# Chapter 4

# Physical Laws in External Gravitational Fields

## 4.1 Motion of particles

### 4.1.1 Action for particles in gravitational fields

In special relativity, the action of a free particle was

$$S = -mc^2 \int_a^b \mathrm{d}\tau = -mc \int_a^b \mathrm{d}s = -mc \int_a^b \sqrt{-\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} \,, \tag{4.1}$$

where we have introduced the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. This can be rewritten as follows: first, we parameterise the trajectory of the particle as a curve $\gamma(\tau)$ and write the four-vector $\mathrm{d}x = u\mathrm{d}\tau$ with the four-velocity $u = \dot{\gamma}$. Second, we use the notation (2.48)

$$\eta(u, u) = \langle u, u\rangle \tag{4.2}$$

to cast the action into the form

$$S = -mc \int_a^b \sqrt{-\langle u, u\rangle}\, \mathrm{d}\tau \,. \tag{4.3}$$

In general relativity, the metric $\eta$ is replaced by the dynamic metric $g$. We thus expect that the motion of a free particle will be described by the action

$$S = -mc \int_a^b \sqrt{-\langle u, u\rangle}\, \mathrm{d}\tau = -mc \int_a^b \sqrt{-g(u, u)}\, \mathrm{d}\tau \,. \tag{4.4}$$

?

Verify that the action (4.1) implies the correct, specially-relativistic equations of motion.

**Caution** Note that the actions (4.1) and (4.4) contain an interpretation of geometry in terms of physics: the line element of the metric is identified with proper time. ◀

### 4.1.2 Equations of motion

To see what this equation implies, we now carry out the variation of $S$ and set it to zero,

$$\delta S = -mc\,\delta \int_a^b \sqrt{-g(u,u)}\,\mathrm{d}\tau = 0\,. \tag{4.5}$$

Since the curve is assumed to be parameterised by the proper time $\tau$, we must have

$$c\mathrm{d}\tau = \mathrm{d}s = \sqrt{-\langle u,u\rangle}\,\mathrm{d}\tau\,, \tag{4.6}$$

which implies that the four-velocity $u$ must satisfy

$$\langle u,u\rangle = -c^2\,. \tag{4.7}$$

This allows us to write the variation (4.5) as

$$\delta S = \frac{mc}{2}\int_a^b \mathrm{d}\tau\left[\partial_\lambda g_{\mu\nu}\delta x^\lambda \dot{x}^\mu \dot{x}^\nu + 2g_{\mu\nu}\delta\dot{x}^\mu\dot{x}^\nu\right] = 0\,. \tag{4.8}$$

We can integrate the second term by parts to find

$$\begin{aligned} 2\int_a^b \mathrm{d}\tau\, g_{\mu\nu}\delta\dot{x}^\mu\dot{x}^\nu &= -2\int_a^b \mathrm{d}\tau\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}\dot{x}^\nu\right)\delta x^\mu \\ &= -2\int_a^b \mathrm{d}\tau\left(\partial_\lambda g_{\mu\nu}\dot{x}^\lambda\dot{x}^\nu + g_{\mu\nu}\ddot{x}^\nu\right)\delta x^\mu\,. \end{aligned} \tag{4.9}$$

Interchanging the summation indices $\lambda$ and $\mu$ and inserting the result into (4.8) yields

$$\left(\partial_\lambda g_{\mu\nu} - 2\partial_\mu g_{\lambda\nu}\right)\dot{x}^\mu\dot{x}^\nu - 2g_{\lambda\nu}\ddot{x}^\nu = 0 \tag{4.10}$$

or, after multiplication with $g^{\alpha\lambda}$,

$$\ddot{x}^\alpha + \frac{1}{2}g^{\alpha\lambda}\left(2\partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu = 0\,. \tag{4.11}$$

?

Convince yourself that (4.11) is correct and agrees with the geodesic equation.

Comparing the result (4.11) to (3.17) and recalling the symmetry of the Christoffel symbols (3.74), we arrive at the following important conclusion:

**Motion of freely falling particles**

The trajectories extremising the action (4.4) are geodesic curves. Freely falling particles thus follow the geodesics of the spacetime.

