Chapter 9

Physics in the Schwarzschild Spacetime

9.1 Orbits in the Schwarzschild spacetime

9.1.1 Lagrange function

According to (4.3), the motion of a particle in the Schwarzschild spacetime is determined by the Lagrangian

$$\mathcal{L} = \sqrt{-\langle u, u\rangle}\ , \tag{9.1}$$

where $u = \mathrm{d}x/\mathrm{d}\tau$ is the four-velocity. The proper-time differential $\mathrm{d}\tau$ is defined by (4.6) to satisfy

$$\mathrm{d}s = c\mathrm{d}\tau = \sqrt{-\langle u, u\rangle}\,\mathrm{d}\tau\ . \tag{9.2}$$

This choice thus requires that the four-velocity u be normalised,

$$\langle u, u\rangle = -c^2\ . \tag{9.3}$$

Note that we have to differentiate and integrate with respect to the proper time τ rather than the coordinate time t because the latter has no invariant physical meaning. In the Newtonian limit, $\tau = t$.

?

Recall the essential arguments for the Lagrange function (9.1) and its physical interpretation.

The constant value of $\langle u, u\rangle$ allows that, instead of varying the action

$$S = -mc\int_a^b \sqrt{-\langle u, u\rangle}\,\mathrm{d}\tau\ , \tag{9.4}$$

we can just as well require that the variation of

$$\bar{S} = \frac{1}{2}\int_a^b \langle u, u\rangle\,\mathrm{d}\tau \tag{9.5}$$

vanish. In fact, from $\delta S = 0$, we have

$$0 = -\delta \int_a^b \sqrt{-\langle u, u\rangle}\, d\tau = \frac{1}{2}\int_a^b \frac{\delta\langle u, u\rangle}{\sqrt{-\langle u, u\rangle}}\, d\tau = \delta\left[\frac{1}{2c}\int_a^b \langle u, u\rangle\, d\tau\right] . \tag{9.6}$$

because of the normalisation condition (9.3).

Thus, we can obtain the equation of motion just as well from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\langle u, u\rangle = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left[-(1-2m/r)c^2\dot{t}^2 + \frac{\dot{r}^2}{1-2m/r} + r^2\left(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2\right)\right] , \tag{9.7}$$

where it is important to recall that the overdot denotes differentiation with respect to proper time τ. In addition, (9.3) immediately implies that $2\mathcal{L} = -c^2$ for material particles, but $2\mathcal{L} = 0$ for light, which will be discussed later.

The Euler-Lagrange equation for ϑ is

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{\vartheta}} - \frac{\partial \mathcal{L}}{\partial \vartheta} = 0 = \frac{d}{d\tau}\left(r^2\dot{\vartheta}\right) - r^2\dot{\varphi}^2 \sin\vartheta\cos\vartheta . \tag{9.8}$$

Suppose the motion starts in the equatorial plane, $\vartheta = \pi/2$ and $\dot{\vartheta} = 0$. Should this not be the case, we can always rotate the coordinate frame so that this is satisfied. Then, (9.8) shows that

$$r^2\dot{\vartheta} = \text{const.} = 0 . \tag{9.9}$$

?

Derive the Lagrangian (9.10) yourself and convince yourself of all steps taken.

Effective Lagrangian

Without loss of generality, we can thus restrict the discussion to motion in the equatorial plane, which simplifies the Lagrangian to

$$\mathcal{L} = \frac{1}{2}\left[-(1-2m/r)c^2\dot{t}^2 + \frac{\dot{r}^2}{1-2m/r} + r^2\dot{\varphi}^2\right] . \tag{9.10}$$

9.1.2 Cyclic coordinates and equation of motion

Obviously, t and φ are cyclic, thus angular momentum

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = r^2\dot{\varphi} \equiv L = \text{const.} \tag{9.11}$$

and energy

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -(1-2m/r)c\dot{t} \equiv E = \text{const.} \tag{9.12}$$

are conserved. We exploit these conservation laws to eliminate

$$\dot{\varphi} = \frac{L}{r^2} \quad \text{and} \quad c\dot{t} = -\frac{E}{1-2m/r} \tag{9.13}$$

from the Lagrangian (9.10), use $2\mathcal{L} = -1$ and find

$$-c^2 = -(1-2m/r)c^2\dot{t}^2 + \frac{\dot{r}^2}{1-2m/r} + r^2\dot{\varphi}^2 = \frac{\dot{r}^2 - E^2}{1-2m/r} + \frac{L^2}{r^2} \; . \tag{9.14}$$

Radial equation of motion

This first integral of the radial equation of motion can be cast into the form

$$\dot{r}^2 + V(r) = E^2 \; , \tag{9.15}$$

where $V(r)$ is the *effective potential*

$$V(r) \equiv \left(1 - \frac{2m}{r}\right)\left(c^2 + \frac{L^2}{r^2}\right) \; . \tag{9.16}$$

Note that the effective potential has (and must have) the dimension of a squared velocity.

