Chapter 7

Weak Gravitational Fields

7.1 Linearised theory of gravity

7.1.1 Linearised field equations

We begin our study of solutions for the field equations with situations in which the metric is almost Minkowskian, writing

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{7.1}$$

where $h_{\mu\nu}$ is considered as a perturbation of the Minkowski metric $\eta_{\mu\nu}$ such that

$$|h_{\mu\nu}| \ll 1 \ . \tag{7.2}$$

This condition is excellently satisfied e.g. in the Solar System, where

$$|h_{\mu\nu}| \approx \frac{\Phi}{c^2} \approx 10^{-6} \ . \tag{7.3}$$

?

How can you most easily confirm the estimate (7.3) for the Solar System and other astronomical objects?

Note that small perturbations of the *metric* do not necessarily imply small perturbations of the *matter density*, as the Solar System illustrates. Also, the metric perturbations may change rapidly in time.

First, we write down the Christoffel symbols for this kind of metric. Starting from (3.74) and ignoring quadratic terms in $h_{\mu\nu}$, we can write

$$\begin{aligned} \Gamma^{\alpha}{}_{\mu\nu} &= \frac{1}{2}\eta^{\alpha\beta}\left(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu}\right) \\ &= \frac{1}{2}\left(\partial_\nu h^{\alpha}_{\mu} + \partial_\mu h^{\alpha}_{\nu} - \partial^{\alpha} h_{\mu\nu}\right) \ . \end{aligned} \tag{7.4}$$

Next, we can ignore the terms quadratic in the Christoffel symbols in the components of the Ricci tensor (3.56) and find

$$R_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\lambda\mu} \ . \tag{7.5}$$

Inserting (7.4) yields

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\lambda\partial_\nu h^\lambda_\mu + \partial_\lambda\partial_\mu h^\lambda_\nu - \partial_\lambda\partial^\lambda h_{\mu\nu} - \partial_\mu\partial_\nu h^\lambda_\lambda\right)$$
$$= \frac{1}{2}\left(\partial_\lambda\partial_\nu h^\lambda_\mu + \partial_\lambda\partial_\mu h^\lambda_\nu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) , \tag{7.6}$$

where we have introduced the d'Alembert operator and abbreviated the trace of the metric perturbation,

Caution Note that the d'Alembert operator is the square of the ordinary partial derivative here, not the covariant derivative. Is this appropriate, and if so, why? ◀

$$\Box = \partial_\lambda\partial^\lambda , \quad h \equiv h^\lambda_\lambda . \tag{7.7}$$

The Ricci scalar is the contraction of $R_{\mu\nu}$,

$$\mathcal{R} = \partial_\lambda\partial_\mu h^{\lambda\mu} - \Box h , \tag{7.8}$$

and the Einstein tensor is

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\lambda\partial_\nu h^\lambda_\mu + \partial_\lambda\partial_\mu h^\lambda_\nu - \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma}\right.$$
$$\left. - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} + \eta_{\mu\nu}\Box h\right) . \tag{7.9}$$

Neglecting terms of order $|h_{\mu\nu}|^2$, the contracted Bianchi identity reduces to

$$\partial_\nu G^{\mu\nu} = 0 , \tag{7.10}$$

which, together with the field equations, implies

$$\partial_\nu T^{\mu\nu} = 0 . \tag{7.11}$$

One could now insert the Minkowski metric in $T^{\mu\nu}$, search for a first solution $h^{(0)}_{\mu\nu}$ of the linearised field equations and iterate replacing $\eta_{\mu\nu}$ by $\eta_{\mu\nu} + h^{(0)}_{\mu\nu}$ in $T^{\mu\nu}$ to find a corrected solution $h^{(1)}_{\mu\nu}$, and so forth. This procedure is useful as long as the back-reaction of the metric on the energy-momentum tensor is small.

If we specialise (7.11) for pressure-less dust and insert (6.82), we find the equation of motion

$$u^\nu\partial_\nu u^\mu = 0 , \tag{7.12}$$

which means that the fluid elements follow straight lines.

?

Compare (7.12) to the geodesic equation for the motion of fluid particles.

7.1.2 Wave equation for metric fluctuations

The field equations simplify considerably when we substitute

$$\gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, h \tag{7.13}$$

for $h_{\mu\nu}$. Since $\gamma \equiv \gamma^\mu_\mu = -h$, we can solve (7.13) for $h_{\mu\nu}$ and insert

$$h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma \tag{7.14}$$

into (7.9) to obtain the linearised field equations

$$\partial^\lambda\partial_\nu\gamma_{\lambda\mu} + \partial^\lambda\partial_\mu\gamma_{\lambda\nu} - \eta_{\mu\nu}\partial^\lambda\partial^\sigma\gamma_{\lambda\sigma} - \Box\gamma_{\mu\nu} = \frac{16\pi\mathcal{G}}{c^4}T_{\mu\nu} \; . \tag{7.15}$$

7.2 Gauge transformations

7.2.1 Diffeomorphism invariance

Diffeomorphism invariance

Let ϕ be a diffeomorphism of M, such that $\phi : M \to N$ in diffeomorphic way. Since ϕ is then bijective and smoothly differentiable and has a smoothly differentiable inverse, M and N can be considered as indistinguishable abstract manifolds. The manifolds M and N then represent the same physical spacetime. In particular, the metric g on M is then physically equivalent to the pulled-back metric ϕ^*g. This *diffeomorphism invariance* is a fundamental property of general relativity.

?

In particular, the diffeomorphism invariance of general relativity implies that coordinate systems can have no physical significance. Use your own words to explain why this is so.

