Chapter 8

The Schwarzschild Solution

8.1 Cartan's structure equations

8.1.1 Curvature forms

This section deals with a generalisation of the connection coefficients, and the torsion and curvature tensor components, to arbitrary bases. This will prove enormously efficient in our further discussion of the Schwarzschild solution.

Let M be a differentiable manifold, $\{e_i\}$ an arbitrary basis for vector fields and $\{\theta^i\}$ an arbitrary basis for dual vector fields, or 1-forms.

Connection forms

In analogy to the Christoffel symbols, we introduce the *connection forms* by

$$\nabla_v e_i = \omega_i^j(v) e_j \ . \tag{8.1}$$

Since $\nabla_v e_i$ is a vector, $\omega_i^j(v) \in \mathbb{R}$ is a real number, and thus $\omega_i^j \in \bigwedge^1$ is a dual vector, or a one-form.

Since, by definition (3.2) of the Christoffel symbols

$$\nabla_{\partial_k}\partial_j = \Gamma^i{}_{kj}\,\partial_i = \omega^i_j(\partial_k)\partial_i \tag{8.2}$$

in the coordinate basis $\{\partial_i\}$, we have in that particular basis,

$$\omega^i_j = \Gamma^i{}_{kj}\,\mathrm{d}x^k \ . \tag{8.3}$$

Since $\langle\theta^i, e_j\rangle$ is a constant (which is either zero or unity if the basis is orthonormal), we must have

$$\begin{aligned} 0 = \nabla_v\langle\theta^i, e_j\rangle &= \langle\nabla_v\theta^i, e_j\rangle + \langle\theta^i, \nabla_v e_j\rangle \\ &= \langle\nabla_v\theta^i, e_j\rangle + \langle\theta^i, \omega^k_j(v)e_k\rangle \\ &= \langle\nabla_v\theta^i, e_j\rangle + \omega^i_j(v) \ . \end{aligned} \tag{8.4}$$

From this result, we can conclude

$$\nabla_v \theta^i = -\omega^i_j(v)\theta^j \tag{8.5}$$

for the covariant derivative of θ^i in the direction of v. Without specifying the vector v, we find the covariant derivative

$$\nabla \theta^i = -\theta^j \otimes \omega^i_j \; . \tag{8.6}$$

Let now $\alpha \in \bigwedge^1$ be a one-form such that $\alpha = \alpha_i \theta^i$ with arbitrary functions α_i. Then, the equations we have derived so far imply

$$\nabla_v \alpha = v(\alpha_i)\theta^i + \alpha_i \nabla_v \theta^i = \langle \mathrm{d}\alpha_i - \alpha_k \omega^k_i, v \rangle \theta^i \; , \tag{8.7}$$

where we have used the differential of the function α_i, defined in (2.35) by $\mathrm{d}\alpha_i(v) = v(\alpha_i)$, together with the notation $\langle w, v \rangle = w(v)$ for a vector v and a dual vector w. More generally, this expression can be written as the covariant derivative

$$\nabla \alpha = \theta^i \otimes (\mathrm{d}\alpha_i - \alpha_k \omega^k_i) \; . \tag{8.8}$$

Similarly, for a vector field $x = x^i e_i$, we find

$$\nabla_v x = \langle \mathrm{d}x^i + x^k \omega^i_k, v \rangle e_i \tag{8.9}$$

or

$$\nabla x = e_i \otimes (\mathrm{d}x^i + \omega^i_k x^k) \tag{8.10}$$

for the covariant derivative of the vector x.

?

Derive the expressions (8.9) and (8.10) yourself, beginning with (8.1).

8.1.2 Torsion and curvature forms

We are now in a position to use the connection forms for defining the torsion and curvature forms.

Torsion and curvature forms

By definition, the torsion $T(x, y)$ is a vector, which can be written in terms of the *torsion forms* Θ^i as

$$T(x, y) = \Theta^i(x, y)e_i \; . \tag{8.11}$$

Obviously, $\Theta^i \in \bigwedge^2$ is a two-form, such that $\Theta^i(x, y) \in \mathbb{R}$ is a real number.
In the same manner, we express the curvature by the *curvature forms* $\Omega^i_j \in \bigwedge^2$,

$$\bar{R}(x, y)e_j = \Omega^i_j(x, y)e_i \; . \tag{8.12}$$

The next important step is now to realise that the torsion and curvature 2-forms satisfy *Cartan's structure equations*:

