Chapter 10

Schwarzschild Black Holes

10.1 The singularity at $r = 2m$

10.1.1 Free fall towards the centre

Before we can continue discussing the physical meaning of the Schwarzschild metric, we need to clarify the nature of the singularity at the Schwarzschild radius, $r = 2m$. Upon closer inspection, it seems to lead to contradictory conclusions.

Let us begin with an astronaut falling freely towards the centre of the Schwarzschild spacetime along a radial orbit. Since $\dot\varphi = 0$, the angular momentum vanishes, $L = 0$, and the equation of motion (9.15) reads

$$\dot r^2 + c^2\left(1 - \frac{2m}{r}\right) = E^2 \; . \tag{10.1}$$

Suppose the astronaut was at rest at $r = R$, then $E^2 = c^2(1 - 2m/R)$ and $E^2 < c^2$, and we have

$$\frac{\dot r^2}{c^2} = \left(1 - \frac{2m}{R}\right) - \left(1 - \frac{2m}{r}\right) = 2m\left(\frac{1}{r} - \frac{1}{R}\right) \; , \tag{10.2}$$

which yields

$$\left[2m\left(\frac{1}{r} - \frac{1}{R}\right)\right]^{-1/2} \mathrm{d}r = c\mathrm{d}\tau \; , \tag{10.3}$$

where τ is the proper time.

This equation admits a parametric solution. Starting from

$$r = \frac{R}{2}(1 + \cos\eta) \; , \quad \mathrm{d}r = -\frac{R}{2}\sin\eta\mathrm{d}\eta \; , \tag{10.4}$$

we first see that

$$\frac{1}{r} - \frac{1}{R} = \frac{2}{R(1+\cos\eta)} - \frac{1}{R} = \frac{1}{R}\left(\frac{1-\cos\eta}{1+\cos\eta}\right)$$
$$= \frac{1}{R}\frac{(1-\cos\eta)^2}{\sin^2\eta}\,, \tag{10.5}$$

where we have used in the last step that $1 - \cos\eta^2 = \sin^2\eta$. This result allows us to translate (10.3) into

$$\frac{\sqrt{R}\sin\eta\mathrm{d}r}{\sqrt{2m}(1-\cos\eta)} = -\frac{R\sqrt{R}}{2\sqrt{2m}}\frac{\sin^2\eta\mathrm{d}\eta}{1-\cos\eta}$$
$$= -\sqrt{\frac{R^3}{8m}}(1+\cos\eta)\mathrm{d}\eta\,. \tag{10.6}$$

Integrating, we find that this solves (10.3) if

$$c\tau = \sqrt{\frac{R^3}{8m}}(\eta + \sin\eta)\,, \quad c\mathrm{d}\tau = \sqrt{\frac{R^3}{8m}}(1+\cos\eta)\,\mathrm{d}\eta\,. \tag{10.7}$$

At $\eta = 0$, the proper time is $\tau = 0$ and $r = R$, i.e. the proper time starts running when the free fall begins. Figure 10.1 shows the radial distance r as a function of the proper time τ for $R = 6m$, i.e. for an astronaut starting at rest at the innermost stable circular orbit.

?

Confirm the solution (10.7) for the proper time by your own calculation.

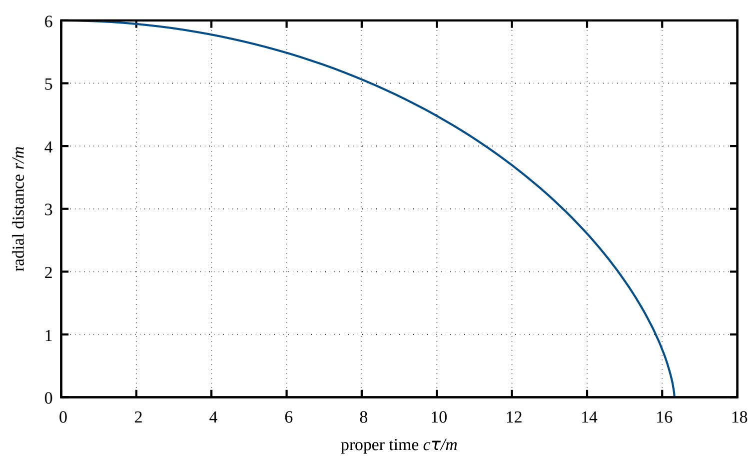

Figure 10.1 Radial distance r as a function of proper time τ for an astronaut falling towards the singularity of the Schwarzschild spacetime beginning at rest at the innermost stable circular orbit, $R = 6m$.

Free-fall time to the centre of a black hole

The centre $r = 0$ is reached when $\eta = \pi$, i.e. after the proper time

$$\tau_0 = \frac{\pi}{c}\sqrt{\frac{R^3}{8m}} \,. \tag{10.8}$$

This indicates that the observer falls freely within finite time "through" the singularity at $r = 2m$ without encountering any (kinematic) problem.

10.1.2 Problems with the Schwarzschild coordinates

However, let us now describe the radial coordinate r as a function of the *coordinate time t*. Using (9.13), we first find

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}\dot{t} = -\frac{\mathrm{d}r}{\mathrm{d}t}\frac{E/c}{1-2m/r} \,. \tag{10.9}$$

?