In particular, this holds for a one-parameter group ϕ_t of diffeomorphisms which represents the (local) flow of some vector field v. By the definition of the Lie derivative, we have, to first order in t,

$$\phi^*g = g + t\mathcal{L}_v g \; . \tag{7.16}$$

Now, set $g = \eta + h$ and define the infinitesimal vector $\xi \equiv tv$. Then, the transformation (7.16) implies

$$h \to \phi^*h = h + t\mathcal{L}_v\eta + t\mathcal{L}_v h = h + \mathcal{L}_\xi\eta + \mathcal{L}_\xi h \; . \tag{7.17}$$

For weak fields, the third term on the right-hand side can be neglected. Using (5.31), we see that

$$(\mathcal{L}_\xi\eta)_{\mu\nu} = \eta_{\lambda\nu}\partial_\mu\xi^\lambda + \eta_{\mu\lambda}\partial_\nu\xi^\lambda = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \; . \tag{7.18}$$

We thus find the following important result:

Gauge transformations of weak metric perturbations

The weak metric perturbation $h_{\mu\nu}$ admits the *gauge transformation*

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \; . \tag{7.19}$$

This gauge transformation changes the tensor $\gamma_{\mu\nu}$ as

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\lambda\xi^\lambda \; . \tag{7.20}$$

7.2.2 Hilbert gauge

We can now arrange matters to enforce the *Hilbert gauge*

$$\partial_\nu \gamma^{\mu\nu} = 0 \;. \tag{7.21}$$

The gauge transformation (7.20) implies that the divergence of $\gamma^{\mu\nu}$ is transformed as

$$\partial_\nu \gamma^{\mu\nu} \to \partial_\nu \gamma^{\mu\nu} + \partial_\nu \partial^\mu \xi^\nu + \Box \xi^\mu - \partial^\mu \partial_\lambda \xi^\lambda = \partial_\nu \gamma^{\mu\nu} + \Box \xi^\mu \;, \tag{7.22}$$

such that, if (7.21) is not satisfied yet, it can be achieved by choosing for ξ^μ a solution of the inhomogeneous wave equation

$$\Box \xi^\mu = -\partial_\nu \gamma^{\mu\nu} \;, \tag{7.23}$$

which, as we know from electrodynamics, can be obtained by means of the retarded Green's function of the d'Alembert operator.

?

How are the retarded and the advanced Green's functions constructed in electrodynamics? Remind yourself of the essential steps.

Wave equation for metric perturbations

Enforcing the Hilbert gauge in this way simplifies the linearised field equation (7.15) dramatically,

$$\Box \gamma^{\mu\nu} = -\frac{16\pi\mathcal{G}}{c^4} T^{\mu\nu} \;. \tag{7.24}$$

These equations are formally identical to Maxwell's equations in Lorenz gauge, and therefore admit the same solutions. Defining the Green's function of the d'Alembert operator $\Box$ by

$$\Box G(x, x') = \Box G(t, t', \vec{x}, \vec{x}') = -4\pi \delta_\mathrm{D}(t - t', \vec{x} - \vec{x}') \tag{7.25}$$

and using $x^0 = ct$ instead of t, we find the retarded Greens function

$$G(x, x') = \frac{1}{|\vec{x} - \vec{x}'|} \delta_\mathrm{D}\left(x^0 - x'^0 - |\vec{x} - \vec{x}'|\right) \;. \tag{7.26}$$

Using it, we arrive at the particular solution

$$\gamma_{\mu\nu}(x) = \frac{4\mathcal{G}}{c^4} \int \frac{T_{\mu\nu}\left(x^0 - |\vec{x} - \vec{x}'|, \vec{x}'\right)}{|\vec{x} - \vec{x}'|} \mathrm{d}^3 x' \tag{7.27}$$

for the linearised field equation. Of course, arbitrary solutions of the homogeneous (vacuum) wave equation can be added.

Thus, similar to electrodynamics, the metric perturbation consists of the field generated by the source plus wave-like vacuum solutions propagating at the speed of light.

7.3 Nearly Newtonian gravity

7.3.1 Newtonian approximation of the metric

A nearly Newtonian source of gravity can be described by the approximations $T_{00} \gg |T_{0j}|$ and $T_{00} \gg |T_{ij}|$, which express that mean velocities are small, and the rest-mass energy dominates the kinetic energy. Then, we can also neglect retardation effects and write

$$\gamma_{00}(\vec{x}) = \frac{4\mathcal{G}}{c^2} \int \frac{\rho(\vec{x}')\,\mathrm{d}^3x'}{|\vec{x} - \vec{x}'|} = -4\frac{\Phi(\vec{x})}{c^2} \,, \tag{7.28}$$

where $\Phi(\vec{x})$ is the ordinary Newtonian gravitational potential. All other components of the metric perturbation $\gamma_{\mu\nu}$ vanish,

$$\gamma_{0j} = 0 = \gamma_{ij} \,. \tag{7.29}$$

Then, the full metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \left(\gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma\right) \tag{7.30}$$

has the components

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right) \,, \quad g_{0j} = 0 \,, \quad g_{ij} = \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij} \,, \tag{7.31}$$

creating the line element

$$\mathrm{d}s^2 = -\left(1 + \frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2\Phi}{c^2}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \,. \tag{7.32}$$

Far away from a source with mass M, the monopole term $-\mathcal{G}M/r$ dominates the gravitational potential Φ in (7.28). Thus, we find:

?

What does it mean to say that the monopole term dominates the potential? Which other terms could contribute?

Metric in the Newtonian limit

In the Newtonian limit, the weakly perturbed metric of a mass M has the line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2\mathcal{G}M}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 + \frac{2\mathcal{G}M}{rc^2}\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \,. \tag{7.33}$$

7.3.2 Gravitational lensing and the Shapiro delay

Two interesting conclusions can be drawn directly from (7.32). Since light follows null geodesics, light propagation is characterised by $\mathrm{d}s^2 = 0$ or

$$\left(1 + \frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 = \left(1 - \frac{2\Phi}{c^2}\right)\mathrm{d}\vec{x}^2 \,, \tag{7.34}$$

which implies that the light speed in a (weak) gravitational field is

$$c' = \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = \left(1 + \frac{2\Phi}{c^2}\right) c \tag{7.35}$$

to first order in Φ.

Since $\Phi \leq 0$ if normalised such that $\Phi \to 0$ at infinity, $c' \leq c$, which we can express by the *index of refraction for a weak gravitational field*,

$$n = \frac{c}{c'} = 1 - \frac{2\Phi}{c^2} \ . \tag{7.36}$$

Index of refraction of a gravitational field

A weak gravitational field with Newtonian gravitational potential Φ has the effective index of refraction

$$n = 1 - \frac{2\Phi}{c^2} \geq 1 \ . \tag{7.37}$$

This can be used to calculate light deflection using Fermat's principle, which asserts that light follows a path along which the light-travel time between a fixed source and a fixed observer is extremal, thus

$$\delta \int \mathrm{d}t = \delta \int \frac{\mathrm{d}x}{c'} \quad \Rightarrow \quad \delta \int n(\vec{x})|\mathrm{d}\vec{x}| = 0 \ . \tag{7.38}$$

Introducing a curve parameter λ, we can write $\vec{x} = \vec{x}(\lambda)$, thus $|\mathrm{d}\vec{x}| = (\dot{\vec{x}}^2)^{1/2}\mathrm{d}\lambda$ and

$$\delta \int n\left(\vec{x}\right)\left(\dot{\vec{x}}^2\right)^{1/2} \mathrm{d}\lambda = 0 \ , \tag{7.39}$$

where the overdot denotes derivation with respect to λ.