?

What form does the effective potential have in Newtonian gravity?

Since it is our primary goal to find the orbit $r(\varphi)$, we use $r' = \mathrm{d}r/\mathrm{d}\varphi = \dot{r}/\dot{\varphi}$ to transform (9.15) to

$$\dot{r}^2 + V(r) = \dot{\varphi}^2 r'^2 + V(r) = \frac{L^2}{r^4} r'^2 + V(r) = E^2 \; . \tag{9.17}$$

Now, we substitute $u \equiv 1/r$ and $u' = -r'/r^2 = -u^2 r'$ and find

$$L^2 u^4 \frac{u'^2}{u^4} + V(1/u) = L^2 u'^2 + (1-2mu)\left(c^2 + L^2 u^2\right) = E^2 \tag{9.18}$$

or, after dividing by L^2 and rearranging terms,

$$u'^2 + u^2 = \frac{E^2 - c^2}{L^2} + \frac{2mc^2}{L^2} u + 2mu^3 \; . \tag{9.19}$$

Differentiation with respect to φ cancels the constant first term on the right-hand side and yields

$$2u'u'' + 2uu' = \frac{2mc^2}{L^2} u' + 6mu^2 u' \; . \tag{9.20}$$

?

Convince yourself by your own calculation that you agree with the result (9.20).

Orbital equation

The trivial solution of this orbital equation is $u' = 0$, which implies a circular orbit. If $u' \neq 0$, this equation can be simplified to read

$$u'' + u = \frac{mc^2}{L^2} + 3mu^2 \; . \tag{9.21}$$

Note that this is the equation of a driven harmonic oscillator.

The fact that t and φ are cyclic coordinates in the Schwarzschild spacetime can be studied from a more general point of view. Let $\gamma(\tau)$ be a geodesic curve with tangent vector $\dot\gamma(\tau)$, and let further ξ be a Killing vector field of the metric. Then, we know from (5.36) that the projection of the Killing vector field on the geodesic is constant along the geodesic,

$$\nabla_{\dot\gamma}\langle\dot\gamma,\xi\rangle = 0 \quad\Rightarrow\quad \langle\dot\gamma,\xi\rangle = \text{constant along}\,\gamma \tag{9.22}$$

Due to its stationarity and the spherical symmetry, the Schwarzschild spacetime has the Killing vector fields ∂_t and ∂_φ. Thus,

$$\langle\dot\gamma,\partial_t\rangle = \langle\dot\gamma^t\partial_t,\partial_t\rangle = \dot\gamma^t\langle\partial_t,\partial_t\rangle = g_{00}\dot\gamma^t = -\left(1-\frac{2m}{r}\right)c\dot t = \text{const.} \tag{9.23}$$

and

$$\langle\dot\gamma,\partial_\varphi\rangle = \dot\gamma^\varphi\langle\partial_\varphi,\partial_\varphi\rangle = g_{\varphi\varphi}\dot\gamma^\varphi = r^2\sin^2\vartheta\,\dot\varphi = r^2\dot\varphi = \text{const.}\;, \tag{9.24}$$

where we have used $\vartheta = \pi/2$ without loss of generality. This reproduces (9.11) and (9.12).

9.2 Comparison to the Kepler problem

9.2.1 Differences in the equation of motion

It is instructive to compare this to the Newtonian case. There, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left(\dot r^2 + r^2\dot\varphi^2\right) - \Phi(r)\;, \tag{9.25}$$

where $\Phi(r)$ is some centrally-symmetric potential and the dots denote the derivative with respect to the *coordinate time* t now instead of the proper time τ. In the Newtonian limit, $\tau = t$. For later comparison of results obtained in this and the previous sections, the overdots can here also be interpreted as derivatives with respect to τ, as in the previous section.

?

Why does the angle ϑ not appear in the Lagrange function (9.25)? Why can it be ignored here?

Since φ is cyclic,

$$\frac{\partial\mathcal{L}}{\partial\dot\varphi} = r^2\dot\varphi \equiv L = \text{const}\;. \tag{9.26}$$

The Euler-Lagrange equation for r is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot r} - \frac{\partial\mathcal{L}}{\partial r} = 0 = \ddot r - r\dot\varphi^2 + \frac{\mathrm{d}\Phi}{\mathrm{d}r}\;. \tag{9.27}$$

Since

$$\dot r = \frac{\mathrm{d}r}{\mathrm{d}t} = r'\dot\varphi = r'\frac{L}{r^2} = -Lu'\;, \tag{9.28}$$

we can write the second time derivative of r as

$$\ddot{r} = -L\frac{\mathrm{d}u'}{\mathrm{d}t} = -L\frac{\mathrm{d}u'}{\mathrm{d}\varphi}\dot{\varphi} = -Lu''\frac{L}{r^2} = -L^2u^2u'' \ . \tag{9.29}$$

Thus, the equation of motion (9.27) can be written as

$$-L^2u^2u'' - r\frac{L^2}{r^4} + \frac{\mathrm{d}\Phi}{\mathrm{d}r} = 0 \tag{9.30}$$

or, after dividing by $-u^2L^2$,

$$u'' + u = \frac{1}{L^2u^2}\frac{\mathrm{d}\Phi}{\mathrm{d}r} \ . \tag{9.31}$$

?