Before you read on, find the function $\bar{r}(r)$ yourself, given (10.10).

Next, we introduce a new, convenient radial coordinate $\bar{r}$ such that

$$\mathrm{d}\bar{r} = \frac{\mathrm{d}r}{1-2m/r} \,. \tag{10.10}$$

This condition can be integrated as follows,

$$\begin{aligned}\frac{\mathrm{d}r}{1-2m/r} &= \frac{r/2m-1+1}{r/2m-1}\mathrm{d}r = \mathrm{d}r + \frac{\mathrm{d}r}{r/2m-1} \\ &= \mathrm{d}r + 2m\,\mathrm{d}\ln\left(\frac{r}{2m}-1\right) \,,\end{aligned} \tag{10.11}$$

giving

$$\bar{r} = r + 2m\ln\left(\frac{r}{2m}-1\right) \,. \tag{10.12}$$

With this, we find

$$\dot{r} = -\frac{E/c}{1-2m/r}\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{E}{c}\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} \,, \tag{10.13}$$

and thus, from the equation of motion (10.1),

$$\frac{E^2}{c^2}\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\right)^2 = E^2 - c^2\left(1-\frac{2m}{r}\right) \,. \tag{10.14}$$

Approaching the Schwarzschild radius from outside, i.e. in the limit $r \to 2m+$, we have from (10.12)

$$\lim_{r\to 2m+}\bar{r} = \lim_{r\to 2m+} 2m\left[1+\ln\left(\frac{r}{2m}-1\right)\right] = -\infty \,, \tag{10.15}$$

However, in the same limit, the equation of motion says

$$\frac{E^2}{c^2}\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\right)^2 \to E^2 \,, \tag{10.16}$$

and thus

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} \to \pm c \,. \tag{10.17}$$

Of the two signs, we have to select the negative because of $\bar{r} \to -\infty$, as (10.15) shows. Therefore, an approximate solution of the equation of motion near the singularity is $\bar{r} \approx c(t-t_0)$ with an arbitrary constant t_0. To be specific, we set $t = 0$ when $r = 6m$, the radius of the innermost stable circular orbit defined in Sect. 9.2. There, $\bar{r}_0 = 2m(3 + \ln 2)$ according to (10.12) and thus

$$\bar{r} \approx -ct + 2m(3 + \ln 2) \,. \tag{10.18}$$

Substituting r for $\bar{r}$,

$$\begin{aligned} -ct + 2m(3 + \ln 2) &= r + 2m \,\ln\left(\frac{r}{2m} - 1\right) \\ &\approx 2m\left[1 + \ln\left(\frac{r}{2m} - 1\right)\right] \,. \end{aligned} \tag{10.19}$$

Free-fall *coordinate* time to the centre of a black hole

Solving the approximate equation (10.19) for r, we find

$$\ln\left(\frac{r}{2m} - 1\right) \approx -\frac{ct}{2m} + 2 + \ln 2 \tag{10.20}$$

or

$$r \approx 2m\left(1 + 2\mathrm{e}^{2-ct/2m}\right) > 2m \,, \tag{10.21}$$

showing that the orbital radius remains larger than the Schwarzschild radius even for $t \to \infty$. Thus, in coordinate time, the Schwarzschild radius is *never* even reached!

Finally, radial light rays are described by radial null geodesics, thus satisfying

$$0 = \mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)c^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2m}{r}} \tag{10.22}$$

or

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm c\left(1 - \frac{2m}{r}\right) \,, \tag{10.23}$$

suggesting that the light cones become infinitely narrow as $r \to 2m+$.

Problems on the horizon

These results appear quite dissatisfactory or confusing: while a freely falling observer reaches the Schwarzschild radius and even the centre of the Schwarzschild spacetime after finite proper time, the coordinate time becomes infinite even for reaching the Schwarzschild radius, and the flattening of the light cones as one approaches the Schwarzschild radius is entirely unwanted because causality cannot be assessed when the light cone degenerates to a line.

10.1.3 Curvature at $r = 2m$

Moreover, consider the components of the Ricci tensor given in (8.57) and (8.58) near the Schwarzschild radius. Since $a = -b$ and

$$b = -\frac{1}{2}\ln\left(1-\frac{2m}{r}\right) = -a \tag{10.24}$$

from (8.64), the required derivatives are

$$a' = \frac{m}{r(r-2m)} = -b'\;, \quad a'' = -\frac{2m(r-m)}{r^2(r-2m)^2} = -b''\;. \tag{10.25}$$

Thus,

$$R_{00} = -\left(a'' + 2a'^2 + \frac{2a'}{r}\right)\left(1-\frac{2m}{r}\right) = 0 = -R_{11} \tag{10.26}$$

and

$$\begin{aligned} R_{22} &= -\frac{2a'}{r}\left(1-\frac{2m}{r}\right) + \frac{1}{r^2}\left(1-\mathrm{e}^{-2b}\right) \\ &= -\frac{2m(r-2m)}{r^3(r-2m)} + \frac{2m}{r^3} = 0 = R_{33}\;, \end{aligned} \tag{10.27}$$

i.e. the components of the Ricci tensor in the Schwarzschild tetrad remain perfectly regular at the Schwarzschild radius!

