Chapter 2

Differential Geometry I

2.1 Differentiable manifolds

By the preceding discussion of how a theory of gravity may be constructed which is compatible with special relativity, we are led to the concept of a spacetime which "looks like" Minkowskian spacetime locally, but may globally be curved. This concept is cast into a mathematically precise form by the introduction of a *manifold*.

Manifolds

An n-dimensional *manifold* M is a topological Hausdorff space with a countable base, which is locally homeomorphic to $\mathbb{R}^n$. This means that for every point $p \in M$, an open neighbourhood U of p exists together with a homeomorphism h which maps U onto an open subset U' of $\mathbb{R}^n$,

$$h : U \to U' \ . \tag{2.1}$$

Caution A topological space is a set M together with a collection $\mathcal{T}$ of open subsets $T_i \subset M$ with the properties (i) $\emptyset \in \mathcal{T}$ and $M \in T$; (ii) $\cap_{i=1}^{n} T_i \in T$ for any finite n; (iii) $\cup_{i=1}^{n} T_i \in T$ for any n. In a Hausdorff space, any two points $x, y \in M$ with $x \neq y$ can be surrounded by disjoint neighbourhoods. ◂

A trivial example for an n-dimensional manifold is the $\mathbb{R}^n$ itself, on which h may be the identity map id. Thus, h is a specialisation of a map ϕ from one manifold M to another manifold N, $\phi : M \to N$.

The homeomorphism h is called a *chart* or a *coordinate system* in the language of physics. U is the *domain* or the *coordinate neighbourhood* of the chart. The image $h(p)$ of a point $p \in M$ under the chart h is expressed by the n real numbers $(x^1, \dots x^n)$, the coordinates of p in the chart h.

Caution A homeomorphism (not to be confused with a homomorphism) is a bijective, continuous map whose inverse is also continuous. ◂

A set of charts h_α is called an *atlas* of M if the domains of the charts cover M completely.

Charts and atlases

Charts are homeomorphisms from an n-dimensional manifold M into $\mathbb{R}^n$. A an *atlas* is a collection of charts whose domains cover M completely.

Example: The sphere as a manifold

An example for a manifold is the n-sphere S^n, for which the two-sphere S^2 is a particular specialisation. It cannot be continuously mapped to $\mathbb{R}^2$, but pieces of it can.
We can embed the two-sphere into $\mathbb{R}^3$ and describe it as the point set

$$S^2 = \left\{(x^1, x^2, x^3) \in \mathbb{R}^3 \,\middle|\, (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\right\} ; \tag{2.2}$$

then, the six half-spheres $U_i^\pm$ defined by

$$U_i^\pm = \left\{(x^1, x^2, x^3) \in S^2 \,\middle|\, \pm x^i > 0\right\} \tag{2.3}$$

can be considered as domains of maps whose union covers S^2 completely, and the charts can be the projections of the half-spheres onto open disks

$$D_{ij} = \left\{(x^i, x^j) \in \mathbb{R}^2 \,\middle|\, (x^i)^2 + (x^j)^2 < 1\right\} , \tag{2.4}$$

such as

$$f_1^+ : U_1^+ \to D_{23} , \quad f_1^+(x^1, x^2, x^3) = (x^2, x^3) . \tag{2.5}$$

Thus, the six charts $f_i^\pm$, $i \in \{1, 2, 3\}$, together form an atlas of the two-sphere. See Fig. 2.1 for an illustration. ◂

Let now h_α and h_β be two charts, and $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta \neq \emptyset$ be the intersection of their domains. Then, the composition of charts $h_\beta \circ h_\alpha^{-1}$ exists and defines a map between two open sets in $\mathbb{R}^n$ which describes the *change of coordinates* or a *coordinate transform* on the intersection of domains U_α and U_β. An atlas of a manifold is called *differentiable* if the coordinate changes between all its charts are differentiable. A manifold, combined with a differentiable atlas, is called a *differentiable manifold*.

Using charts, it is possible to define *differentiable maps between manifolds*. Let M and N be differentiable manifolds of dimension m and n, respectively, and $\phi : M \to N$ be a map from one manifold to the other. Introduce further two charts $h : M \to M' \subset \mathbb{R}^m$ and $k : N \to N' \subset \mathbb{R}^n$ whose domains cover a point $p \in M$ and its image $\phi(p) \in N$. Then, the combination $k \circ \phi \circ h^{-1}$ is a map from the domain M' to the domain N', for which it is clear from advanced calculus what differentiability means. Unless stated otherwise, we shall generally assume that coordinate changes and maps between manifolds are C^∞, i.e. their derivatives of all orders exist and are continuous.

