Chapter 6

Einstein's Field Equations

6.1 The physical meaning of curvature

6.1.1 Congruences of time-like geodesics

Having walked through the introductory chapters, we are now ready to introduce Einstein's field equations, i.e. the equations describing the dynamics of the gravitational field. Einstein searched for these equations essentially between early August 1912, when he moved back from Prague to Zurich, and November 25, 1915, when he published them in their final form, meanwhile in Berlin. We shall give a heuristic argument for the form of the field equations, which should not be mistaken for a derivation, and later show that these equations follow from a suitable Lagrangian.

First, however, we shall investigate into the physical role of the curvature tensor. As we have seen, gravitational fields can locally be transformed away by choosing normal coordinates, in which the Christoffel symbols (the connection coefficients) all vanish. By its nature, this does not hold for the curvature tensor which, as we shall see, is related to the gravitational *tidal field*. Thus, in this sense, the gravitational tidal field has a more profound physical significance as the gravitational field itself.

Let us begin with a *congruence of geodesics*. This is a bundle of time-like geodesics imagined to run through every point of a small environment $U \subset M$ of a point $p \in U$.

Caution Note that the proper time cannot be used for parameterising light rays. In Chapter 13, an affine parameter will be introduced instead. ◂

Let the geodesics be parameterised by the proper time τ along them, and introduce a curve γ transversal to the congruence, parameterised by a curve parameter λ. *Transversal* means that the curve γ is nowhere parallel to the congruence.

When normalised, the tangent vector to one of the time-like geodesics can be written as

$$u = \partial_\tau \quad \text{with} \quad \langle u, u \rangle = -1 \; . \tag{6.1}$$

Since it is tangent to a geodesic, it is parallel-transported along the geodesic,

$$\nabla_u u = 0 \; . \tag{6.2}$$

Similarly, we introduce a unit tangent vector v along the curve γ,

$$v = \dot{\gamma} = \partial_\lambda \; . \tag{6.3}$$

Since the partial derivatives with respect to the curve parameters τ and λ commute, so do the vectors u and v, and thus v is Lie-transported (or Lie-invariant) along u,

$$0 = [u, v] = \mathcal{L}_u v \; . \tag{6.4}$$

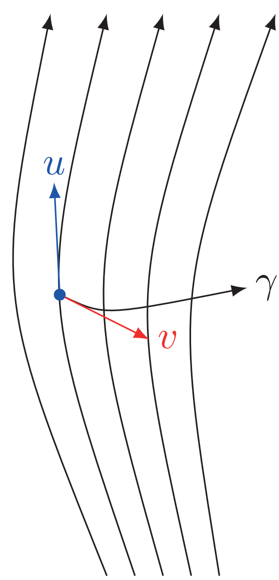

Figure 6.1 Geodesic bundle with tangent vector u of the fiducial geodesic, the curve γ towards a neighbouring geodesic, and tangent vector $v = \dot{\gamma}$.

Now, we project v on u and define a vector n which is perpendicular to u,

$$n = v + \langle v, u \rangle u \; , \tag{6.5}$$

which does indeed satisfy $\langle n, u \rangle = 0$ because of $\langle u, u \rangle = -1$. This vector is also Lie-transported along u, as we shall verify now.

First, we have

$$\mathcal{L}_u n = [u, n] = [u, v] + [u, \langle v, u\rangle u] \\ = u(\langle v, u\rangle)u = (\partial_\tau \langle v, u\rangle)u \;, \tag{6.6}$$

where (6.4) was used in the first step. Since $\langle u, u\rangle = -1$, we have

$$0 = \partial_\lambda \langle u, u\rangle = v\langle u, u\rangle = 2\langle \nabla_v u, u\rangle \;, \tag{6.7}$$

if we use the Ricci identity (3.69).

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Go through all steps leading from (6.6) to (6.9) and convince yourself of them.