10.2 The Kruskal continuation

10.2.1 Construction principle

We shall now try to remove the obvious problems with the Schwarzschild coordinates by transforming (ct, r) to new coordinates (u, v), leaving ϑ and φ, requiring that the metric can be written as

$$g = -f^2(u,v)(\mathrm{d}v^2 - \mathrm{d}u^2) + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2) \tag{10.28}$$

with a function $f(u, v)$ to be determined.

Provided $f(u, v) \neq 0$, radial light rays propagate as in a two-dimensional Minkowski metric according to

$$dv^2 = du^2 \;, \quad \left(\frac{du}{dv}\right)^2 = 1 \;, \tag{10.29}$$

which shows that the light cones remain undeformed in the new coordinates.

Light cones in Kruskal coordinates

The Kruskal coordinates are constructed such that the light cones remain the same everywhere.

The Jacobian matrix of the transformation from the Schwarzschild coordinates $(ct, r, \vartheta, \varphi)$ to the new coordinates $(v, u, \vartheta, \varphi)$ is

$$J^\alpha_\beta = \begin{pmatrix} v_t & u_t & 0 & 0 \\ v_r & u_r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;, \tag{10.30}$$

where subscripts denote derivatives here,

$$v_t = \partial_{ct} v \;, \quad v_r = \partial_r v \tag{10.31}$$

and likewise for u. The metric $\bar{g}$ in the new coordinates,

$$\bar{g} = \mathrm{diag}(-f^2, f^2, r^2, r^2 \sin^2 \vartheta) \;, \tag{10.32}$$

is transformed into the original Schwarzschild coordinates by

$$\begin{aligned} g &= J\bar{g}J^T \\ &= \begin{pmatrix} -f^2\left(v_t^2 - u_t^2\right) & -f^2\left(v_t v_r - u_t u_r\right) & 0 & 0 \\ -f^2\left(v_t v_r - u_t u_r\right) & -f^2\left(v_r^2 - u_r^2\right) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \vartheta \end{pmatrix} \;, \end{aligned} \tag{10.33}$$

?

Beginning with $\bar{g}$, find the matrix representation of the metric g yourself and thus confirm the following result (10.33).

which, by comparison with our requirement (10.28), yields the three equations

$$\begin{aligned} -\left(1 - \frac{2m}{r}\right) &= -f^2\left(v_t^2 - u_t^2\right) \;, \\ \frac{1}{1 - 2m/r} &= -f^2\left(v_r^2 - u_r^2\right) \;, \\ 0 &= v_t v_r - u_t u_r \;. \end{aligned} \tag{10.34}$$

For convenience, we now fall back to the radial coordinates $\bar{r}$ from (10.12) and introduce the function

$$F(\bar{r}) \equiv \frac{1 - 2m/r}{f^2(r)} \;, \tag{10.35}$$

assuming that f will turn out to depend on r only since any dependence on time and on the angles ϑ and φ is forbidden in a static, spherically-symmetric spacetime. Then,

$$v_{\bar{r}} = \frac{\mathrm{d}r}{\mathrm{d}\bar{r}} v_r = \left(1 - \frac{2m}{r}\right) v_r \tag{10.36}$$

and the same for u.

?

Repeat the calculation in (10.36) with the coordinate u.

10.2.2 Transformation to Kruskal coordinates

The equations (10.34) then transform to

$$F(\bar{r}) = v_t^2 - u_t^2 \ , \quad -F(\bar{r}) = v_{\bar{r}}^2 - u_{\bar{r}}^2 \ , \quad v_t v_{\bar{r}} - u_t u_{\bar{r}} = 0 \ . \tag{10.37}$$

Now, we add the two equations containing $F(\bar{r})$ and then add and subtract from the result twice the third equation from (10.37). This yields

$$(v_t \pm v_{\bar{r}})^2 = (u_t \pm u_{\bar{r}})^2 \ . \tag{10.38}$$

Taking the square root of this equation, we can choose the signs. The choice

$$v_t + v_{\bar{r}} = u_t + u_{\bar{r}} \ , \quad v_t - v_{\bar{r}} = -(u_t - u_{\bar{r}}) \tag{10.39}$$

avoids that the Jacobian matrix could become singular, $\det J = 0$.

?

Can you confirm that $\det J \neq 0$ for the choice of sign in (10.39)?

Adding and subtracting the equations (10.39), we find

$$v_t = u_{\bar{r}} \ , \quad u_t = v_{\bar{r}} \ . \tag{10.40}$$

Taking partial derivatives once with respect to t and once with respect to $\bar{r}$ allows us to combine these equations to find the wave equations

$$v_{tt} - v_{\bar{r}\bar{r}} = 0 \ , \quad u_{tt} - u_{\bar{r}\bar{r}} = 0 \ , \tag{10.41}$$

which are solved by any two functions $h_\pm$ propagating with unit velocity,

$$v = h_+(\bar{r} + ct) + h_-(\bar{r} - ct) \ , \quad u = h_+(\bar{r} + ct) - h_-(\bar{r} - ct) \ , \tag{10.42}$$

where the signs were chosen such as to satisfy the sign choice in (10.39).

