\tilde{k}^{\beta}\tilde{k}^{\gamma}E^{\delta}_{j} &= \frac{1}{2}\left(\left\langle \tilde{k},\tilde{k}\right\rangle R\left(E^{i},E_{j}\right) - \left\langle E_{j},\tilde{k}\right\rangle R\left(E^{i},\tilde{k}\right)\right) \; , \\ g_{\alpha[\gamma}g_{\delta]\beta}\, E^{i\alpha}\tilde{k}^{\beta}\tilde{k}^{\gamma}E^{\delta}_{j} &= \frac{1}{2}\left(\left\langle E^{i},\tilde{k}\right\rangle \left\langle E_{j},\tilde{k}\right\rangle - \left\langle E^{i},E_{j}\right\rangle \left\langle \tilde{k},\tilde{k}\right\rangle\right) \; . \end{aligned} \tag{13.15}$$

Defining further the 2×2 matrix C with the components

$$C^{i}_{j} := \bar{C}_{\alpha\beta\gamma\delta}\, E^{i\alpha}\tilde{k}^{\beta}\tilde{k}^{\gamma}E^{\delta}_{j} \; , \tag{13.16}$$

and using $\langle E^{i}, \tilde{k}\rangle = 0 = \langle \tilde{k}, \tilde{k}\rangle$ together with (13.11), we find that we can write (13.12) as

$$\nabla^{2}_{\tilde{k}} N^{i} = \left(-\frac{1}{2}\,\delta^{i}_{j}\, R\left(\tilde{k},\tilde{k}\right) + C^{i}_{j}\right) N^{j} \; . \tag{13.17}$$

13.2 Gravitational lensing

13.2.1 The optical tidal matrix

The evolution of the bundle's perpendicular cross section can thus be described by a matrix $\mathcal{T}$,

$$\nabla^{2}_{\tilde{k}} \begin{pmatrix} N^{1} \\ N^{2} \end{pmatrix} = \mathcal{T} \begin{pmatrix} N^{1} \\ N^{2} \end{pmatrix} \; , \tag{13.18}$$

which, according to (13.17), can be written in the form

$$\mathcal{T} = -\frac{1}{2}\, R\left(\tilde{k},\tilde{k}\right) \mathbb{1}_{2} + C \; . \tag{13.19}$$

Some further insight can be gained by extracting the trace-free part from $\mathcal{T}$. Since the trace is

$$\mathrm{Tr}\, \mathcal{T} = -R\left(\tilde{k},\tilde{k}\right) + \mathrm{Tr}\, C \; , \tag{13.20}$$

the trace-free part of $\mathcal{T}$ is

$$\mathcal{T} - \frac{1}{2}\,\mathrm{Tr}\; \mathcal{T}\; \mathbb{1}_{2} = C - \frac{1}{2}\,\mathrm{Tr}\; C\; \mathbb{1}_{2} =: \Gamma \; , \tag{13.21}$$

?

Why is there a factor of $1/2$ in front of the trace on the left-hand side of (13.21)?

where we have defined the *shear matrix* Γ. Notice that the symmetries (13.14) imply that C is symmetric,

$$C_{ij} = \bar{C}_{\alpha\beta\gamma\delta} E_i^\alpha \tilde{k}^\beta \tilde{k}^\gamma E_j^\delta = \bar{C}_{\delta\gamma\beta\alpha} E_j^\delta \tilde{k}^\gamma \tilde{k}^\beta E_i^\delta = C_{ji} \; . \tag{13.22}$$

Thus, Γ is also symmetric and has only the two independent components

$$\gamma_1 = \bar{C}_{\alpha\beta\gamma\delta} \tilde{k}^\beta \tilde{k}^\gamma \left(E_1^\alpha E_1^\delta - E_2^\alpha E_2^\delta \right) \; , \quad \gamma_2 = \bar{C}_{\alpha\beta\gamma\delta} \tilde{k}^\beta \tilde{k}^\gamma E_1^\alpha E_2^\delta \; . \tag{13.23}$$

Optical tidal matrix

Summarising, we define three scalars, the *shear components* $\gamma_{1,2}$ from (13.23) and the *convergence*

$$\kappa := -\frac{1}{2} \left[R\left(\tilde{k}, \tilde{k}\right) - \mathrm{Tr}\; C \right] \; , \tag{13.24}$$

in terms of which the matrix $\mathcal{T}$ can be brought into the form

$$\mathcal{T} = \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix} \; . \tag{13.25}$$

this is called the *optical tidal matrix*.

