Chapter 11

Charged, Rotating Black Holes

11.1 The Reissner-Nordström solution

11.1.1 Energy-momentum tensor of electric charge

The Schwarzschild solution is a very important exact solution of Einstein's vacuum equations, but we expect that real objects collapsing to become black holes may be charged and rotating. We shall now generalise the Schwarzschild solution into these two directions.

First, we consider a static, axially-symmetric solution of Einstein's equations in the presence of an electromagnetic charge q at the origin of the Schwarzschild coordinates, i.e. at $r = 0$. The electromagnetic field will then also be static and axially symmetric.

Expressing the field tensor in the Schwarzschild tetrad (8.40), we thus expect the Faraday 2-form (5.86) to be

$$F = -\frac{q}{r^2}\, c\mathrm{d}t \wedge \mathrm{d}r = -\frac{q}{r^2}\, \mathrm{e}^{-a-b}\, \theta^0 \wedge \theta^1 \ . \tag{11.1}$$

We shall verify below that $a = -b$ also for a Schwarzschild solution with charge, so that the exponential factor will become unity later.

?

Why would the Faraday-2-form be given by (11.1)? Recall the meaning of the components of the electromagnetic field tensor.

The electromagnetic energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right] \tag{11.2}$$

is now easily evaluated. Since the only non-vanishing component of $F_{\mu\nu}$ is F_{01} and the metric is diagonal in the Schwarzschild tetrad, $g = \mathrm{diag}(-1, 1, 1, 1)$, we have

$$F_{\alpha\beta}F^{\alpha\beta} = F_{01}F^{01} + F_{10}F^{10} = -2F_{01}^2 \ . \tag{11.3}$$

Using this, we find the components of the energy-momentum tensor

$$
\begin{aligned}
T^{00} &= \frac{1}{4\pi}\left[F^{01}F^{0}{}_{1} - \frac{1}{2}F_{01}^2\right] = \frac{1}{8\pi}F_{01}^2 = \frac{q^2}{8\pi r^4}\mathrm{e}^{-2(a+b)} \;, \\
T^{11} &= \frac{1}{4\pi}\left[F^{10}F^{1}{}_{0} + \frac{1}{2}F_{01}^2\right] = -\frac{q^2}{8\pi r^4}\mathrm{e}^{-2(a+b)} = -T^{00} \;, \\
T^{22} &= \frac{1}{8\pi}F_{01}^2 = \frac{q^2}{8\pi r^4}\mathrm{e}^{-2(a+b)} = T^{33} \;.
\end{aligned}
\tag{11.4}
$$

11.1.2 The Reissner-Nordström metric

Inserting these expressions instead of zero into the right-hand side of Einstein's field equations yields, with (8.60),

$$
\begin{aligned}
G_{00} &= \frac{1}{r^2} - \mathrm{e}^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) = \frac{8\pi\mathcal{G}}{c^4}T_{00} = \frac{\mathcal{G}q^2}{c^4 r^4}\mathrm{e}^{-2(a+b)} \;, \\
G_{11} &= -\frac{1}{r^2} + \mathrm{e}^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) = -\frac{8\pi\mathcal{G}}{c^4}T_{00} = -G_{00} \;.
\end{aligned}
\tag{11.5}
$$

Adding these two equations, we find $a' + b' = 0$, which implies $a + b = 0$ because the functions have to tend to zero at infinity. This confirms that we can identify $c\mathrm{d}t \wedge \mathrm{d}r = \theta^0 \wedge \theta^1$ and write $F_{01} = q/r^2$.

Analogous to (8.62), we note that the first of equations (11.5) with $a = -b$ is equivalent to

$$
\left(r\mathrm{e}^{-2b}\right)' = 1 - \frac{\mathcal{G}q^2}{c^4 r^2} \;,
\tag{11.6}
$$

which gives

$$
\mathrm{e}^{-2b} = \mathrm{e}^{2a} = 1 - \frac{2m}{r} + \frac{\mathcal{G}q^2}{c^4 r^2} \;,
\tag{11.7}
$$

if we use $-2m$ as the integration constant as for the neutral Schwarzschild solution.

?

Verify that the term $\mathcal{G}q^2/c^4$ has the unit of a squared length in Gaussian cgs units. Recall that the unit of the electric charge in the cgs system is $\mathrm{g}^{1/2}\mathrm{cm}^{3/2}\mathrm{s}^{-1}$.

Reissner-Nordström solution

Defining

$$
\Delta \equiv r^2 - 2mr + \frac{\mathcal{G}q^2}{c^4} \;,
\tag{11.8}
$$

we thus obtain the line element for the metric of a charged Schwarzschild black hole,

$$
\mathrm{d}s^2 = -\frac{\Delta}{r^2}\mathrm{d}t^2 + \frac{r^2\mathrm{d}r^2}{\Delta} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \;.
\tag{11.9}
$$

This is the *Reissner-Nordström solution*.

Of course, for $q = 0$, the Reissner-Nordström solution returns to the Schwarzschild solution.

Figure 11.1 Hans J. Reissner (right; 1874–1967), German engineer, mathematician and physicist. Source: Wikipedia

Before we proceed, we should verify that Maxwell's equations are indeed satisfied. First, we note that the Faraday 2-form (11.1) is exact because it is the exterior derivative of the 1-form

$$A = -\frac{q}{r}\,c\mathrm{d}t\;, \quad \mathrm{d}A = \frac{q}{r^2}\,\mathrm{d}r\wedge c\mathrm{d}t = -\frac{q}{r^2}\,c\mathrm{d}t\wedge\mathrm{d}r = -\frac{q}{r^2}\,\theta^0\wedge\theta^1\;. \tag{11.10}$$

Thus, since $\mathrm{d}\circ\mathrm{d} = 0$, $\mathrm{d}F = \mathrm{d}^2A = 0$, so that the homogeneous Maxwell equations are satisfied.

Moreover, we notice that

$$*F = \frac{q}{r^2}\,\theta^2\wedge\theta^3\;, \tag{11.11}$$

which is easily verified using (5.75),

$$\begin{aligned} *(\theta^0\wedge\theta^1) &= \frac{1}{2}g^{00}g^{11}\varepsilon_{01\alpha\beta}\,\theta^\alpha\wedge\theta^\beta \\ &= -\frac{1}{2}\left(\theta^2\wedge\theta^3 - \theta^3\wedge\theta^2\right) = -\theta^2\wedge\theta^3\;. \end{aligned} \tag{11.12}$$

Inserting the Schwarzschild tetrad from (8.40) yields

$$*F = q\sin\vartheta\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi = -\mathrm{d}\left(q\cos\vartheta\mathrm{d}\varphi\right)\;, \tag{11.13}$$

which shows, again by $\mathrm{d}\circ\mathrm{d} = 0$, that $\mathrm{d}(*F) = 0$, hence also $(*\mathrm{d}*)F = 0$ and $\delta F = 0$, so that also the inhomogeneous Maxwell equations (in vacuum!) are satisfied.