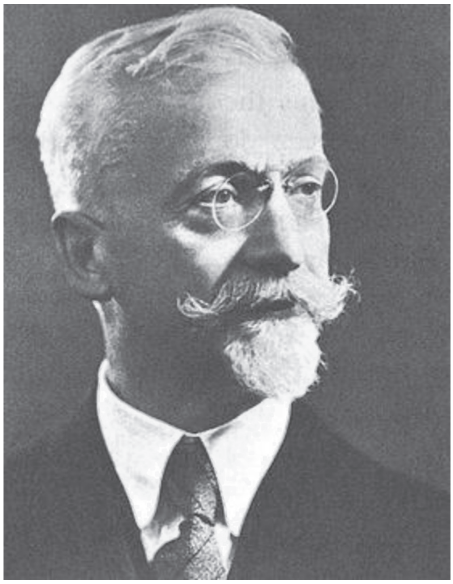

Figure 8.1 Élie Cartan (1869–1951), French mathematician. Source: Wikipedia

Cartan's structure equations

In terms of the connection forms $\omega^i_{\ j}$, the torsion forms Θ^i and the curvature forms $\Omega^i_{\ j}$ are determined by Cartan's structure equations,

$$\begin{aligned} \Theta^i &= \mathrm{d}\theta^i + \omega^i_{\ j} \wedge \theta^j \\ \Omega^i_{\ j} &= \mathrm{d}\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j} \ . \end{aligned} \tag{8.13}$$

Their proof is straightforward. To prove the first structure equation, we insert the definition (3.45) of the torsion to obtain as a first step

$$\begin{aligned} \Theta^i(x,y) &= \nabla_x y - \nabla_y x - [x,y] \\ &= \nabla_x(\theta^i(y)e_i) - \nabla_y(\theta^i(x)e_i) - \theta^i([x,y])e_i \ , \end{aligned} \tag{8.14}$$

where we have expanded the vectors x, y and $[x,y]$ in the basis $\{e_i\}$ according to $x = \langle\theta^i, x\rangle e_i = \theta^i(x)e_i$. Then, we continue by using the connection forms,

$$\begin{aligned} \Theta^i(x,y) &= \nabla_x(\theta^i(y)e_i) - \nabla_y(\theta^i(x)e_i) - \theta^i([x,y])e_i \\ &= x\theta^i(y)e_i + \theta^i(y)\omega^j_{\ i}(x)e_j - y\theta^i(x)e_i - \theta^i(x)\omega^j_{\ i}(y)e_j \\ &\quad - \theta^i([x,y])e_i \\ &= \left[x\theta^i(y) - y\theta^i(x) - \theta^i([x,y])\right]e_i \\ &\quad + \left[\theta^i(y)\omega^j_{\ i}(x) - \theta^i(x)\omega^j_{\ i}(y)\right]e_j \ . \end{aligned} \tag{8.15}$$

According to (5.66), the first term can be expressed by the exterior derivative of the θ^i, and since the second term is antisymmetric in x and y, we can write this as

$$\Theta^i(x,y) = \mathrm{d}\theta^i(x,y)e_i + (\omega^i_{\ j} \wedge \theta^j)(x,y)e_i \ , \tag{8.16}$$

from which the first structure equation follows immediately.

The proof of the second structure equation proceeds similarly, using the definition (3.51) of the curvature. Thus,

$$\begin{aligned}
\Omega^i_{\ j}(x,y)e_i &= \nabla_x\nabla_y e_j - \nabla_y\nabla_x e_j - \nabla_{[x,y]}e_j \\
&= \nabla_x(\omega^i_{\ j}(y)e_i) - \nabla_y(\omega^i_{\ j}(x)e_i) - \omega^i_{\ j}([x,y])e_i \\
&= x\omega^i_{\ j}(y)e_i + \omega^i_{\ j}(y)\nabla_x e_i \\
&\quad - y\omega^i_{\ j}(x)e_i - \omega^i_{\ j}(x)\nabla_y e_i - \omega^i_{\ j}([x,y])e_i \\
&= \left[x\omega^i_{\ j}(y) - y\omega^i_{\ j}(x) - \omega^i_{\ j}([x,y])\right]e_i \\
&\quad + \left[\omega^i_{\ j}(y)\omega^k_{\ i}(x) - \omega^i_{\ j}(x)\omega^k_{\ i}(y)\right]e_k \\
&= \mathrm{d}\omega^i_{\ j}(x,y)e_i + (\omega^k_{\ i} \wedge \omega^i_{\ j})(x,y)e_k \ ,
\end{aligned} \tag{8.17}$$

?

Carry out all steps of the derivations (8.15) and (8.17) yourself and convince yourself that they are correct.

which proves the second structure equation.

