Chapter 1

Introduction

1.1 The idea behind general relativity

There was no need for general relativity when Einstein started working on it. There was no experimental data signalling any failure of the Newtonian theory of gravity, except perhaps for the minute advance of the perihelion of Mercury's orbit by $43''$ per *century*, which researchers at the time tried to explain by perturbations not included yet into the calculations of celestial mechanics in the Solar System.

Essentially, Einstein found general relativity because he was deeply dissatisfied with some of the concepts of the Newtonian theory, in particular the concept of an inertial system, for which no experimental demonstration could be given.

After special relativity, he was convinced quite quickly that trying to build a relativistic theory of gravitation led to conclusions which were in conflict with experiments. Action at a distance is impossible in special relativity because the absolute meaning of space and time had to be given up. The most straightforward way to combine special relativity with Newtonian gravity seemed to start from Poisson's equation for the gravitational potential and to add time derivatives to it so as to make it relativistically invariant.

However, it was then unclear how the law of motion should be modified because, according to special relativity, energy and mass are equivalent and thus the mass of a body should depend on its position in a gravitational field.

This led Einstein to a result which raised his suspicion. In Newtonian theory, the vertical acceleration of a body in a vertical gravitational field is independent of its horizontal motion. In a special-relativistic extension of Newton's theory, this would no longer be the case: the vertical gravitational acceleration would depend on the kinetic energy of a body, and thus not be independent of its horizontal motion.

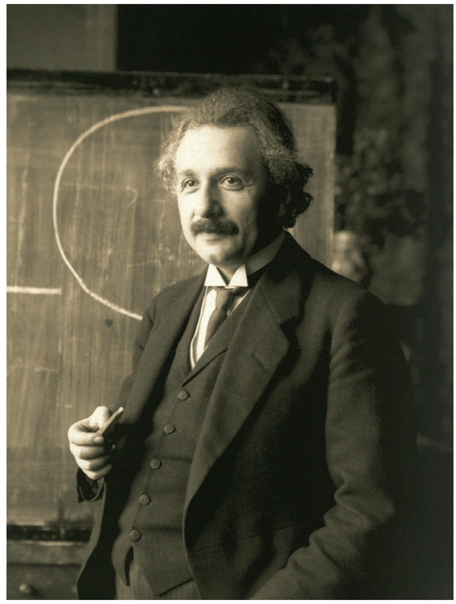

Figure 1.1 Albert Einstein (1879–1955) during a lecture in Vienna, 1921. Source: Wikipedia

This was in striking conflict with experiment, which says that all bodies experience the same gravitational acceleration. At this point, the equivalence of inertial and gravitational mass struck Einstein as a law of deep significance. It became the heuristic guiding principle in the construction of general relativity.

Freely falling frames of reference

This line of thought leads to the fundamental concept of general relativity. It says that it must be possible to introduce local, non-rotating, freely-falling frames of reference in which gravity is locally "transformed away".

The directions of motion of different freely-falling reference frames will generally not be parallel: Einstein elevators released at the same height above the Earth's surface but over different locations will fall towards the Earth's centre and thus approach each other.

Space-time as a manifold

Replacing inertial frames by freely falling, non-rotating frames of references leads to the idea that spacetime is a four-dimensional manifold instead of the "rigid", four-dimensional Euclidean space.

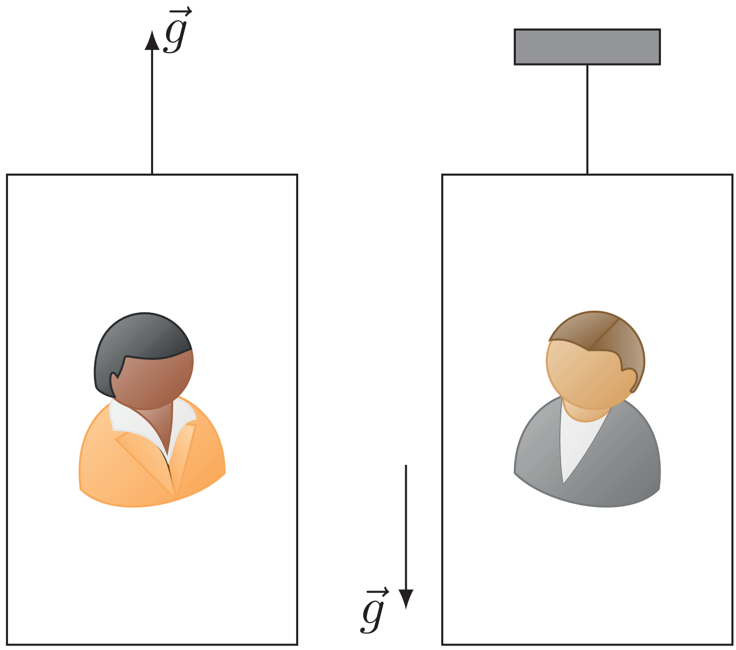

Figure 1.2 Einstein elevators: The left elevator is thought to be placed outside a gravitational field, but accelerated upwards with an acceleration $-\vec{g}$; the right elevator is placed at rest in a gravitational field with gravitational acceleration $\vec{g}$ directed downwards. According to the equivalence principle, their occupants cannot distinguish these situations from each other.