Figure 11.2 Gunnar Nordström (1881–1923), Finnish physicist. Source: Wikipedia

11.2 The Kerr-Newman solution

11.2.1 The Kerr-Newman metric

The formal derivation of the metric of a rotating black hole is a formidable task which we cannot possibly demonstrate during this lecture. We thus start with general remarks on the expected form of the metric and then immediately quote the metric coefficients without deriving them.

In presence of angular momentum, we expect the spherical symmetry of the Schwarzschild solution to be broken. Instead, we expect that the solution must be *axisymmetric*, with the axis fixed by the angular momentum. Moreover, we seek to find a stationary solution.

Then, the group $\mathbb{R} \times SO(2)$ must be an isometry of the metric, where $\mathbb{R}$ represents the stationarity and $SO(2)$ the (two-dimensional) rotations about the symmetry axis. Expressing these symmetries, there must be a time-like Killing vector field k and another Killing vector field m which is tangential to the orbits of $SO(2)$.

These two Killing vector fields span the tangent spaces of the two-dimensional submanifolds which are the orbits of $\mathbb{R} \times SO(2)$, i.e. cylinders.

We can choose adapted coordinates t and φ such that $k = \partial_t$ and $m = \partial_\varphi$. Then, the metric ${}^{(4)}g$ of four-dimensional spacetime can be decomposed as

$$ {}^{(4)}g = g_{ab}(x^i)\,\mathrm{d}x^a \otimes \mathrm{d}x^b + g_{ij}(x^k)\,\mathrm{d}x^i \otimes \mathrm{d}x^j \;, \tag{11.14} $$

where indices $a, b = 0, 1$ indicate the coordinates on the orbits of $\mathbb{R} \times SO(2)$, and indices $i, j, k = 2, 3$ the others. Note that, due to the symmetry imposed, the remaining metric coefficients can only depend on the coordinates x^i.

A stationary, axi-symmetric spacetime (M, g) can thus be foliated into $M = \Sigma \times \Gamma$, where Σ is diffeomorphic to the orbits of $\mathbb{R} \times SO(2)$, and the metric coefficients in adapted coordinates can only depend on the coordinates of Γ. We write

$$ {}^{(4)}g = \sigma + g \tag{11.15} $$

and have

$$ \sigma = \sigma_{ab}(x^i)\,\mathrm{d}x^a \otimes \mathrm{d}x^b \;. \tag{11.16} $$

The coefficients σ_{ab} are scalar products of the two Killing vector fields k and m,

$$ (\sigma_{ab}) = \begin{pmatrix} -\langle k, k\rangle & \langle k, m\rangle \\ \langle k, m\rangle & \langle m, m\rangle \end{pmatrix} , \tag{11.17} $$

and we abbreviate the determinant of σ by

$$ \rho \equiv \sqrt{-\det\sigma} = \sqrt{\langle k, k\rangle\langle m, m\rangle + \langle k, m\rangle^2} \;. \tag{11.18} $$

Without proof, we now give the metric of a stationary, axially-symmetric solution of Einstein's field equations for either vacuum or an electromagnetic field. We first define the auxiliary quantities

$$ \begin{aligned} \Delta &:= r^2 - 2mr + Q^2 + a^2 \;, \quad \rho^2 := r^2 + a^2\cos^2\vartheta \;, \\ \Sigma^2 &:= (r^2 + a^2)^2 - a^2\Delta\sin^2\vartheta \;. \end{aligned} \tag{11.19} $$

Moreover, we need appropriately scaled expressions Q and a for the charge q and the angular momentum L of the black hole, which are given by

$$ Q^2 := \frac{\mathcal{G}q^2}{c^4} \;, \quad a := \frac{L}{Mc} = \frac{\mathcal{G}L}{mc^3} \tag{11.20} $$

and both Q and a have the dimension of a length.

?

Verify that a also has the dimension of a length, like Q.

Kerr-Newman solution

With these definitions, we can write the coefficients of the metric for a charged, rotating black hole in the form

$$
\begin{aligned}
g_{tt} &= -1 + \frac{2mr - Q^2}{\rho^2} = \frac{a^2 \sin^2\vartheta - \Delta}{\rho^2} , \\
g_{t\varphi} &= -\frac{2mr - Q^2}{\rho^2}\, a \sin^2\vartheta = -\frac{r^2 + a^2 - \Delta}{\rho^2}\, a \sin^2\vartheta , \\
g_{rr} &= \frac{\rho^2}{\Delta} , \quad g_{\vartheta\vartheta} = \rho^2 , \quad g_{\varphi\varphi} = \frac{\Sigma^2}{\rho^2} \sin^2\vartheta . \qquad (11.21)
\end{aligned}
$$

Evidently, for $a = 0 = Q$, $\rho = r$, $\Delta = r^2 - 2mr$ and $\Sigma = r^2$ and we return to the Schwarzschild solution (8.67). For $a = 0$, we still have $\rho = r$ and $\Sigma = r^2$, but $\Delta = r^2 - 2mr + Q^2$ as in (11.8), and we return to the Reissner-Nordström solution (11.9). For $Q = 0$, we obtain the *Kerr solution* for a rotating, uncharged black hole, and for $a \neq 0$ and $Q \neq 0$, the solution is called *Kerr-Newman solution*, named after Roy Kerr and Ezra Newman.

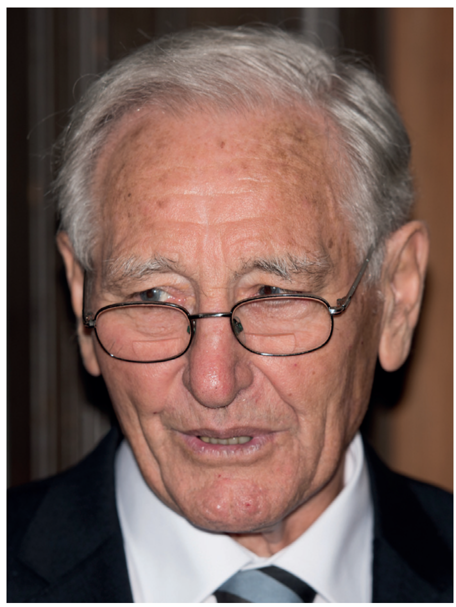

Figure 11.3 Roy Kerr (born 1934), New Zealand mathematician. Source: Wikimedia Commons

Also without derivation, we quote that the vector potential of the rotating, charged black hole is given by the 1-form

$$
A = -\frac{qr}{\rho^2} \left(c\mathrm{d}t - a \sin^2\vartheta \mathrm{d}\varphi \right) , \qquad (11.22)
$$

from which we obtain the Faraday 2-form

?