But the vanishing commutator between u and v and the symmetry of the connection imply $\nabla_v u = \nabla_u v$, and thus

$$\partial_\tau \langle u, v\rangle = u\langle u, v\rangle = \langle \nabla_u u, v\rangle + \langle u, \nabla_u v\rangle = \langle u, \nabla_v u\rangle = 0 \;, \tag{6.8}$$

where the Ricci identity was used in the second step, the geodesic property (6.2) in the third, and (6.7) in the last. Returning to (6.6), this proves that n is Lie-transported,

$$\mathcal{L}_u n = 0 \;. \tag{6.9}$$

The perpendicular separation vector between neighbouring geodesics of the congruence is thus Lie-invariant along the congruence.

6.1.2 The curvature tensor and the tidal field

Now, we take the second derivative of v along u,

$$\nabla_u^2 v = \nabla_u \nabla_u v = \nabla_u \nabla_v u = (\nabla_u \nabla_v - \nabla_v \nabla_u)u \;, \tag{6.10}$$

where we have used again that u and v commute and that u is a geodesic. With $[u, v] = 0$, the curvature (3.51) applied to u and v reads

$$\bar{R}(u, v)u = (\nabla_u \nabla_v - \nabla_v \nabla_u)u \;. \tag{6.11}$$

Jacobi equation

In this way, we see that the second derivative of v along u is determined by the curvature tensor through the *Jacobi equation*

$$\nabla_u^2 v = \bar{R}(u, v)u \;. \tag{6.12}$$

Let us now use this result to find a similar equation for n. First, we observe that

$$\nabla_u n = \nabla_u v + \nabla_u (\langle v, u\rangle u) = \nabla_u v + (\partial_\tau \langle v, u\rangle)u = \nabla_u v \tag{6.13}$$

because of (6.8). Thus $\nabla_u^2 n = \nabla_u^2 v$ and

$$\nabla_u^2 n = \bar{R}(u, v)u \;. \tag{6.14}$$

We then use

$$\bar{R}(u,n) = \bar{R}(u, v + \langle u,v\rangle u) = \bar{R}(u,v) + \langle u,v\rangle\bar{R}(u,u) = \bar{R}(u,v) \quad (6.15)$$

to arrive at the desired result:

Equation of geodesic deviation

The separation vector n between neighbouring geodesics obeys the equation

$$\nabla_u^2 n = \bar{R}(u,n)u \; . \quad (6.16)$$

This is called the *equation of geodesic deviation* because it describes directly how the separation between neighbouring geodesics evolves along the geodesics according to the curvature.

Finally, let us introduce a coordinate basis $\{e_i\}$ in the subspace perpendicular to u which is parallel-transported along u. Since n is confined to this subspace, we can write

$$n = n^i e_i \quad (6.17)$$

and thus

$$\nabla_u n = (un^i)e_i + (n^i\nabla_u)e_i = \frac{\mathrm{d}n^i}{\mathrm{d}\tau}e_i \; . \quad (6.18)$$

Since u is normalised and perpendicular to the space spanned by the triad $\{e_i\}$, we can form a tetrad from the e_i and $e_0 = u$. The equation of geodesic deviation (6.16) then implies

$$\frac{\mathrm{d}^2 n^i}{\mathrm{d}\tau^2}e_i = \bar{R}(e_0, n^j e_j)e_0 = n^j\bar{R}(e_0, e_j)e_0 = n^j\bar{R}^i{}_{00j}\,e_i \; . \quad (6.19)$$

Thus, defining a matrix K by

$$\frac{\mathrm{d}^2 n^i}{\mathrm{d}\tau^2} = \bar{R}^i{}_{00j}\,n^j \equiv K^i_j n^j \; , \quad (6.20)$$

we can write (6.19) in matrix form

$$\frac{\mathrm{d}^2\vec{n}}{\mathrm{d}\tau^2} = K\vec{n} \; . \quad (6.21)$$

Note that K is symmetric because of the symmetries (3.81) of the curvature tensor.

