Chapter 3

Differential Geometry II

3.1 Connections and covariant derivatives

3.1.1 Linear Connections

The *curvature* of the two-dimensional sphere S^2 can be described by embedding the sphere into a Euclidean space of the next-higher dimension, $\mathbb{R}^3$. However, (as far as we know) there is no natural *embedding* of our four-dimensional curved spacetime into $\mathbb{R}^5$, and thus we need a description of curvature which is intrinsic to the manifold.

Caution Whitney's (strong) embedding theorem states that any smooth n-dimensional manifold ($n > 0$) can be smoothly embedded in the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$. Embeddings into lower-dimensional Euclidean spaces may exist, but not necessarily so. An embedding $f : M \to N$ of a manifold M into a manifold N is an injective map such that $f(M)$ is a submanifold of N and $M \to f(M)$ is differentiable. ◀

There is a close correspondence between the curvature of a manifold and the transport of vectors along curves.

As we have seen before, the structure of a manifold does not trivially allow to compare vectors which are elements of tangent spaces at two different points. We will thus have to introduce an additional structure which allows us to meaningfully shift vectors from one point to another on the manifold.

Even before we do so, it is intuitively clear how vectors can be transported along closed paths in flat Euclidean space, say $\mathbb{R}^3$. There, the vector arriving at the starting point after the transport will be identical to the vector before the transport.

However, this will no longer be so on the two-sphere: starting on the equator with a vector pointing north, we can shift it along a meridian to the north pole, then back to the equator along a different meridian, and finally back to its starting point on the equator. There, it will point into a different direction than the original vector.

Curvature can thus be defined from this misalignment of vectors after transport along closed curves. In order to work this out, we thus first need some way for transporting vectors along curves.

We start by generalising the concept of a directional derivative from $\mathbb{R}^n$ by defining a *linear* or *affine connection* or *covariant differentiation* on a manifold as a mapping ∇ which assigns to every pair v, y of C^∞ vector fields another vector field $\nabla_v y$ which is bilinear in v and y and satisfies

$$\begin{aligned} \nabla_{fv} y &= f \nabla_v y \\ \nabla_v (f y) &= f \nabla_v y + v(f) y \,, \end{aligned} \tag{3.1}$$

where $f \in \mathcal{F}$ is a C^∞ function on M.

Figure 3.1 Elwin Bruno Christoffel (1829–1900), German mathematician. Source: Wikipedia

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Why do the Christoffel symbols suffice to specify the connection completely?

In a local coordinate basis $\{e_i\}$, we can describe the linear connection by its action on the basis vectors,

$$\nabla_{\partial_i}(\partial_j) \equiv \Gamma^k{}_{ij}\, \partial_k \,, \tag{3.2}$$

where the n^3 numbers $\Gamma^k{}_{ij}$ are called the *Christoffel symbols* or *connection coefficients* of the connection ∇ in the given chart.

Connection

A connection ∇ generalises the directional derivative of objects on a manifold. The directional derivative of a vector y in the direction of the vector v is the vector $\nabla_v y$. The connection is linear and satisfies the product rule.

The Christoffel symbols are *not* the components of a tensor, which is seen from their transformation under coordinate changes. Let x^i and x'^i

be two different coordinate systems, then we have on the one hand, by definition,

$$\nabla_{\partial'_a}(\partial'_b) = \Gamma'^c{}_{ab}\,\partial'_c = \Gamma'^c{}_{ab}\,\frac{\partial x^k}{\partial x'^c}\,\partial_k = \Gamma'^c{}_{ab}\,J^k_c\,\partial_k\,, \tag{3.3}$$

where J^k_c is the Jacobian matrix of the coordinate transform as defined in (2.26). On the other hand, the axioms (3.1) imply, with f represented by the elements J^k_i of the Jacobian matrix,

$$\begin{aligned}\nabla_{\partial'_a}(\partial'_b) &= \nabla_{J^i_a\partial_i}(J^j_b\partial_j) = J^i_a\nabla_{\partial_i}(J^j_b\partial_j)\\ &= J^i_a\left[J^j_b\nabla_{\partial_i}\partial_j + \partial_i J^j_b\partial_j\right]\\ &= J^i_a J^j_b\Gamma^k{}_{ij}\,\partial_k + J^i_a\partial_i J^k_b\partial_k\,.\end{aligned} \tag{3.4}$$

Comparison of the two results (3.3) and (3.4) shows that

$$\Gamma'^c{}_{ab}\,J^k_c = J^i_a J^j_b\Gamma^k{}_{ij} + J^i_a\partial_i J^k_b\,, \tag{3.5}$$

or, after multiplying with the inverse Jacobian matrix J'^c_k,

$$\Gamma'^c{}_{ab} = J^i_a J^j_b J'^c_k\Gamma^k{}_{ij} + J'^c_k J^i_a\partial_i J^k_b\,. \tag{3.6}$$

While the first term on the right-hand side reflects the tensor transformation law (2.42), the second term differs from it.