Why is it plausible for A to have the form (11.22)? Beginning there, derive the Faraday-2-form (11.23) yourself.

$$F = \mathrm{d}A = \frac{q}{\rho^4}\left(r^2 - a^2\cos^2\vartheta\right)\mathrm{d}r \wedge \left(c\mathrm{d}t - a\sin^2\vartheta\mathrm{d}\varphi\right) + \frac{2qra}{\rho^4}\sin\vartheta\cos\vartheta\mathrm{d}\vartheta \wedge \left[\left(r^2 + a^2\right)\mathrm{d}\varphi - ac\mathrm{d}t\right] . \tag{11.23}$$

For $a = 0$, this trivially returns to the field (11.1) for the Reissner-Nordström solution. Sufficiently far away from the black hole, such that $a \ll r$, we can approximate to first order in a/r and write

$$F = \frac{q}{r^2}\mathrm{d}r \wedge \left(c\mathrm{d}t - a\sin^2\vartheta\mathrm{d}\varphi\right) + \frac{2qa}{r}\sin\vartheta\cos\vartheta\mathrm{d}\vartheta \wedge \mathrm{d}\varphi . \tag{11.24}$$

The field components far away from the black hole can now be read off the result (11.24). Using the orthonormal basis

$$e_t = \partial_{ct} , \quad e_r = \partial_r , \quad e_\vartheta = \frac{1}{r}\partial_\vartheta , \quad e_\varphi = \frac{1}{r\sin\vartheta}\partial_\varphi , \tag{11.25}$$

we find in particular for the radial component B_r of the magnetic field

$$B_r = F(e_\vartheta, e_\varphi) = \frac{2qa}{r^3}\cos\vartheta . \tag{11.26}$$

In the limit of large r, the electric field thus becomes that of a point charge q at the origin, and the magnetic field attains a characteristic dipolar structure.

?

Find the remaining components of the electromagnetic field of the Kerr-Newman solution.

The Biot-Savart law of electrodynamics implies that a charge q with mass M on a circular orbit with angular momentum $\vec{L}$ has the magnetic dipole moment

$$\vec{\mu} = g\,\frac{q\vec{L}}{2Mc} , \tag{11.27}$$

where g is the *gyromagnetic moment*.

A magnetic dipole moment μ creates the dipole field

$$\vec{B} = \frac{3(\vec{\mu}\cdot\vec{e}_r)\vec{e}_r - \vec{\mu}}{r^3} , \tag{11.28}$$

whose radial component is $B_r = \vec{B}\cdot\vec{e}_r = 2\vec{\mu}\cdot\vec{e}_r/r^3$. A comparison of the radial magnetic field from (11.26) with this expression reveals the following interesting result:

Magnetic dipole moment of a charged, rotating black hole

The magnetic dipole moment of a charged, rotating black holes is

$$\vec{\mu} = q\vec{a} = \frac{q\vec{L}}{Mc} = 2\,\frac{q\vec{L}}{2Mc} , \tag{11.29}$$

showing that charged, rotating black holes have a gyromagnetic moment of $g = 2$.

?

What is the gyromagnetic moment of an electron?

11.2.2 Schwarzschild horizon, ergosphere and Killing horizon

By construction, the Kerr-Newman metric (11.21) has the two Killing vector fields $k = \partial_t$, expressing the stationarity of the solution, and $m = \partial_\varphi$, which expresses its axial symmetry.

Since the metric coefficients in adapted coordinates satisfy

$$g_{tt} = \langle k, k\rangle \ , \quad g_{\varphi\varphi} = \langle m, m\rangle \ , \quad g_{t\varphi} = \langle k, m\rangle \ , \tag{11.30}$$

they have an invariant meaning which will now be clarified.

Let us consider an observer moving with r = const. and ϑ = const. with uniform angular velocity ω. If her four-velocity is u, then

$$\omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\dot{\varphi}}{\dot{t}} = \frac{u^\varphi}{u^t} \tag{11.31}$$

for a static observer at infinity, whose proper time can be identified with the coordinate time t. Correspondingly, we can expand the four-velocity as

$$u = u^t\partial_t + u^\varphi\partial_\varphi = u^t\left(\partial_t + \omega\partial_\varphi\right) = u^t(k + \omega m) \ , \tag{11.32}$$

inserting the Killing vector fields. Let

$$|k + \omega m| \equiv \left(-\langle k + \omega m, k + \omega m\rangle\right)^{1/2} \tag{11.33}$$

define the norm of $k + \omega m$, then the four-velocity is

$$u = \frac{k + \omega m}{|k + \omega m|} \ . \tag{11.34}$$

Obviously, $k+\omega m$ is a time-like Killing vector field, at least at sufficiently large distances from the black hole. Since then

$$\begin{aligned}\langle k + \omega m, k + \omega m\rangle &= \langle k, k\rangle + \omega^2\langle m, m\rangle + 2\omega\langle k, m\rangle \\ &= g_{tt} + \omega^2 g_{\varphi\varphi} + 2\omega g_{t\varphi} < 0 \ ,\end{aligned} \tag{11.35}$$

$k + \omega m$ becomes light-like for angular velocities

$$\omega_\pm = \frac{-g_{t\varphi} \pm \sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}{g_{\varphi\varphi}} \ . \tag{11.36}$$

If we define

$$\Omega \equiv -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = -\frac{\langle k, m\rangle}{\langle m, m\rangle} \ , \tag{11.37}$$

we can write (11.36) as

$$\omega_\pm = \Omega \pm \sqrt{\Omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \ . \tag{11.38}$$

For an interpretation of Ω, we note that freely-falling test particles on radial orbits have zero angular momentum and thus $\langle u, m \rangle = 0$. By (11.34), this implies

$$0 = \langle k + \omega m, m \rangle = g_{t\varphi} + \omega g_{\varphi\varphi} \tag{11.39}$$

and thus

$$\omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \Omega \tag{11.40}$$

according to the definition (11.37). This shows that Ω is the angular velocity of a test particle falling freely towards the black hole on a radial orbit.