Differentiable atlases and maps

An atlas is differentiable if all of its coordinate changes are differentiable. Differentiable maps between manifolds are defined by means of differentiable charts.

Example: A differentiable atlas for the 2-sphere

To construct an example for a differentiable atlas, we return to the two-sphere S^2 and the atlas of the six projection charts $\mathcal{A} = \{f_1^\pm, f_2^\pm, f_3^\pm\}$ described above and investigate whether it is differentiable. For doing so, we arbitrarily pick the charts f_3^+ and f_1^+, whose domains are the "northern" and "eastern" half-spheres, respectively, which overlap on the "north-eastern" quarter-sphere. Let therefore $p = (p^1, p^2, p^3)$ be a point in the domain overlap, then

$$\begin{aligned} f_3^+(p) &= (p^1, p^2)\,, \quad f_1^+(p) = (p^2, p^3)\,, \\ (f_3^+)^{-1}(p^1, p^2) &= \left(p^1, p^2, \sqrt{1-(p^1)^2-(p^2)^2}\right)\,, \\ f_1^+ \circ (f_3^+)^{-1}(p^1, p^2) &= \left(p^2, \sqrt{1-(p^1)^2-(p^2)^2}\right)\,, \end{aligned} \tag{2.6}$$

which is obviously differentiable. The same applies to all other coordinate changes between charts of $\mathcal{A}$, and thus S^2 is a differentiable manifold.

As an example for a differentiable map, let $\phi : S^2 \to S^2$ be a map which rotates the sphere by 45° around its z axis. Let us further choose a point p on the positive quadrant of S^2 in which all coordinates are positive. We can combine ϕ with the charts f_3^+ and f_1^+ to define the map

$$\left(f_1^+ \circ \phi \circ (f_3^+)^{-1}\right)(p^1, p^2) = \left(\frac{p^1 + p^2}{\sqrt{2}}, \sqrt{1-(p^1)^2-(p^2)^2}\right)\,, \tag{2.7}$$

which is also evidently differentiable. ◀

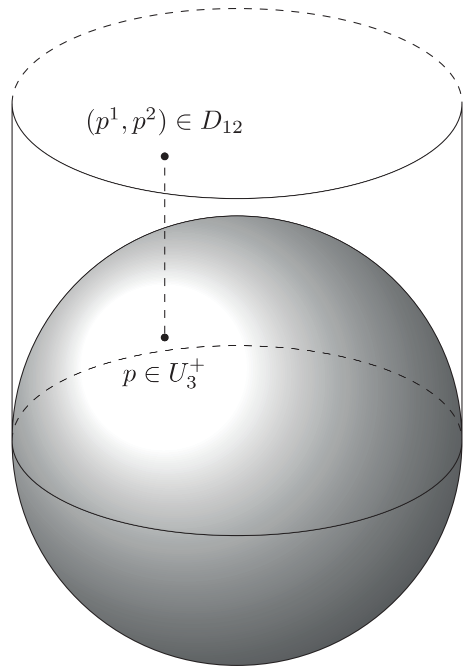

Figure 2.1 Example for a chart, explained in the text: the point p on the half-sphere U_3^+ the two-sphere is projected into the domain $D_{12} \subset \mathbb{R}^2$.

Finally, we introduce *product manifolds* in a straightforward way. Given two differentiable manifolds M and N of dimension m and n, respectively, we can turn the product space $M \times N$ consisting of all pairs (p, q) with $p \in M$ and $q \in N$ into an $(m + n)$-dimensional manifold as follows: if $h : M \to M'$ and $k : N \to N'$ are charts of M and N, a chart $h \times k$ can be defined on $M \times N$ such that

$$h \times k : M \times N \to M' \times N' \ , \quad (h \times k)(p, q) = \big[h(p), k(q)\big] \ . \tag{2.8}$$

In other words, pairs of points from the product manifold are mapped to pairs of points from the two open subsets $M' \subset \mathbb{R}^m$ and $N' \subset \mathbb{R}^n$.