The variation leads to the Euler-Lagrange equation

$$\frac{\partial L}{\partial \vec{x}} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{\vec{x}}} = 0 \quad \text{with} \quad L \equiv n\left(\vec{x}\right)\left(\dot{\vec{x}}^2\right)^{1/2} \ . \tag{7.40}$$

Thus, we find

$$\left(\dot{\vec{x}}^2\right)^{1/2} \vec{\nabla} n - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[n\dot{\vec{x}}\left(\dot{\vec{x}}^2\right)^{-1/2}\right] = 0 \ . \tag{7.41}$$

We can simplify this expression by choosing the curve parameter such that $\dot{\vec{x}}$ is a unit vector $\vec{e}$, hence

$$\vec{\nabla} n - \left(\vec{e}\cdot\vec{\nabla} n\right)\vec{e} - n\dot{\vec{e}} = 0 \ . \tag{7.42}$$

?

Perform the calculation yourself leading from (7.39) to (7.42).

The first two terms are the component of $\vec{\nabla} n$ perpendicular to $\vec{e}$, and $\dot{\vec{e}}$ is the change of direction of the tangent vector along the light ray. Thus,

$$\dot{\vec{e}} = \vec{\nabla}_\perp \ln n = -\frac{2}{c^2}\vec{\nabla}_\perp \Phi \tag{7.43}$$

to first order in Φ. The total deflection angle is obtained by integrating $\dot{\vec{e}}$ along the light path.

As a second consequence, we see that the light travel time along an infinitesimal path length dl is

$$\mathrm{d}t = \frac{\mathrm{d}l}{c'} = n\,\frac{\mathrm{d}l}{c} = \left(1 - \frac{2\Phi}{c^2}\right)\frac{\mathrm{d}l}{c}\;. \tag{7.44}$$

Shapiro delay

Compared to light propagation in vacuum, there is thus a time delay

$$\Delta(\mathrm{d}t) = \mathrm{d}t - \frac{\mathrm{d}l}{c} = -\frac{2\Phi}{c^3}\,\mathrm{d}l\;, \tag{7.45}$$

which is called the *Shapiro delay*.

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How could the Shapiro delay be measured?

Example: Time delay in a gravitationally-lensed quasar

Gravitational bending of light can lead to multiple light paths, or null geodesics, leading from a single source to the observer. Then, the observer sees the source multiply imaged. If the source is variable, the Shapiro delay, together with the different geometrical lengths of the light paths, leads to a measureable time shift between the images: shifted copies of the light curves are then seen in the individual images. Many such time delays caused by gravitational lensing have been observed. A recent example is the time delay of (111.3 ± 3) days measured in the doubly-imaged quasar SDSS 1206 + 4332. Such measurements are important for cosmology because the allow determinations of the Hubble constant, i.e. the relative expansion rate of the Universe. ◂

7.3.3 The gravitomagnetic field

At next order in powers of c^{-1}, the current terms in the energy-momentum tensor appear, but no stresses yet. That is, we now approximate $T_{ij} = 0$ and use the field equations

$$\Box\gamma_{ij} = 0\;,\quad \Box\gamma_{0\mu} = -\frac{16\pi\mathcal{G}}{c^4}T_{0\mu}\;. \tag{7.46}$$

?

How could boundary conditions be taken into account in solving equation (7.46)? Which boundary conditions could be appropriate? Compare with electrodynamics.

Now, we set $A_\mu \equiv \gamma_{0\mu}/4$ and obtain the Maxwell-type equations

$$\Box A_\mu = -\frac{4\pi}{c^2}\,j_\mu\;, \tag{7.47}$$

where the current density $j_\mu \equiv \mathcal{G}T_{0\mu}/c^2$ was introduced. According to our earlier result (7.28), $A_0 = -\Phi/c^2$. This similarity to electromagnetic theory naturally leads to the introduction of "electric" and "magnetic" components of the gravitational field.

Suppose now that the field is quasi-stationary, so that time derivatives of the metric $\gamma_{\mu\nu}$ can be neglected. Then, $\vec{\nabla}^2\gamma_{ij} = 0$ everywhere because $T_{ij} = 0$ was assumed, thus $\gamma_{ij} = 0$, and the potentials A_μ determine the field completely. They are

$$A_0 = -\frac{\Phi}{c^2}\,, \quad A_i = \frac{\mathcal{G}}{c^4}\int \frac{T_{0i}(\vec{x}\,')\,\mathrm{d}^3x'}{|\vec{x}-\vec{x}\,'|}\,, \tag{7.48}$$

and the components of the metric g are, according to (7.30),

$$g_{00} = -1 + 2A_0\,, \quad g_{0i} = \gamma_{0i} = 4A_i\,, \quad g_{ij} = (1+2A_0)\delta_{ij}\,. \tag{7.49}$$

Gravitomagnetic potential

Matter currents create a magnetic gravitational potential similar to the electromagnetic vector potential.

The most direct approach to the equations of motion starts from the variational principle (4.5), or

$$\delta\int\left(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)^{1/2}\mathrm{d}t = 0\,, \tag{7.50}$$

where the dot now denotes the derivative with respect to the coordinate time t. The radicand is

$$c^2 - 2c^2A_0 - 8c\vec{A}\cdot\vec{v} - \vec{v}^{\,2}\,, \tag{7.51}$$

where we have neglected terms of order $\Phi\vec{v}^{\,2}$ since the velocities are assumed to be small compared to the speed of light.

Using (7.51), we can reduce the least-action principle (7.51) to the Euler-Lagrange equations with the effective Lagrangian

$$\mathcal{L} = \frac{\vec{v}^{\,2}}{2} + A_0c^2 + 4c\vec{A}\cdot\vec{v}\,. \tag{7.52}$$

We first find

$$\frac{\partial\mathcal{L}}{\partial\vec{v}} = \vec{v} + 4c\vec{A} \quad\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\vec{v}} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + 4c(\vec{v}\cdot\vec{\nabla})\vec{A}\,. \tag{7.53}$$

?