Can you agree with the result (9.31)?

Orbital equation in Newtonian gravity

In the Newtonian limit of the Schwarzschild solution, the potential and its radial derivative are

$$\Phi = -\frac{\mathcal{G}M}{r} \ , \quad \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{\mathcal{G}M}{r^2} = \mathcal{G}Mu^2 = mc^2u^2 \ , \tag{9.32}$$

so that the orbital equation becomes

$$u'' + u = \frac{mc^2}{L^2} \ . \tag{9.33}$$

Compared to this, the equation of motion in the Schwarzschild case (9.21) has the additional term $3mu^2$. We have seen in (8.66) that $m \approx 1.5\,\mathrm{km}$ in the Solar System. There, the ratio of the two terms on the right-hand side of (9.21) is

$$\frac{3mu^2}{mc^2/L^2} = \frac{3u^2L^2}{c^2} = \frac{3r^4\dot{\varphi}^2}{r^2c^2} = \frac{3}{c^2}(r\dot{\varphi})^2 = \frac{3v_\perp^2}{c^2} \approx 7.7 \cdot 10^{-8} \tag{9.34}$$

for the innermost planet Mercury. Here, $v_\perp$ is the tangential velocity along the orbit, $v_\perp = r\dot{\varphi}$.

9.2.2 Effective potential

The equation of motion (9.21) in the Schwarzschild spacetime can thus be reduced to a Kepler problem with a potential which, according to (9.31), is given by

$$\frac{1}{L^2u^2}\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{mc^2}{L^2} + 3mu^2 \tag{9.35}$$

or

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = mc^2u^2 + 3mL^2u^4 = \frac{mc^2}{r^2} + \frac{3mL^2}{r^4} \ , \tag{9.36}$$

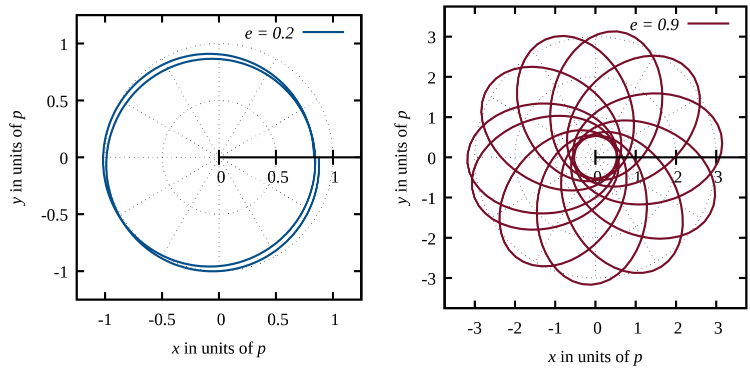

Figure 9.1 Numerical solutions of the orbital equation (9.21) for test particles, for different values of the orbital eccentricity e. All lengths, including the mass $m = 0.025$, are scaled by the orbital parameter p. The orbits shown begin at $u = 1 + e$ with $u' = 0$. For $e = 0.2$ (*left*), two orbits are shown, and twelve orbits for $e = 0.9$ (*right*).

which leads to

$$\Phi(r) = -\frac{mc^2}{r} - \frac{mL^2}{r^3} \tag{9.37}$$

if we set the integration constant such that $\Phi(r) \to 0$ for $r \to \infty$.

As a function of $x \equiv r/R_s = r/2m$, the effective potential $V(r)$ from (9.16) depends in an interesting way on $L/(cR_s) = L/(2mc \equiv \lambda)$. The dimensionless function

$$v(x) := \frac{V(x)}{c^2} = \left(1 - \frac{1}{x}\right)\left(1 + \frac{\lambda^2}{x^2}\right) \tag{9.38}$$

corresponding to the effective potential (9.16) asymptotically behaves as $v(x) \to 1$ for $x \to \infty$ and $v(x) \to -\infty$ for $x \to 0$.

For the potential to have a minimum, $v(x)$ must have a vanishing derivative, $v'(x) = 0$. This is the case where

$$0 = v'(x) = \frac{1}{x^2}\left(1 + \frac{\lambda^2}{x^2}\right) - \left(1 - \frac{1}{x}\right)\frac{2\lambda^2}{x^3} \tag{9.39}$$

or, after multiplication with x^4,

$$x^2 - 2\lambda^2 x + 3\lambda^2 = 0 \quad \Rightarrow \quad x_\pm = \lambda^2 \pm \lambda\sqrt{\lambda^2 - 3}\,. \tag{9.40}$$

Real solutions require $\lambda \geq \sqrt{3}$. If $\lambda < \sqrt{3}$, particles with $E^2 < 1$ will crash directly towards $r = R_s$.