3.1.2 Covariant derivative

Let now y and v be vector fields on M and w a dual vector field, then the *covariant derivative* ∇y is a tensor field of rank $(1,1)$ which is defined by

$$\nabla y(v,w) \equiv w[\nabla_v(y)]\,. \tag{3.7}$$

In a coordinate basis $\{\partial_i\}$, we write

$$y = y^i\partial_i \quad\text{and}\quad \nabla y \equiv y^i_{;j}\,\mathrm{d}x^j\otimes\partial_i\,, \tag{3.8}$$

and obtain the tensor components

$$\begin{aligned}y^i_{;j} &= \nabla y(\partial_j, \mathrm{d}x^i) = \mathrm{d}x^i\left(\nabla_{\partial_j}(y^k\partial_k)\right)\\ &= \mathrm{d}x^i\left(y^k_{,j}\partial_k + y^k\Gamma^l{}_{jk}\,\partial_l\right)\\ &= y^k_{,j}\delta^i_k + y^k\Gamma^l{}_{jk}\,\delta^i_l\\ &= y^i_{,j} + y^k\Gamma^i{}_{jk}\,.\end{aligned} \tag{3.9}$$

An affine connection is symmetric if

$$\nabla_v w - \nabla_w v = [v,w]\,, \tag{3.10}$$

which a short calculation shows to be equivalent to the symmetry property

$$\Gamma^k{}_{ij} = \Gamma^k{}_{ji} \tag{3.11}$$

of the Christoffel symbols in a coordinate basis.

Caution Indices separated by a comma denote ordinary partial differentiations with respect to coordinates, $y_{,i} \equiv \partial_i y$. ◂

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Carry out the calculation in (3.9) yourself and verify (3.11) for a symmetric connection. How many Christoffel symbols do you need for a symmetric connection on S^2?

3.2 Geodesics

3.2.1 Parallel transport and geodesics

Given a linear connection, it is now straightforward to introduce parallel transport . To begin, let $\gamma : I \to M$ with $I \subset \mathbb{R}$ a curve in M with tangent vector $\dot{\gamma}(t)$. A vector field v is called *parallel along* γ if

$$\nabla_{\dot{\gamma}} v = 0 \; . \tag{3.12}$$

The vector $\nabla_{\dot{\gamma}} v$ is the covariant derivative of v along γ, and it is often denoted by

$$\nabla_{\dot{\gamma}} v = \frac{\mathrm{D}v}{\mathrm{d}t} = \frac{\nabla v}{\mathrm{d}t} \; . \tag{3.13}$$

In the coordinate basis $\{\partial_i\}$, the covariant derivative along γ reads

$$\begin{aligned} \nabla_{\dot{\gamma}} v &= \nabla_{\dot{x}^i \partial_i}(v^j \partial_j) = \dot{x}^i \nabla_{\partial_i}(v^j \partial_j) \\ &= \dot{x}^i \left[v^j \nabla_{\partial_i}(\partial_j) + \partial_i v^j \partial_j \right] = \left(\dot{v}^k + \Gamma^k{}_{ij}\, \dot{x}^i v^j \right) \partial_k \; , \end{aligned} \tag{3.14}$$

and if this is to vanish identically, (3.12) and (3.14) imply the components

$$\dot{v}^k + \Gamma^k{}_{ij}\, \dot{x}^i v^j = 0 \; . \tag{3.15}$$

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Convince yourself of the results (3.14) and (3.15).

The existence and uniqueness theorems for ordinary differential equations imply that (3.15) has a unique solution once v is given at one point along the curve $\gamma(t)$. The *parallel transport* of a vector along a curve is then uniquely defined.

If the tangent vector $\dot{\gamma}$ of a curve γ is *autoparallel* along γ,

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \; , \tag{3.16}$$

the curve is called a *geodesic*. In a local coordinate system, this condition reads

$$\ddot{x}^k + \Gamma^k{}_{ij}\, \dot{x}^i \dot{x}^j = 0 \; . \tag{3.17}$$

In flat Euclidean space, geodesics are straight lines. Quite intuitively, the condition (3.16) generalises the concept of straight lines to manifolds.

Parallel transport and geodesics

A vector v is parallel transported along a curve γ if the geodesic equation

$$\nabla_{\dot{\gamma}} v = 0 \tag{3.18}$$

holds. Geodesics are autoparallel curves,

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \; . \tag{3.19}$$

3.2.2 Normal Coordinates

Geodesics allow the introduction of a special coordinate system in the neighbourhood of a point $p \in M$. First, given a point $p = \gamma(0)$ and a vector $\dot{\gamma}(0) \in T_pM$ from the tangent space in p, the existence and uniqueness theorems for ordinary differential equations ensure that (3.17) has a unique solution, which implies that a unique geodesic exists through p into the direction $\dot{\gamma}(0)$.