Now, let us use the curvature forms $\Omega^i_{\ j}$ to define tensor components $\bar{R}^i_{\ jkl}$ by

$$\Omega^i_{\ j} \equiv \frac{1}{2}\bar{R}^i_{\ jkl}\,\theta^k \wedge \theta^l \ , \tag{8.18}$$

whose antisymmetry in the last two indices is obvious by definition,

$$\bar{R}^i_{\ jkl} = -\bar{R}^i_{\ jlk} \ . \tag{8.19}$$

In an arbitrary basis $\{e_i\}$, we then have

$$\langle\theta^i, \bar{R}(e_k,e_l)e_j\rangle = \langle\theta^i, \Omega^s_{\ j}(e_k,e_l)e_s\rangle = \Omega^i_{\ j}(e_k,e_l) = \bar{R}^i_{\ jkl} \ . \tag{8.20}$$

Comparing this to the components of the curvature tensor in the coordinate basis $\{\partial_i\}$ given by (3.56) shows that the functions $\bar{R}^i_{\ jkl}$ are indeed the components of the curvature tensor in the arbitrary basis $\{e_i\}$.

A similar operation shows that the functions $T^i_{\ jk}$ defined by

$$\Theta^i \equiv \frac{1}{2}T^i_{\ jk}\,\theta^j \wedge \theta^k \tag{8.21}$$

are the elements of the torsion tensor in the basis $\{e_i\}$, since

$$\langle\theta^i, T(e_j,e_k)\rangle = \langle\theta^i, \Theta^s(e_j,e_k)e_s\rangle = \Theta^i(e_j,e_k) = T^i_{\ jk} \ . \tag{8.22}$$

Thus, Cartan's structure equations allow us to considerably simplify the computation of curvature and torsion for an arbitrary metric, provided we find a base in which the metric appears simple (e.g. diagonal and constant).

Symmetry of the connection forms

We mention without proof that the connection ∇ is metric if and only if

$$\omega_{ij} + \omega_{ji} = \mathrm{d}g_{ij} , \tag{8.23}$$

where the definitions

$$\omega_{ij} \equiv g_{ik}\omega^k_j \quad \text{and} \quad g_{ij} \equiv g(e_i, e_j) \tag{8.24}$$

were used, i.e. the g_{ij} are the components of the metric in the arbitrary basis $\{e_i\}$.

8.2 Stationary and static spacetimes

Stationary spacetimes (M, g) are defined to be spacetimes which have a time-like Killing vector field K. This means that observers moving along the integral curves of K do not notice any change.

?

What exactly are Killing vector fields? How are they defined, and what do they mean?

This definition implies that we can introduce coordinates in which the components $g_{\mu\nu}$ of the metric do not depend on time. To see this, suppose we choose a space-like hypersurface $\Sigma \subset M$ and construct the integral curves of K through Σ.

We further introduce arbitrary coordinates on Σ and carry them into M as follows: let ϕ_t be the flow of K, $p_0 \in \Sigma$ and $p = \phi_t(p_0)$, then the coordinates of p are chosen as $(t, x^1(p_0), x^2(p_0), x^3(p_0))$. These are the so-called *Lagrange coordinates* of p.

In these coordinates, $K = \partial_0$, i.e. $K^\mu = \delta^\mu_0$. From the derivation of the Killing equation (5.34), we further have that the components of the Lie derivative of the metric are

$$\begin{aligned}(\mathcal{L}_K g)_{\mu\nu} &= K^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu K^\lambda + g_{\mu\lambda}\partial_\nu K^\lambda \\ &= \partial_0 g_{\mu\nu} = 0 ,\end{aligned} \tag{8.25}$$

which proves that the $g_{\mu\nu}$ do not depend on time in these so-called *adapted* coordinates.

We can straightforwardly introduce a one-form ω corresponding to the Killing vector K by $\omega = K^\flat$. This one-form obviously satisfies

$$\omega(K) = \langle K, K\rangle \neq 0 . \tag{8.26}$$

Suppose that we now have a *stationary* spacetime in which we have introduced adapted coordinates and in which also $g_{0i} = 0$. Then, the Killing vector field is orthogonal to the spatial sections, for which $t =$ const. Then, the one-form ω is quite obviously

$$\omega = g_{00}\, c\mathrm{d}t = \langle K, K\rangle\, c\mathrm{d}t , \tag{8.27}$$

because $K = \partial_0$. This then trivially implies the *Frobenius condition*

$$\omega \wedge \mathrm{d}\omega = 0 \tag{8.28}$$

because the exterior derivative d satisfies $\mathrm{d} \circ \mathrm{d} \equiv 0$.