Moreover, the trace of K is

$$\mathrm{Tr}\,K = \bar{R}^i{}_{00i} = \bar{R}^\mu{}_{00\mu} = -R_{00} = -R_{\mu\nu}u^\mu u^\nu \; , \quad (6.22)$$

where we have inserted $\bar{R}^0{}_{000} = 0$ and the definition of the Ricci tensor (3.57).

Let us now compare this result to the motion of test bodies in Newtonian theory. At two neighbouring points $\vec{x}$ and $\vec{x}+\vec{n}$, we have the equations of motion

$$\ddot{x}^i = -\,(\partial_i\Phi)|_{\vec{x}} \tag{6.23}$$

and, to first order in a Taylor expansion,

$$\ddot{x}^i + \ddot{n}^i = -\,(\partial_i\Phi)|_{\vec{x}+\vec{n}} \approx -\,(\partial_i\Phi)|_{\vec{x}} - (\partial_i\partial_j\Phi)\Big|_{\vec{x}}\,n^j\ . \tag{6.24}$$

Subtracting (6.23) from (6.24) yields the evolution equation for the separation vector.

Relative acceleration in Newtonian gravity

In Newtonian gravity, the separation vector between any two particle trajectories changes due to the tidal field according to

$$\ddot{n}^i = -(\partial_i\partial_j\Phi)n^j\ . \tag{6.25}$$

This equation can now be compared to the result (6.21).

Taking into account that

$$\frac{\mathrm{d}^2 n^i}{\mathrm{d}\tau^2} = \frac{\ddot{n}^i}{c^2} = -\left(\frac{\partial_i\partial_j\Phi}{c^2}\right)n^j\ , \tag{6.26}$$

we see that the matrix K in Newton's theory is

$$K^{(\mathrm{N})}_{ij} = -\frac{\partial_i\partial_j\Phi}{c^2}\ , \tag{6.27}$$

and its trace is

$$\mathrm{Tr}\,K^{(\mathrm{N})} = -\frac{\vec{\nabla}^2\Phi}{c^2} = -\frac{\Delta\Phi}{c^2}\ , \tag{6.28}$$

i.e. the negative Laplacian of the Newtonian potential, scaled by the squared light speed.

Tidal field and curvature

The essential results of this discussion are the correspondences

$$\bar{R}^i{}_{0j0} \leftrightarrow \frac{\partial_i\partial_j\Phi}{c^2} \tag{6.29}$$

and

$$R_{\mu\nu}u^\mu u^\nu \leftrightarrow \frac{\vec{\nabla}^2\Phi}{c^2}\ . \tag{6.30}$$

These confirm the assertion that the curvature represents the gravitational *tidal* field, describing the *relative* accelerations of freely-falling test bodies; (6.29) and (6.30) will provide useful guidance in guessing the field equations.

6.2 Einstein's field equations

6.2.1 Heuristic "derivation"

We start from the field equation from Newtonian gravity, i.e. the Poisson equation

$$4\pi\mathcal{G}\rho = \vec{\nabla}^2\Phi = -c^2\,\mathrm{Tr}\,K^{(\mathrm{N})}\ . \tag{6.31}$$

The density ρ can be expressed by the energy-momentum tensor T. For an ideal fluid, we have

$$T = (\rho c^2 + p)u^\flat \otimes u^\flat + pg\ , \tag{6.32}$$

from which we find because of $\langle u, u\rangle = -1$

$$T(u, u) = \rho c^2\ . \tag{6.33}$$

Moreover, its trace is

$$\mathrm{Tr}\,T = -\rho c^2 + 3p \approx -\rho c^2 \tag{6.34}$$

because $p \ll \rho c^2$ under Newtonian conditions (the pressure is much less than the *energy* density).

?

Write equations (6.32) and (6.33) in components. At what level are the indices?