Convince yourself of the vector identity (7.54).

Using the vector identity

$$\vec{\nabla}(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{a} + \vec{a}\times\vec{\nabla}\times\vec{b} + \vec{b}\times\vec{\nabla}\times\vec{a}\,, \tag{7.54}$$

we further obtain

$$\frac{\partial\mathcal{L}}{\partial\vec{x}} = c^2\vec{\nabla}A_0 + 4c\left[(\vec{v}\cdot\vec{\nabla})\vec{A} + \vec{v}\times(\vec{\nabla}\times\vec{A})\right]\,, \tag{7.55}$$

from which we obtain the equations of motion

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \equiv \vec{f} = c^2\vec{\nabla}A_0 + 4c\vec{v}\times(\vec{\nabla}\times\vec{A}) , \tag{7.56}$$

in which the (specific) force term on the right-hand side corresponds to the Lorentz force in electrodynamics.

Let us consider now a small body characterised by its density suspended in a gravitomagnetic field; "small" means that the field can be considered constant across it. It experiences the torque about its centre-of-mass

$$\begin{aligned}\vec{M} &= \int \mathrm{d}^3x\,\vec{x}\times\rho\vec{f} \qquad (7.57)\\ &= -c^2\vec{\nabla}A_0\times\int \mathrm{d}^3x\,\vec{x}\rho + 4c\int \mathrm{d}^3x\,\vec{x}\times\left(\vec{j}\times\vec{B}\right) ,\end{aligned}$$

where $\vec{j} = \rho\vec{v}$ is the matter current density and $\vec{B} = \vec{\nabla}\times\vec{A}$ is the gravitomagnetic field. With the coordinates' being centred on the centre-of-mass, the first term vanishes. A non-trivial calculation carried out in the In-depth box "Spin in a gravitomagnetic field" shows that the second term gives

$$\vec{M} = 2c\left(\int \mathrm{d}^3x\,\vec{x}\times\vec{j}\right)\times\vec{B} = 2c\vec{s}\times\vec{B} , \tag{7.58}$$

where $\vec{s}$ is the intrinsic angular momentum of the body, i.e. its spin.

?

Can you prove (7.64) in the In-depth box "Spin in a gravitomagnetic field" with partial integration in one of the two terms? Do you need any further conditions for doing so?

Thus, the body's spin changes according to

$$\dot{\vec{s}} = \vec{M} = 2c\vec{s}\times\vec{B} . \tag{7.59}$$

Let us now orient the coordinate frame such that $\vec{B} = B\vec{e}_3$, i.e. $B_1 = 0 = B_2$. Then,

$$\dot{s}_1 = 2cBs_2 , \quad \dot{s}_2 = -2cs_1B . \tag{7.60}$$

Introducing $\sigma = s_1 + \mathrm{i}s_2$ turns this into the single equation

$$\dot{\sigma} = -2cB\mathrm{i}\sigma , \tag{7.61}$$

which is solved by the ansatz $\sigma = \sigma_0\exp(\mathrm{i}\omega t)$ if $\omega = -2cB$. This shows that:

Lense-Thirring effect

A spinning body in a gravitomagnetic field will experience *spin precession* with the angular frequency

$$\vec{\omega} = -2c\vec{B} = -2c\vec{\nabla}\times\vec{A} , \tag{7.62}$$

which is called the *Lense-Thirring* effect.

Example: Measurement of spin precession near the Earth

On April 20th, 2004, the satellite *Gravity Probe B* was launched in order to measure the combined geodetic and Lense-Thirring precessions of four spinning quartz spheres. For the orbit of the satellite, general relativity predicts a geodetic precession of $-6606.1\,\mathrm{mas\,yr^{-1}}$ and a Lense-Thirring precession of $-39.2\,\mathrm{mas\,yr^{-1}}$. The data taken between August 28th, 2004, and August 14th, 2005, were analysed until mid-2011 and resulted in a geodetic precession of $(-6601.8 \pm 18.3)\,\mathrm{mas\,yr^{-1}}$ (cf. Eq. 9.94) and a Lense-Thirring precession of $(-37.2 \pm 7.2)\,\mathrm{mas\,yr^{-1}}$, confirming the predictions, albeit less precisely than planned (Phys. Rev. Lett. 106 (2011) 221101). ◂

7.4 Gravitational waves

7.4.1 Polarisation states

As shown in (7.24), the linearised field equations in vacuum are

$$\Box\gamma^{\mu\nu} = 0\,, \tag{7.69}$$

if the Hilbert gauge condition (7.21) is enforced,

$$\partial_\nu\gamma^{\mu\nu} = 0\,. \tag{7.70}$$

Within the Hilbert gauge class, we can further require that the trace of $\gamma^{\mu\nu}$ vanish,

$$\gamma = \gamma^\mu_\mu = 0\,. \tag{7.71}$$

To see this, we return to the gauge transformation (7.20), which implies

$$\gamma \to \gamma + 2\partial_\mu\xi^\mu - 4\partial_\mu\xi^\mu = \gamma - 2\partial_\mu\xi^\mu\,, \tag{7.72}$$

i.e. if $\gamma \neq 0$, we can choose the vector ξ^μ such that

$$2\partial_\mu\xi^\mu = \gamma\,. \tag{7.73}$$

Moreover, (7.22) shows that the Hilbert gauge condition remains preserved if ξ^μ satisfies the d'Alembert equation

$$\Box\xi^\mu = 0 \tag{7.74}$$

at the same time. It can be generally shown that vector fields ξ^μ can be constructed which indeed satisfy (7.74) and (7.74) at the same time. If we arrange things in this way, (7.14) shows that then $h_{\mu\nu} = \gamma_{\mu\nu}$.

?

How would you construct a solution to both equations (7.73) and (7.73)?