Conversely, it can be shown that if the Frobenius condition holds, the one-form ω can be written in the form (8.27). For a vector field v tangential to a spacelike section defined by t = const., we have

$$\langle K, v\rangle = \omega(v) = \langle K, K\rangle\, c\mathrm{d}t(v) = \langle K, K\rangle v(t) = 0 \tag{8.29}$$

because t = const., and thus K is then perpendicular to the spatial section. Thus, $K = \partial_0$ and

$$g_{0i} = \langle \partial_0, \partial_i\rangle = \langle K, \partial_i\rangle = 0\ . \tag{8.30}$$

Stationary and static spacetimes

Thus, in a stationary spacetime with time-like Killing vector field K, the Frobenius condition (8.28) for the one-form $\omega = K^\flat$ is equivalent to the condition $g_{0i} = 0$ in adapted coordinates. Such spacetimes are called *static*. In other words, stationary spacetimes are static if and only if the Frobenius condition holds.

In *static* spacetimes, the metric can thus be written in the form

$$g = g_{00}(\vec{x})c^2\mathrm{d}t^2 + g_{ij}(\vec{x})\mathrm{d}x^i\mathrm{d}x^j\ . \tag{8.31}$$

8.3 The Schwarzschild solution

8.3.1 Form of the metric

Formally speaking, the Schwarzschild solution is a static, spherically symmetric solution of Einstein's field equations for vacuum spacetime.

From our earlier considerations, we know that a *static* spacetime is a *stationary* spacetime whose (time-like) Killing vector field satisfies the Frobenius condition (8.28).

As the spacetime is (globally) stationary, we know that we can introduce spatial hypersurfaces Σ perpendicular to the Killing vector field which, in adapted coordinates, is $K = \partial_0$. The manifold (M, g) can thus be *foliated* as $M = \mathbb{R} \times \Sigma$.

?

How does a product space composed of two manifolds attains the structure of a product manifold.

From (8.31), we then know that, also in adapted coordinates, the metric acquires the form

$$g = -\phi^2\, c^2\mathrm{d}t^2 + h\ , \tag{8.32}$$

where ϕ is a smoothly varying function on Σ and h is the metric of the spatial sections Σ. Under the assumption that K is the only time-like Killing vector field which the spacetime admits, t is a uniquely distinguished time coordinate, and we can write

$$-\phi^2 = \langle K, K\rangle \ . \tag{8.33}$$

The stationarity of the spacetime, expressed by the existence of the single Killing vector field K, thus allows a convenient foliation of the spacetime into spatial hypersurfaces or foils Σ and a time coordinate.

Furthermore, the spatial hypersurfaces Σ are expected to be spherically symmetric. This means that the group $SO(3)$ (i.e. the group of rotations in three dimensions) must be an isometry group of the metric h. The orbits of $SO(3)$ are two-dimensional, space-like surfaces in Σ. Thus, $SO(3)$ foliates the spacetime (Σ, h) into invariant two-spheres.

Let the surface of these two-spheres be A, then we *define* a radial coordinate for the Schwarzschild metric requiring

$$4\pi r^2 = A \tag{8.34}$$

as in Euclidean geometry. Moreover, the spherical symmetry implies that we can introduce spherical polar coordinates (ϑ, φ) on one particular orbit of $SO(3)$ which can then be transported along geodesic lines perpendicular to the orbits. Then, the spatial metric h can be written in the form

$$h = \mathrm{e}^{2b(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \ , \tag{8.35}$$

where the exponential factor was introduced to allow a scaling of the radial coordinate.

?

Why could it be useful to represent the coefficient of $\mathrm{d}r^2$ in h by an exponential?

Due to the stationarity of the metric and the spherical symmetry of the spatial sections, $\langle K, K\rangle$ can only depend on r. We set

$$\phi^2 = -\langle K, K\rangle = \mathrm{e}^{2a(r)} \ . \tag{8.36}$$

The full metric g is thus characterised by two radial functions $a(r)$ and $b(r)$ which we need to determine. The exponential functions in (8.35) and (8.36) are chosen to ensure that the prefactors e^a and e^b are always positive.

The spatial sections Σ are now foliated according to

$$\Sigma = I \times S^2 \ , \quad I \subset \mathbb{R}^+ \ , \tag{8.37}$$

with coordinates $r \in I$ and $(\vartheta, \varphi) \in S^2$.

Metric for static, spherically-symmetric spacetimes

In the *Schwarzschild coordinates* $(t, r, \vartheta, \varphi)$, the metric of a static, spherically-symmetric spacetime has the form

$$g = -e^{2a(r)}\,c^2 dt^2 + e^{2b(r)}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\ . \tag{8.38}$$

The functions $a(r)$ and $b(r)$ are constrained by the requirement that the metric should asymptotically turn flat, which means

$$a(r) \to 0\ ,\quad b(r) \to 0 \quad \text{for} \quad r \to \infty\ . \tag{8.39}$$

They must be determined by inserting the metric (8.38) into the vacuum field equations, $G = 0$.