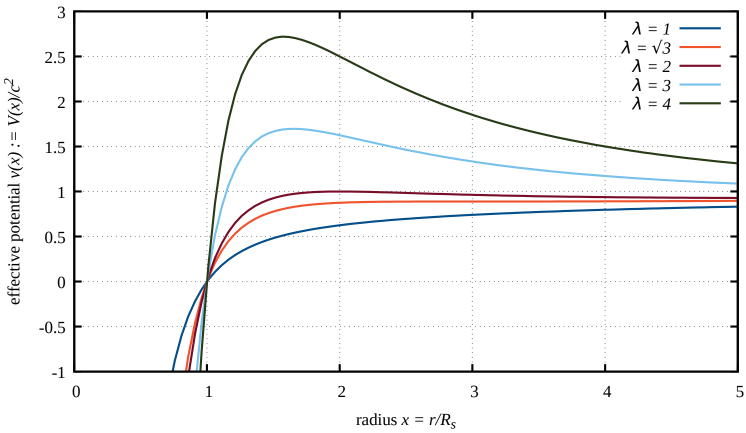

Figure 9.2 Dimensionless effective potential $v(x)$ for a test particle in the Schwarzschild spacetime for various scaled angular momenta λ.

Last stable orbit in the Schwarzschild metric

The last stable orbit, or more precisely the innermost stable circular orbit or ISCO, must thus have $\lambda = \sqrt{3}$ and is therefore located at $x_\pm = 3$, i.e. at $r = 6m = 3R_s$, or three Schwarzschild radii. There, the dimensionless effective potential is

$$v(x = 3) = \frac{2}{3}\left(1 + \frac{3}{9}\right) = \frac{8}{9} \,. \tag{9.41}$$

For $\lambda > \sqrt{3}$, the effective potential has a minimum at x_+ and a maximum at x_- which reaches the height $v = 1$ for $\lambda = 2$ at $x_- = 2$ and is higher for larger λ. This means that particles with $E \geq 1$ and $L < 2cR_s$ will fall unimpededly towards $r = R_s$.

9.3 Perihelion shift and light deflection

9.3.1 The perihelion shift

The treatment of the Kepler problem in classical mechanics shows that closed orbits in the Newtonian limit are described by

$$u_0(\varphi) = \frac{1}{p}\,(1 + e\cos\varphi) \,, \tag{9.42}$$

where the parameter p is related to the angular momentum L by

$$p = a\left(1 - e^2\right) = \frac{L^2}{m} \tag{9.43}$$

in terms of the semi-major axis a and the eccentricity e of the orbit.

Assuming that the perturbation $3mu^2$ in the equation of motion (9.21) is small, we can approximate it by $3mu_0^2$, thus

$$u'' + u = \frac{mc^2}{L^2} + \frac{3m}{p^2}(1 + e\cos\varphi)^2 \ . \tag{9.44}$$

The solution of this equation turns out to be simple because differential equations of the sort

$$u'' + u = \begin{cases} A \\ B\cos\varphi \\ C\cos^2\varphi \end{cases} , \tag{9.45}$$

which are driven harmonic-oscillator equations, have the particular solutions

$$u_1 = A \ , \quad u_2 = \frac{B}{2}\varphi\sin\varphi \ , \quad u_3 = \frac{C}{2}\left(1 - \frac{1}{3}\cos 2\varphi\right) \ . \tag{9.46}$$

?

Verify the particular solutions (9.46) of the driven harmonic oscillator equations (9.45).

Orbits in the Schwarzschild spacetime

Since the unperturbed equation $u'' + u = mc^2/L^2$ has the Keplerian solution $u = u_0$, the complete solution is thus the sum

$$\begin{aligned} u &= u_0 + u_1 + u_2 + u_3 \\ &= \frac{1}{p}(1 + e\cos\varphi) + \frac{3m}{p^2}\left[1 + e\varphi\sin\varphi + \frac{e^2}{2}\left(1 - \frac{1}{3}\cos 2\varphi\right)\right] \ . \end{aligned} \tag{9.47}$$

This solution of (9.44) has its perihelion at $\varphi = 0$ because the unperturbed solution u_0 was chosen to have it there. This can be seen by taking the derivative with respect to φ,

$$u' = -\frac{e}{p}\sin\varphi + \frac{3me}{p^2}\left[\sin\varphi + \varphi\cos\varphi + \frac{e}{3}\sin 2\varphi\right] \tag{9.48}$$

and verifying that $u' = 0$ at $\varphi = 0$, i.e. the orbital radius $r = 1/u$ still has an extremum at $\varphi = 0$.