The minimum angular velocity ω_- from (11.38) vanishes if and only if $g_{tt} = \langle k, k \rangle = 0$, i.e. if the Killing vector field k turns light-like. With (11.21), this is so where

$$0 = a^2 \sin^2 \vartheta - \Delta = 2mr - r^2 - Q^2 - a^2 \cos^2 \vartheta \,, \tag{11.41}$$

i.e. at the radius

$$r_0 = m + \sqrt{m^2 - Q^2 - a^2 \cos^2 \vartheta} \,. \tag{11.42}$$

Static limit in Kerr spacetime

The radius r_0 marks the *static limit* of Kerr spacetime: for an observer at this radius to remain static with respect to observers at infinity (i.e. with respect to the "fixed stars"), she would have to move with the speed of light. At smaller radii, observers cannot remain static against the drag of the rotating black hole.

We have seen in (4.48) that the light emitted by a source with four-velocity u_s is seen by an observer with four-velocity u_o with a redshift

$$\frac{\nu_\mathrm{o}}{\nu_\mathrm{s}} = \frac{\langle \tilde{k}, u_\mathrm{o} \rangle}{\langle \tilde{k}, u_\mathrm{s} \rangle} \,, \tag{11.43}$$

where $\tilde{k}$ is the wave vector of the light.

Observers at rest in a stationary spacetime have four-velocities proportional to the Killing vector field k,

$$u = \frac{k}{\sqrt{-\langle k, k \rangle}} \,, \quad \text{hence} \quad k = \sqrt{-\langle k, k \rangle}\, u \,. \tag{11.44}$$

We have seen in (5.36) that the projection of a Killing vector K on a geodesic γ is constant along that geodesic, $\nabla_{\dot\gamma} \langle \dot\gamma, K \rangle = 0$. The light ray propagating from the source to the observer is a null geodesic with $\dot\gamma = \tilde{k}$, hence

$$\nabla_{\tilde{k}} \langle \tilde{k}, k \rangle = 0 \tag{11.45}$$

and $\langle\tilde{k}, k\rangle_\mathrm{s} = \langle\tilde{k}, k\rangle_\mathrm{o}$. Using this in a combination of (11.43) and (11.44), we obtain

$$\frac{\nu_\mathrm{o}}{\nu_\mathrm{s}} = \frac{\langle\tilde{k}, k\rangle_\mathrm{o}}{\langle\tilde{k}, k\rangle_\mathrm{s}} \frac{\sqrt{-\langle k, k\rangle_\mathrm{s}}}{\sqrt{-\langle k, k\rangle_\mathrm{o}}} = \frac{\sqrt{-\langle k, k\rangle_\mathrm{s}}}{\sqrt{-\langle k, k\rangle_\mathrm{o}}} \,. \tag{11.46}$$

For an observer at rest far away from the black hole, $\langle k, k\rangle_\mathrm{o} \approx -1$, and the redshift becomes

$$1 + z = \frac{\nu_\mathrm{s}}{\nu_\mathrm{o}} \approx \frac{1}{\sqrt{-\langle k, k\rangle_\mathrm{s}}} = (-g_{tt})^{-1/2} \,, \tag{11.47}$$

which tends to infinity as the source approaches the static limit.

The minimum and maximum angular velocities $\omega_\pm$ from (11.38) both become equal to Ω for

$$\Omega^2 = \left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)^2 = \frac{g_{tt}}{g_{\varphi\varphi}} \quad \Rightarrow \quad g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi} = 0 \,. \tag{11.48}$$

This equation means that the Killing field $\xi \equiv k + \Omega m$ turns light-like,

$$\begin{aligned} \langle\xi, \xi\rangle &= \langle k, k\rangle + 2\Omega\langle k, m\rangle + \Omega^2\langle m, m\rangle \\ &= g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi} = g_{tt} - 2\frac{g_{t\varphi}^2}{g_{\varphi\varphi}} + \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} \\ &= \frac{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}{g_{\varphi\varphi}} = 0 \,. \end{aligned} \tag{11.49}$$

Interestingly, writing the expression from (11.48) with the metric coefficients (11.21) leads to the simple result

$$g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi} = \Delta \sin^2\vartheta \,, \tag{11.50}$$

so that the condition (11.48) is equivalent to

$$0 = \Delta = r^2 - 2mr + Q^2 + a^2 \,, \tag{11.51}$$

which describes a spherical hypersurface with radius

$$r_H = m + \sqrt{m^2 - Q^2 - a^2} \,, \tag{11.52}$$

for which we choose the larger of the two solutions of (11.51).

By its definition (11.37), the angular velocity Ω on this hypersurface H can be written as

$$\Omega_H = -\left.\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right|_H = \left.\frac{a(2mr - Q^2)}{\Sigma^2}\right|_H = \frac{a(2mr_H - Q^2)}{(r_H^2 + a^2)^2} \,, \tag{11.53}$$

since $\Sigma^2 = (r^2 + a^2)^2$ because of $\Delta = 0$ at r_H. Because of (11.51), the numerator is $a(r_H^2 + a^2)$, and we find the following remarkable result:

Angular frequency of H

The hypersurface H is rotating with the constant angular velocity

$$\Omega_H = \frac{a}{r_H^2 + a^2} , \tag{11.54}$$

like a solid body.

Since the hypersurface H is defined by the condition $\Delta = 0$, its normal vectors are given by

$$\mathrm{grad}\Delta = \mathrm{d}\Delta^\sharp , \quad \mathrm{d}\Delta = 2(r - m)\mathrm{d}r . \tag{11.55}$$

Thus, the norm of the normal vectors is

$$\langle \mathrm{grad}\Delta, \mathrm{grad}\Delta \rangle = 4g^{rr}(r - m)^2 , \tag{11.56}$$

now, according to (11.21), $g^{rr} \propto \Delta = 0$ on the hypersurface, showing that H is a *null hypersurface*. Because of this fact, the tangent space to the null hypersurface H at any of its points is orthogonal to a null vector, and hence it does not contain time-like vectors.

Killing horizon and ergosphere

The surface H is called a *Killing horizon*. The hypersurface defined by the static limit is time-like, which means that it can be crossed in both directions, in contrast to the horizon H. The region in between the static limit and the Killing horizon is the *ergosphere*, in which k is space-like and no observer can be prevented from following the rotation of the black hole.