Obviously, if $\gamma_v(t)$ is a geodesic with "initial velocity" $v = \dot{\gamma}(0)$, then $\gamma_v(at)$ is also a geodesic with initial velocity $av = a\dot{\gamma}(0)$, or

$$\gamma_{av}(t) = \gamma_v(at) \ . \tag{3.20}$$

Thus, given some neighbourhood $U \subset T_pM$ of $p = \gamma(0)$, unique geodesics $\gamma(t)$ with $t \in [0, 1]$ can be constructed through p into any direction $v \in U$, i.e. such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v \in U$.

Using this, we define the *exponential map* at p,

$$\exp_p : T_pM \supset U \to M \ , \quad v \mapsto \exp_p(v) = \gamma_v(1) \ , \tag{3.21}$$

which maps any vector v from $U \subset T_pM$ into a point along the geodesic through p into direction v at distance $t = 1$.

Now, we choose a coordinate basis $\{e_i\}$ of T_pM and use the n basis vectors in the exponential mapping (3.21). Then, the neighbourhood of p can uniquely be represented by the exponential mapping along the basis vectors, $\exp_p(x^i e_i)$, and the x^i are called *normal coordinates*.

Since $\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$, the curve $\gamma_v(t)$ has the normal coordinates $x^i = tv^i$, with $v = v^i e_i$. In these coordinates, x^i is linear in t, thus $\ddot{x}^i = 0$, and (3.17) implies

$$\Gamma^k{}_{ij} v^i v^j = 0 \ , \tag{3.22}$$

and thus

$$\Gamma^k{}_{ij} + \Gamma^k{}_{ji} = 0 \ . \tag{3.23}$$

If the connection is symmetric as defined in (3.11), the connection coefficients must vanish,

$$\Gamma^k{}_{ij} = 0 \ . \tag{3.24}$$

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What could the exponential map have to do with physics, in view of the equivalence principle?

Normal coordinates

Thus, at every point $p \in M$, local coordinates can uniquely be introduced by means of the exponential map, the *normal coordinates*, in which the coefficients of a symmetric connection vanish. This will turn out to be important shortly.

3.2.3 Covariant derivative of tensor fields

Extending the concept of the covariant derivative to tensor fields, we start with a simple tensor of rank (1, 1) which is the tensor product of a vector field v and a dual vector field w,

$$t = v \otimes w \;, \tag{3.25}$$

and we require that ∇_x satisfy the Leibniz rule,

$$\nabla_x(v \otimes w) = \nabla_x v \otimes w + v \otimes \nabla_x w \;, \tag{3.26}$$

and commute with the contraction,

$$C\left[\nabla_x(v \otimes w)\right] = \nabla_x[w(v)] \;. \tag{3.27}$$

We now contract (3.26) and use (3.27) to find

$$\begin{aligned} C\left[\nabla_x(v \otimes w)\right] &= C\left(\nabla_x v \otimes w\right) + C\left(v \otimes \nabla_x w\right) \\ &= w(\nabla_x v) + (\nabla_x w)(v) \\ &= \nabla_x[w(v)] = x w(v) \;, \end{aligned} \tag{3.28}$$

where (3.1) was used in the final step (note that $w(v)$ is a real-valued function). Thus, we find an expression for the covariant derivative of a dual vector,

$$(\nabla_x w)(v) = x w(v) - w(\nabla_x v) \;. \tag{3.29}$$

Introducing the coordinate basis $\{\partial_i\}$, it is straightforward to show (and a useful exercise!) that this result can be expressed as

$$(\nabla_x w)(v) = \left(w_{j,i} - \Gamma^k{}_{ij}\, w_k\right) x^i v^j \;. \tag{3.30}$$

Specialising $x = \partial_i$, $w = \mathrm{d}x^j$ and $v = \partial_k$, hence $x^a = \delta^a_i$, $w_b = \delta^j_b$ and $v^c = \delta^c_k$, we see that this implies for the covariant derivatives of the dual basis vectors $\mathrm{d}x^j$

$$(\nabla_{\partial_i} \mathrm{d}x^j)(\partial_k) = -\Gamma^j{}_{ik} \quad \text{or} \quad \nabla_{\partial_i} \mathrm{d}x^j = -\Gamma^j{}_{ik}\, \mathrm{d}x^k \;. \tag{3.31}$$

?

Verify equations (3.30) and (3.31) yourself.

As before, we now define the covariant derivative ∇t of a tensor field as a map from the tensor fields of rank (r, s) to the tensor fields of rank $(r, s+1)$,

$$\nabla : \mathcal{T}^r_s \to \mathcal{T}^r_{s+1} \tag{3.32}$$

by setting

$$\begin{aligned} &(\nabla t)(w_1, \ldots, w_r, v_1, \ldots, v_s, v_{s+1}) \equiv \\ &(\nabla_{v_{s+1}} t)(w_1, \ldots, w_r, v_1, \ldots, v_s) \;, \end{aligned} \tag{3.33}$$

where the v_i are vector fields and the w_j dual vector fields.