# 4.2 Motion of light

## 4.2.1 Maxwell's Equations in a Gravitational Field

As an example for how physical laws can be carried from special to general relativity, we now formulate the equations of classical electrodynamics in a gravitational field. For a summary of classical electrodynamics, see Appendix A.

In terms of the field tensor $F$, Maxwell's equations read

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0\ ,$$
$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu\ , \tag{4.12}$$

where $j^\mu$ is the current four-vector. The homogeneous equations are identically satisfied introducing the potentials $A^\mu$, in terms of which the field tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\ . \tag{4.13}$$

We can impose a gauge condition, such as the Lorenz gauge

$$\partial_\mu A^\mu = 0\ , \tag{4.14}$$

which allows to write the inhomogeneous Maxwell equation in the form

$$\Box A^\mu = -\frac{4\pi}{c} j^\mu \tag{4.15}$$

of the d'Alembert equation.

Indices are raised with the (inverse) Minkowski metric,

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}\ . \tag{4.16}$$

Finally, the equation for the Lorentz force can be written as

$$m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{q}{c} F^\mu{}_\nu u^\nu\ , \tag{4.17}$$

where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ is the four-velocity.

Moving to general relativity, we first replace the partial by covariant derivatives in Maxwell's equations and find

$$\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = 0\ ,$$
$$\nabla_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu\ . \tag{4.18}$$

However, it is easy to see that the identity

$$\nabla_\lambda F_{\mu\nu} + \text{cyclic} \equiv \partial_\lambda F_{\mu\nu} + \text{cyclic} \tag{4.19}$$

holds because of the antisymmetry of the field tensor $F$ and the symmetry of the connection $\nabla$.

Indices have to be raised with the inverse metric $g^{-1}$ now,

$$F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} \; . \tag{4.20}$$

Equation (4.17) for the Lorentz force has to be replaced by

$$m\left(\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\alpha\beta}\,u^\alpha u^\beta\right) = \frac{q}{c}\,F^\mu{}_\nu\,u^\nu \; . \tag{4.21}$$

We thus arrive at the following *general rule*:

**Porting physical laws into general relativity**

In the presence of a gravitational field, the physical laws of special relativity are changed simply by substituting the covariant derivative for the partial derivative, $\partial \to \nabla$, by raising indices with $g^{\mu\nu}$ instead of $\eta^{\mu\nu}$ and by lowering them with $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$, and by replacing the motion of free particles along straight lines by the motion along geodesics.

Note that this is a rule, not a law, because ambiguities may occur in presence of second derivatives, as we shall see shortly.

We can impose a gauge condition such as the generalised Lorenz gauge

$$\nabla_\mu A^\mu = 0 \; , \tag{4.22}$$

but now the inhomogeneous wave equation (4.15) becomes more complicated. We first note that

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.23}$$

identically. Inserting (4.23) into the inhomogeneous Maxwell equation first yields

$$\nabla_\nu\left(\nabla^\mu A^\nu - \nabla^\nu A^\mu\right) = \frac{4\pi}{c}\,j^\mu \; , \tag{4.24}$$

but now the term $\nabla_\nu\nabla^\mu A^\nu$ does not vanish despite the Lorenz gauge condition because the *covariant derivatives do not commute*.

**Caution** Applying the rule given above, it has to be taken into account that covariant derivatives do not generally commute. ◂

Instead, we have to use

$$\left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right)A^\alpha = R^\alpha{}_{\beta\mu\nu}\,A^\beta \tag{4.25}$$

by definition of the curvature tensor, and thus

$$\nabla_\nu\nabla^\mu A^\nu = \nabla^\mu\nabla_\nu A^\nu + R^\mu{}_\beta\,A^\beta = R^\mu{}_\beta\,A^\beta \tag{4.26}$$

inserting the Lorenz gauge condition.

**Electromagnetic wave equation in a curved spacetime**

Thus, the inhomogeneous wave equation for an electromagnetic field in general relativity reads

$$\nabla_\nu \nabla^\nu A^\mu - R^\mu{}_\nu A^\nu = -\frac{4\pi}{c} j^\mu \,. \tag{4.27}$$

Had we started directly from the wave equation (4.15) from special relativity, we would have missed the curvature term! This illustrates the ambiguities that may occur applying the rule $\partial \to \nabla$ when second derivatives are involved.