Now, since

$$\begin{aligned} v_t &= h'_+ - h'_- \ , \quad u_t = h'_+ + h'_- \ , \\ v_{\bar{r}} &= h'_+ + h'_- \ , \quad u_{\bar{r}} = h'_+ - h'_- \ , \end{aligned} \tag{10.43}$$

where the primes denote derivatives with respect to the functions' arguments, we find from (10.37)

$$F(\bar{r}) = (h'_+ - h'_-)^2 - (h'_+ + h'_-)^2 = -4h'_+ h'_- \ . \tag{10.44}$$

We start from outside the Schwarzschild radius, assuming $r > 2m$, where also $F(\bar{r}) > 0$ according to (10.35). The derivative of (10.44) with respect to $\bar{r}$ yields

$$F'(\bar{r}) = -4\left(h''_+ h'_- + h'_+ h''_-\right) \tag{10.45}$$

or, with (10.44),

$$\frac{F'}{F} = \frac{h''_+}{h'_+} + \frac{h''_-}{h'_-} \ . \tag{10.46}$$

the derivative of (10.44) with respect to time yields

$$0 = -4\left(h''_+ h'_- - h'_+ h''_-\right) \quad \Rightarrow \quad \frac{h''_+}{h'_+} - \frac{h''_-}{h'_-} = 0 \ . \tag{10.47}$$

The sum of these two equations gives

$$(\ln F)' = 2(\ln h'_+)' \ . \tag{10.48}$$

Now, the left-hand side depends on $\bar{r}$, the right-hand side on the independent variable $\bar{r} + t$. Thus, the two sides of this equation must equal the same constant, which we call $2C$:

$$(\ln F)' = 2C = 2(\ln h'_+)' \ . \tag{10.49}$$

The left of these equations yields

$$\ln F = 2C\bar{r} + \text{const.} \quad \Rightarrow \quad F = \text{const.}\mathrm{e}^{2C\bar{r}} \ , \tag{10.50}$$

while the right equation gives

$$\ln h'_+ = C(\bar{r} + ct) + \text{const.} \tag{10.51}$$

or

$$h_+ = \text{const.}\mathrm{e}^{C(\bar{r}+ct)} \ . \tag{10.52}$$

For later convenience, we choose the remaining constants in (10.50) and (10.52) such that

$$F(\bar{r}) = C^2 \mathrm{e}^{2C\bar{r}} \ , \quad h_+(\bar{r} + ct) = \frac{1}{2}\mathrm{e}^{C(\bar{r}+ct)} \ , \tag{10.53}$$

and (10.47) gives

$$h_-(\bar{r} - ct) = -\frac{1}{2}\mathrm{e}^{C(\bar{r}-ct)} \ , \tag{10.54}$$

where the negative sign must be chosen to satisfy both (10.44) and $F > 0$.

Working our way back, we find

$$\begin{aligned} u = h_+(\bar{r} + ct) - h_-(\bar{r} - ct) &= \frac{1}{2}\left[\mathrm{e}^{C(\bar{r}+ct)} + \mathrm{e}^{C(\bar{r}-ct)}\right] \\ &= \mathrm{e}^{C\bar{r}} \cosh(Cct) = \left(\frac{r}{2m} - 1\right)^{2mC} \mathrm{e}^{Cr} \cosh(Cct) \ , \end{aligned} \tag{10.55}$$

using (10.12) for $\bar{r}$. Similarly, we find

$$v = \left(\frac{r}{2m} - 1\right)^{2mC} \mathrm{e}^{Cr} \sinh(Cct) , \tag{10.56}$$

and the function f follows from (10.35),

$$\begin{aligned} f^2 &= \frac{1 - 2m/r}{F} = \frac{1 - 2m/r}{C^2} \mathrm{e}^{-2C\bar{r}} \\ &= \frac{1 - 2m/r}{C^2} \mathrm{e}^{-2Cr} \exp\left[-4mC \ln\left(\frac{r}{2m} - 1\right)\right] \\ &= \frac{2m}{rC^2} \left(\frac{r}{2m} - 1\right)^{1-4mC} \mathrm{e}^{-2Cr} . \end{aligned} \tag{10.57}$$

Caution Note that we could equally well choose $h_+ < 0$ and $h_- > 0$ in (10.53) and (10.54). This possible alternative choice is important for our later discussion. ◄

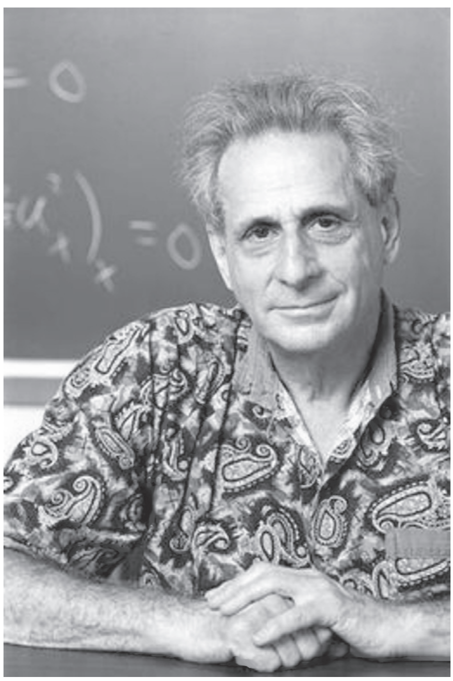

Figure 10.2 Martin D. Kruskal (1925–2006), US-American mathematician and physicist. Source: Wikipedia