The effect of the optical tidal matrix becomes obvious if we start with a light bundle with circular cross section, for which the components N^i of the distance vector can be written as

$$\begin{pmatrix} N^1 \\ N^2 \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \; , \tag{13.26}$$

where φ is the polar angle on the screen spanned by the vectors $E_{1,2}$. Before we apply the optical tidal matrix, we rotate it into its principal-axis frame,

$$\begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix} \to \begin{pmatrix} \kappa + \gamma & 0 \\ 0 & \kappa - \gamma \end{pmatrix} \tag{13.27}$$

with $\gamma^2 = \gamma_1^2 + \gamma_2^2$, which shows that it maps the circle onto a curve outlined by the vector

$$\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} (\kappa + \gamma)\cos\varphi \\ (\kappa - \gamma)\sin\varphi \end{pmatrix} \; . \tag{13.28}$$

This is an ellipse with semi-major axis $\kappa + \gamma$ and semi-minor axis $\kappa - \gamma$, because obviously

$$\frac{x^2}{(\kappa + \gamma)^2} + \frac{y^2}{(\kappa - \gamma)^2} = \cos^2\varphi + \sin^2\varphi = 1 \; . \tag{13.29}$$

Thus, for $\gamma = 0$, the originally circular cross section remains circular, with κ being responsible for isotropically expanding or shrinking it, while the light bundle is elliptically deformed if $\gamma \neq 0$.

13.2.2 Homogeneous and isotropic spacetimes

In an isotropic spacetime, it must be impossible to single out preferred directions. This implies that $\gamma = 0$ then, because otherwise the principal-axis frame of the optical tidal matrix would break isotropy. If the spacetime is homogeneous, this must hold everywhere, so that we can specialise

$$\mathcal{T} = \kappa \mathbb{1}_2 \, , \tag{13.30}$$

with κ defined in (13.24). Since the propagation equation (13.18) for the light bundle is then isotropic, we replace $N^i = D$ for $i = 1, 2$ and write

$$\nabla_{\tilde{k}}^2 D = \kappa D \, . \tag{13.31}$$

Moreover, we see that

$$G\left(\tilde{k}, \tilde{k}\right) = \left(R - \frac{\mathcal{R}}{2} g\right)\left(\tilde{k}, \tilde{k}\right) = R\left(\tilde{k}, \tilde{k}\right) \tag{13.32}$$

because $\tilde{k}$ is a null vector. Thus, we can put

$$\kappa = -\frac{1}{2} G\left(\tilde{k}, \tilde{k}\right) = -\frac{4\pi\mathcal{G}}{c^4} T\left(\tilde{k}, \tilde{k}\right) \, , \tag{13.33}$$

using Einstein's field equations in the second step.

?

Why does the cosmological constant Λ not appear in the expression (13.33) for the convergence?

Next, we can insert the energy-momentum tensor (12.53) for a perfect fluid,

$$T = \left(\rho + \frac{p}{c^2}\right) u^\flat \otimes u^\flat + p g \, , \tag{13.34}$$

and use the fact that fundamental observers (i.e. observers for whom the universe appears isotropic) have $u = c\partial_t$ and $u^\flat = -c\mathrm{d}t$.

The frequency of the light measured by a hypothetical fundamental observer moving with four-velocity u and placed between the source and the final observer is $\langle k, u \rangle$. Due to our definition of $\tilde{k} = k/\omega_\mathrm{o}$, and because of the cosmological redshift (12.73), we can write

$$\left\langle \tilde{k}, u \right\rangle = \frac{\langle k, u \rangle}{\omega_\mathrm{o}} = \frac{\omega}{\omega_\mathrm{o}} = 1 + z \, , \tag{13.35}$$

where ω_o is the frequency measured by the final observer, and z is the redshift relative to the final observer.

Thus, we find

$$\kappa = -\frac{4\pi\mathcal{G}}{c^2}\left(\rho + \frac{p}{c^2}\right)(1 + z)^2 = -\frac{4\pi\mathcal{G}}{c^2} \frac{\rho + p/c^2}{a^2} \, , \tag{13.36}$$

where a is the scale factor of the metric inserted according to (12.73), setting $a = 1$ at the time of observation.