### 4.2.2 Geometrical Optics

We now study how light rays propagate in a gravitational field. As usual in geometrical optics, we assume that the wavelength $\lambda$ of the electromagnetic field is very much smaller compared to the scale $L$ of the space within which we study light propagation. In a gravitational field, which causes spacetime to curve on another scale $R$, we have to further assume that $\lambda$ is also very small compared to $R$, thus

$$\lambda \ll L \quad \text{and} \quad \lambda \ll R \,. \tag{4.28}$$

**Example: Geometrical optics in curved space**

An example could be an astronomical source at a distance of several million light-years from which light with optical wavelengths travels to the observer. The scale $L$ would then be of order $10^{24}$ cm or larger, the scale $R$ would be the curvature radius of the Universe, of order $10^{28}$ cm, while the light would have wavelengths of order $10^{-6}$ cm. ◀

Consequently, we introduce an expansion of the four-potential in terms of a small parameter $\varepsilon \equiv \lambda/\min(L, R) \ll 1$ and write the four-potential as a product of a slowly varying amplitude and a quickly varying phase,

$$A^\mu = \mathrm{Re}\left\{(a^\mu + \varepsilon b^\mu)\mathrm{e}^{\mathrm{i}\psi/\varepsilon}\right\} \,, \tag{4.29}$$

where the amplitude is understood as the two leading-order terms in the expansion, and the phase $\psi$ carries the factor $\varepsilon^{-1}$ because it is inversely proportional to the wave length. The real part is introduced because the amplitude is complex.

As in ordinary geometrical optics, the wave vector is the gradient of the phase, thus $k_\mu = \partial_\mu \psi$. We further introduce the scalar amplitude $a \equiv (a_\mu a^{*\mu})^{1/2}$, where the asterisk denotes complex conjugation, and the *polarisation vector* $e_\mu \equiv a_\mu / a$.

We first impose the Lorenz gauge and find the condition

$$\mathrm{Re}\left\{\left[\nabla_\mu(a^\mu + \varepsilon b^\mu) + (a^\mu + \varepsilon b^\mu)\frac{\mathrm{i}}{\varepsilon}k_\mu\right]\mathrm{e}^{\mathrm{i}\psi/\varepsilon}\right\} = 0\ . \tag{4.30}$$

To leading order ($\varepsilon^{-1}$), this implies

$$k_\mu a^\mu = 0\ , \tag{4.31}$$

which shows that the wave vector is perpendicular to the polarisation vector. The next-higher order yields

$$\nabla_\mu a^\mu + \mathrm{i}k_\mu b^\mu = 0\ . \tag{4.32}$$

Next, we insert the *ansatz* (4.29) into Maxwell's equation (4.27) in vacuum, i.e. setting the right-hand side to zero. This yields

$$\begin{aligned} \mathrm{Re}\Bigg\{\Bigg[&\nabla_\nu\nabla^\nu(a^\mu + \varepsilon b^\mu) + \frac{2\mathrm{i}}{\varepsilon}k^\nu\nabla_\nu(a^\mu + \varepsilon b^\mu) \\ &+ \frac{\mathrm{i}}{\varepsilon}(a^\mu + \varepsilon b^\mu)\nabla_\nu k^\nu - \frac{1}{\varepsilon^2}k_\nu k^\nu(a^\mu + \varepsilon b^\mu) \\ &- R^\mu{}_\nu(a^\nu + \varepsilon b^\nu)\Bigg]\mathrm{e}^{\mathrm{i}\psi/\varepsilon}\Bigg\} = 0\ . \end{aligned} \tag{4.33}$$