2.2 The tangent space

2.2.1 Tangent vectors

Now we have essentially introduced ways how to construct local coordinate systems, or charts, on a manifold, how to change between them, and how to use charts to define what differentiable functions on the manifold are. We now proceed to see how vectors can be introduced on a manifold.

Example: Product manifold

Many manifolds which are relevant in General Relativity can be expressed as product manifolds of the Euclidean space $\mathbb{R}^m$ with spheres S^n. For example, we can construct the product manifold $\mathbb{R} \times S^2$ composed of the real line and the two-sphere. Points on this product manifold can be mapped onto $\mathbb{R} \times \mathbb{R}^2$ for instance using the identical chart id on $\mathbb{R}$ and the chart f_3^+ on the "northern" half-sphere of S^2,

$$(\mathrm{id} \times f_3^+) : \mathbb{R} \times S^2 \to \mathbb{R} \times D_{12} , \quad (p, q) \to (p, q^2, q^3) . \tag{2.9}$$

◀

Recall the definition of a *vector space*: a set V, combined with a *field* (*Körper* in German) F, an addition,

$$+ : V \times V \to V , \quad (v, w) \mapsto v + w , \tag{2.10}$$

and a multiplication,

$$\cdot : F \times V \to V , \quad (\lambda, v) \mapsto \lambda v , \tag{2.11}$$

is an F-vector space if V is an Abelian group under the addition + and the multiplication is distributive and associative. In other words, a vector space is a set of elements which can be added and multiplied with scalars (i.e. numbers from the field F).

?

What are the defining properties of a field?

On a curved manifold, this vector space structure is lost because it is not clear how vectors at different points on the manifold should be added. However, it still makes sense to define vectors locally in terms of infinitesimal displacements within a sufficiently small neighbourhood of a point p, which are "tangential" to the manifold at p.

This leads to the concept of the *tangential space* of a manifold, whose elements are tangential vectors, or directional derivatives of functions. We denote by $\mathcal{F}$ the set of C^∞ functions f from the manifold into $\mathbb{R}$.

Example: Functions on a manifold

Examples for functions on the manifold $S^2 \to \mathbb{R}$ could be the average temperature on Earth or the height of the Earth's surface above sea level. ◀

Tangent space

Generally, the tangent space T_pM of a differentiable manifold M at a point p is the set of *derivations* of $\mathcal{F}(p)$. A derivation v is a map from $\mathcal{F}(p)$ into $\mathbb{R}$,

$$v : \mathcal{F}(p) \to \mathbb{R} \ , \tag{2.12}$$

which is linear,

$$v(\lambda f + \mu g) = \lambda v(f) + \mu v(g) \tag{2.13}$$

for $f, g \in \mathcal{F}(p)$ and $\lambda, \mu \in \mathbb{R}$, and satisfies the product rule (or Leibniz rule)

$$v(fg) = v(f)g + fv(g) \ . \tag{2.14}$$

See Fig. 2.2 for an illustration of the tangent space to a 2-sphere.

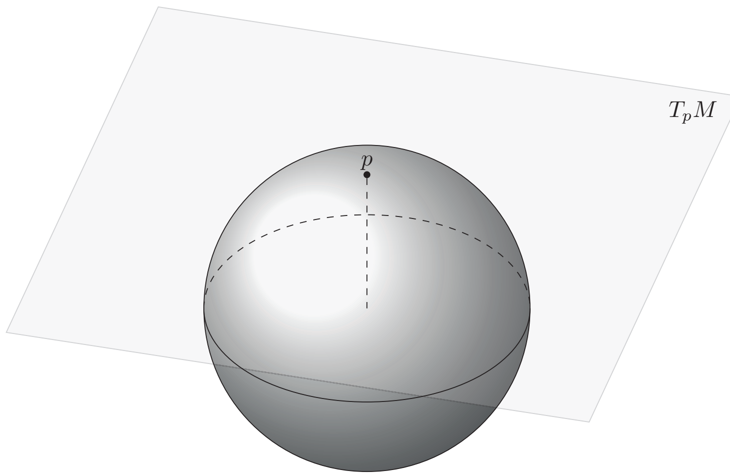

Figure 2.2 Illustration of the tangent space T_pM at point p on the 2-sphere.