If $p \ll \rho c^2$, the density scales like a^{-3} as shown in (12.62), and then

$$\kappa = -\frac{4\pi\mathcal{G}}{c^2}\rho_0 a^{-5} \; . \tag{13.37}$$

We still need to choose a suitable affine curve parameter λ along the fiducial light ray. In terms of λ, the tangent vector $\tilde{k}$ is given by

$$\tilde{k} = \frac{\mathrm{d}x}{\mathrm{d}\lambda} \; . \tag{13.38}$$

Since we have normalised $\tilde{k}$ such that $\langle\tilde{k}, u\rangle = 1 + z$ as shown in (13.35), we must have

$$\left\langle\frac{\mathrm{d}x}{\mathrm{d}\lambda}, u\right\rangle = \frac{\mathrm{d}x^0}{\mathrm{d}\lambda} = \frac{c\mathrm{d}t}{\mathrm{d}\lambda} = 1 + z = a^{-1} \; , \tag{13.39}$$

where $u = \partial_t$ was used in the second step. Thus, the curve parameter must be related to the cosmic time by $\mathrm{d}\lambda = ca\mathrm{d}t$. Then, observing that $\mathrm{d}a = \dot{a}\mathrm{d}t$, we find

$$\mathrm{d}\lambda = ca\mathrm{d}t = \frac{ca\mathrm{d}a}{\dot{a}} = \frac{c\mathrm{d}a}{\dot{a}/a} \; . \tag{13.40}$$

With this result, we can rewrite

$$\nabla_{\tilde{k}} D = \tilde{k}^\alpha \nabla_\alpha D = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\partial_\alpha D = \frac{\mathrm{d}D}{\mathrm{d}\lambda} \; , \tag{13.41}$$

and the propagation equation (13.31) becomes

$$D'' = \kappa D \; , \tag{13.42}$$

where the prime indicates the derivative with respect to the affine parameter λ.

Equation (13.42) can be further simplified to reveal its very intuitive meaning. From the metric in the form (12.83), we see that radially propagating light rays must satisfy

$$c\mathrm{d}t = \pm a\mathrm{d}w \; , \tag{13.43}$$

where w is the radial distance coordinate defined in (12.81). The sign can be chosen depending on whether the distance should grow with increasing time (i.e. into the future) or with decreasing time (i.e. into the past), but it is irrelevant for our consideration. We choose $c\mathrm{d}t = a\mathrm{d}w$ and therefore, with (13.40),

$$\mathrm{d}\lambda = ca\mathrm{d}t = a^2\mathrm{d}w \; . \tag{13.44}$$

In addition to replacing the affine parameter λ by the comoving radial coordinate w, we consider now the propagation of the *comoving* diameter

D/a, i.e. the diameter with the expansion of the universe divided out. Substituting $\mathrm{d}w$ for $\mathrm{d}\lambda$, we first see that

$$\frac{\mathrm{d}^2}{\mathrm{d}w^2}\left(\frac{D}{a}\right) = a^2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[a^2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{D}{a}\right)\right] = a^2\frac{\mathrm{d}}{\mathrm{d}\lambda}(aD' - a'D)$$
$$= a^2(aD'' - a''D) \ . \tag{13.45}$$

Next, we use (13.40) to write

$$a'' = \frac{\mathrm{d}a'}{\mathrm{d}\lambda} = \frac{\dot{a}}{ca}\frac{\mathrm{d}a'}{\mathrm{d}a} = \frac{\dot{a}}{c^2 a}\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\dot{a}}{a}\right) = \frac{1}{2c^2}\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\dot{a}}{a}\right)^2 \ , \tag{13.46}$$

which enables us to insert Friedmann's equation (12.55) in the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\mathcal{G}}{3}\frac{\rho_0}{a^3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2} \tag{13.47}$$

to find

$$a'' = \frac{\mathrm{d}a'}{\mathrm{d}\lambda} = -\frac{4\pi\mathcal{G}}{c^2}\rho_0 a^{-4} + ka^{-3} = \kappa a + ka^{-3} \ , \tag{13.48}$$

inserting κ from (13.37).

Finally, we substitute $D'' = \kappa D$ from (13.31) and a'' from (13.48) into (13.45) and obtain an intuitive result.