To leading order ($\varepsilon^{-2}$), this implies

$$k_\nu k^\nu = 0\ , \tag{4.34}$$

which yields the general-relativistic *eikonal equation*

$$g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi = 0\ . \tag{4.35}$$

Trivially, (4.34) implies

$$0 = \nabla_\mu(k_\nu k^\nu) = 2k^\nu\nabla_\mu k_\nu\ . \tag{4.36}$$

Recall that the wave vector is the gradient of the scalar phase $\psi$. The second covariant derivatives of $\psi$ commute,

$$\nabla_\mu\nabla_\nu\psi = \nabla_\nu\nabla_\mu\psi \tag{4.37}$$

as is easily seen by direct calculation, using the symmetry of the connection. Thus,

$$\nabla_\mu k_\nu = \nabla_\nu k_\mu\ , \tag{4.38}$$

which, inserted into (4.36), leads to

$$k^\nu\nabla_\nu k^\mu = 0 \quad \text{or} \quad \nabla_k k = 0\ . \tag{4.39}$$

In other words, we arrive at the following important result:

**Light rays in curved spacetime**

In the limit of geometrical optics, Maxwell's equations imply that *light rays follow null geodesics*.

The next-higher order ($\varepsilon^{-1}$) gives

$$2\mathrm{i}\left(k^\nu\nabla_\nu a^\mu + \frac{1}{2}a^\mu\nabla_\nu k^\nu\right) - k_\nu k^\nu b^\mu = 0 \tag{4.40}$$

and, with (4.34), this becomes

$$k^\nu\nabla_\nu a^\mu + \frac{1}{2}a^\mu\nabla_\nu k^\nu = 0\;. \tag{4.41}$$

We use this to derive a propagation law for the amplitude $a$. Obviously, we can write

$$2ak^\nu\partial_\nu a = 2ak^\nu\nabla_\nu a = k^\nu\nabla_\nu(a^2) = k^\nu\left(a^{*\mu}\nabla_\nu a_\mu + a_\mu\nabla_\nu a^{*\mu}\right)\;. \tag{4.42}$$

By (4.41), this can be transformed to

$$k^\nu\left(a^{*\mu}\nabla_\nu a^\mu + a^\mu\nabla_\nu a^{*\mu}\right) = -\frac{1}{2}\nabla_\nu k^\nu(a^{*\mu}a_\mu + a_\mu a^{*\mu}) = -a^2\nabla_\nu k^\nu\;. \tag{4.43}$$

Combining (4.43) with (4.42) yields

$$k^\nu\partial_\nu a = -\frac{a}{2}\nabla_\nu k^\nu\;, \tag{4.44}$$

which shows how the amplitude is transported along light rays: the change of the amplitude in the direction of the wave vector is proportional to the negative divergence of the wave vector, which is a very intuitive result.

?

Why and in what sense is the result (4.44) called intuitive here? What does it mean?

Finally, we obtain a law for the propagation of the polarisation. Using $a^\mu = ae^\mu$ in (4.41) gives

$$\begin{aligned} 0 &= k^\nu\nabla_\nu(ae^\mu) + \frac{1}{2}ae^\mu\nabla_\nu k^\nu \\ &= ak^\nu\nabla_\nu e^\mu + e^\mu\left(k^\nu\partial_\nu a + \frac{a}{2}\nabla_\nu k^\nu\right) = ak^\nu\nabla_\nu e^\mu\;, \end{aligned} \tag{4.45}$$

where (4.44) was used in the last step. This shows that

$$k^\nu\nabla_\nu e^\mu = 0 \quad\text{or}\quad \nabla_k e = 0\;, \tag{4.46}$$

or in other words:

**Transport of polarisation**

The polarisation of electromagnetic waves is parallel-transported along light rays.

### 4.2.3 Redshift

Suppose now that a light source moving with four-velocity $u_\mathrm{s}$ is sending a light ray to an observer moving with four-velocity $u_\mathrm{o}$, and another light ray after a proper-time interval $\delta\tau_\mathrm{s}$. The phases of the first and second light rays be $\psi_1$ and $\psi_2 = \psi_1 + \delta\psi$, respectively.