8.3.2 Connection and curvature forms

In order to evaluate Einstein's field equations for the Schwarzschild metric, we now need to compute the Riemann, Ricci, and Einstein tensors. Traditionally, one would begin this step with computing all Christoffel symbols of the metric (8.38). This very lengthy and error-prone procedure can be considerably shortened using Cartan's structure equations (8.13) for the torsion and curvature forms Θ^i and $\Omega^i_{\ j}$.

To do so, we need to introduce a suitable basis, or *tetrad* $\{e_i\}$, or alternatively a dual tetrad $\{\theta^i\}$. Guided by the form of the metric (8.38), we choose

$$\theta^0 = e^a c dt\ ,\quad \theta^1 = e^b dr\ ,\quad \theta^2 = r d\vartheta\ ,\quad \theta^3 = r\sin\vartheta d\varphi\ . \tag{8.40}$$

In terms of these, the metric attains the simple diagonal, Minkowskian form

$$g = g_{\mu\nu}\,\theta^\mu \otimes \theta^\nu\ ,\quad g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)\ . \tag{8.41}$$

Obviously, $dg = 0$, and thus (8.23) implies that the connection forms $\omega_{\mu\nu}$ need to be antisymmetric,

$$\omega_{\mu\nu} = -\omega_{\nu\mu}\ . \tag{8.42}$$

Given the dual tetrad $\{\theta^\mu\}$, we must take their exterior derivatives. For this purpose, we apply the expression (**??**) and find, for $d\theta^0$,

$$d\theta^0 = de^a \wedge c dt = -a' e^a\, c dt \wedge dr\ . \tag{8.43}$$

because $de^a = a' e^a\, dr$. Similarly, we find

$$d\theta^1 = 0 \tag{8.44}$$

because $\mathrm{d}r \wedge \mathrm{d}r = 0$, further

$$\mathrm{d}\theta^2 = \mathrm{d}r \wedge \mathrm{d}\vartheta \tag{8.45}$$

and

$$\mathrm{d}\theta^3 = \sin\vartheta\, \mathrm{d}r \wedge \mathrm{d}\varphi + r\cos\vartheta\, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi\ . \tag{8.46}$$

Using (8.40), we can also express the coordinate differentials by the dual tetrad,

$$c\mathrm{d}t = \mathrm{e}^{-a}\theta^0\ , \quad \mathrm{d}r = \mathrm{e}^{-b}\theta^1\ , \quad \mathrm{d}\vartheta = \frac{\theta^2}{r}\ , \quad \mathrm{d}\varphi = \frac{\theta^3}{r\sin\vartheta}\ , \tag{8.47}$$

so that we can write the exterior derivatives of the dual tetrad as

$$\begin{aligned} \mathrm{d}\theta^0 = a'\mathrm{e}^{-b}\,\theta^1 \wedge \theta^0\ , \quad \mathrm{d}\theta^1 = 0\ , \quad \mathrm{d}\theta^2 = \frac{\mathrm{e}^{-b}}{r}\,\theta^1 \wedge \theta^2\ , \\ \mathrm{d}\theta^3 = \frac{\mathrm{e}^{-b}}{r}\,\theta^1 \wedge \theta^3 + \frac{\cot\vartheta}{r}\,\theta^2 \wedge \theta^3\ . \end{aligned} \tag{8.48}$$

?

Test by independent calculation whether you can confirm the differentials (8.48).

Since the torsion must vanish, $\Theta^i = 0$, Cartan's first structure equation from (8.13) implies

$$\mathrm{d}\theta^\mu = -\omega^\mu_\nu \wedge \theta^\nu\ . \tag{8.49}$$

Connection forms

With (8.48), this suggests that the connection forms of a static, spherically-symmetric metric are

$$\begin{aligned} \omega^0_1 = \omega^1_0 = \frac{a'\theta^0}{\mathrm{e}^b}\ , \quad \omega^0_2 = \omega^2_0 = 0\ , \quad \omega^0_3 = \omega^3_0 = 0\ , \\ \omega^2_1 = -\omega^1_2 = \frac{\theta^2}{r\mathrm{e}^b}\ , \quad \omega^3_1 = -\omega^1_3 = \frac{\theta^3}{r\mathrm{e}^b}\ , \\ \omega^3_2 = -\omega^2_3 = \frac{\cot\vartheta\,\theta^3}{r}\ . \end{aligned} \tag{8.50}$$

They satisfy the antisymmetry condition (8.42) and Cartan's first structure equation (8.49) for a torsion-free connection.