Now, since we want f to be non-zero and regular at $r = 2m$, we must require $4mC = 1$, which finally fixes the *Kruskal transformation* of the Schwarzschild metric, found by Martin Kruskal in 1960. The coordinates $(v, u, \vartheta, \varphi)$ are also called Kruskal-Szekeres coordinates, including George (György) Szekeres (1911–2005), who found them independently in 1961.

Kruskal-Szekeres coordinates

The Kruskal-Szekeres coordinates (u, v) are related to the Schwarzschild coordinates (ct, r) by

$$u = \sqrt{\frac{r}{2m} - 1}\, \mathrm{e}^{r/4m} \cosh\left(\frac{ct}{4m}\right) ,$$
$$v = \sqrt{\frac{r}{2m} - 1}\, \mathrm{e}^{r/4m} \sinh\left(\frac{ct}{4m}\right) , \tag{10.58}$$

and the scale function f is

$$f^2 = \frac{32m^3}{r}\mathrm{e}^{-r/2m} . \tag{10.59}$$

We have (or rather, Martin Kruskal has) thus achieved our goal to replace the Schwarzschild coordinates by others in which the Schwarzschild metric remains prefectly regular at $r = 2m$. Appendix C shows how space-times can be compactly represented in Penrose-Carter diagrams.

10.3 Physical meaning of the Kruskal continuation

10.3.1 Regions in the Kruskal spacetime

Since $\cosh^2(x) - \sinh^2(x) = 1$, eqs. (10.58) imply

$$u^2 - v^2 = \left(\frac{r}{2m} - 1\right)\mathrm{e}^{r/2m} , \quad \frac{v}{u} = \tanh\left(\frac{ct}{4m}\right) . \tag{10.60}$$

This means $u = |v|$ for $r = 2m$, which is reached for $t \to \pm\infty$. Lines of constant coordinate time t are straight lines through the origin in the (u, v) plane with slope $\tanh(ct/4m)$, and lines of constant radial coordinate r are hyperbolae.

The metric in Kruskal coordinates (10.28) is regular as long as $r(u, v) > 0$, which is the case for

$$u^2 - v^2 > -1 , \tag{10.61}$$

as (10.61) shows. The hyperbola limiting the regular domain in the Kruskal manifold is thus given by $v^2 - u^2 = 1$. If (10.61) is satisfied, r is uniquely defined, because the equation

$$\rho(x) \equiv (x - 1)\mathrm{e}^x = u^2 - v^2 > -1 \tag{10.62}$$

is monotonic for $x > 0$:

$$\rho'(x) = \mathrm{e}^x(x - 1) + \mathrm{e}^x = x\mathrm{e}^x > 0 \quad (x > 0) . \tag{10.63}$$

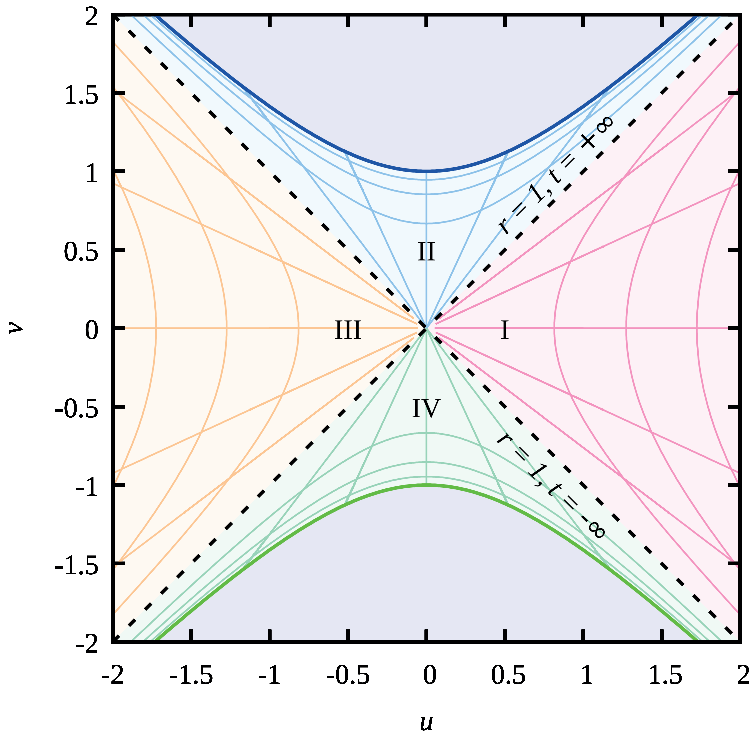

Figure 10.3 Illustration of the Kruskal continuation in the u-v plane. The Schwarzschild domain $r > 2m$ is shaded in red and marked with I, the forbidden region $r < 0$ is shaded in gray.