Propagation equation for the bundle diameter

In a spatially homogeneous and isotropic spacetime, the *comoving* diameter D of a light bundle obeys the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}w^2}\left(\frac{D}{a}\right) = a^3\kappa D - a^2 D\left(\kappa a + ka^{-3}\right)$$
$$= -k\left(\frac{D}{a}\right) \ , \tag{13.49}$$

which is a simple oscillator equation.

?

The derivation of the behaviour of light bundles from Friedmann's equation suggests that it should be possible to derive Friedmann's equation from the behaviour of light bundles. Is it?

Equation (13.49) is now easily solved. We set the boundary conditions such that the bundle emerges from a source point, hence $D = 0$ at the source, and that it initially expands linearly with the radial distance w, hence $\mathrm{d}(D/a)/\mathrm{d}w = 1$ there. Then, the solution of (13.49) is

$$D = af_k(w) = a\begin{cases} k^{-1/2}\sin\left(k^{1/2}w\right) & (k > 0) \\ w & (k = 0) \\ |k|^{-1/2}\sinh\left(|k|^{1/2}w\right) & (k < 0) \end{cases} \ , \tag{13.50}$$

with $f_k(w)$ defined in (12.82).

This shows that the diameter of the bundle increases linearly if space is flat, diverges hyperbolically if space is negatively curved, and expands and shrinks as a sine if space is positively curved.

13.3 The Tolman-Oppenheimer-Volkoff solution

13.3.1 Relativistic hydrostatics

We now consider an axially symmetric, static solution of Einstein's field equations in presence of matter. As usual for an axisymmetric solution, we can work in the Schwarzschild tetrad (8.40), in which the energy-momentum tensor of a perfect fluid,

$$T = T_{\mu\nu}\,\theta^\mu \otimes \theta^\nu \quad \text{with} \quad T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu + p g_{\mu\nu} \,, \tag{13.51}$$

simplifies to

$$T_{\mu\nu} = \mathrm{diag}(\rho c^2, p, p, p) \tag{13.52}$$

because $u = u^0 e_0 = e_0$ in the static situation we are considering.

It has been shown in the In-depth box "Ideal hydrodynamics in general relativity" on page 175 that the relativistic Euler equation is

$$(\rho c^2 + p)\nabla_u u = -c^2 \mathrm{d}p^\sharp - u\nabla_u p \,, \tag{13.53}$$

which had been derived by contracting the local conservation equation

$$\nabla \cdot T = 0 \tag{13.54}$$

with the perpendicular projection tensor $\pi^\perp = \mathbb{1}_4 + u \otimes u^\flat$.

Specialising (13.53) to our situation, we use (8.9) to see that

$$\nabla_u u = \langle \mathrm{d}u^\alpha + u^\beta \omega^\alpha_\beta, u \rangle e_\alpha = \langle u^0 \omega^1_0, u \rangle e_1 = c^2 a' \mathrm{e}^{-b} e_1 \,, \tag{13.55}$$

because the only non-vanishing of the connection forms ω^α_0 for a static, axially symmetric spacetime is

$$\omega^1_0 = a' \mathrm{e}^{-b}\,\theta^0 \,, \tag{13.56}$$

as shown in (8.50). Moreover, in the static situation, $\nabla_u p = 0$.

The pressure gradient $\mathrm{grad}p = \mathrm{d}p^\sharp$ is

$$\mathrm{grad}p = \mathrm{d}p^\sharp = p' \mathrm{d}r^\sharp = p' \mathrm{e}^{-b}\,(\theta^1)^\sharp = p' \mathrm{e}^{-b}\, e_1 \,. \tag{13.57}$$

?

Compare the relativistic hydrostatic equation to its non-relativistic counterpart in Newtonian gravity.

Relativistic hydrostatic equation

Substituting (13.55) and (13.57) into (13.53) yields

$$(\rho c^2 + p) a' = -p' \quad \Rightarrow \quad a' = -\frac{p'}{\rho c^2 + p} \,, \tag{13.58}$$

which is the relativistic hydrostatic equation.