Hence, let us take a constant $\lambda \in \mathbb{R}$, put

$$\rho c^2 = \lambda T(u, u) + (1-\lambda)\ \mathrm{Tr}\,T\,g(u, u) \tag{6.35}$$

and insert this into the field equation (6.31), using (6.22) for the trace of K. We thus obtain

$$R(u, u) = \frac{4\pi\mathcal{G}}{c^4}\left[\lambda T + (1-\lambda)\ \mathrm{Tr}\,T\,g\right](u, u)\ . \tag{6.36}$$

Since this equation should hold for any observer and thus for arbitrary four-velocities u, we find

$$R = \frac{4\pi\mathcal{G}}{c^4}\left[\lambda T + (1-\lambda)\ \mathrm{Tr}\,T\,g\right]\ , \tag{6.37}$$

where $\lambda \in \mathbb{R}$ remains to be determined.

We take the trace of (6.37), obtain

$$\mathrm{Tr}\,R = \mathcal{R} = \frac{4\pi\mathcal{G}}{c^4}\left[\lambda + 4(1-\lambda)\right]\mathrm{Tr}\,T \tag{6.38}$$

and combine this with (6.37) to assemble the Einstein tensor (3.90),

$$\begin{aligned} G &= R - \frac{\mathcal{R}}{2}g \\ &= \frac{4\pi\mathcal{G}}{c^4}\left(\lambda T - \frac{2-\lambda}{2}\ \mathrm{Tr}\,T\,g\right)\ . \end{aligned} \tag{6.39}$$

We have seen in (3.91) that the Einstein tensor G satisfies the contracted Bianchi identity

$$\nabla \cdot G = 0 \,. \tag{6.40}$$

Likewise, the divergence of the energy-momentum tensor must vanish in order to guarantee local energy-momentum conservation,

$$\nabla \cdot T = 0 \,. \tag{6.41}$$

These two conditions are generally compatible with (6.39) only if we choose $\lambda = 2$, which specifies the field equations.

Einstein's field equations

Einstein's field equations, published on November 25th, 1915, relate the Einstein tensor G to the energy-momentum tensor T as

$$G = \frac{8\pi \mathcal{G}}{c^4} T \,. \tag{6.42}$$

An equivalent form follows from (6.37),

$$R = \frac{8\pi \mathcal{G}}{c^4} \left(T - \frac{1}{2} \,\mathrm{Tr}\, T \, g \right) . \tag{6.43}$$

?

Convince yourself that equations (6.42) and (6.43) are equivalent.

6.2.2 Uniqueness

In the appropriate limit, Einstein's equations satisfy Newton's theory by construction and are thus *one possible* set of gravitational field equations. A remarkable theorem due to David Lovelock (1938–) states that they are the *only possible* field equations under certain very general conditions.

It is reasonable to assume that the gravitational field equations can be written in the form

$$\mathcal{D}[g] = T \,, \tag{6.44}$$

where the tensor $\mathcal{D}[g]$ is a functional of the metric tensor g and T is the energy-momentum tensor. This equation says that the source of the gravitational field is assumed to be expressed by the energy-momentum tensor of all matter and energy contained in spacetime. Now, Lovelock's theorem states:

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Express the coefficients α and β in terms of κ and Λ.

Lovelock's theorem

If $\mathcal{D}[g]$ depends on g and its derivatives only up to second order, then it must be a linear combination of the Einstein and metric tensors,

$$\mathcal{D}[g] = \alpha G + \beta g \;, \tag{6.45}$$

with $\alpha, \beta \in \mathbb{R}$. This absolutely remarkable theorem says that G must be of the form

$$G = \kappa T - \Lambda g \;, \tag{6.46}$$

with κ and Λ are constants. The correct Newtonian limit then requires that $\kappa = 8\pi G c^{-4}$, and Λ is the "cosmological constant" introduced by Einstein for reasons which will become clear later.