The domain of the original Schwarzschild solution is restricted to $u > 0$ and $|v| < u$ (i.e. to the blue area I in Fig. 10.2), but this is a consequence of our choice for the relative signs of $h_\pm$ in (10.54). We could as well have chosen $h_+ < 0$ and $h_- > 0$, which would correspond to the replacement $(u, v) \to (-u, -v)$.

The original Schwarzschild solution for $r < 2m$ also satisfies Einstein's vacuum field equations. There, the Schwarzschild metric shows that r then behaves like a time coordinate because $g_{rr} < 0$, and t behaves like a spatial coordinate.

Looking at the definition of $F(\bar{r})$ in (10.35), we see that $r < 2m$ corresponds to $F < 0$, which implies that h_+ and h_- must have the same (rather than opposite) signs because of (10.44). This interchanges the functions u and v from (10.58), i.e. $u \to v$ and $v \to u$. Then, the condition $|v| < u$ derived for $r > 2m$ changes to $|v| > u$.

Domains in the Kruskal spacetime

In summary, the exterior of the Schwarzschild radius corresponds to the domain $u > 0$, $|v| < u$, and its interior is bounded in the (u, v) plane by the lines $u > 0$, $|v| = u$ and $v^2 - u^2 = 1$.
Radial light rays propagate according to $\mathrm{d}s^2 = 0$ or $\mathrm{d}v = \mathrm{d}u$, i.e. they are straight diagonal lines in the (u, v) plane. This shows that light rays can propagate freely into the region $r < 2m$, but there is no causal connection from within $r < 2m$ to the outside.

Non-static interior of the Schwarzschild horizon

The Killing vector field $K = \partial_t$ for the Schwarzschild spacetime outside $r = 2m$ becomes space-like for $r < 2m$, which means that the spacetime cannot be static any more inside the Schwarzschild radius.

10.3.2 Eddington-Finkelstein coordinates

We now want to study the collapse of an object, e.g. a star. For this purpose, coordinates originally introduced by Arthur S. Eddington and re-discovered by David R. Finkelstein are convenient, which are defined by

$$r = r' \,, \quad \vartheta = \vartheta' \,, \quad \varphi = \varphi'$$
$$ct = ct' - 2m \ln\left(\pm\frac{r}{2m} \mp 1\right) \tag{10.64}$$

in analogy to the radial coordinate $\bar{r}$ from (10.12), where the upper and lower signs in the second line are valid for $r > 2m$ and $r < 2m$, respectively.

?

What do the light cones look like in the coordinates given in (10.64)?

Since

$$\mathrm{e}^{\pm ct/4m} = \mathrm{e}^{\pm ct'/4m} \begin{cases} \left(\frac{r}{2m} - 1\right)^{\mp 1/2} & (r > 2m) \\ \left(-\frac{r}{2m} + 1\right)^{\mp 1/2} & (r < 2m) \end{cases} \,, \tag{10.65}$$

inserting these expressions into the Kruskal-Szekeres coordinates (10.58) shows that they are related to the Eddington-Finkelstein coordinates by

$$u = \frac{e^{r/4m}}{2}\left(\mathrm{e}^{ct'/4m} + \frac{r-2m}{2m}\mathrm{e}^{-ct'/4m}\right)$$
$$v = \frac{e^{r/4m}}{2}\left(\mathrm{e}^{ct'/4m} - \frac{r-2m}{2m}\mathrm{e}^{-ct'/4m}\right) \,, \tag{10.66}$$

such that

$$\frac{r-2m}{2m}\mathrm{e}^{r/2m} = u^2 - v^2 \,, \quad \mathrm{e}^{ct'/2m} = \frac{r-2m}{2m}\frac{u+v}{u-v} \,. \tag{10.67}$$

The first of these equations shows again that r can be uniquely determined from u and v if $u^2 - v^2 > -1$. The second equation determines t' uniquely

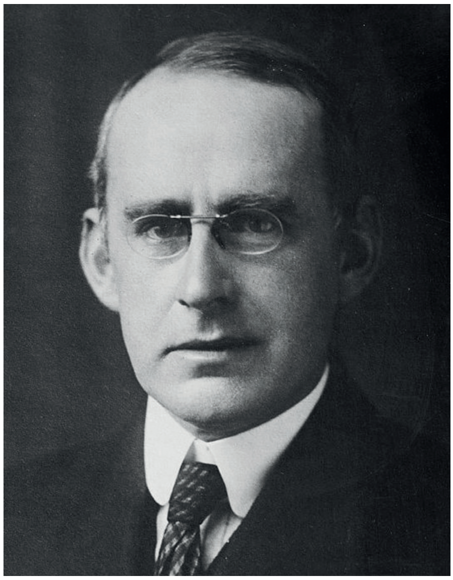

Figure 10.4 Sir Arthur Stanley Eddington (1882–1944), British astrophysicist. Source: Wikipedia

provided $r > 2m$ and $(u+v)/(u-v) > 0$, or $r < 2m$ and $(u+v)/(u-v) < 0$. This is possible if $v > -u$.