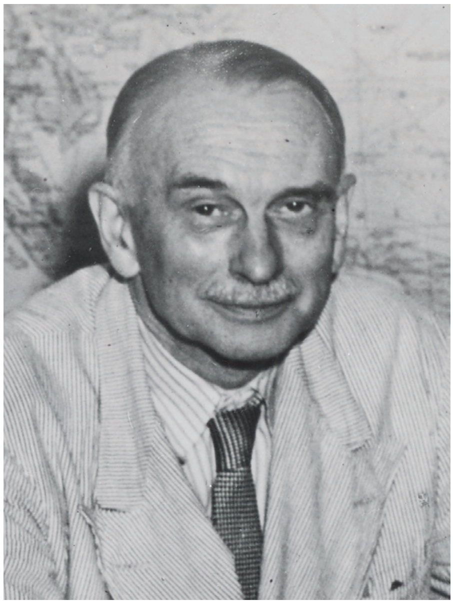

Figure 13.1 Richard C. Tolman (1881–1948), US-American physicist. Source: Wikipedia

13.3.2 The Tolman-Oppenheimer-Volkoff equation

With the components of the Einstein tensor given in (8.60) and the energy-momentum tensor (13.52), the two independent field equations read

$$\begin{aligned} -\frac{1}{r^2} + \mathrm{e}^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) &= -\frac{8\pi\mathcal{G}}{c^2}\rho \\ -\frac{1}{r^2} + \mathrm{e}^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) &= \frac{8\pi\mathcal{G}}{c^4}p\ . \end{aligned} \tag{13.59}$$

The first of these equations is equivalent to

$$\left(r\mathrm{e}^{-2b}\right)' = 1 - \frac{8\pi\mathcal{G}}{c^2}\rho r^2\ . \tag{13.60}$$

Integrating, and using the mass

$$M(r) = 4\pi\int_0^r \rho(r')r'^2\mathrm{d}r'\ , \tag{13.61}$$

shows that the function b is determined by

$$\mathrm{e}^{-2b} = 1 - \frac{2m}{r}\ , \quad m := \frac{\mathcal{G}M(r)}{c^2}\ . \tag{13.62}$$

If we subtract the first from the second field equation (13.59), we find

$$\frac{2\mathrm{e}^{-2b}}{r}(a' + b') = \frac{8\pi\mathcal{G}}{c^4}(\rho c^2 + p) \tag{13.63}$$

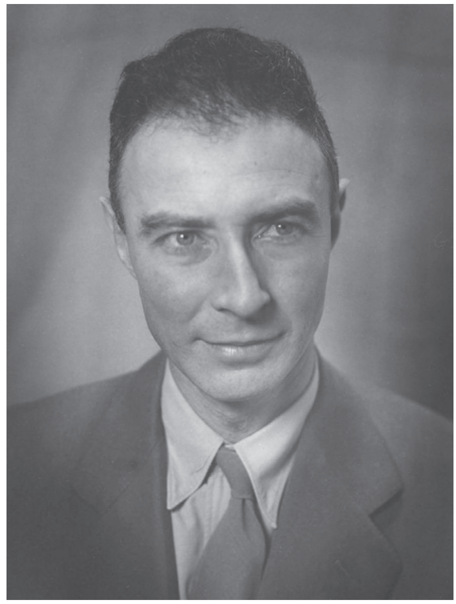

Figure 13.2 J. Robert Oppenheimer (1904–1967), US-American physicist. Source: Wikimedia Commons

or

$$a' = -b' + \frac{4\pi\mathcal{G}}{c^4} e^{2b} (\rho c^2 + p) r \, . \tag{13.64}$$

On the other hand, (13.62) gives

$$-2b' e^{-2b} = \frac{2m}{r^2} - \frac{2m'}{r} = \frac{2m}{r^2} - \frac{8\pi\mathcal{G}}{c^2} \rho r \, , \tag{13.65}$$

or

$$b' = \left(\frac{4\pi\mathcal{G}}{c^2} \rho r - \frac{m}{r^2} \right) e^{2b} \, , \tag{13.66}$$

which allows us to write (13.64) as

$$a' = \left(\frac{m}{r^2} + \frac{4\pi\mathcal{G}}{c^4} p r \right) e^{2b} = \frac{m + 4\pi\mathcal{G} p r^3 / c^4}{r(r - 2m)} \, . \tag{13.67}$$

Tolman-Oppenheimer-Volkoff equation

But the hydrostatic equation demands (13.58), which we combine with (13.67) to find

$$-p' = \frac{(\rho c^2 + p)(m + 4\pi\mathcal{G} p r^3 / c^4)}{r(r - 2m)} \, . \tag{13.68}$$

This is the *Tolman-Oppenheimer-Volkoff equation* for the pressure gradient in a relativistic star.