We find a general expression for ∇t, with $t \in \mathcal{T}^r_s$, by taking the tensor product of t with s vector fields v_i and r dual vector fields w_j and applying ∇_x to the result, using the Leibniz rule,

$$\begin{aligned}
&\nabla_x (w_1 \otimes \ldots \otimes w_r \otimes v_1 \otimes \ldots \otimes v_s \otimes t) \\
&= (\nabla_x w_1) \otimes \ldots \otimes t + \ldots w_1 \otimes \ldots \otimes (\nabla_x v_1) \otimes \ldots \otimes t \\
&+ w_1 \otimes \ldots \otimes (\nabla_x t) \ ,
\end{aligned} \tag{3.34}$$

and then taking the total contraction, using that it commutes with the covariant derivative, which yields

$$\begin{aligned}
&\nabla_x [t(w_1, \ldots, w_r, v_1, \ldots, v_s)] \\
&= t(\nabla_x w_1, \ldots, w_r, v_1, \ldots, v_s) + \ldots + t(w_1, \ldots, w_r, v_1, \ldots, \nabla_x v_s) \\
&+ (\nabla_x t)(w_1, \ldots, w_r, v_1, \ldots, v_s) \ .
\end{aligned} \tag{3.35}$$

Therefore, the covariant derivative $\nabla_x t$ of t is

$$\begin{aligned}
&(\nabla_x t)(w_1, \ldots, w_r, v_1, \ldots, v_s) \\
&= x t(w_1, \ldots, w_r, v_1, \ldots, v_s) \\
&- t(\nabla_x w_1, \ldots, v_s) - \ldots - t(w_1, \ldots, \nabla_x v_s) \ .
\end{aligned} \tag{3.36}$$

We now work out the last expression for the covariant derivative of a tensor field in a local coordinate basis $\{\partial_i\}$ and its dual basis $\{\mathrm{d}x^j\}$ for the special case of a tensor field t of rank $(1, 1)$. The result for tensor fields of higher rank are then easily found by induction.

We can write the tensor field t as

$$t = t^i{}_j \, (\partial_i \otimes \mathrm{d}x^j) \ , \tag{3.37}$$

and the result of its application to $w_1 = \mathrm{d}x^a$ and $v_1 = \partial_b$ is

$$t(\mathrm{d}x^a, \partial_b) = t^i{}_j \, \mathrm{d}x^a(\partial_i)\mathrm{d}x^j(\partial_b) = t^a{}_b \ . \tag{3.38}$$

Therefore, we can write (3.36) as

$$\begin{aligned}
&(\nabla_x t)(\mathrm{d}x^a, \partial_b) \\
&= x^c \partial_c t^a{}_b - t^i{}_j \, (\nabla_x \mathrm{d}x^a)(\partial_i)\mathrm{d}x^j(\partial_b) - t^i{}_j \, \mathrm{d}x^a(\partial_i)\mathrm{d}x^j(\nabla_x \partial_b) \ .
\end{aligned} \tag{3.39}$$

According to (3.31), the second term on the right-hand side is

$$t^i{}_j \, \delta^j_b x^c (\nabla_{\partial_c} \mathrm{d}x^a)(\partial_i) = -x^c t^i{}_b \, \Gamma^a{}_{ci} \ , \tag{3.40}$$

while the third term is

$$t^i{}_j \, \delta^a_i x^c (\nabla_{\partial_c} \partial_b)(x^j) = x^c t^a{}_j \, \Gamma^k{}_{cb} \, \partial_k x^j = x^c t^a{}_j \, \Gamma^j{}_{cb} \ . \tag{3.41}$$

Summarising, the components of $\nabla_x t$ are

$$t^a{}_{b;c} = t^a{}_{b,c} + \Gamma^a{}_{ci} \, t^i{}_b - \Gamma^j{}_{cb} \, t^a{}_j \ , \tag{3.42}$$

showing that the covariant indices are transformed with the negative, the contravariant indices with the positive Christoffel symbols.

In particular, the covariant derivatives of tensors of rank $(0, 1)$ (dual vectors w) and of tensors of rank $(1, 0)$ (vectors v) have components

$$\begin{aligned} w_{i;k} &= w_{i,k} - \Gamma^j_{\ ki}\, w_j \,, \\ v^i_{;k} &= v^i_{,k} + \Gamma^i_{\ kj}\, v^j \,. \end{aligned} \tag{3.43}$$

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Convince yourself of the results (3.42) and (3.43).

3.3 Curvature

3.3.1 The Torsion and Curvature Tensors

Torsion

The *torsion* T maps two vector fields x and y into another vector field,

$$T : TM \times TM \to TM \,, \tag{3.44}$$

such that

$$T(x, y) = \nabla_x y - \nabla_y x - [x, y] \,. \tag{3.45}$$

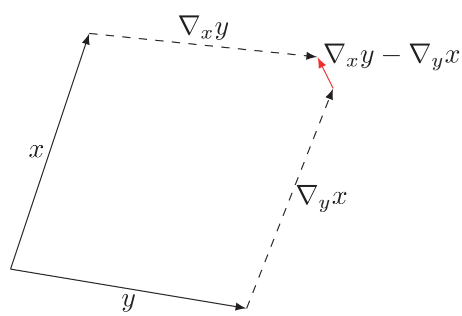

Figure 3.2 Torsion quantifies by how much parallelograms do not close.