All functions propagating with the light speed satisfy the d'Alembert equation (7.69). In particular, we can describe them as superpositions of plane waves

$$\gamma_{\mu\nu} = h_{\mu\nu} = \mathrm{Re}\left(\varepsilon_{\mu\nu}\mathrm{e}^{\mathrm{i}\langle k,x\rangle}\right) \tag{7.75}$$

In depth: Spin in a gravitomagnetic field

On the precession frequency of angular momentum

We begin by noting that $\vec{x} \times (\vec{j} \times \vec{B}) \neq (\vec{x} \times \vec{j}) \times \vec{B}$ because the vector product is not associative, but rather satisfies the Jacobi identity (2.33). The double vector product can be expressed by two scalar products,

$$\vec{x} \times \left(\vec{j} \times \vec{B}\right) = \left(\vec{x} \cdot \vec{B}\right)\vec{j} - \left(\vec{x} \cdot \vec{j}\right)\vec{B} \, , \tag{7.63}$$

which is also known as the *Grassmann identity*. For a body rotating with an angular frequency $\vec{\omega}$, the matter-current density is $\vec{j} = \rho\vec{\omega} \times \vec{x}$, thus $\vec{x} \perp \vec{j}$, making the second term on the right-hand side of (7.63) vanish. For evaluating the first term, it is important to realise that $\vec{\nabla} \cdot \vec{j} = 0$, which is guaranteed here by the continuity equation. Then, for arbitrary functions f and g,

$$\int \mathrm{d}^3x \left(f\vec{j} \cdot \vec{\nabla}g + g\vec{j} \cdot \vec{\nabla}f\right) = 0 \, . \tag{7.64}$$

The proof is straightforward, integrating the second term by parts. Setting $f = x_i$ and $g = x_k$ in (7.64) gives

$$\int \mathrm{d}^3x \,(x_i j_k + x_k j_i) = 0 \tag{7.65}$$

and thus allows us to write

$$\int \mathrm{d}^3x \left(\vec{x} \cdot \vec{B}\right) j_i = B_k \int \mathrm{d}^3x \, x_k j_i = \frac{1}{2}B_k \int \mathrm{d}^3x \,(x_k j_i - x_i j_k) \tag{7.66}$$

or

$$\int \mathrm{d}^3x \left(\vec{x} \cdot \vec{B}\right)\vec{j} = \frac{1}{2}\int \mathrm{d}^3x \left[\left(\vec{B} \cdot \vec{x}\right)\vec{j} - \left(\vec{B} \cdot \vec{j}\right)\vec{x}\right] \, . \tag{7.67}$$

Reading the Grassmann identity (7.63) backwards finally enables us to bring the right-hand side of (7.67) into the form

$$\int \mathrm{d}^3x \left(\vec{x} \cdot \vec{B}\right)\vec{j} = \frac{1}{2}\vec{B} \times \underbrace{\int \mathrm{d}^3x \, \vec{j} \times \vec{x}}_{=-\vec{s}} = \frac{1}{2}\vec{s} \times \vec{B} \, , \tag{7.68}$$

as used in (7.58).

with amplitudes given by the so-called *polarisation tensor* $\varepsilon_{\mu\nu}$. They satisfy the d'Alembert equation if

$$k^2 = \langle k, k\rangle = k_\mu k^\mu = 0 \; . \tag{7.76}$$

The Hilbert gauge condition then requires

$$0 = \partial_\nu h^{\mu\nu} \quad \Rightarrow \quad k_\nu \varepsilon^{\mu\nu} = 0 \; , \tag{7.77}$$

and (7.71) is satisfied if the trace of $\varepsilon_{\mu\nu}$ vanishes,

$$\varepsilon^\mu_\mu = 0 \; . \tag{7.78}$$

The five conditions (7.78) and (7.78) imposed on the originally ten independent components of $\varepsilon_{\mu\nu}$ leave five independent components. Without loss of generality, suppose the wave propagates into the positive z direction, then

$$k^\mu = (k, 0, 0, k) \; , \tag{7.79}$$

and (7.77) implies

$$\varepsilon^{0\mu} = \varepsilon^{3\mu} \; ; \tag{7.80}$$

specifically,

$$\varepsilon^{00} = \varepsilon^{30} = \varepsilon^{03} = \varepsilon^{33} \quad \text{and} \quad \varepsilon^{01} = \varepsilon^{31} \; , \quad \varepsilon^{02} = \varepsilon^{32} \; , \tag{7.81}$$

while (7.78) means

$$-\,\varepsilon^{00} + \varepsilon^{11} + \varepsilon^{22} + \varepsilon^{33} = 0 \; . \tag{7.82}$$

Since $\varepsilon^{33} = \varepsilon^{00}$, this last equation means

$$\varepsilon^{11} + \varepsilon^{22} = 0 \; . \tag{7.83}$$

Therefore, all components of $\varepsilon^{\mu\nu}$ can be expressed by five of them, as follows:

$$\varepsilon^{\mu\nu} = \begin{pmatrix} \varepsilon^{00} & \varepsilon^{01} & \varepsilon^{02} & \varepsilon^{00} \\ \varepsilon^{01} & \varepsilon^{11} & \varepsilon^{12} & \varepsilon^{01} \\ \varepsilon^{02} & \varepsilon^{12} & -\varepsilon^{11} & \varepsilon^{02} \\ \varepsilon^{00} & \varepsilon^{01} & \varepsilon^{02} & \varepsilon^{00} \end{pmatrix} . \tag{7.84}$$

Now, a gauge transformation belonging to a vector field

$$\xi^\mu = \mathrm{Re}\left(\mathrm{i}\varepsilon^\mu \mathrm{e}^{\mathrm{i}\langle k,x\rangle}\right) \tag{7.85}$$

which keeps the metric perturbation $h_{\mu\nu}$ trace-less,

$$\partial_\mu \xi^\mu = 0 \; , \tag{7.86}$$

changes the polarisation tensor according to

$$\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu \tag{7.87}$$

for the k vector specified in (7.79), we thus have

$$\varepsilon^{00} \to \varepsilon^{00} + 2k\varepsilon^{0}\,, \quad \varepsilon^{01} \to \varepsilon^{01} + k\varepsilon^{1}\,, \quad \varepsilon^{02} \to \varepsilon^{02} + k\varepsilon^{2}\,,$$
$$\varepsilon^{11} \to \varepsilon^{11}\,, \quad \varepsilon^{12} \to \varepsilon^{12}\,. \qquad (7.88)$$

The condition (7.86) implies that $k_\mu\varepsilon^\mu = 0$, hence $\varepsilon^0 = \varepsilon^3$. We can then use (7.88) to make ε^{00}, ε^{01} and ε^{02} vanish, and only the gauge-invariant components ε^{11} and ε^{12} are left. Then

$$\varepsilon^\mu = \frac{1}{2k}\left(-\varepsilon^{00}, -2\varepsilon^{01} - 2\varepsilon^{02}, -\varepsilon^{00}\right) \qquad (7.89)$$

fixes the gauge transformation, and the polarisation tensor is reduced to

$$\varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon^{11} & \varepsilon^{12} & 0 \\ 0 & \varepsilon^{12} & -\varepsilon^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \qquad (7.90)$$

?