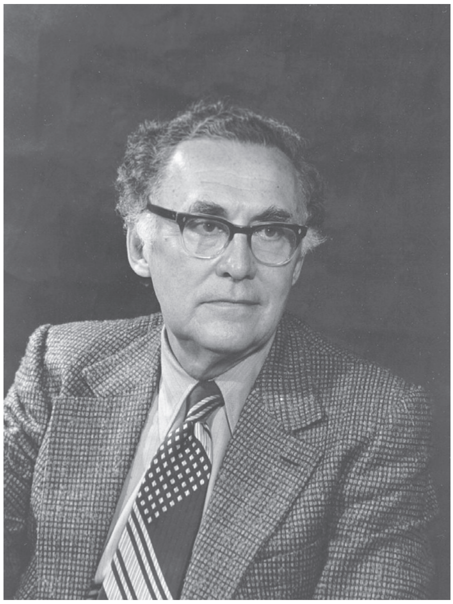

Figure 13.3 George M. Volkoff (1914–2000), Canadian physicist. Source: Wikipedia

This equation generalises the hydrostatic Euler equation in Newtonian physics, which reads for a spherically-symmetric configuration

$$-p' = \frac{\mathcal{G}M\rho}{r^2} = \frac{m\rho c^2}{r^2} \; . \tag{13.69}$$

This shows that gravity acts on $\rho c^2 + p$ instead of ρ alone, the pressure itself adds to the source of gravity, and gravity increases more strongly than $\propto r^{-2}$ towards the centre of the star.

13.4 The mass of non-rotating neutron stars

Neutron stars are a possible end product of the evolution of massive stars. When such stars explode as supernovae, they may leave behind an object with a density so high that protons and electrons combine to neutrons in the process of inverse β decay. Objects thus form which consist of matter with nuclear density

$$\rho_0 \approx 5 \cdot 10^{14}\,\mathrm{g\,cm^{-3}} \; . \tag{13.70}$$

A greatly simplified, yet instructive solution to the Tolman-Oppenheimer-Volkoff equation can be found assuming a constant density

$$\rho(r) = \begin{cases} \rho_0 & (r \le r_0) \\ 0 & (r > r_0) \end{cases} \tag{13.71}$$

throughout the star, with r_0 representing the stellar radius. Introducing the length scale

$$\lambda_0 := \left(\frac{4\pi G}{c^2}\rho_0\right)^{-1/2} , \tag{13.72}$$

scaling the pressure p with the central energy density

$$q := \frac{p}{\rho_0 c^2} \tag{13.73}$$

and introducing $x := r/\lambda_0$, we can transform the Tolman-Oppenheimer-Volkoff equation (13.68) to

$$-\frac{\mathrm{d}q}{\mathrm{d}x} = \frac{(1+q)(1+3q)x}{3-2x^2} . \tag{13.74}$$

Separating variables, setting the scaled pressure to $q = q_0$ at $x = 0$, and adopting $q_0 = 1/3$ as appropriate for an ultrarelativistic gas, we can integrate (13.74) to find

$$q(x) = \frac{2-\sqrt{9-6x^2}}{\sqrt{9-6x^2}-6} . \tag{13.75}$$

?

Introducing the length scale λ_0 from (13.72) and the dimensionless pressure q from (13.75) into the Tolman-Oppenheimer-Volkoff equation (13.68), confirm (13.74) and solve it to arrive at (13.75).

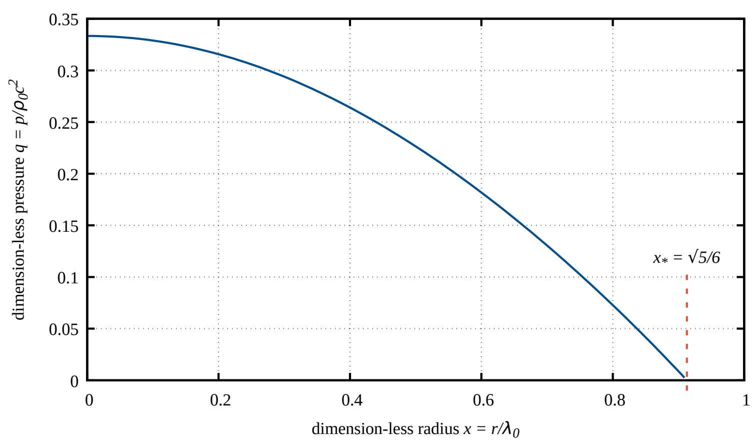

Figure 13.4 Pressure profile obtained from the Tolman-Oppenheimer-Volkoff equation for a homogeneous star.