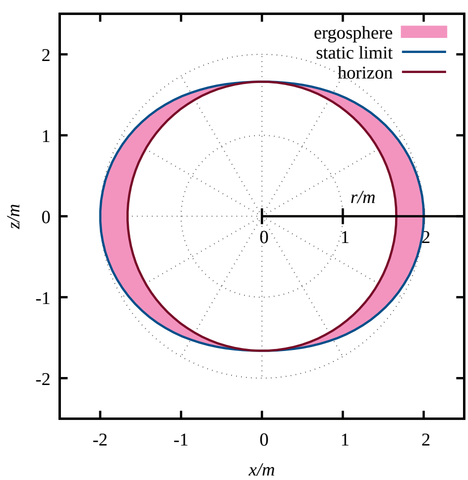

Figure 11.4 Static limit, horizon, and ergosphere for a Kerr black hole with $a = 0.75$.

Formally, the Kerr solution is singular where $\Delta = 0$, but this singularity can be lifted by a transformation to coordinates similar to the Eddington-Finkelstein coordinates for a Schwarzschild black hole.

11.3 Motion near a Kerr black hole

11.3.1 Kepler's third law

We shall now assume $q = 0$ and consider motion on a circular orbit in the equatorial plane. Thus $\dot{r} = 0$ and $\vartheta = \pi/2$, and

$$\Delta = r^2 - 2mr + a^2 \quad \text{and} \quad \rho = r\,, \tag{11.57}$$

further

$$\Sigma^2 = \left(r^2 + a^2\right)^2 - a^2\Delta = r^4 + a^2r^2 + 2ma^2r\,, \tag{11.58}$$

and the coefficients of the metric (11.21) become

$$\begin{aligned} g_{tt} &= -1 + \frac{2m}{r}\,, \quad g_{t\varphi} = -\frac{2ma}{r}\,, \\ g_{rr} &= \frac{r^2}{\Delta}\,, \quad g_{\vartheta\vartheta} = r^2\,, \\ g_{\varphi\varphi} &= \frac{\Sigma^2}{r^2} = r^2 + a^2 + \frac{2ma^2}{r}\,. \end{aligned} \tag{11.59}$$

?

If a particle starts orbiting in the equatorial plane, does it stay there?

Since $\dot{\vartheta} = 0$ and $\dot{r} = 0$, the Lagrangian reduces to

$$2\mathcal{L} = -\left(1 - \frac{2m}{r}\right)c^2\dot{t}^2 - \frac{4mac}{r}\dot{t}\dot{\varphi} + \left(r^2 + a^2 + \frac{2ma^2}{r}\right)\dot{\varphi}^2\,. \tag{11.60}$$

By the Euler-Lagrange equation for r and due to $\dot{r} = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{r}} = 0 = \frac{\partial\mathcal{L}}{\partial r}\,, \tag{11.61}$$

which yields, after multiplying with $r^2/\dot{t}^2$,

$$-mc^2 + 2mac\omega + (r^3 - ma^2)\omega^2 = 0 \tag{11.62}$$

where we have introduced the angular frequency ω according to (11.31).

Kepler's third law

Noticing that

$$r^3 - ma^2 = (r^{3/2} - m^{1/2}a)(r^{3/2} + m^{1/2}a)\,, \tag{11.63}$$

we can write the solutions as

$$\omega_\pm = \pm\frac{cm^{1/2}}{r^{3/2} \pm m^{1/2}a}\,. \tag{11.64}$$

This is Kepler's third law for a Kerr black hole: The angular velocity of a test particle depends on whether it is co-rotating with or counter-rotating against the black hole.

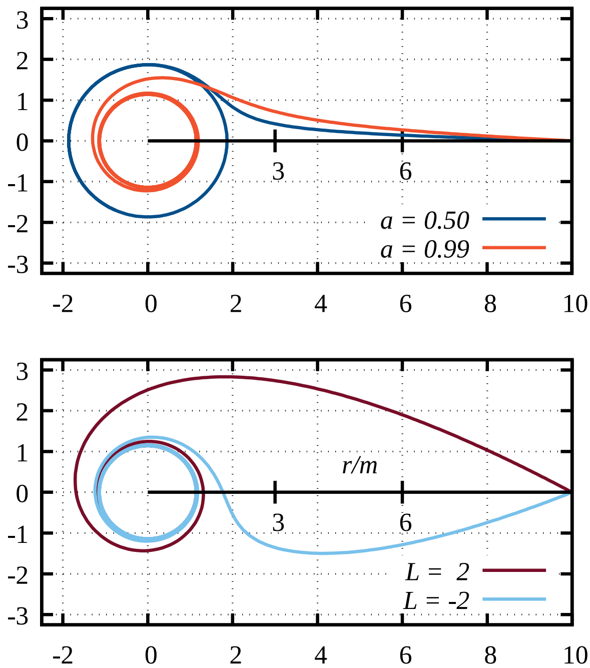

Figure 11.5 Trajectories of test particles in the equatorial plane of the Kerr metric. All orbits begin at $r = 10m$ and $\varphi = 0$. *Top*: Orbits with angular momentum $L = 0$ for $a = 0.5$ and $a = 0.9$. *Bottom*: orbits with angular momenta $L = \pm 2$ for $a = 0.99$.

11.3.2 Accretion flow onto a Kerr black hole

We now consider a stationary, axially-symmetric flow of a perfect fluid onto a Kerr black hole. Because of the symmetry constraints, the Lie derivatives of all physical quantities in the direction of the Killing vector fields $k = \partial_t$ and $m = \partial_\varphi$ need to vanish.

As in (11.32), the four-velocity of the flow is

$$u = u^t(k + \omega m) \ . \tag{11.65}$$

We introduce

$$e \equiv -\langle u, k\rangle = -u_t \ , \qquad j \equiv \langle u, m\rangle = u_\varphi \ , \qquad l \equiv \frac{j}{e} = -\frac{u_\varphi}{u_t} \tag{11.66}$$

and use

$$\begin{aligned} u_\varphi &= g_{t\varphi}u^t + g_{\varphi\varphi}u^\varphi = u^t\left(g_{t\varphi} + g_{\varphi\varphi}\omega\right) \\ u_t &= g_{tt}u^t + g_{t\varphi}u^\varphi = u^t\left(g_{tt} + g_{t\varphi}\omega\right) \end{aligned} \tag{11.67}$$

to see that

$$l = -\frac{g_{t\varphi} + \omega g_{\varphi\varphi}}{g_{tt} + \omega g_{t\varphi}} \quad \Leftrightarrow \quad \omega = -\frac{g_{t\varphi} + l g_{tt}}{g_{\varphi\varphi} + l g_{t\varphi}} \ . \tag{11.68}$$

Moreover, by the definition of l in (11.66) and ω in (11.31), and using $\langle u, u\rangle = u^t u_t + u^\varphi u_\varphi = -c^2$, we see that

$$\omega l = -\frac{u^\varphi u_\varphi}{u^t u_t} \quad \Rightarrow \quad u_t u^t = \frac{c^2}{\omega l - 1} \ . \tag{11.69}$$

Finally, using (11.67) and (11.69), we have

$$-u_t^2 = -u_t u^t (g_{tt} + g_{t\varphi}\omega) = c^2 \frac{g_{tt} + g_{t\varphi}\omega}{1 - \omega l} \ . \tag{11.70}$$

If we substitute ω from (11.68) here, we obtain after a short calculation

$$e^2 = u_t^2 = c^2 \frac{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}{g_{\varphi\varphi} + 2lg_{t\varphi} + l^2 g_{tt}} \ . \tag{11.71}$$

?