6.3 Lagrangian formulation

6.3.1 The action of general relativity

The remarkable uniqueness of the tensor $\mathcal{D}$ shown by Lovelock's theorem lets us suspect that a Lagrangian formulation of general relativity should be possible starting from a scalar constructed from $\mathcal{D}$, most naturally its contraction $\mathrm{Tr}\,\mathcal{D}$, which is simply proportional to the Ricci scalar $\mathcal{R}$ if we ignore the cosmological term proportional to Λ for now.

Writing down the action, we have to take into account that we require an invariant volume element, which we obtain from the canonical volume form η introduced in (5.69). Then, according to (5.92) and (5.93), we can represent volume integrals as

$$\int_M \eta = \int_U \sqrt{-g}\,\mathrm{d}^4x \;, \tag{6.47}$$

where $\sqrt{-g}$ is the square root of the determinant of g, and $U \subset M$ admits a single chart. Recall that, if we need to integrate over a domain covered by multiple charts, a sum over the domains of the individual charts is understood.

Thus, we suppose that the action of general relativity in a compact region $D \subset M$ with smooth boundary ∂D is

$$S_{\mathrm{GR}}[g] = \int_D \mathcal{R}[g]\eta = \int_D \mathcal{R}[g]\sqrt{-g}\,\mathrm{d}^4x \;. \tag{6.48}$$

6.3.2 Variation of the action

Working out the variation of this action with respect to the metric components $g_{\mu\nu}$, we write explicitly

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu} \tag{6.49}$$

and thus

$$\begin{aligned}\delta S_{\mathrm{GR}} &= \int_D \delta\left(g^{\mu\nu} R_{\mu\nu} \sqrt{-g}\right) \mathrm{d}^4 x \\ &= \int_D \delta R_{\mu\nu}\, g^{\mu\nu} \sqrt{-g}\, \mathrm{d}^4 x + \int_D R_{\mu\nu} \delta\left(g^{\mu\nu} \sqrt{-g}\right) \mathrm{d}^4 x \,.\end{aligned} \tag{6.50}$$

We evaluate the variation of the Ricci tensor first, using its expression (3.57) in terms of the Christoffel symbols. Matters simplify considerably if we introduce normal coordinates, which allow us to ignore the terms in (3.57) which are quadratic in the Christoffel symbols. Then, the Ricci tensor specialises to

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} \;, \tag{6.51}$$

and its variation is

$$\delta R_{\mu\nu} = \partial_\alpha (\delta \Gamma^\alpha{}_{\mu\nu}) - \partial_\nu (\delta \Gamma^\alpha{}_{\mu\alpha}) \,. \tag{6.52}$$

Although the Christoffel symbols do not transform as tensors, their variation does, as the transformation law (3.6) shows. Thus, we can locally replace the partial by the covariant derivatives and write

$$\delta R_{\mu\nu} = \nabla_\alpha (\delta \Gamma^\alpha{}_{\mu\nu}) - \nabla_\nu (\delta \Gamma^\alpha{}_{\mu\alpha}) \,, \tag{6.53}$$

which is a tensor identity, called the *Palatini identity*, and thus holds in all coordinate systems everywhere. It implies

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla_\alpha \left(g^{\mu\nu} \delta \Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha} \delta \Gamma^\nu{}_{\mu\nu}\right) \,, \tag{6.54}$$

where the indices α and ν were swapped in the last term. Thus, the variation of the Ricci tensor, contracted with the metric, can be expressed by the divergence of a vector W,

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla_\alpha W^\alpha \;, \tag{6.55}$$

whose components W^α are defined by the term in parentheses on the right-hand side of (6.54).