Note that this definition immediately implies that the derivation of a constant vanishes: let $h \in \mathcal{F}$ be a constant function, $h(p) = c$ for all $p \in M$, then $v(h^2) = 2cv(h)$ from (2.14) and $v(h^2) = v(ch) = cv(h)$ from (2.13), which is possible only if $v(h) = 0$.

Together with the real numbers $\mathbb{R}$ and their addition and multiplication laws, T_pM does indeed have the structure of a vector space, with

$$(v + w)(f) = v(f) + w(f) \quad \text{and} \quad (\lambda v)(f) = \lambda v(f) \tag{2.15}$$

for $v, w \in T_pM$, $f \in \mathcal{F}$ and $\lambda \in \mathbb{R}$.

2.2.2 Coordinate basis

We now construct a basis for the vector space T_pM, i.e. we provide a complete set $\{e_i\}$ of linearly independent basis vectors. For doing so, let $h : U \to U' \subset \mathbb{R}^n$ be a chart with $p \in U$ and $f \in \mathcal{F}(p)$ a function. Then, $f \circ h^{-1} : U' \to \mathbb{R}$ is C^∞ by definition, and we introduce n vectors $e_i \in T_pM$, $1 \leq i \leq n$, by

$$e_i(f) := \frac{\partial}{\partial x^i}(f \circ h^{-1})\bigg|_{h(p)} , \tag{2.16}$$

where x^i are the usual cartesian coordinates of $\mathbb{R}^n$.

The function $(f \circ h^{-1})$ is applied to the image $h(p) \in \mathbb{R}^n$ of p under the chart h, i.e. $(f \circ h^{-1})$ "carries" the function f from the manifold M to the locally isomorphic manifold $\mathbb{R}^n$.

To show that these vectors span T_pM, we first state that for any C^∞ function $F : U' \to \mathbb{R}$ defined on an open neighbourhood U' of the origin of $\mathbb{R}^n$, there exist n C^∞ functions $H_i : U' \to \mathbb{R}$ such that

$$F(x) = F(0) + \sum_{i=1}^{n} x^i H_i(x) . \tag{2.17}$$

Note the equality! This is *not* a Taylor expansion. This is easily seen using the identity

$$\begin{aligned} F(x) - F(0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} F(tx^1, \dots, tx^n)\mathrm{d}t \\ &= \sum_{i=1}^{n} x^i \int_0^1 D_i F(tx^1, \dots, tx^n)\mathrm{d}t , \end{aligned} \tag{2.18}$$

where D_i is the partial derivative with respect to the i-th argument of F. Thus, it suffices to set

$$H_i(x) = \int_0^1 D_i F(tx^1, \dots, tx^n)\mathrm{d}t \tag{2.19}$$

to prove (2.17). For $x = 0$ in particular, we find

$$H_i(0) = \int_0^1 \frac{\partial F}{\partial x^i}\bigg|_0 \mathrm{d}t = \frac{\partial F}{\partial x^i}\bigg|_0 . \tag{2.20}$$

Now we substitute $F = f \circ h^{-1}$ and choose a chart $h : U \to U'$ such that $h(q) = x$ and $h(p) = 0$, i.e. $q = h^{-1}(x)$. Then, we first obtain from (2.17)

$$f(q) = f(p) + \sum_{i=1}^{n} (x^i \circ h)(q)\, (H_i \circ h)(q) , \tag{2.21}$$

and from (2.20)

$$H_i(0) = (H_i \circ h)(p) = \frac{\partial}{\partial x^i}(f \circ h^{-1})\bigg|_{h(p)} = e_i(f) . \tag{2.22}$$

Next, we apply a tangent vector $v \in T_pM$ to (2.21),

$$v(f) = v[f(p)] + \sum_{i=1}^{n} \left[v(x^i \circ h)\,(H_i \circ h)|_p + (x^i \circ h)|_p\, v(H_i \circ h) \right]$$
$$= \sum_{i=1}^{n} v(x^i \circ h)\, e_i(f)\ , \tag{2.23}$$

where we have used that v applied to the constant $f(p)$ vanishes, that $(x^i \circ h)(p) = 0$ and that $H_i(0) = e_i(f)$ according to (2.22). Thus, setting $v^i = v(x^i \circ h)$, we find that any $v \in T_pM$ can be written as a linear combination of the basis vectors e_i. This also demonstrates that the dimension of the tangent space T_pM equals that of the manifold itself.