Carry out all steps yourself leading from (7.77) to (7.90).

Gauge-invariant polarisation states

As for electromagnetic waves, there are only two gauge-invariant, linearly independent polarisation states for gravitational waves.

?

How are polarisation states of electromagnetic waves being described? Recall the Stokes parameters and their meaning.

Under rotations about the z axis by an arbitrary angle ϕ, the polarisation tensor changes according to

$$\varepsilon'^{\mu\nu} = R^\mu_\alpha R^\nu_\beta \varepsilon^{\alpha\beta}\,, \qquad (7.91)$$

where R is the rotation matrix with the components

$$R(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \qquad (7.92)$$

Carrying out the matrix multiplication yields

$$\begin{aligned} \varepsilon'^{11} &= \varepsilon^{11}\cos 2\phi + \varepsilon^{12}\sin 2\phi \\ \varepsilon'^{12} &= -\varepsilon^{11}\sin 2\phi + \varepsilon^{12}\cos 2\phi\,. \end{aligned} \qquad (7.93)$$

Defining $\varepsilon_\pm \equiv \varepsilon^{11} \mp \mathrm{i}\varepsilon^{12}$, this can be written as

$$\varepsilon'_\pm = \mathrm{e}^{\pm 2\mathrm{i}\phi}\varepsilon_\pm\,, \qquad (7.94)$$

which shows that the two polarisation states $\varepsilon_\pm$ have helicity ± 2, and thus that they correspond to left and right-handed circular polarisation.

7.4.2 Generation of gravitational waves

We return to (7.27) to see how gravitational waves can be emitted. From the start, we introduce the two simplifications that the source is far away and changing with a velocity small compared to the speed-of-light. Then, we can replace the distance $|\vec{x} - \vec{x}'\,|$ by

$$|\vec{x} - \vec{x}'\,| \approx |\vec{x}| = r \tag{7.95}$$

because "far away" means that the source is small compared to its distance. Moreover, we can approximate the retarded time coordinate x^0 as follows:

$$\begin{aligned} x^0 - |\vec{x} - \vec{x}'\,| &= x^0 - \sqrt{(\vec{x} - \vec{x}'\,)^2} = x^0 - \sqrt{\vec{x}^2 + \vec{x}'^2 - 2\vec{x} \cdot \vec{x}'} \\ &\approx x^0 - r + \vec{x}' \cdot \vec{e}_r \ , \end{aligned} \tag{7.96}$$

where $\vec{e}_r$ is the unit vector in radial direction. Then, we obtain

$$\gamma_{\mu\nu}(t, \vec{x}) = -\frac{4\mathcal{G}}{rc^4} \int T_{\mu\nu}\left(t - \frac{r - \vec{x}' \cdot \vec{e}_r}{c}, \vec{x}'\right) \mathrm{d}^3 x' \ . \tag{7.97}$$

Under the assumption of slow motion, we can further ignore the directional dependence of the retarted time, thus approximate $\vec{x}' \cdot \vec{e}_r = 0$, and write

$$\gamma_{\mu\nu}(t, \vec{x}) = -\frac{4\mathcal{G}}{rc^4} \int T_{\mu\nu}\left(t - \frac{r}{c}, \vec{x}'\right) \mathrm{d}^3 x' \ . \tag{7.98}$$

While this is already the essential result, a sequence of transformations of the right-hand side now leads to further important insight.

By means of the local conservation law $\partial_\nu T^{\mu\nu} = 0$, we can begin by simplifying the integral on the right-hand side of (7.98):

$$\begin{aligned} 0 = \int x^k \partial_\nu T^{\mu\nu} \mathrm{d}^3 x &= \frac{1}{c}\partial_t \int x^k T^{0\mu} \mathrm{d}^3 x + \int x^k \partial_l T^{l\mu} \mathrm{d}^3 x \\ &= \frac{1}{c}\partial_t \int x^k T^{0\mu} \mathrm{d}^3 x - \int T^{l\mu} \delta^k_l \mathrm{d}^3 x \ , \end{aligned} \tag{7.99}$$

where the second term on the right-hand side was partially integrated, assuming that boundary terms vanish (i.e. enclosing the source completely in the integration boundary). Thus, we see that the volume integral over the energy-momentum tensor can be written as a time derivative,

$$\int T^{k\mu} \mathrm{d}^3 x = \frac{1}{c}\partial_t \int x^k T^{0\mu} \mathrm{d}^3 x \ . \tag{7.100}$$

From Gauß' theorem, we infer that the volume integral over the divergence of a vector field equals the integral of the vector field over the boundary of the volume and must vanish if the field disappears on the surface,

$$\int \partial_j \left(T^{j0} x^l x^k\right) \mathrm{d}^3 x = 0 \ . \tag{7.101}$$

This result, together with $\partial_\nu T^{\mu\nu} = 0$, enables us to write

$$\begin{aligned}\frac{1}{c}\partial_t \int T^{00} x^l x^k \mathrm{d}^3 x &= \int \partial_\nu \left(T^{\nu 0} x^l x^k\right) \mathrm{d}^3 x = \int T^{\nu 0} \partial_\nu \left(x^l x^k\right) \mathrm{d}^3 x \\ &= \int T^{\nu 0} \left(\delta^k_\nu x^l + x^k \delta^l_\nu\right) \mathrm{d}^3 x \\ &= \int \left(T^{k0} x^l + T^{l0} x^k\right) \mathrm{d}^3 x \ . \end{aligned} \tag{7.102}$$

Taking a further partial time derivative of (7.102) and using (7.100) results in

$$\begin{aligned}\frac{1}{2c^2}\partial_t^2 \int (T^{00} x^k x^l) \mathrm{d}^3 x &= \frac{1}{2c}\partial_t \int (T^{k0} x^l + T^{l0} x^k) \mathrm{d}^3 x \\ &= \frac{1}{2}\int (T^{kl} + T^{lk}) \mathrm{d}^3 x = \int T^{kl} \mathrm{d}^3 x \ . \end{aligned} \tag{7.103}$$

The spatial components of the metric perturbation $\gamma_{\mu\nu}$ thus turn out to be given by the second time derivative

$$\gamma^{jk}(t, \vec{x}) = -\frac{2\mathcal{G}}{rc^6}\partial_t^2 \int T^{00}\left(t - \frac{r}{c}, \vec{x}'\right) x'^j x'^k \mathrm{d}^3 x' \ . \tag{7.104}$$

?