Obviously, the torsion vanishes if and only if the connection is symmetric, cf. (3.10).

The torsion is antisymmetric,

$$T(x, y) = -T(y, x) \,, \tag{3.46}$$

and satisfies

$$T(fx, gy) = fg\, T(x, y) \tag{3.47}$$

with arbitrary C^∞ functions f and g.

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Confirm the statement (3.47).

The map

$$T^*M \times TM \times TM \to \mathbb{R} \,, \quad (w, x, y) \to w[T(x, y)] \tag{3.48}$$

with $w \in T^*M$ and $x, y \in TM$ is a tensor of rank $(1, 2)$ called the *torsion tensor* .

Caution In alternative, but equivalent representations of general relativity, the torsion does not vanish, but the curvature does. This is the teleparallel or Einstein-Cartan version of general relativity. ◀

According to (3.48), the components of the torsion tensor in the coordinate basis $\{\partial_i\}$ and its dual basis $\{\mathrm{d}x^i\}$ are

$$T^k{}_{ij} = \mathrm{d}x^k \left[T(\partial_i, \partial_j) \right] = \Gamma^k{}_{ij} - \Gamma^k{}_{ji} \; . \tag{3.49}$$

Curvature

The *curvature* $\bar{R}$ maps three vector fields x, y and v into a vector field,

$$\bar{R} : TM \times TM \times TM \to TM \; , \tag{3.50}$$

such that

$$\bar{R}(x, y)v = \nabla_x(\nabla_y v) - \nabla_y(\nabla_x v) - \nabla_{[x,y]} v \; . \tag{3.51}$$

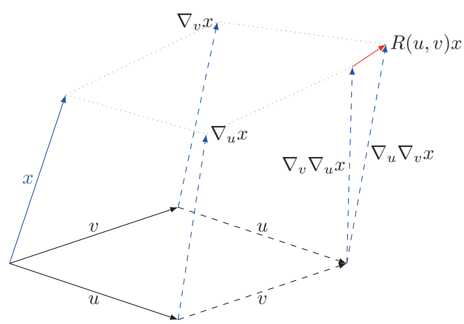

Figure 3.3 Curvature quantifies by how much second covariant derivatives do not commute.

Since the covariant derivatives ∇_x and ∇_y represent the infinitesimal parallel transports along the integral curves of the vector fields x and y, the curvature $\bar{R}$ directly quantifies the change of the vector v when it is parallel-transported around an infinitesimal, closed loop.

Exchanging x and y and using the antisymmetry of the commutator $[x, y]$, we see that $\bar{R}$ is antisymmetric in x and y,

$$\bar{R}(x, y) = -\bar{R}(y, x) \; . \tag{3.52}$$

Also, if f, g and h are C^∞ functions on M,

$$\bar{R}(fx, gy)hv = fgh \, \bar{R}(x, y)v \; , \tag{3.53}$$

which follows immediately from the defining properties (3.1) of the connection.

Curvature or Riemann tensor

Obviously, the map

$$T^*M \times TM \times TM \times TM \to \mathbb{R}\,, \quad (w, x, y, v) = w[\bar{R}(x, y)v] \quad (3.54)$$

with $w \in T^*M$ and $x, y, v \in TM$ defines a tensor of rank $(1, 3)$. It is called the *curvature tensor* or *Riemann tensor*.

To work out the components of $\bar{R}$ in a local coordinate basis $\{\partial_i\}$, we first note that

$$\begin{aligned}\nabla_{\partial_i}(\nabla_{\partial_j}\partial_k) &= \nabla_{\partial_i}\left[\nabla_{\partial_j}(\partial_k)\right] = \nabla_{\partial_i}\left(\Gamma^l{}_{jk}\,\partial_l\right) \\ &= \Gamma^l{}_{jk,i}\,\partial_l + \Gamma^l{}_{jk}\,\Gamma^m{}_{il}\,\partial_m\,. \end{aligned} \quad (3.55)$$

Interchanging i and j yields the coordinate expression for $\nabla_y(\nabla_x v)$. Since the commutator of the basis vectors vanishes, $[\partial_i, \partial_j] = 0$, the components of the curvature tensor are

$$\begin{aligned}\bar{R}^i{}_{jkl} &= \mathrm{d}x^i[\bar{R}(\partial_k, \partial_l)\partial_j] \\ &= \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \Gamma^m{}_{lj}\,\Gamma^i{}_{km} - \Gamma^m{}_{kj}\,\Gamma^i{}_{lm}\,. \end{aligned} \quad (3.56)$$

?

Verify the statement (3.53) and the coordinate representation (3.56).