Coordinate basis of T_pM

The basis $\{e_i\}$, which is often simply denoted as $\{\partial/\partial x^i\}$ or $\{\partial_i\}$, is called a *coordinate basis* of T_pM. Vectors $v \in T_pM$ can thus be written as

$$v = v^i e_i = v^i \partial_i\ . \tag{2.24}$$

If we choose a different chart h' instead of h, we obtain of course a different coordinate basis $\{e'_i\}$. Denoting the i-th coordinate of the map $h' \circ h^{-1}$ with x', the chain rule applied to $f \circ h^{-1} = (f \circ h'^{-1}) \circ (h' \circ h^{-1})$ yields

$$e_i = \sum_{j=1}^{n} \frac{\partial x'^j}{\partial x^i}\, e'_j =: J'^j_i\, e'_j\ . \tag{2.25}$$

which shows that the two different coordinate bases are related by the *Jacobian matrix* of the coordinate change, which has the elements $J'^j_i = \partial x'^j/\partial x^i$. Its inverse has the elements $J^i_j = \partial x^i/\partial x'^j$.

This relates the present definition of a tangent vector to the traditional definition of a vector as a quantity whose components transform as

$$v'^i = v(x^i \circ h') = \sum_{j=1}^{n} v^j e_j(x^i \circ h') = \sum_{j=1}^{n} \frac{\partial x'^i}{\partial x^j}\, v^j = J'^i_j\, v^j\ . \tag{2.26}$$

Repeating the construction of a tangent space at another point $q \in M$, we obtain a tangent space T_qM which cannot be identified in any way with the tangent space T_pM given only the structure of a differentiable manifold that we have so far.

Consequently, a *vector field* is defined as a map $v : p \mapsto v_p$ which assigns a tangent vector $v_p \in T_pM$ to every point $p \in M$. If we apply a vector field v to a C^∞ function f, its result $(v(f))(p)$ is a number for each point p. The vector field is called smooth if the function $(v(f))(p)$ is also smooth.

Since we can write $v = v^i \partial_i$ with components v^i in a local coordinate neighbourhood, the function $v(f)$ is

$$(v(f))(p) = v^i(p) \partial_i f(p)\ , \tag{2.27}$$

and thus it is called the *derivative of f with respect to the vector field v*.

2.2.3 Curves and infinitesimal transformations

We can give a geometrical meaning to tangent vectors as "infinitesimal displacements" on the manifold. First, we define a *curve* on M through $p \in M$ as a map from an open interval $I \subset \mathbb{R}$ with $0 \in I$ into M,

$$\gamma : I \to M \,, \tag{2.28}$$

such that $\gamma(0) = p$.

Next, we introduce a *one-parameter group of diffeomorphisms* γ_t as a C^∞ map,

$$\gamma_t : \mathbb{R} \times M \to M \,, \tag{2.29}$$

such that for a fixed $t \in \mathbb{R}$, $\gamma_t : M \to M$ is a diffeomorphism and, for all $t, s \in \mathbb{R}$, $\gamma_t \circ \gamma_s = \gamma_{t+s}$. Note the latter requirement implies that γ_0 is the identity map.

Caution A diffeomorphism is a continuously differentiable, bijective map with a continously differentiable inverse. ◂

For a fixed t, γ_t maps points $p \in M$ to other points $q \in M$ in a differentiable way. As an example on the two-sphere S^2, γ_t could be the map which rotates the sphere about an (arbitrary) z axis by an angle parameterised by t, such that γ_0 is the rotation by zero degrees.

We can now associate a vector field v to γ_t as follows: For a fixed point $p \in M$, the map $\gamma_t : \mathbb{R} \to M$ is a curve as defined above which passes through p at $t = 0$. This curve is called an *orbit of γ_t*. Then, we assign to p the tangent vector v_p to this curve at $t = 0$. Repeating this operation for all points $p \in M$ defines a vector field v on M which is associated with γ_t and can be considered as the *infinitesimal generator* of the transformations γ_t.