Clearly, the phase difference measured at the source and at the observer must equal, thus

$$u_\mathrm{s}^\mu(\partial_\mu\psi)_\mathrm{s}\delta\tau_\mathrm{s} = \delta\psi = u_\mathrm{o}^\mu(\partial_\mu\psi)_\mathrm{o}\delta\tau_\mathrm{o} \ . \tag{4.47}$$

Using $k_\mu = \partial_\mu\psi$, and assigning frequencies $\nu_\mathrm{s}$ and $\nu_\mathrm{o}$ to the light rays which are indirectly proportional to the time intervals $\delta\tau_\mathrm{s}$ and $\delta\tau_\mathrm{o}$, we find

$$\frac{\nu_\mathrm{o}}{\nu_\mathrm{s}} = \frac{\delta\tau_\mathrm{s}}{\delta\tau_\mathrm{o}} = \frac{\langle k, u\rangle_\mathrm{o}}{\langle k, u\rangle_\mathrm{s}} \ , \tag{4.48}$$

which gives the combined gravitational redshift and the Doppler shift of the light rays. Any distinction between Doppler shift and gravitational redshift has no invariant meaning in general relativity.

?

Beginning with (4.48), can you derive the specially-relativistic Doppler formula?

## 4.3 Energy-momentum (non-)conservation

### 4.3.1 Contracted Christoffel Symbols

From (3.74), we see that the contracted Christoffel symbol can be written as

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{2}g^{\mu\alpha}\left(g_{\alpha\nu,\mu} + g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha}\right) \ . \tag{4.49}$$

Exchanging the arbitrary dummy indices $\alpha$ and $\mu$ and using the symmetry of the metric, we can simplify this to

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{2}g^{\mu\alpha}g_{\mu\alpha,\nu} \ . \tag{4.50}$$

We continue by using Cramer's rule from linear algebra, which states that the inverse of a matrix $A$ has the components

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A} \ , \tag{4.51}$$

where the $C_{ji}$ are the cofactors (signed minors) of the matrix $A$. Thus, the cofactors are

$$C_{ji} = \det A(A^{-1})_{ij} \ . \tag{4.52}$$

The determinant of $A$ can be expressed using the cofactors as

$$\det A = \sum_{j=1}^{n} C_{ji}A_{ji} \tag{4.53}$$

for any fixed $i$, where $n$ is the dimension of the (square) matrix $A$. This so-called *Laplace expansion* of the determinant follows after multiplying (4.52) with the matrix $A_{jk}$.

By definition of the cofactors, any cofactor $C_{ji}$ does not contain the element $A_{ji}$ of the matrix $A$. Therefore, we can use (4.52) and the Laplace expansion (4.53) to conclude

$$\frac{\partial \det A}{\partial A_{ji}} = C_{ji} = \det A (A^{-1})_{ij} \ . \tag{4.54}$$

The metric is represented by the matrix $g_{\mu\nu}$, its inverse by $g^{\mu\nu}$. We abbreviate its determinant by $g$ here. Cramer's rule (4.52) then implies that the cofactors of $g_{\mu\nu}$ are $C^{\mu\nu} = g\, g^{\mu\nu}$, and we can immediately conclude from (4.54) that

$$\frac{\partial g}{\partial g_{\mu\nu}} = g g^{\mu\nu} \tag{4.55}$$

and thus

$$\partial_\lambda g = \frac{\partial g}{\partial g_{\mu\nu}} \partial_\lambda g_{\mu\nu} = g g^{\mu\nu} \partial_\lambda g_{\mu\nu} \ . \tag{4.56}$$

**Contracted Christoffel symbols**

Comparing this with the expression (4.50) for the contracted Christoffel symbol, we see that

$$\begin{aligned} g g^{\mu\nu} g_{\mu\nu,\lambda} &= 2g\Gamma^\mu{}_{\mu\lambda} \ , \\ \Gamma^\mu{}_{\mu\lambda} &= \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\lambda} = \frac{1}{2g} g_{,\lambda} = \frac{1}{\sqrt{-g}} \partial_\lambda \sqrt{-g} \ , \end{aligned} \tag{4.57}$$

which is a very convenient expression for the contracted Christoffel symbol, as we shall see.

Using (4.51) and (4.53), calculate the inverse and the determinant of $2 \times 2$ and $3 \times 3$ matrices.