We now use equation (9.48) in the following way. Starting at the perihelion at $\varphi = 0$, we wait for approximately one revolution at $\varphi = 2\pi + \delta\varphi$ and see what $\delta\varphi$ has to be for u' to vanish again. Thus, the condition for the next perihelion is

$$0 = -\sin\delta\varphi + \frac{3m}{p}\left[\sin\delta\varphi + (2\pi + \delta\varphi)\cos\delta\varphi + \frac{e}{3}\sin 2\delta\varphi\right] \tag{9.49}$$

or, to first order in the small angle $\delta\varphi$,

$$\delta\varphi \approx \frac{3m}{p}\left[2\delta\varphi + 2\pi + \frac{2e}{3}\delta\varphi\right] \ . \tag{9.50}$$

Sorting terms, we find

$$\delta\varphi\left[1-\frac{6m}{p}\left(1+\frac{e}{3}\right)\right]\approx\frac{6\pi m}{p}=\frac{6\pi m}{a(1-e^2)} \tag{9.51}$$

for the perihelion shift $\delta\varphi$.

Perihelion shift

Substituting the Schwarzschild radius from (8.68), we can write this result as

$$\delta\varphi\approx\frac{3\pi R_\mathrm{s}}{a(1-e^2)} \,. \tag{9.52}$$

This turns out to be −6 times the result (1.45) from the scalar theory of gravity discussed in § 1.4.2, or

$$\delta\varphi\approx 43'' \tag{9.53}$$

per century for Mercury's orbit, which reproduces the measurement extremely well.

?

Can you confirm (9.51) beginning with (9.49)?

9.3.2 Light deflection

For light rays, the condition $2\mathcal{L}=-c^2$ that we had before for material particles is replaced by $2\mathcal{L}=0$. Then, (9.14) changes to

$$\frac{\dot r^2-E^2}{1-2m/r}+\frac{L^2}{r^2}=0 \tag{9.54}$$

or

$$\dot r^2+\frac{L^2}{r^2}\left(1-\frac{2m}{r}\right)=E^2 \,. \tag{9.55}$$

Changing again the independent variable to φ and substituting $u=1/r$ leads to the equation of motion for light rays in the Schwarzschild spacetime

$$u'^2+u^2=\frac{E^2}{L^2}+2mu^3 \,, \tag{9.56}$$

which should be compared to the equation of motion for material particles, (9.19). Differentiation finally yields the orbital equation for light rays in the Schwarzschild spacetime.

?

Derive the orbital equation (9.56) yourself.

Light rays in the Schwarzschild spacetime

Light rays (null geodesics) in the Schwarzschild spacetime follow the orbital equation

$$u''+u=3mu^2 \,. \tag{9.57}$$

Compared to u on the left-hand side, the term $3mu^2$ is very small. In the Solar System,

$$\frac{3mu^2}{u}=3mu=\frac{3R_\mathrm{s}}{2r}\le\frac{R_\mathrm{s}}{R_\odot}\approx 10^{-6} \,. \tag{9.58}$$

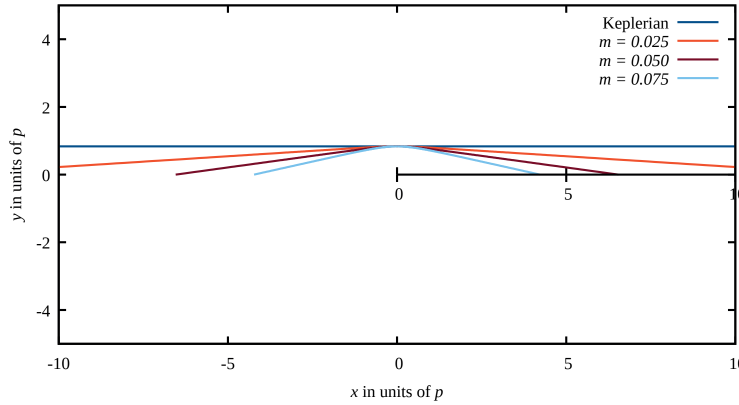

Figure 9.3 Numerical solutions of the orbital equation (9.57) for light rays, compared to the Keplerian straight line, for different values of m. All lengths, including the mass m, are scaled by the orbital parameter p. The orbits shown begin at $u = 1$ with $u' = 0$.

Thus, the light ray is almost given by the homogeneous solution of the harmonic-oscillator equation $u'' + u = 0$, which is $u_0 = A\sin\varphi + B\cos\varphi$. We require that the closest impact at $u_0 = 1/b$ be reached when $\varphi = \pi/2$, which implies $B = 0$ and $A = 1/b$, or

$$u_0 = \frac{\sin\varphi}{b} \quad \Rightarrow \quad r_0 = \frac{b}{\sin\varphi} \,. \tag{9.59}$$

Note that this is a straight line in plane polar coordinates, as it should be!

Inserting this lowest-order solution as a perturbation into the right-hand side of (9.57) gives

$$u'' + u = \frac{3m}{b^2}\sin^2\varphi = \frac{3m}{b^2}\left(1 - \cos^2\varphi\right) \,, \tag{9.60}$$

for which particular solutions can be found using (9.45) and (9.46). Combining this with the unperturbed solution (9.59) gives

$$u = \frac{\sin\varphi}{b} + \frac{3m}{b^2} - \frac{3m}{2b^2}\left(1 - \frac{1}{3}\cos 2\varphi\right) \,. \tag{9.61}$$

?