Ricci tensor

The *Ricci tensor* R is the contraction $C^1_3\bar{R}$ of the curvature tensor $\bar{R}$. Its components are

$$R_{jl} = \bar{R}^i{}_{jil} = \Gamma^i{}_{lj,i} - \Gamma^i{}_{ij,l} + \Gamma^m{}_{lj}\,\Gamma^i{}_{im} - \Gamma^m{}_{ij}\,\Gamma^i{}_{lm}\,. \quad (3.57)$$

3.3.2 The Bianchi Identities

The curvature and the torsion together satisfy the two Bianchi identities.

Bianchi identities

The *first Bianchi identity* is

$$\sum_{\text{cyclic}}\left[\bar{R}(x, y)z\right] = \sum_{\text{cyclic}}\{T[T(x, y), z] + (\nabla_x T)(y, z)\}\,, \quad (3.58)$$

where the sums extend over all cyclic permutations of the vectors x, y and z. The *second Bianchi identity* is

$$\sum_{\text{cyclic}}\left\{(\nabla_x\bar{R})(y, z) + \bar{R}[T(x, y), z]\right\} = 0\,. \quad (3.59)$$

They are important because they define symmetry relations of the curvature and the curvature tensor. In particular, for a symmetric connection, $T = 0$ and the Bianchi identities reduce to

$$\sum_{\text{cyclic}}\left[\bar{R}(x, y)z\right] = 0\,, \quad \sum_{\text{cyclic}}(\nabla_x\bar{R})(y, z) = 0\,. \quad (3.60)$$

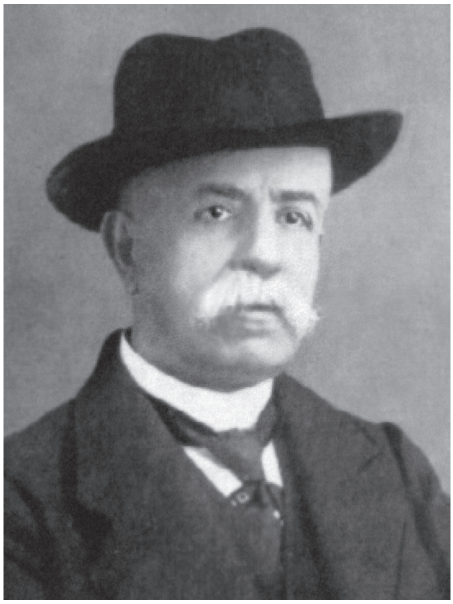

Figure 3.4 Gregorio Ricci-Curbastro (1853–1925), Italian mathematician. Source: Wikipedia

Before we go on, we have to clarify the meaning of the covariant derivatives of the torsion and the curvature. We have seen that T defines a tensor field $\tilde{T}$ of rank $(1, 2)$. Given a dual vector field $w \in T^*M$, we define the covariant derivative of the torsion T such that

$$w\,[\nabla_v T(x, y)] = (\nabla_v \tilde{T})(w, x, y)\ . \tag{3.61}$$

Using (3.36), we can write the right-hand side as

$$\begin{aligned}(\nabla_v \tilde{T})(w, x, y) = v\tilde{T}(w, x, y) - \tilde{T}(\nabla_v w, x, y)\\ - \tilde{T}(w, \nabla_v x, y) - \tilde{T}(w, x, \nabla_v y)\ .\end{aligned} \tag{3.62}$$

The first two terms on the right-hand side can be combined using (3.29),

$$\begin{aligned}\tilde{T}(\nabla_v w, x, y) &= \nabla_v w[T(x, y)] = vw[T(x, y)] - w[\nabla_v T(x, y)]\\ &= v\tilde{T}(w, x, y) - w[\nabla_v T(x, y)]\ ,\end{aligned} \tag{3.63}$$

which yields

$$(\nabla_v \tilde{T})(w, x, y) = w[\nabla_v T(x, y)] - \tilde{T}(w, \nabla_v x, y) - \tilde{T}(w, x, \nabla_v y) \tag{3.64}$$

or, dropping the common argument w from all terms,

$$(\nabla_v T)(x, y) = \nabla_v[T(x, y)] - T(\nabla_v x, y) - T(x, \nabla_v y)\ . \tag{3.65}$$

Similarly, we find that

$$(\nabla_v \bar{R})(x, y) = \nabla_v[\bar{R}(x, y)] - \bar{R}(\nabla_v x, y) - \bar{R}(x, \nabla_v y) - \bar{R}(x, y)\nabla_v\ . \tag{3.66}$$

For symmetric connections, $T = 0$, the first Bianchi identity is easily proven. Its left-hand side reads

$$\begin{aligned}
&\nabla_x\nabla_y z - \nabla_y\nabla_x z + \nabla_y\nabla_z x - \nabla_z\nabla_y x + \nabla_z\nabla_x y - \nabla_x\nabla_z y \\
&- \nabla_{[x,y]}z - \nabla_{[y,z]}x - \nabla_{[z,x]}y \\
&= \nabla_x(\nabla_y z - \nabla_z y) + \nabla_y(\nabla_z x - \nabla_x z) + \nabla_z(\nabla_x y - \nabla_y x) \\
&- \nabla_{[x,y]}z - \nabla_{[y,z]}x - \nabla_{[z,x]}y \\
&= \nabla_x[y,z] - \nabla_{[y,z]}x + \nabla_y[z,x] - \nabla_{[z,x]}y + \nabla_z[x,y] - \nabla_{[x,y]}z \\
&= [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \ , \qquad (3.67)
\end{aligned}$$

where we have used the relation (3.10) and the Jacobi identity (2.33).