Convince yourself of all the steps leading from (7.99) to (7.103). Verify that the expression (7.104) for the metric perturbation has the correct units.

If we further use that the T^{00} component of the energy-momentum tensor is well approximated by the matter density if the source's material is moving slowly, we arrive at the main result of this sequence of transformations:

Source of gravitational waves

Wave-like metric perturbations in vacuum are created by the second time derivative of a matter distribution with density ρ,

$$\gamma^{jk}(t, \vec{x}) = -\frac{2\mathcal{G}}{rc^4}\partial_t^2 \int \rho\left(t - \frac{r}{c}, \vec{x}'\right) x'^j x'^k \mathrm{d}^3 x' \ . \tag{7.105}$$

?

Why can electromagnetic waves be created by a time-dependent *dipole* moment instead?

Finally, we can further simplify the physical interpretation of this result by introducing the source's *quadrupole tensor*, which is defined by

$$Q^{jk} = \int \left(3x^j x^k - r^2 \delta^{jk}\right) \rho(\vec{x}) \mathrm{d}^3 x \ . \tag{7.106}$$

It allows us to rewrite the metric perturbation from (7.105) as

$$\gamma_{jk} = -\frac{2\mathcal{G}}{3rc^4}\left[\partial_t^2 Q_{jk}\left(t - \frac{r}{c}, \vec{x}\right) + \delta_{jk}\partial_t^2 \int r'^2 \rho\left(t - \frac{r}{c}, \vec{x}'\right) \mathrm{d}^3 x'\right] \ . \tag{7.107}$$

?

Calculate the quadrupole moment of a binary star with components having masses M_1 and M_2, orbiting each other on a circular orbit with radius r.

Generation of gravitational waves

In order to generate gravitational waves, a mass distribution needs to have a quadrupole moment with a non-vanishing second time derivative.

7.4.3 Energy transport by gravitational waves

The energy current density of electromagnetic waves is given by the time-space components $T^{0i}_{\rm GW}$ of their energy-momentum tensor. The 01-component, i.e. the energy current density propagating into the x^1 direction, can be shown to be

$$T^{01}_{\rm GW} = \frac{c^3}{32\pi\mathcal{G}}\left\langle 2\dot\gamma_{23}^2 + \frac{1}{2}\left(\dot\gamma_{22} - \dot\gamma_{33}\right)^2\right\rangle \ , \tag{7.108}$$

which can be written with the help of (7.107) as

$$T^{01}_{\rm GW} = \frac{\mathcal{G}}{72\pi r^2 c^5}\left\langle 2\dddot{Q}_{23}^2 + \frac{1}{2}\left(\dddot{Q}_{22} - \dddot{Q}_{33}\right)^2\right\rangle \ , \tag{7.109}$$

?

Does the expression (7.109) have the correct units?

showing one of the rare cases of a third time derivative in physics.

The transversal quadrupole tensor is

$$Q^{\rm T} = \begin{pmatrix} Q_{22} & Q_{23} \\ Q_{32} & Q_{33} \end{pmatrix} \tag{7.110}$$

because the direction of propagation was chosen as the x^1 axis. Defining the *transversal trace-free quadrupole tensor* by

$$Q^{\rm TT} := Q^{\rm T} - \frac{\mathcal{I}}{2}\,{\rm Tr}\,Q^{\rm T} = \frac{1}{2}\begin{pmatrix} Q_{22} - Q_{33} & 2Q_{23} \\ 2Q_{23} & -(Q_{22} - Q_{33}) \end{pmatrix} \ , \tag{7.111}$$

we see that an invariant expression for the right-hand side of (7.109) is given by

$${\rm Tr}\left(Q^{\rm TT}Q^{\rm TT}\right) = \frac{1}{2}\left(Q_{22} - Q_{33}\right)^2 + 2Q_{23}^2 \ , \tag{7.112}$$

and thus the energy current density in gravitational waves has the components

$$T^{0i}_{\rm GW} = \frac{\mathcal{G}}{72\pi r^2 c^5}\left\langle {\rm Tr}\left(Q^{\rm TT}Q^{\rm TT}\right)\right\rangle \ . \tag{7.113}$$

Einstein's quadrupole formula

A final integration over a sphere with radius r yields Einstein's famous *quadrupole formula* for the gravitational-wave "luminosity",

$$L_{\rm GW} = \frac{\mathcal{G}}{5c^5}\left\langle {\rm Tr}\left(\dddot{Q}^2\right)\right\rangle \ . \tag{7.114}$$

Example: First direct detection of gravitational waves

On September 14th, 2015, the LIGO interferometers at Hanford (Washington, USA) and Livingston (Louisiana, USA) registered the gravitational-wave signal summarised in Fig. 7.1. The figure shows that the frequency f increased from $\approx 50\,\mathrm{Hz}$ to $\approx 100\,\mathrm{Hz}$ within $\approx 40\,\mathrm{ms}$. Inserting $f \approx 75\,\mathrm{Hz}$ and $\dot{f} \approx 50\,\mathrm{Hz}/0.04\,\mathrm{s} \approx 1250\,\mathrm{Hz\,s}^{-1}$ into the formula (7.128) derived in the In-depth box "The chirp mass of a binary star" for the chirp mass $\mathcal{M}$ gives

$$\mathcal{M} \approx 30\,M_\odot\,. \tag{7.115}$$

For two equal masses $m_1 = m_2 =: m$, $M = 2m$ and $\mu = m/2$, thus

$$\mathcal{M} = \frac{m}{2^{1/5}}\,, \quad m \approx 1.15\,\mathcal{M} \approx 35\,M_\odot\,. \tag{7.116}$$

At an orbital frequency of $\omega = \pi f \approx 240\,\mathrm{Hz}$, Kepler's third law (7.123) requires the two masses to be separated by

$$R \approx \left(\frac{2\mathcal{G}m}{\omega^2}\right)^{1/3} \approx 550\,\mathrm{km}\,, \tag{7.117}$$

less than a thousandth of the Solar radius. No ordinary stars could ever come as close. Objects of mass m closer than R must be black holes. The merging black-hole binary became known as GW150914.