Beginning with (9.60), confirm the deflection angle (9.63).

Given the orientation of our coordinate system, i.e. with the closest approach at $\varphi = \pi/2$, we have $\varphi \approx 0$ for a ray incoming from the left at large distances. Then, $\sin\varphi \approx \varphi$ and $\cos 2\varphi \approx 1$, and (9.61) yields

$$u \approx \frac{\varphi}{b} + \frac{2m}{b^2} \,. \tag{9.62}$$

In the asymptotic limit $r \to \infty$, or $u \to 0$, this gives the angle

$$|\varphi| \approx \frac{2m}{b} \,. \tag{9.63}$$

Deflection angle for light rays

The total deflection angle of light rays is then

$$\alpha = 2|\varphi| \approx \frac{4m}{b} = 2\frac{R_s}{b} \approx 1.74'' \ . \tag{9.64}$$

This is *twice* the result from our simple consideration leading to (4.90) which did not take the field equations into account yet.

9.4 Spins in the Schwarzschild spacetime

9.4.1 Equations of motion

Let us now finally study how a gyroscope with spin s is moving along a geodesic γ in the Schwarzschild spacetime. Without loss of generality, we assume that the orbit falls into the equatorial plane $\vartheta = \pi/2$, and we restrict the motion to circular orbits.

Then, the four-velocity of the gyroscope is characterised by $u^1 = 0 = u^2$ because both $r = x^1$ and $\vartheta = x^2$ are constant.

The equations that the spin s and the tangent vector $u = \dot{\gamma}$ of the orbit have to satisfy are

$$\langle s, u \rangle = 0 \ , \quad \nabla_u s = 0 \ , \quad \nabla_u u = 0 \ . \tag{9.65}$$

The first is because s falls into a spatial hypersurface perpendicular to the time-like four-velocity u, the second because the spin is parallel transported, and the third because the gyroscope is moving along a geodesic curve.

We work in the same tetrad $\{\theta^\mu\}$ introduced in (8.40) that we used to derive the Schwarzschild solution. From (8.9), we know that

$$\begin{aligned}(\nabla_u s)^\mu &= \langle \mathrm{d}s^\mu + s^\nu \omega^\mu_\nu, u \rangle = u(s^\mu) + \omega^\mu_\nu(u) s^\nu \\ &= \dot{s}^\mu + \omega^\mu_\nu(u) s^\nu = 0 \ ,\end{aligned} \tag{9.66}$$

where the overdot marks the derivative with respect to the proper time τ.

With the connection forms in the Schwarzschild tetrad given in (8.50), and taking into account that $a = -b$ and $\cot\vartheta = 0$, we find for the components of $\dot{s}$

$$\begin{aligned}\dot{s}^0 &= -\omega^0_1(u) s^1 = b' \mathrm{e}^{-b} u^0 s^1 \ , \\ \dot{s}^1 &= -\omega^1_0(u) s^0 - \omega^1_2(u) s^2 - \omega^1_3(u) s^3 = b' \mathrm{e}^{-b} u^0 s^0 + \frac{\mathrm{e}^{-b}}{r} u^3 s^3 \ , \\ \dot{s}^2 &= -\omega^2_1(u) s^1 - \omega^2_3(u) s^3 = 0 \ , \\ \dot{s}^3 &= -\omega^3_1(u) s^1 - \omega^3_2(u) s^2 = -\frac{\mathrm{e}^{-b}}{r} u^3 s^1 \ ,\end{aligned} \tag{9.67}$$

where we have repeatedly used that

$$\theta^1(u) = u^1 = 0 = u^2 = \theta^2(u) \tag{9.68}$$

and $\omega^2_3 = 0$ because $\cot\vartheta = 0$.

Similarly, the geodesic equation $\nabla_u u = 0$, specialised to $u^1 = 0 = u^2$, leads to

$$\begin{aligned}
\dot u^0 &= b' \mathrm{e}^{-b} u^0 u^1 = 0 \,, \\
\dot u^1 &= -b' \mathrm{e}^{-b} (u^0)^2 - \frac{\mathrm{e}^{-b}}{r} (u^3)^2 = 0 \,, \\
\dot u^2 &= 0 \,, \\
\dot u^3 &= -\frac{\mathrm{e}^{-b}}{r} u^1 u^3 = 0 \,.
\end{aligned} \tag{9.69}$$

The second of these equations implies

$$\left(\frac{u^0}{u^3}\right)^2 = -\frac{1}{b' r} \,. \tag{9.70}$$

?

What is the physical meaning of equation (9.70)?