?

Convince yourself of the result (3.67).

3.4 Riemannian connections

3.4.1 Definition and Uniqueness

Up to now, the affine connection ∇ has not yet been uniquely defined. We shall now see that a unique connection can be introduced on each pseudo-Riemannian manifold (M, g).

A connection is called *metric* if the parallel transport along any smooth curve γ in M leaves the inner product of two autoparallel vector fields x and y unchanged. This is the case if and only if the covariant derivative ∇ of g vanishes,

$$\nabla g = 0 \ . \qquad (3.68)$$

Because of (3.36), this condition is equivalent to the *Ricci identity*

$$xg(y,z) = g(\nabla_x y, z) + g(y, \nabla_x z) \ , \qquad (3.69)$$

where x, y, z are vector fields.

Caution In a third, equivalent representation of general relativity, curvature and torsion both vanish, but the metricity (3.68) is given up. ◂

It can now be shown that a unique connection ∇ can be introduced on each pseudo-Riemannian manifold such that ∇ is symmetric or torsion-free, and metric, i.e. $\nabla g = 0$. Such a connection is called the *Riemannian* or *Levi-Civita connection*.

Suppose first that such a connection exists, then (3.69) and the symmetry of ∇ allow us to write

$$xg(y,z) = g(\nabla_y x, z) + g([x,y], z) + g(y, \nabla_x z) \ . \qquad (3.70)$$

Taking the cyclic permutations of this equation, summing the second and the third and subtracting the first (3.70), we obtain the *Koszul formula*

$$\begin{aligned}
2g(\nabla_z y, x) = &-xg(y,z) + yg(z,x) + zg(x,y) \qquad (3.71) \\
&+ g([x,y],z) - g([y,z],x) - g([z,x],y) \ .
\end{aligned}$$

Figure 3.5 Tullio Levi-Civita (1873–1941), Italian mathematician. Source: Wikipedia

Since the right-hand side is independent of ∇, and g is non-degenerate, this result implies the *uniqueness* of ∇. The *existence* of an affine, symmetric and metric connection can be proven by explicit construction.

The Christoffel symbols for a Riemannian connection can now be determined specialising the Koszul formula (3.71) to the basis vectors $\{\partial_i\}$ of a local coordinate system. We choose $x = \partial_k$, $y = \partial_j$ and $z = \partial_i$ and use that their commutator vanishes, $[\partial_i, \partial_j] = 0$, and that $g(\partial_i, \partial_j) = g_{ij}$.

Then, (3.71) implies

$$2g(\nabla_{\partial_i}\partial_j, \partial_k) = -\partial_k g_{ij} + \partial_j g_{ik} + \partial_i g_{jk} \,, \tag{3.72}$$

thus

$$g_{mk}\Gamma^m{}_{ij} = \frac{1}{2}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) \,. \tag{3.73}$$

If (g^{ij}) denotes the matrix inverse to (g_{ij}), we can write

$$\Gamma^l{}_{ij} = \frac{1}{2}g^{lk}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) \,. \tag{3.74}$$

Levi-Civita-connection

On a pseudo-Riemannian manifold (M, g) with metric g, a unique connection exists which is symmetric and metric, $\nabla g = 0$. It is called *Levi-Civita connection*.

3.4.2 Symmetries. The Einstein Tensor

In addition to (3.52), the curvature tensor of a Riemannian connection has the following symmetry properties:

$$\langle \bar{R}(x,y)v,w\rangle = -\langle \bar{R}(x,y)w,v\rangle\ , \quad \langle \bar{R}(x,y)v,w\rangle = \langle \bar{R}(v,w)x,y\rangle\ . \tag{3.75}$$

The first of these relations is easily seen noting that the antisymmetry is equivalent to

$$\langle \bar{R}(x,y)v,v\rangle = 0\ . \tag{3.76}$$

From the definition of $\bar{R}$ and the antisymmetry (3.52), we first have

$$\langle v,\bar{R}(x,y)v\rangle = \langle v,\nabla_x\nabla_y v - \nabla_y\nabla_x v - \nabla_{[x,y]}v\rangle\ . \tag{3.77}$$

Replacing y by $\nabla_y v$ and z by v, the Ricci identity (3.69) allows us to write

$$\langle v,\nabla_x\nabla_y v\rangle = x\langle\nabla_y v,v\rangle - \langle\nabla_y v,\nabla_x v\rangle \tag{3.78}$$

and, replacing x by y and both y and z by v,

$$\langle\nabla_y v,v\rangle = \frac{1}{2}\,y\langle v,v\rangle\ . \tag{3.79}$$

Hence, the first two terms on the right-hand side of (3.77) yield

$$\begin{aligned}\langle v,\nabla_x\nabla_y v - \nabla_y\nabla_x v\rangle &= \langle v,x\langle\nabla_y v,v\rangle - y\langle\nabla_x v,v\rangle\rangle\\ &= \frac{1}{2}\langle v,xy\langle v,v\rangle - yx\langle v,v\rangle\rangle\\ &= \frac{1}{2}\langle v,[x,y]\langle v,v\rangle\rangle\ .\end{aligned} \tag{3.80}$$

By (3.79), this is the negative of the third term on the right-hand side of (3.77), which proves (3.76).