Example: Transformation of S^2

In our example on S^2, we fix a point p on the sphere whose orbit under the map γ_t is a part of the "latitude circle" through p. The tangent vector to this curve in p defines the local "direction of motion" under the rotation expressed by γ_t. Applying this to all points $p \in S^2$ defines a vector field v on S^2. ◂

Conversely, given a vector field v on M, we can construct curves through all points $p \in M$ whose tangent vectors are v_p. This is most easily seen in a local coordinate neighbourhood, $h(p) = (x^1, \ldots x^n)$, in which the curves are the unique solutions of the system

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = v^i(x^1, \ldots x^n) \tag{2.30}$$

of ordinary, first-order differential equations. Thus, tangent vectors can be identified with infinitesimal transformations of the manifold.

Given two vector fields v, w and a function f on M, we can define the *commutator* of the two fields as

$$[v, w](f) = vw(f) - wv(f) . \tag{2.31}$$

In coordinates, we can write $v = v^i \partial_i$ and $w = w^j \partial_j$, and the commutator can be written as

$$[v, w] = \left(v^i \partial_i w^j - w^i \partial_i v^j\right) \partial_j . \tag{2.32}$$

It can easily be shown to have the following properties (where v, w, x are vector fields and f, g are functions on M):

$$\begin{aligned} [v + w, x] &= [v, x] + [w, x] \\ [v, w] &= -[w, v] \\ [fv, gw] &= fg[v, w] + fv(g)w - gw(f)v \\ [v, [w, x]] + [x, [v, w]] + [w, [x, v]] &= 0 , \end{aligned} \tag{2.33}$$

where the latter equation is called the *Jacobi identity*.

2.3 Dual vectors and tensors

2.3.1 Dual space

We had introduced the tangent space TM as the set of derivations of functions $\mathcal{F}$ on M, which were certain linear maps from $\mathcal{F}$ into $\mathbb{R}$. We now introduce the *dual vector space* T^*M to TM as the set of linear maps

$$T^*M : TM \to \mathbb{R} \tag{2.34}$$

from TM into $\mathbb{R}$. Defining addition of elements of T^*M and their multiplication with scalars in the obvious way, T^*M obtains the structure of a vector space; the elements of T^*M are called *dual vectors*.

Let now f be a C^∞ function on M and $v \in TM$ an arbitrary tangent vector. Then, we define the *differential* of f by

$$\mathrm{d}f : TM \to \mathbb{R} , \quad \mathrm{d}f(v) = v(f) . \tag{2.35}$$

It is obvious that, by definition of the dual space T^*M, $\mathrm{d}f$ is an element of T^*M and thus a dual vector. Choosing a coordinate representation, we see that

$$\mathrm{d}f(v) = v^i \partial_i f . \tag{2.36}$$

Specifically letting $f = x^i$ be the i-th coordinate function, we see that

$$\mathrm{d}x^i(\partial_j) = \partial_j x^i = \delta^i_j , \tag{2.37}$$

which shows that the n-tuple $\{e^{*i}\} = \{\mathrm{d}x^i\}$ forms a basis of T^*M, which is called the *dual basis* to the basis $\{e_i\} = \{\partial_i\}$ of the tangent space TM.

Dual vectors

Dual vectors map vectors to the real numbers. If $\{\partial_i\}$ is a coordinate basis of TM, the dual basis of T^*M is given by the coordinate differentials $\{\mathrm{d}x^i\}$. Dual vectors can thus be written as

$$w = w_i \mathrm{d}x^i \; . \tag{2.38}$$

Starting the same operation leading from TM to the dual space T^*M with T^*M instead, we arrive at the double-dual vector space $T^{**}M$ as the vector space of all linear maps from $T^*M \to \mathbb{R}$. It can be shown that $T^{**}M$ is isomorphic to TM and can thus be identified with TM.

2.3.2 Tensors

Tensors T of rank (r, s) can now be defined as *multilinear maps*

$$T : \underbrace{T^*M \times \ldots \times T^*M}_{r} \times \underbrace{TM \times \ldots \times TM}_{s} \to \mathbb{R} \; , \tag{2.39}$$

in other words, given r dual vectors and s tangent vectors, T returns a real number, and if all but one vector or dual vector are fixed, the map is linear in the remaining argument. If a tensor of rank (r, s) is assigned to every point $p \in M$, we have a *tensor field* of rank (r, s) on M.

Tensors

Tensors of rank (r, s) are multilinear maps of r dual vectors and s vectors into the real numbers.