9.4.2 Spin precession

We now introduce a set of basis vectors orthogonal to u, namely

$$\bar e_1 = e_1 \,, \quad \bar e_2 = e_2 \,, \quad \bar e_3 = \frac{u^3}{c} e_0 + \frac{u^0}{c} e_3 \,. \tag{9.71}$$

The orthogonality of $\bar e_1$ and $\bar e_2$ to u is obvious because of $u^1 = 0 = u^2$, and

$$\langle u, \bar e_3 \rangle = u^3 u_0 + u^0 u_3 = 0 \tag{9.72}$$

shows the orthogonality of u and $\bar e_3$. Recall that $u_0 = -u^0$, but $u_3 = u^3$ because the metric is $g = \mathrm{diag}(-1, 1, 1, 1)$ in this basis.

?

Carry out the calculations leading to equations (9.74) and (9.75) yourself.

Since the basis $\{\bar e_i\}$ spans the three-space orthogonal to u, the spin s of the gyroscope can be expanded into this basis as $s = \bar s^i \bar e_i$. We find

$$s^0 = \langle \bar s^i \bar e_i, e_0 \rangle = \frac{u^3}{c} \bar s^3 \,, \quad s^1 = \bar s^1 \,, \quad s^2 = \bar s^2 \,, \quad s^3 = \frac{u^0}{c} \bar s^3 \,, \tag{9.73}$$

which we can insert into (9.67) to find

$$\begin{aligned}
\dot u^3 \bar s^3 + u^3 \dot{\bar s}^3 = u^3 \dot{\bar s}^3 &= c b' \mathrm{e}^{-b} u^0 \bar s^1 \,, \\
\dot{\bar s}^1 &= \left(b' \mathrm{e}^{-b} + \frac{\mathrm{e}^{-b}}{r} \right) \frac{u^0 u^3}{c} \bar s^3 \,, \\
\dot{\bar s}^2 &= 0 \,, \\
\dot u^0 \bar s^3 + u^0 \dot{\bar s}^3 = u^0 \dot{\bar s}^3 &= -\frac{c \mathrm{e}^{-b}}{r} u^3 \bar s^1 \,.
\end{aligned} \tag{9.74}$$

Note that $\dot{u}^\mu = 0$ for all μ according to (9.69). Using (9.69) and the normalisation relation $(u^0)^2 - (u^3)^2 = c^2$, we obtain

$$\begin{aligned}\dot{\bar{s}}^1 &= b' \mathrm{e}^{-b} \left[1 - \frac{(u^0)^2}{(u_3)^2}\right] \frac{u^0 u^3}{c} \bar{s}^3 = -cb' \mathrm{e}^{-b} \frac{u^0}{u^3} \bar{s}^3 \;, \\ \dot{\bar{s}}^2 &= 0 \;, \\ \dot{\bar{s}}^3 &= cb' \mathrm{e}^{-b} \frac{u^0}{u^3} \bar{s}^1 \;. \end{aligned} \tag{9.75}$$

From now on, we shall drop the overbar, understanding that the s^i denote the components of the spin with respect to the basis $\bar{e}_i$.

Next, we transform the time derivative from the proper time τ to the coordinate time t. Since

$$u^0 = \theta^0(u) = \mathrm{e}^a c \mathrm{d}t(u) = c\dot{t}\, \mathrm{e}^a \;, \tag{9.76}$$

we have

$$\dot{t} = \frac{u^0}{c} \mathrm{e}^{-a} = \frac{u^0}{c} \mathrm{e}^b \;, \tag{9.77}$$

or

$$\frac{\mathrm{d}s^i}{\mathrm{d}t} = \frac{\dot{s}^i}{\dot{t}} = \frac{c\dot{s}^i}{u^0} \mathrm{e}^{-b} \;. \tag{9.78}$$

Inserting this into (9.75) yields

$$\frac{\mathrm{d}s^1}{\mathrm{d}t} = -\frac{c^2 b'}{u^3} \mathrm{e}^{-2b} s^3 \;, \quad \frac{\mathrm{d}s^2}{\mathrm{d}t} = 0 \;, \quad \frac{\mathrm{d}s^3}{\mathrm{d}t} = \frac{c^2 b'}{u^3} \mathrm{e}^{-2b} s^1 \;. \tag{9.79}$$

Finally, using (8.40), we have

$$u^3 = \theta^3(u) = r \sin\vartheta\, \mathrm{d}\varphi(u) = r u^\varphi = r\dot{\varphi} \tag{9.80}$$

at $\vartheta = \pi/2$, which yields the angular frequency

$$\omega \equiv \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\dot{\varphi}}{\dot{t}} = \frac{c\mathrm{e}^{-b}}{r} \frac{u^3}{u^0} \;, \tag{9.81}$$

which can be rewritten by means of (9.70),

$$\omega^2 = \left(\frac{u^3}{u^0}\right)^2 \frac{\mathrm{e}^{-2b}}{r^2} = -\frac{c^2 b'}{r} \mathrm{e}^{-2b} = \frac{c^2}{2r} \left(\mathrm{e}^{-2b}\right)' \;. \tag{9.82}$$

Now, since the exponential factor was

$$\mathrm{e}^{-2b} = \left(1 - \frac{2m}{r}\right) \quad \Rightarrow \quad \left(\mathrm{e}^{-2b}\right)' = \frac{2m}{r^2} \;, \tag{9.83}$$

we obtain the well-known intermediate result

$$\omega^2 = \frac{mc^2}{r^3} = \frac{\mathcal{G}M}{r^3} \;, \tag{9.84}$$

which is Kepler's third law.