The pressure falls to zero at $x_* = \sqrt{5/6}$, which defines the stellar radius $r_* = x_*\lambda_0$ and a stellar mass M_* of

$$M_* = \frac{4\pi}{3} r_*^3 \rho_0 . \tag{13.76}$$

With the nuclear density (13.75), we find

$$\lambda_0 = 14.7\,\mathrm{km} , \quad r_* = 13.4\,\mathrm{km} \quad \text{and} \quad M_* = 2.5\,M_\odot . \tag{13.77}$$

These are approximate results obtained under simplifying assumptions, which show however that at most a few solar masses can be stabilised by a relativistic gas with nuclear density. Masses exceeding this limit will collapse into black holes.

Instead of a postface

As mentioned instead of a preface, these lectures aim at introducing the theory of general relativity, but cannot replace a comprehensive textbook. They can be summarised as follows:

- The main concern of the introduction in Chap. 1 is the equivalence principle and the consequence drawn from it that the light-cone structure, commonly expressed by the metric, needs to be flexible. Locally, in a freely-falling reference frame, special relativity must hold with its light-cone defined by the Minkowski metric. Since the directions of free fall will generally differ at different locations in spacetime, the metric needs to vary from place to place. Sufficiently flexible spacetimes are represented by differentiable manifolds.
- The mathematics on differentiable manifolds, i.e. differential geometry, is thus the adequate mathematical language for general relativity. Tangent and dual spaces provide vectors and dual vectors. Connections, or covariant derivatives, define how vectors can be moved along curves from one tangent space to another. Having chosen a connection, torsion and curvature can be defined. Chapters 2 and 3 serve this purpose.
- With these tools at hand, concepts of physics can be ported from Minkowskian spacetime to manifolds. The essential choice here is the identification of the line element of the metric with the proper time interval measured by an observer. Motion of test particles and light rays on geodesics follows from this choice, as described in Chap. 4.
- The Lie derivative defines how objects on a manifold change as the manifold itself is transformed. It is most important for specifying symmetry transformations of manifolds, generated by Killing vector fields. Differential forms allow coordinate-free differentiation and integration on manifolds. Chapter 5 introduces these concepts.
- Einstein's field equations are then motivated in two ways in Chap. 6, first via the gravitational tidal field and its relation to curvature, second via an action principle. Lovelock's theorems reveal the remarkable uniqueness of the field equations derived therefrom.
- In the remainder of the lecture, several classes of solutions of Einstein's field equations are discussed. In Chap. 7, the field equations are linearised, leading to the various effects of weak gravitational fields, among them gravitational light deflection, gravitomagnetic frame-dragging and gravitational waves. The

diffeomorphism invariance of general relativity and the gauge freedom following from it are an important mathematical side-line of this chapter.

- The Schwarzschild solution, its derivation, properties, its maximal continuation, and its causal structure are the subjects of Chapters 8, 9, and 10. Chapter 11 adds charge and angular momentum to the solution and offers a first look into the consequences. Thermodynamics of black holes is briefly introduced there.

- Chapter 12 shows how the Friedmann equations of spatially homogeneous and isotropic cosmology follow from Einstein's field equations. It thus describes the root of a specialised cosmology lecture which typically begins with these equations and their premises. Similarly, Chap. 13 begins with light propagation through general space-times and later focuses on the evolution of light bundles in Friedmann cosmologies. Cosmic gravitational lensing by large-scale structures begins with the optical tidal matrix defined there and is typically again the subject of more specialised lectures. Finally, the Tolman-Oppenheimer-Volkoff equation is derived as the generally-relativistic analog to the hydrostatic equation of hydrostatic stellar models in Newtonian gravity.

If these lectures lay the foundation for studying more detailed textbooks and reading the research literature, they serve their intended purpose.