## 4.3.2 Covariant Divergences

The covariant derivative of a vector with components $v^\mu$ has the components

$$\nabla_\nu v^\mu = \partial_\nu v^\mu + \Gamma^\mu{}_{\nu\alpha} v^\alpha \ . \tag{4.58}$$

Using (4.57), the covariant divergence of this vector can thus be written

$$\nabla_\mu v^\mu = \partial_\mu v^\mu + \frac{1}{\sqrt{-g}} v^\mu \partial_\mu \sqrt{-g} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, v^\mu) \ . \tag{4.59}$$

Similarly, for a tensor $A$ of rank $(2, 0)$ with components $A^{\mu\nu}$, we have

$$\nabla_\nu A^{\mu\nu} = \partial_\nu A^{\mu\nu} + \Gamma^\mu{}_{\alpha\nu} A^{\alpha\nu} + \Gamma^\nu{}_{\nu\alpha} A^{\mu\alpha} \ . \tag{4.60}$$

Again, by means of (4.57), we can combine the first and third terms on the right-hand side to write

$$\nabla_\nu A^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}A^{\mu\nu}) + \Gamma^\mu{}_{\alpha\nu}A^{\alpha\nu} \ . \tag{4.61}$$

**Tensor divergences**

If the tensor $A^{\mu\nu}$ is *antisymmetric*, the second term on the right-hand side of the divergence (4.61) vanishes because then the symmetric Christoffel symbol $\Gamma^\mu{}_{\alpha\nu}$ is contracted with the antisymmetric tensor $A^{\alpha\nu}$. If $A^{\mu\nu}$ is *symmetric*, however, this final term remains, with important consequences.

### 4.3.3 Charge Conservation

Since the electromagnetic field tensor $F^{\mu\nu}$ is antisymmetric, (4.61) implies

$$\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}F^{\mu\nu}) \ . \tag{4.62}$$

On the other hand, replacing the vector $v^\mu$ by $\nabla_\nu F^{\mu\nu}$ in (4.59), we see that

$$\nabla_\mu\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\nabla_\nu F^{\mu\nu}) = \frac{1}{\sqrt{-g}}\partial_\mu\partial_\nu(\sqrt{-g}F^{\mu\nu}) \ , \tag{4.63}$$

where we have used (4.62) in the final step. But the partial derivatives commute, so that once more the antisymmetric tensor $F^{\mu\nu}$ is contracted with the symmetric symbol $\partial_\mu\partial_\nu$. Thus, the result must vanish, allowing us to conclude

$$\nabla_\mu\nabla_\nu F^{\mu\nu} = 0 \ . \tag{4.64}$$

However, by Maxwell's equation (4.18),

$$\nabla_\mu\nabla_\nu F^{\mu\nu} = \frac{4\pi}{c}\nabla_\mu j^\mu \ , \tag{4.65}$$

which implies, by (4.59)

$$\partial_\mu(\sqrt{-g}j^\mu) = 0 \ . \tag{4.66}$$

**Charge conservation**

Equation (4.66) is the continuity equation of the electric four-current, implying charge conservation. We thus see that the antisymmetry of the electromagnetic field tensor is necessary for charge conservation.

### 4.3.4 Energy-Momentum "Conservation"

In special relativity, energy-momentum conservation can be expressed by the vanishing four-divergence of the energy-momentum tensor $T$,

$$\partial_\nu T^{\mu\nu} = 0 \; . \tag{4.67}$$

**Example: Energy conservation in an electromagnetic field**

For example, the energy-momentum tensor of the electromagnetic field is, in special relativity

$$T^{\mu\nu} = \frac{1}{4\pi}\left[-F^{\mu\lambda}F^{\nu}{}_{\lambda} + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right] , \tag{4.68}$$

and for $\mu = 0$, the vanishing divergence (4.67) yields the energy conservation equation

$$\frac{\partial}{\partial t}\left(\frac{\vec{E}^2 + \vec{B}^2}{8\pi}\right) + \vec{\nabla}\cdot\left[\frac{c}{4\pi}(\vec{E}\times\vec{B})\right] = 0 \; , \tag{4.69}$$

in which the Poynting vector

$$\vec{S} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right) \tag{4.70}$$

represents the energy current density. ◂

According to our general rule for moving results from special relativity to general relativity, we can replace the partial derivative in (4.67) by the covariant derivative,

$$\nabla_\nu T^{\mu\nu} = 0 \; , \tag{4.71}$$

and obtain an equation which is covariant and thus valid in all reference frames. Moreover, we would have to replace the Minkowski metric $\eta$ in (4.68) by the metric $g$ if we wanted to consider the energy-momentum tensor of the electromagnetic field.