Taking another time derivative of (9.79), we can use $\dot{r} = 0$ for circular orbits and $\dot{u}^3 = 0$ from (9.69). Thus,

$$\frac{\mathrm{d}^2 s^1}{\mathrm{d}t^2} = -\frac{c^2 b'}{u^3}\mathrm{e}^{-2b}\,\frac{\mathrm{d}s^3}{\mathrm{d}t} = -\frac{c^4 b'^2 \mathrm{e}^{-4b}}{(u^3)^2}\,s^1 \tag{9.85}$$

? Verify the calculation leading to the squared angular frequency Ω^2 in (9.86).

and likewise for s^3. This is an oscillator equation for s^1 with the squared angular frequency

$$\begin{aligned}\Omega^2 &= \frac{c^4 b'^2 \mathrm{e}^{-4b}}{(u^3)^2} = c^2 b'^2 \mathrm{e}^{-4b}\,\frac{(u^0)^2 - (u^3)^2}{(u^3)^2} \\ &= c^2 b'^2 \mathrm{e}^{-4b}\left(-1 - \frac{1}{b'r}\right) = -\frac{c^2 b' \mathrm{e}^{-4b}}{r}\,(1 + b'r)\ .\end{aligned} \tag{9.86}$$

Now, we use (9.82) to substitute the factor out front the final expression and find the relation

$$\Omega^2 = \omega^2 \mathrm{e}^{-2b}\,(1 + b'r) \tag{9.87}$$

between the angular frequencies Ω and ω. From (8.62), we further know that

$$rb' = \frac{1}{2}\left(1 - \mathrm{e}^{2b}\right) = \frac{1}{2}\left(1 - \frac{1}{1 - 2m/r}\right) = -\frac{m}{r}\,\frac{1}{1 - 2m/r}\ , \tag{9.88}$$

thus

$$rb' + 1 = \frac{r - 3m}{r - 2m} = \frac{1 - 3m/r}{1 - 2m/r} \tag{9.89}$$

and

$$\Omega^2 = \omega^2 \mathrm{e}^{-2b}\,\frac{1 - 3m/r}{1 - 2m/r} = \omega^2\left(1 - \frac{3m}{r}\right)\ . \tag{9.90}$$

In vector notation, we can write (9.79) as

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = \vec{\Omega} \times \vec{s}\,, \quad \vec{\Omega} = \begin{pmatrix} 0 \\ \Omega \\ 0 \end{pmatrix}\ . \tag{9.91}$$

Recall that we have projected the spin into the three-dimensional space perpendicular to the direction of motion. Thus, the result (9.91) shows that $\vec{s}$ precesses retrograde in that space about an axis perpendicular to the plane of the orbit, since $u^2 = 0$.

After a complete orbit, i.e. after the orbital time $\tau = 2\pi/\omega$, the projection of $\vec{s}$ onto the plane of the orbit has advanced by an angle

$$\phi = \Omega\tau = 2\pi\,\frac{\Omega}{\omega} = 2\pi\sqrt{1 - \frac{3m}{r}} < 2\pi\ , \tag{9.92}$$

according to (9.90). The spin thus falls behind the orbital motion; its precession is *retrograde*. The *geodetic precession frequency* is

$$\omega_s = \frac{\phi - 2\pi}{\tau} = \omega\left(\sqrt{1 - \frac{3m}{r}} - 1\right) \approx -\left(\frac{\mathcal{G}M}{r^3}\right)^{1/2} \frac{3\mathcal{G}M}{2rc^2} = -\frac{3}{2}\frac{(\mathcal{G}M)^{3/2}}{c^2 r^{5/2}} \tag{9.93}$$

to first-order Taylor approximation in m/r, with ω from Kepler's third law (9.84).

Geodetic precession near the Earth

If we insert the Earth's mass and radius here, $M_{\text{Earth}} = 5.97 \cdot 10^{27}$ g and $R_{\text{Earth}} = 6.38 \cdot 10^8$ cm, we find a geodetic precession near the Earth of

$$\omega_s \approx -\left(2.66 \cdot 10^{-7}\right)'' \, \text{s}^{-1} \left(\frac{R_{\text{Earth}}}{r}\right)^{5/2} = -8.4'' \, \text{year}^{-1} \left(\frac{R_{\text{Earth}}}{r}\right)^{5/2} . \tag{9.94}$$

In this context, see also the Example box "Measurement of spin precession near the Earth" following the discussion of the Lense-Thirring effect leading to (7.62).