The symmetries (3.52) and (3.75) imply

$$\bar{R}_{ijkl} = -\bar{R}_{jikl} = -\bar{R}_{ijlk}\ , \quad \bar{R}_{ijkl} = \bar{R}_{klij}\ , \tag{3.81}$$

where $\bar{R}_{ijkl} \equiv g_{im}\bar{R}^m{}_{jkl}$. Of the $4^4 = 256$ components of the Riemann tensor in four dimensions, the first symmetry relation (3.81) leaves $6\times 6 = 36$ independent components, while the second symmetry relation (3.81) reduces their number to $6+5+4+3+2+1 = 21$.

In a coordinate basis, the Bianchi identities (3.60) for the curvature tensor of a Riemannian connection read

$$\sum_{(jkl)}\bar{R}^i{}_{jkl} = 0\ , \quad \sum_{(klm)}\bar{R}^i{}_{jkl;m} = 0\ , \tag{3.82}$$

where (jkl) denotes the cyclic permutations of the indices enclosed in parentheses. In four dimensions, the first Bianchi identity establishes one further relation between the components of the Riemann tensor which is not covered yet by the symmetry relations (3.81), namely

$$\bar{R}_{0123} + \bar{R}_{0231} + \bar{R}_{0312} = 0 \,, \tag{3.83}$$

and thus leaves 20 independent components of the Riemann tensor. These are

$$\begin{pmatrix} \bar{R}_{0101} & \bar{R}_{0102} & \bar{R}_{0103} & \bar{R}_{0112} & \bar{R}_{0113} & \\ & \bar{R}_{0202} & \bar{R}_{0203} & \bar{R}_{0212} & \bar{R}_{0213} & \bar{R}_{0223} \\ & & \bar{R}_{0303} & \bar{R}_{0312} & \bar{R}_{0313} & \bar{R}_{0323} \\ & & & \bar{R}_{1212} & \bar{R}_{1213} & \bar{R}_{1223} \\ & & & & \bar{R}_{1313} & \bar{R}_{1323} \\ & & & & & \bar{R}_{2323} \end{pmatrix} , \tag{3.84}$$

where $\bar{R}_{0123}$ is determined by (3.83).

Using the symmetries (3.81) and the second Bianchi identity from (3.82), we can obtain an important result. We first contract

$$\bar{R}^i{}_{jkl;m} + \bar{R}^i{}_{jlm;k} + \bar{R}^i{}_{jmk;l} = 0 \tag{3.85}$$

by multiplying with δ^k_i and use the symmetry relations (3.81) to find

$$R_{jl;m} + \bar{R}_j{}^i{}_{ml;i} - R_{jm;l} = 0 \tag{3.86}$$

for the Ricci tensor. Next, we contract again by multiplying with g^{jm}, which yields

$$R^m{}_{l;m} + R^i{}_{l;i} - \mathcal{R}_{;l} = 0 \,, \tag{3.87}$$

where R_{ij} are the components of the Ricci tensor and $\mathcal{R} = R^i{}_i$ is the *Ricci scalar* or the *scalar curvature*. Renaming dummy indices, the last equation can be brought into the form

$$\left(R^i{}_j - \frac{\mathcal{R}}{2}\delta^i_j \right)_{;i} = 0 \,, \tag{3.88}$$

which is the *contracted Bianchi identity*. Moreover, the Ricci tensor can easily be shown to be symmetric,

$$R_{ij} = R_{ji} \,. \tag{3.89}$$

We finally introduce the symmetric *Einstein tensor* by

$$G_{ij} \equiv R_{ij} - \frac{\mathcal{R}}{2} g_{ij} \,, \tag{3.90}$$

which has vanishing divergence because of the contracted Bianchi identity,

$$G^i{}_{j;i} = 0 \,. \tag{3.91}$$

Riemann, Ricci, and Einstein tensors

The Ricci tensor is the only non-vanishing contraction of the Riemann tensor. Its components are

$$R_{ij} = \bar{R}^{a}{}_{iaj} \ . \tag{3.92}$$

The Ricci scalar is the only contraction (the trace) of the Ricci tensor, $\mathcal{R} = \mathrm{Tr}\, R$. The Ricci tensor, the Ricci scalar and the metric together define the *Einstein tensor*, which has the components

$$G_{ij} = R_{ij} - \frac{\mathcal{R}}{2}\, g_{ij} \tag{3.93}$$

and is divergence-free, $G^{i}{}_{j;i} = 0$.