Repeat the calculations leading to (11.76) and (11.78) in components. Why can (11.77) be called a perpendicular projector? Can you confirm the non-relativistic limit (11.79)?

It is shown in the In-depth box "Ideal hydrodynamics in general relativity" on page 175 that the relativistic Euler equation reads

$$\left(\rho c^2 + p\right) \nabla_u u = -c^2 \mathrm{d}p^\sharp - u(p)u \ , \tag{11.80}$$

where ρc^2 and p are the density and the pressure of the ideal fluid. Applying this equation to the present case of a stationary flow, we first observe that

$$0 = \mathcal{L}_u p = u(p) \ , \tag{11.81}$$

thus the second term on the right-hand side of (11.80) vanishes.

Next, we introduce the dual vector $u^\flat$ belonging to the four-velocity u. In components, $(u^\flat)_\mu = g_{\mu\nu}u^\nu = u_\mu$. Then, from (5.32),

$$\left(\mathcal{L}_u u^\flat\right)_\mu = u^\nu \partial_\nu u_\mu + u_\nu \partial_\mu u^\nu = u^\nu \nabla_\nu u_\mu + u_\nu \nabla_\mu u^\nu \ , \tag{11.82}$$

where we have employed the symmetry of the connection ∇. This shows that

$$\mathcal{L}_u u^\flat = \nabla_u u^\flat \ . \tag{11.83}$$

Now, we introduce $f \equiv 1/u^t$ and compute $\mathcal{L}_{fu}u^\flat$ in two different ways. First, a straightforward calculation beginning with (5.24) shows that

$$\mathcal{L}_{fx} w = f \mathcal{L}_x w + w(x) \mathrm{d}f \ . \tag{11.84}$$

Specialising this result to $x = u$ and $w = u^\flat$ gives

$$\mathcal{L}_{fu} u^\flat = f \mathcal{L}_u u^\flat - c^2 \mathrm{d}f = f \nabla_u u^\flat - c^2 \mathrm{d}f \ , \tag{11.85}$$

making use of (11.83) in the last step.

?

Can you confirm (11.84)?

On the other hand, $fu = u/u^t = k + \omega m$ because of (11.65), which allows us to write

$$\mathcal{L}_{fu} u^\flat = \underbrace{\mathcal{L}_k u^\flat}_{=0} + \mathcal{L}_{\omega m} u^\flat \ . \tag{11.86}$$

In depth: Ideal hydrodynamics in general relativity

The relativistic continuity and Euler equations

Relativistic hydrodynamics begins with the vanishing divergence of the energy-momentum tensor, $\nabla \cdot T = 0$, demanded by Einstein's equations. Specialising the energy-momentum tensor to that of an ideal fluid with energy density ρc^2, pressure p and four-velocity u,

$$T = \left(\rho + \frac{p}{c^2}\right) u \otimes u + p g^{-1} , \tag{11.72}$$

we first find

$$0 = \left[u\left(\rho + \frac{p}{c^2}\right) + \left(\rho + \frac{p}{c^2}\right)\nabla \cdot u\right] u + \left(\rho + \frac{p}{c^2}\right)\nabla_u u + \mathrm{d}p^\sharp . \tag{11.73}$$

The first terms in brackets are proportional to the four-velocity u. Projecting $\nabla \cdot T$ on u, and taking $\langle u, u\rangle = -c^2$ into account, leads to

$$0 = -\left[u\left(\rho c^2 + p\right) + \left(\rho c^2 + p\right)\nabla \cdot u\right] + \left(\rho + \frac{p}{c^2}\right)\langle u, \nabla_u u\rangle + u(p) . \tag{11.74}$$

Now, since the connection is metric,

$$\nabla_u \langle u, u\rangle = 0 = 2\langle \nabla_u u, u\rangle , \tag{11.75}$$

and (11.74) turns into the *relativistic continuity equation*

$$u\left(\rho c^2\right) + \left(\rho c^2 + p\right)\nabla \cdot u = 0 . \tag{11.76}$$

If we project (11.73) instead into the three-space perpendicular to u by applying the perpendicular projector

$$\pi^\perp := \mathbb{1}_4 + c^{-2} u \otimes u^\flat , \tag{11.77}$$

the terms proportional to u drop out by construction. Further using (11.75) once more, we retain the *relativistic Euler equation*

$$\left(\rho c^2 + p\right)\nabla_u u + c^2 \mathrm{d}p^\sharp + u(p) u = 0 . \tag{11.78}$$

In the non-relativistic limit, equations (11.76) and (11.78) simplify to the familiar expressions

$$\begin{aligned} \partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) &= 0 , \\ \partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} + \frac{\vec{\nabla} P}{\rho} &= 0 . \end{aligned} \tag{11.79}$$

Applying (11.83) once more gives

$$\mathcal{L}_{\omega m} u^\flat = \omega \mathcal{L}_m u^\flat + u^\flat(m)\mathrm{d}\omega \; . \tag{11.87}$$

Since the Lie derivative of $u^\flat$ in the direction m must vanish because of the axisymmetry, this means

$$\mathcal{L}_{fu} u^\flat = \mathcal{L}_{\omega m} u^\flat = u^\flat(m)\mathrm{d}\omega = \langle u, m\rangle \mathrm{d}\omega = j\mathrm{d}\omega \; . \tag{11.88}$$

Equating this to (11.85) gives

$$f\nabla_u u^\flat = c^2 \mathrm{d}f + j\mathrm{d}\omega \; . \tag{11.89}$$

However, we know from (11.69) that

$$f = (u^t)^{-1} = \frac{u_t(\omega l - 1)}{c^2} = \frac{e(1-\omega l)}{c^2} \; . \tag{11.90}$$