From our general result (4.61), we know that we can rephrase (4.71) as

$$\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}T^{\mu\nu}) + \Gamma^{\mu}{}_{\lambda\nu}T^{\lambda\nu} = 0 \; . \tag{4.72}$$

If the second term on the left-hand side was absent, this equation would imply a conservation law. It remains there, however, because the energy-momentum tensor is symmetric. In presence of this term, we cannot convert (4.72) to a conservation law any more. This result expresses the following important fact:

**Energy non-conservation**

Energy is not generally conserved in general relativity. This is not surprising because energy can now be exchanged with the gravitational field.

## 4.4 The Newtonian limit

### 4.4.1 Metric and Gravitational Potential

Finally, we want to see under which conditions for the metric the Newtonian limit for the equation of motion in a gravitational field is reproduced, which is

$$\ddot{\vec{x}} = -\vec{\nabla}\Phi \tag{4.73}$$

to very high precision in the Solar System.

We first restrict the gravitational field to be weak and to vary slowly with time. This implies that the Minkowski metric of flat space is perturbed by a small amount,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \tag{4.74}$$

with $|h_{\mu\nu}| \ll 1$.

Moreover, we restrict the consideration to bodies moving much slower than the speed of light, such that

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Does it matter with respect to which coordinate frame the velocity is assumed to be much less than the speed of light?

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll \frac{\mathrm{d}x^0}{\mathrm{d}\tau} \approx 1 . \tag{4.75}$$

Under these conditions, the geodesic equation for the $i$-th spatial coordinate reduces to

$$\frac{\mathrm{d}^2x^i}{c^2\mathrm{d}t^2} \approx \frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} = -\Gamma^i{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \approx -\Gamma^i{}_{00} . \tag{4.76}$$

By definition (3.74), the remaining Christoffel symbols read

$$\Gamma^i{}_{00} = h_0{}^i{}_{,0} - \frac{1}{2}h_{00}{}^{,i} \approx -\frac{1}{2}h_{00}{}^{,i} \tag{4.77}$$

due to the assumption that the metric changes slowly in time so that its time derivative can be ignored compared to its spatial derivatives. Equation (4.76) can thus be reduced to

$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} \approx \frac{c^2}{2}\vec{\nabla}h_{00} , \tag{4.78}$$

which agrees with the Newtonian equation of motion (4.73) if we identify

$$h_{00} \approx -\frac{2\Phi}{c^2} + \text{const.} \tag{4.79}$$

The constant can be set to zero because both the deviation from the Minkowski metric and the gravitational potential vanish at large distance from the source of gravity. Therefore, the metric in the Newtonian limit has the 0-0 element

$$g_{00} \approx -1 - \frac{2\Phi}{c^2} \ . \tag{4.80}$$

### 4.4.2 Gravitational Light Deflection

Based on this result, we might speculate that the metric in Newtonian approximation could be written as

$$g = \mathrm{diag}\left[-\left(1 + \frac{2\Phi}{c^2}\right), 1, 1, 1\right] \ . \tag{4.81}$$

We shall now work out the gravitational light deflection by the Sun in this metric, which was one of the first observational tests of general relativity.

Since light rays propagate along null geodesics, we have

$$\nabla_k k = 0 \quad \text{or} \quad k^\nu \partial_\nu k^\mu + \Gamma^\mu{}_{\nu\lambda} k^\nu k^\lambda = 0 \ , \tag{4.82}$$

where $k = (\omega/c, \vec{k})$ is the wave four-vector which satisfies

$$\langle k, k \rangle = 0 \quad \text{thus} \quad \omega = c|\vec{k}| \ , \tag{4.83}$$

which is the ordinary dispersion relation for electromagnetic waves in vacuum. We introduce the unit vector $\vec{e}$ in the direction of $\vec{k}$ by $\vec{k} = |\vec{k}|\,\vec{e} = \omega\,\vec{e}/c$.