From Cramer's rule in the form (4.55), we see that

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = g g^{\mu\nu} \delta g_{\mu\nu} \,. \tag{6.56}$$

Moreover, since $g^{\mu\nu} g_{\mu\nu} = \text{const.} = 4$, we conclude

$$g^{\mu\nu} \delta g_{\mu\nu} = -g_{\mu\nu} \delta g^{\mu\nu} \,. \tag{6.57}$$

Using these expressions, we obtain for the variation of $\sqrt{-g}$

$$\delta \sqrt{-g} = -\frac{\delta g}{2\sqrt{-g}} = \frac{g g_{\mu\nu} \delta g^{\mu\nu}}{2\sqrt{-g}} = -\frac{\sqrt{-g}}{2} g_{\mu\nu} \delta g^{\mu\nu} \,, \tag{6.58}$$

or, in terms of the canonical volume form η,

$$\delta\eta = -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\,\eta = \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}\,\eta\ . \tag{6.59}$$

Now, we put (6.58) and (6.55) back into (6.50) and obtain

$$\begin{aligned}\delta S_{\mathrm{GR}} &= \int_D \nabla_\alpha W^\alpha \eta + \int_D \left(R_{\mu\nu} - \frac{\mathcal{R}}{2}\,g_{\mu\nu}\right)\delta g^{\mu\nu}\eta \\ &= \int_D G_{\mu\nu}\delta g^{\mu\nu}\eta + \int_D \nabla_\alpha W^\alpha \eta \overset{!}{=} 0\ . \end{aligned}\tag{6.60}$$

Varying $g^{\mu\nu}$ only in the interior of D, the divergence term vanishes by Gauß' theorem, and admitting arbitrary variations $\delta g^{\mu\nu}$ implies

$$G_{\mu\nu} = 0\ . \tag{6.61}$$

Caution Notice that we have ignored a possible boundary term here which needs to be taken into account if the manifold has a boundary. It has become known as the Gibbons-Hawking-York boundary term, which plays a central role e.g. in calculations of black-hole entropy. ◂

Vacuum field equations from a variational principle

Including the cosmological constant and using (6.58) once more, we see that Einstein's vacuum equations, $G + \Lambda g = 0$, follow from the variational principle

$$\delta \int_D (\mathcal{R} - 2\Lambda)\,\eta = 0\ . \tag{6.62}$$

The complete Einstein equations including the energy momentum tensor cannot yet be obtained here because no matter or energy contribution to the Lagrange density has been included yet into the action.

6.4 The energy-momentum tensor

6.4.1 Matter fields in the action

In order to include matter (where "matter" summarises all kinds of matter and non-gravitational energy) into the field equations, we assume that the matter fields ψ are described by a Lagrangian $\mathcal{L}$ depending on ψ, its gradient $\nabla\psi$ and the metric g,

$$\mathcal{L}(\psi, \nabla\psi, g)\ , \tag{6.63}$$

where ψ may be a scalar or tensor field.

The field equations are determined by the variational principle

$$\delta \int_D \mathcal{L}\eta = 0\ , \tag{6.64}$$

where the Lagrangian is varied with respect to the fields ψ and their derivatives $\nabla\psi$. Thus,

$$\delta \int_D \mathcal{L}\eta = \int_D \left(\frac{\partial \mathcal{L}}{\partial \psi}\delta\psi + \frac{\partial \mathcal{L}}{\partial \nabla\psi}\delta\nabla\psi \right)\eta = 0\ . \tag{6.65}$$

As usual, we can express the second term by the difference

$$\frac{\partial \mathcal{L}}{\partial \nabla\psi}\delta\nabla\psi = \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial \nabla\psi}\delta\psi \right) - \left(\nabla\cdot\frac{\partial \mathcal{L}}{\partial \nabla\psi} \right)\delta\psi\ , \tag{6.66}$$

of which the first term is a divergence which vanishes according to Gauß' theorem upon volume integration. Combining (6.66) with (6.65), and allowing arbitrary variations $\delta\psi$ of the matter fields, then yields the Euler-Lagrange equations for the matter fields,

$$\frac{\partial \mathcal{L}}{\partial \psi} - \nabla\cdot\frac{\partial \mathcal{L}}{\partial \nabla\psi} = 0\ . \tag{6.67}$$