According to this definition, tensors of rank $(0, 1)$ are simply dual vectors, and tensors of rank $(1, 0)$ are elements of V^{**} and can thus be identified with tangent vectors.

Example: Tensor field of rank $(1, 1)$

For one specific example, a tensor of rank $(1, 1)$ is a bilinear map from $T^*M \times TM \to \mathbb{R}$. If we fix a vector $v \in TM$, $T(\cdot, v)$ is a linear map $T^*M \to \mathbb{R}$ and thus an element of $T^{**}M$, which can be identified with a vector. In this way, given a vector $v \in TM$, a tensor of rank $(1, 1)$ produces another vector $\in TM$, and vice versa for dual vectors. Thus, tensors of rank $(1, 1)$ can be seen as linear maps from $TM \to TM$, or from $T^*M \to T^*M$. ◀

With the obvious rules for adding linear maps and multiplying them with scalars, the set of tensors $\mathcal{T}^r_s$ of rank (r, s) attains the structure of a vector space of dimension n^{r+s}.

Given a tensor t of rank (r, s) and another tensor t' of rank (r', s'), we can construct a tensor of rank $(r + r', s + s')$ called the *outer product* $t \otimes t'$ of t and t' by simply multiplying their results on the $r + r'$ dual vectors w^i and the $s + s'$ vectors v_j, thus

$$(t \otimes t')(w^1, \dots, w^{r+r'}, v_1, \dots, v_{s+s'}) = \\ t(w^1, \dots, w^r, v_1, \dots, v_s)\, t'(w^{r+1}, \dots, w^{r+r'}, v_{s+1}, \dots, v_{s+s'}) \ . \tag{2.40}$$

In particular, it is thus possible to construct a basis for tensors of rank (r, s) out of the bases $\{e_i\}$ of the tangent space and $\{e^{*j}\}$ of the dual space by taking the tensor products. Thus, a tensor of rank (r, s) can be written in the form

$$t = t^{i_1 \dots i_r}_{j_1 \dots j_s} \left(\partial_{i_1} \otimes \dots \otimes \partial_{i_r}\right) \otimes \left(\mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_s}\right) \ , \tag{2.41}$$

where the numbers $t^{i_1 \dots i_r}_{j_1 \dots j_s}$ are its components with respect to the coordinate system h.

The transformation law (2.25) for the basis vectors under coordinate changes implies that the tensor components transform as

$$t'^{i_1 \dots i_r}_{j_1 \dots j_s} = J'^{i_1}_{k_1} \dots J'^{i_r}_{k_r} J^{l_1}_{j_1} \dots J^{l_s}_{j_s}\, t^{k_1 \dots k_r}_{l_1 \dots l_s} \ , \tag{2.42}$$

a property which is often used to define tensors in the first place.

Contraction

The *contraction* $C^i_j t$ of a tensor of rank (r, s) is a map which reduces both r and s by unity,

$$C^i_j t : \mathcal{T}^r_s \to \mathcal{T}^{r-1}_{s-1} \ , \quad C^i_j t = t(\dots, e^{*k}, \dots, e_k, \dots) \ , \tag{2.43}$$

where $\{e_k\}$ and $\{e^{*k}\}$ are bases of the tangent and dual spaces, as before, and the summation over all $1 \le k \le n$ is implied. The basis vectors e^{*k} and e_k are inserted as the i-th and j-th arguments of the tensor t.

Expressing the tensor in a coordinate basis, we can write the tensor in the form (2.41), and thus its contraction with respect to the i_a-th and j_b-th arguments reads

$$\begin{aligned} C^{i_a}_{j_b} t = {} & t^{i_1 \dots i_r}_{j_1 \dots j_s} \mathrm{d}x^{i_k}(\partial_{j_k}) \\ & (\partial_{i_1} \otimes \dots \otimes \partial_{i_a-1} \otimes \partial_{i_a+1} \otimes \dots \otimes \partial_{i_r}) \\ & \left(\mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_b-1} \otimes \mathrm{d}x^{j_b+1} \otimes \dots \otimes \mathrm{d}x^{j_s}\right) \\ & = t^{i_1 \dots i_{a-1} i_k i_{a+1} \dots i_r}_{j_1 \dots j_{b-1} i_k j_{b+1} \dots j_s} \\ & (\partial_{i_1} \otimes \dots \otimes \partial_{i_a-1} \otimes \partial_{i_a+1} \otimes \dots \otimes \partial_{i_r}) \\ & \left(\mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_b-1} \otimes \mathrm{d}x^{j_b+1} \otimes \dots \otimes \mathrm{d}x^{j_s}\right) \ . \end{aligned} \tag{2.44}$$