?

Why can none of the connection forms in (8.50) depend on φ?

For evaluating the curvature forms Ω^μ_ν, we first recall that the exterior derivative of a one-form ω multiplied by a function f is

$$\begin{aligned} \mathrm{d}(f\omega) &= \mathrm{d}f \wedge \omega + f\mathrm{d}\omega \\ &= (\partial_i f)\mathrm{d}x^i \wedge \omega + f\mathrm{d}\omega \end{aligned} \tag{8.51}$$

according to the (anti-)Leibniz rule (**??**).

Thus, we have for $\mathrm{d}\omega^0_1$

$$\begin{aligned} \mathrm{d}\omega^0_1 &= (a'\mathrm{e}^{-b})'\,\mathrm{d}r \wedge \theta^0 + a'\mathrm{e}^{-b}\,\mathrm{d}\theta^0 \\ &= (a''\mathrm{e}^{-b} - a'b'\mathrm{e}^{-b})\mathrm{e}^{-b}\,\theta^1 \wedge \theta^0 + (a'\mathrm{e}^{-b})^2\,\theta^1 \wedge \theta^0 \\ &=: A\,\theta^0 \wedge \theta^1 \end{aligned} \tag{8.52}$$

where we have used (8.47) and (8.48) and abbreviated $A := (a'' - a'b' + a'^2)E$ with $E := \exp(-2b)$.

In much the same way and using this definition of E, we find

$$\begin{aligned}
\mathrm{d}\omega^2_1 &= -\frac{b'E}{r}\,\theta^1 \wedge \theta^2 \ , \\
\mathrm{d}\omega^3_1 &= -\frac{b'E}{r}\,\theta^1 \wedge \theta^3 + \frac{\cot\vartheta}{r^2 \mathrm{e}^b}\,\theta^2 \wedge \theta^3 \ , \\
\mathrm{d}\omega^3_2 &= -\frac{1}{r^2}\,\theta^2 \wedge \theta^3 \ .
\end{aligned} \tag{8.53}$$

This yields the curvature two-forms according to (8.13).

Curvature forms of a static, spherically-symmetric metric

The curvature forms of a static, spherically-symmetric metric are

$$\begin{aligned}
\Omega^0_1 &= \mathrm{d}\omega^0_1 = -A\,\theta^0 \wedge \theta^1 = \Omega^1_0 \\
\Omega^0_2 &= \omega^0_1 \wedge \omega^1_2 = -\frac{a'E}{r}\,\theta^0 \wedge \theta^2 = \Omega^2_0 \\
\Omega^0_3 &= \omega^0_1 \wedge \omega^1_3 = -\frac{a'E}{r}\,\theta^0 \wedge \theta^3 = \Omega^3_0 \\
\Omega^1_2 &= \mathrm{d}\omega^1_2 + \omega^1_3 \wedge \omega^3_2 = \frac{b'E}{r}\,\theta^1 \wedge \theta^2 = -\Omega^2_1 \\
\Omega^1_3 &= \mathrm{d}\omega^1_3 + \omega^1_2 \wedge \omega^2_3 = \frac{b'E}{r}\,\theta^1 \wedge \theta^3 = -\Omega^3_1 \\
\Omega^2_3 &= \mathrm{d}\omega^2_3 + \omega^2_1 \wedge \omega^1_3 = \frac{1-E}{r^2}\,\theta^2 \wedge \theta^3 = -\Omega^3_2 \ .
\end{aligned} \tag{8.54}$$

?

Perform the calculations leading to the curvature forms (8.54) yourself and see whether you can confirm them.

The remaining curvature two-forms follow from antisymmetry since

$$\Omega_{\mu\nu} = g_{\mu\lambda}\Omega^\lambda_\nu = -\Omega_{\nu\mu} \ , \tag{8.55}$$

because of the (anti-)symmetries of the curvature.