Assuming that the gravitational potential $\Phi$ does not vary with time, $\partial_0\Phi = 0$, the only non-vanishing Christoffel symbols of the metric (4.81) are

$$\Gamma^0{}_{0i} \approx \frac{1}{c^2}\partial_i\Phi \approx \Gamma^i{}_{00} \ . \tag{4.84}$$

For $\mu = 0$, (4.82) yields

$$\left(\frac{1}{c}\partial_t + \vec{e}\cdot\vec{\nabla}\right)\omega + \omega\frac{\vec{e}\cdot\vec{\nabla}\Phi}{c^2} = 0 \ , \tag{4.85}$$

which shows that the frequency changes with time only because the light path can run through a spatially varying gravitational potential. Thus, if the potential is constant in time, the frequencies of the incoming and the outgoing light must equal.

Using this result, the spatial components of (4.82) read

$$\left(\frac{1}{c}\partial_t + \vec{e}\cdot\vec{\nabla}\right)\vec{e} = \frac{\mathrm{d}\vec{e}}{c\mathrm{d}t} = -\frac{1}{c^2}\left[\vec{\nabla} - \vec{e}(\vec{e}\cdot\vec{\nabla})\right]\Phi = -\frac{\vec{\nabla}_\perp\Phi}{c^2} \ ; \tag{4.86}$$

in other words, the total time derivative of the unit vector in the direction of the light ray equals the negative perpendicular gradient of the gravitational potential.

For calculating the light deflection, we need to know the total change of $\vec{e}$ as the light ray passes the Sun. This is obtained by integrating (4.86) along the *actual (curved)* light path, which is quite complicated. However, due to the weakness of the gravitational field, the deflection will be very small, and we can evaluate the integral along the *unperturbed (straight)* light path.

**Caution** Note that this approximation is conceptually identical to Born's approximation in quantum-mechanical scattering problems. ◀

We choose a coordinate system centred on the Sun and rotated such that the light ray propagates parallel to the $z$ axis from $-\infty$ to $\infty$ at an impact parameter $b$. Outside the Sun, its gravitational potential is

$$\frac{\Phi}{c^2} = -\frac{GM_\odot}{c^2 r} = -\frac{GM_\odot}{c^2\sqrt{b^2+z^2}} \ . \tag{4.87}$$

The perpendicular gradient of $\Phi$ is

$$\vec{\nabla}_\perp \Phi = \frac{\partial \Phi}{\partial b}\vec{e}_b = \frac{GM\,b}{c^2(b^2+z^2)^{3/2}}\vec{e}_b \ , \tag{4.88}$$

where $\vec{e}_b$ is the radial unit vector in the $x$-$y$ plane from the Sun to the light ray.

**Light deflection in (incomplete) Newtonian approximation**

Thus, under the present assumptions, the deflection angle is

$$\delta\vec{e} = -\vec{e}_b \int_{-\infty}^{\infty} \mathrm{d}z \, \frac{GMb}{c^2(b^2+z^2)^{3/2}} = -\frac{2GM}{c^2 b}\,\vec{e}_b \ . \tag{4.89}$$

Evaluating (4.89) at the rim of the Sun, we insert $M_\odot = 2 \cdot 10^{33}$ g and $R_\odot = 7 \cdot 10^{10}$ cm to find

$$|\delta\vec{e}| = 0.87'' \ . \tag{4.90}$$

For several reasons, this is a remarkable result. First, it had already been derived by the German astronomer Soldner in the 19th century who had assumed that light was a stream of material particles to which celestial mechanics could be applied just as well as to planets. Before general relativity, a strict physical meaning could not be given to the trajectory of light in the presence of a gravitational field because the interaction between electromagnetic fields and gravity was entirely unclear. The statement of general relativity that light propagates along null geodesics for the first time provided a physical law for the propagation of light rays in gravitational fields.

Second, the result (4.90) is experimentally found to be *incorrect*. In fact, the measured value is twice as large. This is a consequence of our assumption that the metric in the Newtonian limit is given by (4.81), while the line element in the complete Newtonian limit is

$$\mathrm{d}s^2 = -\left(1 + \frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2\Phi}{c^2}\right)\mathrm{d}\vec{x}^2 \ . \tag{4.91}$$