Inserting this into (11.89) yields

$$\begin{aligned} \frac{e(1-\omega l)}{c^2}\nabla_u u^\flat &= (1-\omega l)\mathrm{d}e - el\mathrm{d}\omega - e\omega \mathrm{d}l + j\mathrm{d}\omega \\ &= (1-\omega l)\mathrm{d}e - e\omega\mathrm{d}l \; , \end{aligned} \tag{11.91}$$

where we have used $el = j$ in the final step. Thus,

$$\nabla_u u^\flat = c^2\left(\mathrm{d}\ln e - \frac{\omega \mathrm{d}l}{1-\omega l}\right) \; . \tag{11.92}$$

Returning with this result to Euler's equation (11.80), we obtain

$$\frac{\mathrm{d}p}{\rho c^2 + p} = -\mathrm{d}\ln e + \frac{\omega\mathrm{d}l}{1-\omega l} \; , \tag{11.93}$$

which shows that surfaces of constant pressure are given by

$$\ln e - \int \frac{\omega \mathrm{d}l}{1-\omega l} = \text{const.} \tag{11.94}$$

Setting $\mathrm{d}l = 0$, i.e. defining a surface of constant l, makes the second term on the left-hand side vanish. In this case, find from (11.71)

$$\frac{g_{\varphi\varphi} + 2lg_{t\varphi} + l^2 g_{tt}}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}} = \frac{c^2}{e^2} = \text{const.} \tag{11.95}$$

Accretion tori

We now insert the metric coefficients (11.21) for the Kerr-Newman solution to obtain the surfaces of constant pressure and constant l. Assuming further $a = 0$, we obtain the isobaric surfaces of the accretion flow onto a Schwarzschild black hole. With

$$g_{tt} = -1 + \frac{2m}{r} , \quad g_{\varphi\varphi} = r^2 \sin^2 \vartheta , \quad g_{t\varphi} = 0 , \tag{11.96}$$

we find

$$\frac{r}{r - 2m} - \frac{l^2}{r^2 \sin^2 \vartheta} = \text{const.} \tag{11.97}$$

This describes toroidal surfaces around black holes, the so-called accretion tori.

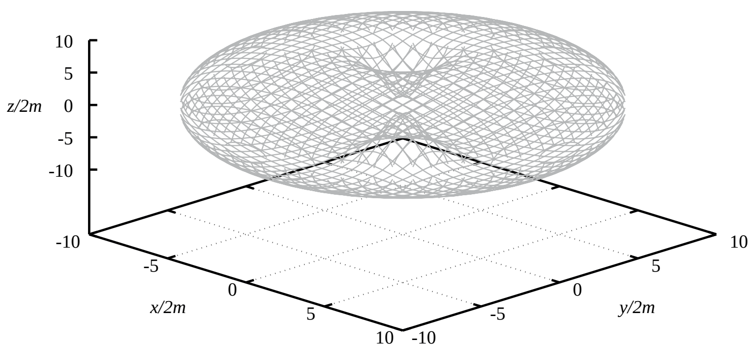

Figure 11.6 Accretion torus around a Schwarzschild black hole. The constants l and e were set to $l = 0.45$ and $e = 0.95\,c$ here.

11.4 Entropy and temperature of a black hole

It was realised by Stephen Hawking, Roger Penrose and Demetrios Christodoulou that the area of a possibly charged and rotating black hole, defined by

$$A := 4\pi\alpha := 4\pi \left(r_+^2 + a^2\right) \tag{11.98}$$

cannot shrink. Here, r_+ is the positive branch of the two solutions of (11.51),

$$r_\pm = m \pm \sqrt{m^2 - Q^2 - a^2} . \tag{11.99}$$

This led Jacob Bekenstein (1973) to the following consideration. If A cannot shrink, it reminds of the entropy as the only other quantity known in physics that cannot shrink. Could the area A have anything to do with an entropy that could be assigned to a black hole? In fact, this is much more plausible than it may appear at first sight. Suppose radiation disappears in a black hole. Without accounting for a possible entropy of

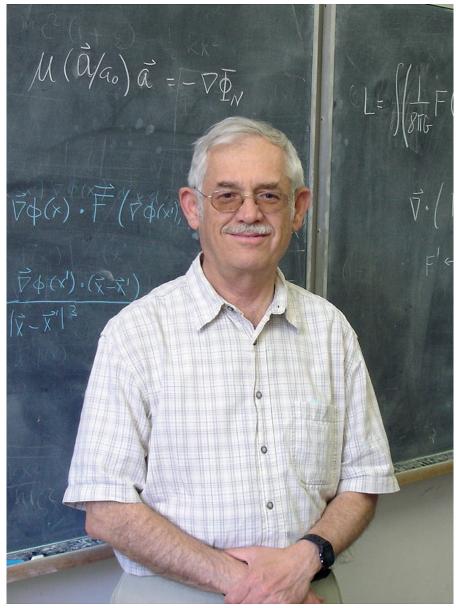

Figure 11.7 Jacob D. Bekenstein (1947–2015), Israeli-US-American physicist. Source: Wikipedia

the black hole, its entropy would be gone, violating the second law of thermodynamics. The same holds for gas accreted by the black hole: Its entropy would be removed from the outside world, leaving the entropy there lower than before.

If, however, the increased mass of the black hole led to a suitably increased entropy of the black hole itself, this violation of the second law could be remedied.

Analogy between area and entropy

Any mass and angular momentum swallowed by a black hole leads to an increase of the area (11.98), which makes it appear plausible that the area of a black hole might be related to its entropy.

Following Bekenstein (1973), we shall now work out this relation.