Using

$$c\mathrm{d}t = c\mathrm{d}t' - 2m\frac{1}{\pm r/2m \mp 1}\frac{\pm\mathrm{d}r}{2m} = c\mathrm{d}t' - \frac{2m}{r}\frac{\mathrm{d}r}{1-2m/r}\ , \tag{10.68}$$

we find

$$-\left(1-\frac{2m}{r}\right)c^2\mathrm{d}t^2 = -\left(1-\frac{2m}{r}\right)c^2\mathrm{d}t'^2 + \frac{4mc}{r}\mathrm{d}t'\mathrm{d}r - \frac{4m^2}{r^2}\frac{\mathrm{d}r^2}{1-2m/r}\ , \tag{10.69}$$

and thus the line element of the metric in Eddington-Finkelstein coordinates reads

$$\begin{aligned}\mathrm{d}s^2 &= -\left(1-\frac{2m}{r}\right)c^2\mathrm{d}t'^2 + \left(1-\frac{4m^2}{r^2}\right)\frac{\mathrm{d}r^2}{1-2m/r} \\ &\quad + \frac{4mc}{r}\mathrm{d}t'\mathrm{d}r + r^2\mathrm{d}\Omega^2 \\ &= -\left(1-\frac{2m}{r}\right)c^2\mathrm{d}t'^2 + \left(1+\frac{2m}{r}\right)\mathrm{d}r^2 + \frac{4mc}{r}\mathrm{d}t'\mathrm{d}r + r^2\mathrm{d}\Omega^2\ .\end{aligned} \tag{10.70}$$

Thus, the metric acquires off-diagonal elements such that it no longer depends on t' and r separately.

For radial light rays, $\mathrm{d}\Omega = 0$ and $\mathrm{d}s^2 = 0$, which implies from (10.70)

$$\left(1-\frac{2m}{r}\right)c^2\mathrm{d}t'^2 - \left(1+\frac{2m}{r}\right)\mathrm{d}r^2 - \frac{4mc}{r}\mathrm{d}t'\mathrm{d}r = 0\ , \tag{10.71}$$

which can be factorised as

$$\left[\left(1-\frac{2m}{r}\right)c\mathrm{d}t' - \left(1+\frac{2m}{r}\right)\mathrm{d}r\right](c\mathrm{d}t' + \mathrm{d}r) = 0\ . \tag{10.72}$$

Light cones in Eddington-Finkelstein coordinates

Light cones in Eddington-Finkelstein coordinates are defined either by

$$\frac{\mathrm{d}r}{\mathrm{d}t'} = -c \quad \Rightarrow \quad r = -t' + \text{const.} \tag{10.73}$$

or by

$$\frac{\mathrm{d}r}{\mathrm{d}t'} = c\,\frac{r-2m}{r+2m}\ . \tag{10.74}$$

This shows that $\mathrm{d}r/\mathrm{d}t' \to -c$ for $r \to 0$, $\mathrm{d}r/\mathrm{d}t' = 0$ for $r = 2m$, and $\mathrm{d}r/\mathrm{d}t' = c$ for $r \to \infty$. Due to the vanishing derivative of r with respect to t' at $r = 2m$, geodesics cannot cross the Schwarzschild radius from inside, but they can from outside because of (10.73).

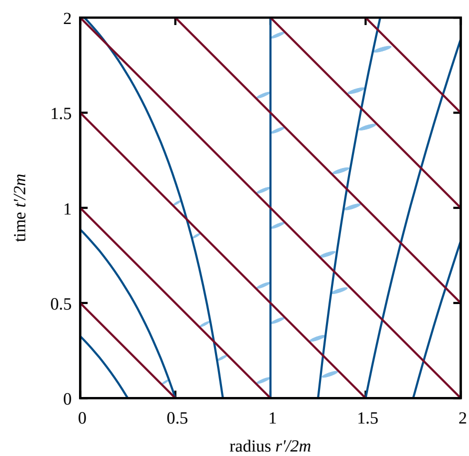

Figure 10.5 Light cones in the Schwarzschild spacetime in Eddington-Finkelstein coordinates. The blue lines mark outgoing, the red lines incoming radial light rays. The blue ellipses emphasise the light cones.

10.4 Redshift approaching the Schwarzschild radius

Suppose a light-emitting source (e.g. an astronaut with a torch) is falling towards a (Schwarzschild) black hole, what does a distant observer see? Let v and u be the four-velocities of the astronaut and the observer, respectively. Then according to (4.48) the redshift of the light from the torch as seen by the observer is

$$1 + z = \frac{\nu_{\rm em}}{\nu_{\rm obs}} = \frac{\langle k, v\rangle}{\langle k, u\rangle}\ , \tag{10.75}$$

where k is the wave vector of the light.