Example: Tensor contraction

For a simple example, let $v \in TM$ be a tangent vector and $w \in T^*M$ a dual vector, and $t = v \otimes w$ a tensor of rank $(1, 1)$. Its contraction results in a tensor of rank $(0, 0)$, i.e. a real number, which is

$$Ct = C(v \otimes w) = \mathrm{d}x^k(v)\, w(\partial_k) = v^k w_k \ . \tag{2.45}$$

At the same time, this can be written as

$$\begin{aligned} Ct &= C(v \otimes w) = w(v) \\ &= (w_j \mathrm{d}x^j)(v^i \partial_i) = w_j v^i \mathrm{d}x^j(\partial_i) = w_j v^i \partial_i x^j = w_j v^i \delta^j_i = w_i v^i \ . \end{aligned} \tag{2.46}$$

In this sense, the contraction amounts to applying the tensor (partially) "on itself". ◂

2.4 The metric

We need some way to define and measure the "distance" between two points on a manifold. A *metric* is introduced via the infinitesimal squared distance between two neighbouring points on the manifold.

We have seen above that tangent vectors $v \in T_pM$ are closely related to infinitesimal displacements around a point p on the manifold. Moreover, the infinitesimal squared distance between two neighbouring points p and q should be quadratic in the displacement leading from one point to the other. Thus, we construct the metric g as a bi-linear map

$$g : TM \times TM \to \mathbb{R} \ , \tag{2.47}$$

which means that the g is a tensor of rank $(0, 2)$. The metric thus assigns a number to two elements of a vector field TM on M. The metric g thus defines to two vectors their scalar product, which is *not necessarily positive*. We abbreviate the scalar product of two vectors $v, w \in TM$ by

$$g(v, w) \equiv \langle v, w \rangle \ . \tag{2.48}$$

In addition, we require that the metric be symmetric and non-degenerate, which means

$$\begin{aligned} &g(v, w) = g(w, v) \quad \forall \quad v, w \in T_pM \ , \\ &g(v, w) = 0 \quad \forall \quad v \in T_pM \quad \Leftrightarrow \quad w = 0 \ . \end{aligned} \tag{2.49}$$

Metric

A metric is a rank-$(0, 2)$ tensor field which is symmetric and non-degenerate.

In a coordinate basis, the metric can be written in components as

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j \ . \tag{2.50}$$

The line element ds is the metric applied to an infinitesimal distance vector dx with components dx^i,

$$\mathrm{d}s^2 = g(\mathrm{d}x, \mathrm{d}x) = g_{ij}\mathrm{d}x^i\mathrm{d}x^j \ . \tag{2.51}$$

Given a coordinate basis $\{e_i\}$, the metric g can always be chosen such that

$$g(e_i, e_j) = \langle e_i, e_j \rangle = \pm\delta_{ij} \ , \tag{2.52}$$

where the number of positive and negative signs is independent of the coordinate choice and is called the *signature* of the metric. Positive-(semi-) definite metrics, which have only positive signs, are called *Riemannian*, and *pseudo-Riemannian* metrics have positive and negative signs.

Figure 2.3 Georg Friedrich Bernhard Riemann (1826–1866), German mathematician. Source: Wikipedia

Example: Minkowski metric

Perhaps the most common pseudo-Riemannian metric is the Minkowski metric known from special relativity, which can be chosen to have the signature $(-, +, +, +)$ and has the line element

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 \ . \tag{2.53}$$

A metric with the same signature as for the spacetime is called *Lorentzian*. ◂

Given a tangent vector v, the metric can also be seen as a linear map from TM into T^*M,

$$g : TM \to T^*M \ , \quad v \mapsto g(\cdot, v) \ . \tag{2.54}$$

This is an element of T^*M because it linearly maps vectors into $\mathbb{R}$. Since the metric is non-degenerate, the inverse map g^{-1} also exists, and the metric can be used to establish a one-to-one correspondence between vectors and dual vectors, and thus between the tangent space TM and its dual space T^*M.