Example: Lagrangian of a scalar field

To give an example, suppose we describe a neutral scalar field ψ with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\langle\nabla\psi, \nabla\psi\rangle - \frac{m^2}{2}\psi^2 = -\frac{1}{2}\nabla_\mu\psi\nabla^\mu\psi - \frac{m^2}{2}\psi^2\ , \tag{6.68}$$

where $m\psi^2/2$ is a mass term with constant parameter m. The Euler-Lagrange equations then imply the field equations

$$\left(-\Box + m^2\right)\psi = \left(-\nabla_\mu\nabla^\mu + m^2\right)\psi = 0\ , \tag{6.69}$$

which can be interpreted as the Klein-Gordon equation for a particle with mass m. ◀

Similarly, we can vary the action with respect to the *metric*, which requires care because the Lagrangian may depend on the metric explicitly and implicitly through the covariant derivatives $\nabla\psi$ of the fields, and the canonical volume form η depends on the metric as well because of (6.59). Thus,

$$\delta \int_D \mathcal{L}\eta = \int_D \left[(\delta\mathcal{L})\eta + \mathcal{L}\delta\eta\right] = \int_D \left(\delta\mathcal{L} - \frac{1}{2}g_{\mu\nu}\mathcal{L}\delta g^{\mu\nu} \right)\eta\ . \tag{6.70}$$

6.4.2 Field equations with matter

If the Lagrangian does not implicitly depend on the metric, we can write

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}}\delta g^{\mu\nu}\ . \tag{6.71}$$

If there is an implicit dependence on the metric, we can introduce normal coordinates to evaluate the variation of the Christoffel symbols,

$$\delta\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}\left(\nabla_\nu\delta g_{\sigma\mu} + \nabla_\mu\delta g_{\sigma\nu} - \nabla_\sigma\delta g_{\mu\nu}\right) , \quad (6.72)$$

which is a tensor, as remarked above, whence (6.72) holds in all coordinate frames everywhere. The derivatives can then be moved away from the variations of the metric by partial integration, and expressions proportional to $\delta g^{\mu\nu}$ remain.

Energy-momentum tensor

Thus, it is possible to write the variation of the action with respect to the *metric* in the form

$$\delta\int_D \mathcal{L}\eta = -\frac{1}{2}\int_D T_{\mu\nu}\delta g^{\mu\nu}\eta , \quad (6.73)$$

in which the tensor T is the energy-momentum tensor. If there are no implicit dependences on the metric, its components are

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + \mathcal{L}g_{\mu\nu} . \quad (6.74)$$

?

Derive the energy-momentum tensor for the matter field described by the Lagrangian (6.68).

Example: Energy-momentum tensor of the electromagnetic field

Let us show by an example that this identification does indeed make sense. We start from the Lagrangian of the free electromagnetic field,

$$\mathcal{L} = -\frac{1}{16\pi}F^{\alpha\beta}F_{\alpha\beta} = -\frac{1}{16\pi}F_{\alpha\beta}F_{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta} . \quad (6.75)$$

We know from (4.23) that the covariant derivatives in the field tensor F can be replaced by partial derivatives, thus there is no implicit dependence on the metric. Then, the variation $\delta\mathcal{L}$ is

$$\delta\mathcal{L} = -\frac{1}{8\pi}F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta}\delta g^{\mu\nu} \quad (6.76)$$

With (6.70), this implies

$$\delta\int_D \mathcal{L}\eta = \frac{1}{8\pi}\int_D\left(F_{\mu\alpha}F^{\alpha}{}_{\nu} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\right)\delta g^{\mu\nu}\eta \quad (6.77)$$

and, from (6.73), the familiar energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \quad (6.78)$$

of the electromagnetic field. ◂

Therefore, Einstein's field equations *and* the matter equations follow from the variational principle

$$\delta \int_D \left(\mathcal{R} - 2\Lambda + \frac{16\pi\mathcal{G}}{c^4} \mathcal{L} \right) \eta = 0 \tag{6.79}$$