We transform to the retarded time $ct_{\rm ret} \equiv ct - \bar{r}$, with $\bar{r}$ given by (10.12). Then, (10.10) implies that

$$c\mathrm{d}t_{\rm ret} = c\mathrm{d}t - \frac{\mathrm{d}r}{1 - 2m/r}\ , \tag{10.76}$$

thus

$$c^2\mathrm{d}t^2 = c^2\mathrm{d}t_{\rm ret}^2 + \frac{\mathrm{d}r^2}{(1 - 2m/r)^2} + \frac{2c\,\mathrm{d}t_{\rm ret}\mathrm{d}r}{1 - 2m/r} \tag{10.77}$$

and the line element of the Schwarzschild metric transforms to

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)c^2\mathrm{d}t_{\rm ret}^2 - 2c\,\mathrm{d}t_{\rm ret}\mathrm{d}r + r^2\mathrm{d}\Omega^2\ . \tag{10.78}$$

For radial light rays, $\mathrm{d}\Omega = 0$, this means

$$0 = -\left(1 - \frac{2m}{r}\right)c^2\mathrm{d}t_{\rm ret}^2 - 2c\,\mathrm{d}t_{\rm ret}\mathrm{d}r\ , \tag{10.79}$$

which is possible for *outgoing* light rays only if $\mathrm{d}t_{\rm ret} = 0$. This shows that such light rays must propagate along r, or $k \propto \partial_r$, which is of course a consequence of our using the retarded time $t_{\rm ret}$. We set the amplitude of k such that $k = \kappa\partial_r$. Since $\langle\partial_r, \partial_r\rangle = 0$ in the coordinates of the line element (10.78), the null condition on k is satisfied for any κ.

For a distant observer at a fixed distance $r \gg 2m$, the line element (10.78) simplifies to

$$\mathrm{d}s^2 \approx -c^2\mathrm{d}t_{\rm ret}^2\ , \tag{10.80}$$

which shows that the retarded time $t_{\rm ret}$ is also the distant observer's proper time.

Expanding now the astronaut's velocity as

$$v = \dot{t}_{\rm ret}\partial_{t_{\rm ret}} + \dot{r}\partial_r\ , \tag{10.81}$$

we find

$$\langle k, v\rangle = \kappa\langle\partial_r, v\rangle = \kappa\, g_{t_{\rm ret}r}\dot{t}_{\rm ret} = -\kappa\dot{t}_{\rm ret} \tag{10.82}$$

because $\langle\partial_r, \partial_r\rangle = 0$ according to the metric with the line element (10.78). The dots in these equations indicate derivatives with respect to the astronaut's proper time.

Far away from the black hole, the metric can be assumed to be Minkowskian. For a distant observer at rest, the four-velocity is $u = \partial_t$. In Minkowski coordinates, the wave vector of the light ray must be

$$k = \kappa\,(\partial_t + \partial_r)\ , \tag{10.83}$$

which is required by $c\mathrm{d}t_{\mathrm{ret}}(k) = (c\mathrm{d}t - \mathrm{d}r)(k) = 0$, valid for $r \gg 2m$, together with $k = \kappa\partial_r$. Thus,

$$\langle k, u\rangle = \langle \kappa\,(\partial_t + \partial_r)\,, \partial_t\rangle = -\kappa\ . \tag{10.84}$$

This gives the redshift

$$1 + z \approx \dot{t}_{\mathrm{ret}} = \dot{t} - \frac{\dot{r}/c}{1 - 2m/r}\ . \tag{10.85}$$

When restricted to radial orbits, $\dot{\varphi} = 0 = L$, the equation of motion (9.15) is

$$\dot{r}^2 + c^2\left(1 - \frac{2m}{r}\right) = E^2\ , \tag{10.86}$$

where E was defined as

$$E = -c\dot{t}\left(1 - \frac{2m}{r}\right)\ , \tag{10.87}$$

see (9.12). To be specific, we set the constant E such that the astronaut is at rest at infinite radius, $E^2 = c^2$. Requiring that the astronaut's proper time increases with the coordinate time, $\dot{t} > 0$ and $E < 0$, hence we must set $E = -c$. Since $\dot{r} < 0$ for the infalling astronaut,

$$\dot{r} = -c\,\sqrt{1 - \delta}\ , \tag{10.88}$$

with $\delta \equiv 1 - 2m/r$.

The redshift (10.85) can now be written

$$1 + z = \frac{1}{\delta}\left(1 + \sqrt{1 - \delta}\right) \approx \frac{2}{\delta} \tag{10.89}$$

to leading order close to the Schwarzschild radius, where $\delta \to 0+$. We have seen in (10.21) that the radial coordinate of the falling astronaut is well approximated by $r \approx 2m(1 + 2\mathrm{e}^{2-ct/2m})$ near the Schwarzschild radius if the coordinate clock is set to zero at $r = 6m$. This enables us to approximate δ by

$$\delta = 1 - \frac{2m}{r} \approx \frac{r - 2m}{2m} = 2\mathrm{e}^{2-ct/2m} \tag{10.90}$$

and the redshift by

$$1 + z \approx \mathrm{e}^{ct/2m-2}\ . \tag{10.91}$$

Redshift approaching the Schwarzschild horizon

Summarizing, this calculation shows that the astronaut's redshift

$$1 + z \approx \frac{2}{\delta} = e^{ct/2m-2} \tag{10.92}$$

grows exponentially to infinity as he approaches the Schwarzschild radius.

This resolves the apparent contradiction that, while the astronaut has long reached the singularity as measured by his own watch, the distant observer never even sees him reach the Schwarzschild radius: The signal of the astronaut's passing the Schwarzschild radius is infinitely delayed and thus never reaches the distant observer.