8.4 Solution of the field equations

8.4.1 Components of the Ricci and Einstein tensors

The components of the curvature tensor are given by (8.20), and thus the components of the Ricci tensor in the tetrad $\{e_\alpha\}$ are

$$R_{\mu\nu} = \bar{R}^\lambda{}_{\mu\lambda\nu} = \Omega^\lambda_\mu(e_\lambda, e_\nu) \ . \tag{8.56}$$

Thus, the components of the Ricci tensor in the Schwarzschild tetrad are

$$\begin{aligned}
R_{00} &= \Omega^1_0(e_1, e_0) + \Omega^2_0(e_2, e_0) + \Omega^3_0(e_3, e_0) = A + \frac{2a'E}{r} \ , \\
R_{11} &= \Omega^0_1(e_0, e_1) + \Omega^2_1(e_2, e_1) + \Omega^3_1(e_3, e_1) = -A + \frac{2b'E}{r}
\end{aligned} \tag{8.57}$$

and, with $B := (b' - a')E/r$,

$$
\begin{aligned}
R_{22} &= \Omega^0_2(e_0, e_2) + \Omega^1_2(e_1, e_2) + \Omega^3_2(e_3, e_2) =: B + \frac{1-E}{r^2} \\
R_{33} &= \Omega^0_3(e_0, e_3) + \Omega^1_3(e_1, e_3) + \Omega^2_3(e_2, e_3) = R_{22}
\end{aligned}
\tag{8.58}
$$

The Ricci scalar becomes

$$
\mathcal{R} = -2A + 4B + 2\frac{1-E}{r^2} \;, \tag{8.59}
$$

such that we can now determine the components of the Einstein tensor in the tetrad $\{e_\alpha\}$:

Einstein tensor for a static, spherically-symmetric metric

The Einstein tensor of a static, spherically-symmetric metric has the components

$$
\begin{aligned}
G_{00} &= R_{00} - \frac{\mathcal{R}}{2} g_{00} = \frac{1}{r^2} - E\left(\frac{1}{r^2} - \frac{2b'}{r}\right) \\
G_{11} &= -\frac{1}{r^2} + E\left(\frac{1}{r^2} + \frac{2a'}{r}\right) \\
G_{22} &= E\,(A - B) = G_{33} \;.
\end{aligned}
\tag{8.60}
$$

All off-diagonal components of $G_{\mu\nu}$ vanish identically.

?

Convince yourself of the components (8.60) of the Einstein tensor.

8.4.2 The Schwarzschild metric

The vacuum field equations now require that all components of the Einstein tensor vanish. In particular, then,

$$
0 = G_{00} + G_{11} = \frac{2E}{r}(a' + b') \tag{8.61}
$$

shows that $a' + b' = 0$. Since $a + b \to 0$ asymptotically for $r \to \infty$, integrating $a + b$ from $r \to \infty$ indicates that $a + b = 0$ everywhere, or $b = -a$.

After multiplying with r^2, equation $G_{00} = 0$ itself implies that

$$
E(1 - 2rb') = 1 \quad \Leftrightarrow \quad (rE)' = 1 \;. \tag{8.62}
$$

Therefore, (8.62) is equivalent to

$$
rE = r + C \quad \Leftrightarrow \quad E = 1 + \frac{C}{r} \;, \tag{8.63}
$$

with an integration constant C to be determined.

Figure 8.2 Karl Schwarzschild (1873–1916), German astronomer and physicist. Source: Wikipedia

Since $a = -b$, this also allows to conclude that

$$e^{2a} = E = 1 + \frac{C}{r} \; . \tag{8.64}$$

The integration constant C is finally determined by the Newtonian limit. We have seen before in (4.80) that the 0-0 element of the metric must be related to the Newtonian gravitational potential as $g_{00} = -(1 + 2\Phi/c^2)$ in order to meet the Newtonian limit. The Newtonian potential of a point mass M at a distance r is

$$\Phi = -\frac{\mathcal{G}M}{r} \; . \tag{8.65}$$

Together with (8.64), this shows that the Newtonian limit is reached by the Schwarzschild solution if the integration constant C is set to

$$C = -\frac{2\mathcal{G}M}{c^2} =: -2m \quad \text{with} \quad m = \frac{\mathcal{G}M}{c^2} \approx 1.5\,\mathrm{km}\left(\frac{M}{M_\odot}\right) \; . \tag{8.66}$$

Schwarzschild metric

We thus obtain the *Schwarzschild solution* for the metric,

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)c^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \dfrac{2m}{r}} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \; . \tag{8.67}$$

The Schwarzschild metric (8.67) has an (apparent) singularity at $r = 2m$ or

$$r = R_\mathrm{s} \equiv \frac{2\mathcal{G}M}{c^2} \; , \tag{8.68}$$

the so-called *Schwarzschild radius*. We shall clarify the meaning of this singularity later.

In order to illustrate the geometrical meaning of the spatial part of the Schwarzschild metric, we need to find a geometrical interpretation for its radial dependence. Specialising to the equatorial plane of the Schwarzschild solution, $\vartheta = \pi/2$ and $t = 0$, we find the induced spatial line element

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{1 - 2m/r} + r^2\mathrm{d}\varphi^2 \tag{8.69}$$

on that plane.