Since, as we have seen before, the variation of the first two terms yields $G + \Lambda g$, and the variation of the third term yields minus one-half of the energy-momentum tensor, $-T/2$. In components, the variation yields

$$G_{\mu\nu} = \frac{8\pi\mathcal{G}}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu} \, . \tag{6.80}$$

This shows that the cosmological constant can be considered as part of the energy-momentum tensor,

$$T_{\mu\nu} \to T_{\mu\nu} + T^{\Lambda}_{\mu\nu} \, , \quad T^{\Lambda}_{\mu\nu} \equiv -\frac{\Lambda c^4}{8\pi\mathcal{G}} g_{\mu\nu} \, . \tag{6.81}$$

6.4.3 Equations of motion

Suppose spacetime is filled with an ideal fluid whose pressure p can be neglected compared to the energy density ρc^2. Then, the energy-momentum tensor (6.32) can be reduced to

$$T = \rho c^2 \, u^\flat \otimes u^\flat \, . \tag{6.82}$$

Conservation of the fluid can be expressed in the following way: the amount of matter contained in a domain D of spacetime must remain the same, even if the domain is mapped into another domain $\phi_t(D)$ by the flow ϕ_t of the vector field u with the time t. Thus

$$\int_D \rho\eta = \int_{\phi_t(D)} \rho\eta \, . \tag{6.83}$$

This expression just says that, if the domain D is mapped along the flow lines of the fluid flow, it will encompass a constant amount of material independent of time t.

?

Convince yourself recalling the definition of the pull-back that (6.84) is correct.

Now, we can use the pull-back to write

$$\int_{\phi_t(D)} \rho\eta = \int_D \phi_t^*(\rho\eta) \, , \tag{6.84}$$

and take the limit $t \to 0$ to see the equivalence of (6.83) and (6.84) with the vanishing Lie derivative of $\rho\eta$ along u,

$$\mathcal{L}_u(\rho\eta) = 0 \, . \tag{6.85}$$

The Leibniz rule (5.14) yields

$$(\mathcal{L}_u\rho)\eta + \rho\mathcal{L}_u\eta = 0\ . \tag{6.86}$$

Due to (5.15), the first term yields

$$(\mathcal{L}_u\rho)\eta = (u\rho)\eta = (u^i\partial_i\rho)\eta = (u^i\nabla_i\rho)\eta = (\nabla_u\rho)\eta\ . \tag{6.87}$$

For the second term, we can apply equation (5.30) for the components of the Lie derivative of a rank-$(0, 4)$ tensor, and use the antisymmetry of η to see that

$$\mathcal{L}_u\eta = (\nabla_\mu u^\mu)\eta = (\nabla\cdot u)\eta\ . \tag{6.88}$$

Accordingly, (6.86) can be written as

$$0 = (\nabla_u\rho + \rho\nabla\cdot u)\,\eta = \nabla\cdot(\rho u)\eta\ , \tag{6.89}$$

or

$$\nabla_\mu(\rho u^\mu) = 0\ . \tag{6.90}$$

?

Give a physical interpretation of equation (6.90). What does it mean?

At the same time, the divergence of T must vanish, hence

$$0 = \nabla_\nu T^{\mu\nu} = \nabla_\nu\left(\rho u^\mu u^\nu\right) = \nabla_\nu\left(\rho u^\nu\right)u^\mu + \rho u^\nu\nabla_\nu u^\mu\ . \tag{6.91}$$

The first term vanishes because of (6.90), and the second implies

$$u^\nu\nabla_\nu u^\mu = 0 \quad \Leftrightarrow \quad \nabla_u u = 0\ . \tag{6.92}$$

In other words, the flow lines have to be geodesics. For an ideal fluid, the equation of motion thus follows directly from the vanishing divergence of the energy-momentum tensor, which is required in general relativity by the contracted Bianchi identity (3.91).