Inserting the quadrupole tensor (7.122) into (7.105) leads to

$$\left|\gamma^{jk}\right| \le 4\left(\frac{\mathcal{G}M}{rc^2}\right)\left(\frac{\mathcal{G}\mu}{Rc^2}\right) \tag{7.118}$$

which, for equal masses, turns into the intuitive expression

$$\left|\gamma^{jk}\right| \le \frac{R_\mathrm{s}^2}{Rr} \tag{7.119}$$

in terms of the Schwarzschild radius $R_\mathrm{s} = 2\mathcal{G}m/c^2$. With $R_\mathrm{s} \approx 100\,\mathrm{km}$ for $m \approx 35\,M_\odot$ and $|\gamma^{jk}| \lesssim 10^{-21}$, the distance of the merging black holes can be estimated to be

$$r \approx 2 \cdot 10^{27}\,\mathrm{cm} \approx 600\,\mathrm{Mpc}\,. \tag{7.120}$$

◀

In depth: The chirp mass of a binary star

A Newtonian estimate

Two stars of masses $m_{1,2}$ separated by a distance R orbit their centre-of-mass at distances $R_{1,2}$ with an angular frequency ω. They obey Kepler's third law,

$$\omega^2 = \frac{\mathcal{G}M}{R^3}\,, \quad M := m_1 + m_2\,. \tag{7.121}$$

Assuming circular orbits with radii $R_i = m_i R/M$ according to the definition of the centre-of-mass, their quadrupole tensor is

$$Q = \mu R^2 \left[\begin{pmatrix} \cos^2\omega t & \sin\omega t\cos\omega t & 0 \\ \sin\omega t\cos\omega t & \sin^2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}\mathbb{1}_3\right]\,, \tag{7.122}$$

where $\mu := m_1 m_2/M$ is the reduced mass. Straightforward calculation gives

$$\mathrm{Tr}\left(\dddot{Q}^2\right) = 32\omega^6\mu^2R^4\,, \tag{7.123}$$

leading us with (7.114) to the gravitational-wave luminosity

$$L_{\mathrm{GW}} = \frac{32\,\mathcal{G}}{5\,c^5}\omega^6\mu^2R^4\,. \tag{7.124}$$

According to the virial theorem, the total energy of the binary star is

$$E = -\frac{1}{2}E_{\mathrm{pot}} = \frac{1}{2}\frac{\mathcal{G}m_1m_2}{R} = \frac{1}{2}\frac{\mathcal{G}\mu M}{R}\,. \tag{7.125}$$

Its absolute time derivative must equal the gravitational-wave luminosity, $|\dot{E}| = L_{\mathrm{GW}}$. Since $R \propto \omega^{-2/3}$ from (7.121),

$$\frac{\dot{R}}{R} = -\frac{2}{3}\frac{\dot{\omega}}{\omega} \tag{7.126}$$

and thus

$$|E_{\mathrm{pot}}| = \frac{1}{3}\frac{\mathcal{G}\mu M}{R}\frac{\dot{\omega}}{\omega}\,. \tag{7.127}$$

Equating this to (7.124), using (7.121) to eliminate the radius via the angular frequency, taking into account that the frequency f of the gravitational waves emitted by the binary is $f = \omega/\pi$, and sorting terms leads to the *chirp mass*

$$\mathcal{M} := \left(M^{2/3}\mu\right)^{3/5} = \frac{c^3}{8\mathcal{G}}\left(\frac{5}{3\pi^{8/3}}\frac{\dot{f}}{f^{11/3}}\right)^{3/5}\,. \tag{7.128}$$

Although this estimate is based on three grossly simplifying assumptions: Newtonian gravity, circular orbits, and negligible energy loss per orbit, the qualitative expression for the chirp mass and its numerical value are close to the relativistic result in leading-order calculation.

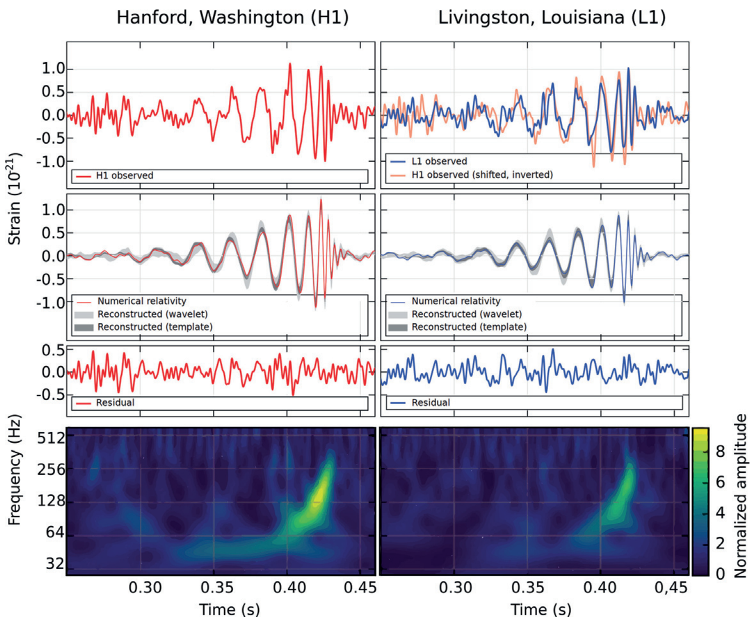

Figure 7.1 Wave forms and frequency diagrams of the gravitational-wave signals registered on September 14th, 2015, by the LIGO interferometers at Hanford and Livingston (USA). This was the first direct detection of a gravitational wave. The figure shows the strain $|\gamma^{jk}|$ measured by the two interferometers, the reconstruction of the signal by comparison to the signal expected from a merging black-hole binary, and the frequency of the gravitational waves as a function of time. Since the frequency is increasing during the event, an acoustic representation resembles a chirp. Source: Wikipedia