On the other hand, consider a surface of rotation in the three-dimensional Euclidean space E^3. If we introduce the adequate cylindrical coordinates (r, ϕ, z) on E^3 and rotate a curve $z(r)$ about the z axis, we find the induced line element

$$\begin{aligned} \mathrm{d}l^2 &= \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 = \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2 \mathrm{d}r^2 + \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 \\ &= (1 + z'^2)\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 \,. \end{aligned} \tag{8.70}$$

We can now try and identify the two induced line elements from (8.69) and (8.70) and find that this is possible if

$$z' = \left(\frac{1}{1 - 2m/r} - 1\right)^{1/2} = \sqrt{\frac{2m}{r - 2m}} \,, \tag{8.71}$$

which is readily integrated to yield

$$z = \sqrt{8m(r - 2m)} + \text{const.} \quad \text{or} \quad z^2 = 8m(r - 2m) \,, \tag{8.72}$$

if we set the integration constant to zero.

This shows that the geometry on the equatorial plane of the spatial section of the Schwarzschild solution can be identified with a rotational paraboloid in E^3. In other words, the dependence of radial distances on the radius r is equivalent to that on a rotational paraboloid (cf. Fig. 8.3).

8.4.3 Birkhoff's theorem

Suppose now we had started from a spherically symmetric vacuum spacetime, but with explicit time dependence of the functions a and b, such that the spacetime could either expand or contract. Then, a repetition of the derivation of the connection and curvature forms, and

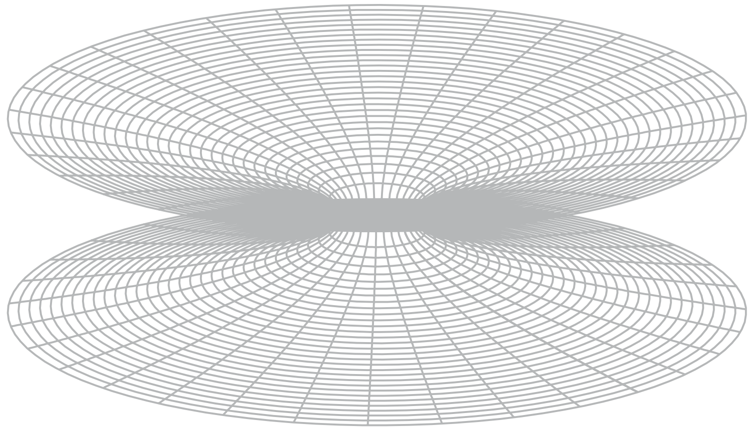

Figure 8.3 Surface of rotation illustrating the spatial part of the Schwarzschild metric.

the components of the Einstein tensor following from them, had resulted in the new components $\bar{G}_{\mu\nu}$

$$\begin{aligned} \bar{G}_{00} &= G_{00}\,, \quad \bar{G}_{11} = G_{11} \\ \bar{G}_{22} &= G_{22} - \mathrm{e}^{-2a}\left(\dot{b}^2 - \dot{a}\dot{b} - \ddot{b}\right) = \bar{G}_{33} \\ \bar{G}_{10} &= \frac{2\dot{b}}{r}\mathrm{e}^{-a-b} \end{aligned} \tag{8.73}$$

and $\bar{G}_{\mu\nu} = 0$ for all other components.

The vacuum field equations imply $\bar{G}_{10} = 0$ and thus $\dot{b} = 0$, hence b must be independent of time. From $\bar{G}_{00} = 0$, we can again conclude (8.63), i.e. b retains the same form as before. Similarly, since $\bar{G}_{00} + \bar{G}_{11} = G_{00} + G_{11}$, the requirement $a' + b' = 0$ must continue to hold, but now the time dependence of a allows us to conclude only that

$$a = -b + f(t)\,, \tag{8.74}$$

where $f(t)$ is an otherwise unconstrained function of time only. Thus, the line element then reads

$$\mathrm{d}s^2 = -\mathrm{e}^{2f}\left(1 - \frac{2m}{r}\right)c^2\mathrm{d}t^2 + \mathrm{d}l^2\,, \tag{8.75}$$

where $\mathrm{d}l^2$ is the unchanged line element of the spatial sections.

Introducing the new time coordinate t' by

$$t' = \int \mathrm{e}^f\,\mathrm{d}t \tag{8.76}$$

converts (8.75) back to the original form (8.67) of the Schwarzschild metric.

Birkhoff's theorem

This is *Birkhoff's theorem*, which states that a spherically symmetric solution of Einstein's vacuum equations is *necessarily static* for $r > 2m$.