Beginning with the scaled area $\alpha = r_+^2 + a^2$ from (11.98), we have

$$d\alpha = 2\left(r_+ dr_+ + \vec{a}\cdot d\vec{a}\right) \ . \tag{11.100}$$

Inserting r_+ from (11.99) and using that

$$r_+ - r_- =: \delta r = 2\sqrt{m^2 - Q^2 - a^2} \ , \tag{11.101}$$

we find directly

$$d\alpha = 2\left[\frac{r_+\delta r + 2r_+ m}{\delta r}dm - \frac{2r_+ Q}{\delta r}dQ + \left(1 - \frac{2r_+}{\delta r}\right)\vec{a}\cdot d\vec{a}\right] \ . \tag{11.102}$$

The coefficients of dm and d$\vec{a}$ can be further simplified. Noting that

$$\delta r + 2m = r_+ - r_- + (r_+ + r_-) = 2r_+ \tag{11.103}$$

and

$$\delta r - 2r_+ = -(r_+ + r_-) = -2m\,, \tag{11.104}$$

we can bring (11.102) into the form

$$\mathrm{d}\alpha = \frac{4r_+^2}{\delta r}\mathrm{d}m - \frac{4r_+Q}{\delta r}\mathrm{d}Q - \frac{2m}{\delta r}\vec{a}\cdot\mathrm{d}\vec{a}\,. \tag{11.105}$$

Now, we need to take into account that the scaled angular momentum a can change by changing the angular momentum L or the mass m. From the definition (11.20), we have

$$\mathrm{d}\vec{a} = \frac{\mathcal{G}}{c^3}\left(\frac{\mathrm{d}\vec{L}}{m} - \frac{\vec{L}}{m^2}\mathrm{d}m\right) = \frac{\mathcal{G}}{c^3}\frac{\mathrm{d}\vec{L}}{m} - \frac{\vec{a}\mathrm{d}m}{m}\,. \tag{11.106}$$

Substituting this expression for da in (11.105), we find

$$\mathrm{d}\alpha = \frac{4\alpha}{\delta r}\mathrm{d}m - \frac{4r_+Q}{\delta r}\mathrm{d}Q - \frac{4\mathcal{G}}{c^3}\frac{\vec{a}\cdot\mathrm{d}\vec{L}}{\delta r}\,, \tag{11.107}$$

?

Confirm equation (11.107) by your own calculation.

Solving equation (11.107) for dm yields

$$\mathrm{d}m = \Theta\mathrm{d}\alpha + \Phi\mathrm{d}Q + \vec{\Omega}\cdot\mathrm{d}\vec{L} \tag{11.108}$$

with the definitions

$$\Theta := \frac{\delta r}{4\alpha}\,, \quad \Phi := \frac{r_+Q}{\alpha}\,, \quad \vec{\Omega} := \frac{\mathcal{G}}{c^3}\frac{\vec{a}}{\alpha}\,. \tag{11.109}$$

This reminds of the first law of thermodynamics if we tentatively associate m with the internal energy, α with the entropy and the remaining terms with external work.

Let us now see whether a linear relation between the entropy S and the area α will lead to consistent results. Thus, assume $S = \gamma\alpha$ with some constant γ to be determined. Then, a change $\delta\alpha$ will lead to a change $\delta S = \gamma\delta\alpha$ in the entropy.

Bekenstein showed that the minimal change of the effective area is twice the squared Planck length (1.5), thus

$$\delta\alpha = \frac{2\hbar\mathcal{G}}{c^3}\,. \tag{11.110}$$

On the other hand, he identified the minimal entropy change of the black hole with the minimal change of the Shannon entropy, which is derived from information theory and is

$$\delta S = k_\mathrm{B}\ln 2\,, \tag{11.111}$$

where the Boltzmann constant k_B was inserted to arrive at conventional units for the entropy. This could e.g. correspond to the minimal information loss when a single particle disappears in a black hole. Requiring

$$k_B \ln 2 = \delta S = \gamma \delta\alpha = \gamma \frac{2\hbar\mathcal{G}}{c^3} \tag{11.112}$$

fixes the constant γ to

$$\gamma = \frac{\ln 2}{2} \frac{k_B c^3}{\hbar\mathcal{G}} . \tag{11.113}$$

? Look up Bekenstein's arguments leading to the normalisation (11.113) of the black-hole entropy (Bekenstein, J., *Black Holes and Entropy*. Phys. Rev. D 7 (1973) 2333).

Bekenstein entropy

The Bekenstein entropy of a black hole is

$$S = \frac{\ln 2}{8\pi} \frac{c^3 k_B}{\hbar\mathcal{G}} A , \tag{11.114}$$

where A is the area of the black hole.

The quantity Θ defined in (11.109) must then correspond to the temperature of the black hole. From (11.108), we have on the one hand

$$\Theta = \left(\frac{\partial m}{\partial\alpha}\right)_{Q,L} . \tag{11.115}$$

If the association of a temperature should be consistent, it must on the other hand agree with the thermodynamic definition of temperature,

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_V . \tag{11.116}$$

For E, we can use the mass or rather

$$E = Mc^2 = \frac{mc^4}{\mathcal{G}} . \tag{11.117}$$

Then,

$$\left(\frac{\partial S}{\partial E}\right)_V = \frac{\mathcal{G}}{c^4}\left(\frac{\partial S}{\partial m}\right)_{Q,L} = \frac{\ln 2}{2} \frac{k_B}{\hbar c} \left(\frac{\partial\alpha}{\partial m}\right)_{Q,L} . \tag{11.118}$$

Inserting (11.115) now leads to an expression for the temperature.

Black-hole temperature

The analogy between the area of a black hole and entropy implies that black holes can be assigned the temperature

$$T = \frac{2}{\ln 2} \frac{\hbar c}{k_B} \Theta = \frac{2\pi}{\ln 2} \frac{\hbar c}{k_B} \frac{\delta r}{A} . \tag{11.119}$$

This result leads to a remarkable conclusion. If black holes have a temperature, they will radiate and thus lose energy or its mass equivalent. They can therefore evaporate. By the Stefan-Boltzmann law, the luminosity radiated by a black body of area A and temperature T is

$$L = \sigma A T^4 \ , \quad \sigma = \frac{\pi^2 k_\mathrm{B}^4}{60\hbar^3 c^2} \ . \tag{11.120}$$

For an uncharged and non-rotating black hole, $\delta r = 2m$ and $A = 16\pi m^2$, thus its temperature is

$$T = \frac{1}{4\ln 2}\frac{\hbar c}{k_\mathrm{B} m} = \frac{1}{4\ln 2}\frac{\hbar c^3}{k_\mathrm{B}\mathcal{G}M} \ . \tag{11.121}$$

Defining the *Planck temperature* by

$$T_\mathrm{Pl} := \frac{M_\mathrm{Pl}c^2}{k_\mathrm{B}} = 1.42 \cdot 10^{32}\,\mathrm{K} \tag{11.122}$$

in terms of the Planck mass $M_\mathrm{Pl} = 2.2 \cdot 10^{-5}$ g from (1.4), we can write

$$T = \frac{T_\mathrm{Pl}}{4\ln 2}\frac{M_\mathrm{Pl}}{M} \ . \tag{11.123}$$

For a black hole of solar mass, $M = M_\odot = 2.0 \cdot 10^{33}$ g, the temperature is

$$T = 5.6 \cdot 10^{-7}\,\mathrm{K} \ . \tag{11.124}$$