As will be explained in the following two chapters, manifolds can locally be mapped onto Euclidean space. In a freely-falling reference frame, special relativity must hold, which implies that the Minkowskian metric of special relativity must locally be valid. The same operation must be possible in all freely-falling reference frames individually, but not globally, as is illustrated by the example of the Einstein elevators falling towards the Earth.

Thus, general relativity considers the metric of the spacetime manifold as a dynamical field. The necessity to match it with the Minkowski metric in freely-falling reference frames means that the signature of the metric must be $(-, +, +, +)$ or $(+, -, -, -)$. A manifold with a metric which is not positive definite is called *pseudo-Riemannian*, or *Lorentzian* if the metric has the signature of the Minkowski metric.

The lecture starts with an introductory chapter describing the fundamental characteristics of gravity, their immediate consequences and the failure of a specially-relativistic theory of gravity. It then introduces in two chapters the mathematical apparatus necessary for general relativity, which are the basics of differential geometry, i.e. the geometry on manifolds. After this necessary mathematical digressions, we shall return to physics when we discuss the motion of test particles in given gravitational fields in Chap. 4 and later introduce Einstein's field equations in Chap. 6.

1.2 Fundamental properties of gravity

1.2.1 Scales

Caution We are using Gaussian cgs units throughout, in which the electrostatic potential of a charge q is simply

$$\Phi(r) = -\frac{q}{r}\ .$$

In these units, the elementary charge is

$$e = 4.80 \cdot 10^{-10}\,\frac{\mathrm{g}^{1/2}\,\mathrm{cm}^{3/2}}{\mathrm{s}}\ .$$

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The first remarkable property of gravity is its weakness. It is by far the weakest of the four known fundamental interactions. To see this, compare the gravitational and electrostatic forces acting between two protons at a distance r. We have

$$\frac{\text{gravity}}{\text{electrostatic force}} = \left(\frac{\mathcal{G}m_\mathrm{p}^2}{r^2}\right)\left(\frac{e^2}{r^2}\right)^{-1} = \frac{\mathcal{G}m_\mathrm{p}^2}{e^2} = 8.1 \cdot 10^{-37}\ ! \tag{1.1}$$

This leads to an interesting comparison of scales. In quantum physics, a particle of mass m can be assigned the *Compton wavelength*

$$\lambda = \frac{\hbar}{mc}\ , \tag{1.2}$$

where Planck's constant h is replaced by $\hbar$ merely for conventional reasons. We ask what the mass of the particle must be such that its gravitational potential energy equals its rest mass mc^2, and set

$$\frac{\mathcal{G}m^2}{\lambda} \stackrel{!}{=} mc^2\ . \tag{1.3}$$

The result is the *Planck mass*,

$$m = M_\mathrm{Pl} = \sqrt{\frac{\hbar c}{\mathcal{G}}} = 2.2 \cdot 10^{-5}\,\mathrm{g} = 1.2 \cdot 10^{19}\,\frac{\mathrm{GeV}}{c^2}\ , \tag{1.4}$$

which, inserted into (1.2), yields the *Planck length*

$$\lambda_\mathrm{Pl} = \sqrt{\frac{\hbar\mathcal{G}}{c^3}} = 1.6 \cdot 10^{-33}\,\mathrm{cm} \tag{1.5}$$

and the *Planck time*

$$t_\mathrm{Pl} = \frac{\lambda_\mathrm{Pl}}{c} = \sqrt{\frac{\hbar\mathcal{G}}{c^5}} = 5.3 \cdot 10^{-44}\,\mathrm{s}\ . \tag{1.6}$$

As Max Planck noted already in 1900[1], these are the only scales for mass, length and time that can be assigned an objective meaning.

[1] Über irreversible Strahlungsvorgänge, Annalen der Physik 306 (1900) 69

The Planck mass is huge in comparison to the mass scales of elementary particle physics. The Planck length and time are commonly interpreted as the scales where our "classical" description of spacetime is expected to break down and must be replaced by an unknown theory combining relativity and quantum physics.

Using the Planck mass, the ratio from (1.1) can be written as

$$\frac{\mathcal{G}m_\mathrm{p}^2}{e^2} = \frac{1}{\alpha}\frac{m_\mathrm{p}^2}{M_\mathrm{Pl}^2} , \tag{1.7}$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine-structure constant.

Dominance of gravity

These comparisons suggest that gravity will dominate all other interactions once the mass of an object is sufficiently large. A mass scale important for the astrophysics of stars is set by the ratio

$$M_\mathrm{Pl}\frac{M_\mathrm{Pl}^2}{m_\mathrm{p}^2} = 1.7 \cdot 10^{38}\, M_\mathrm{Pl} = 3.7 \cdot 10^{33}\,\mathrm{g} , \tag{1.8}$$

which is almost two solar masses.

We shall see at the end of this lecture that stellar cores of this mass cannot be stabilised against gravitational collapse.

1.2.2 The Equivalence Principle

The observation that inertial and gravitational mass cannot be experimentally distinguished is a highly remarkable finding. It is by no means obvious that the ratio between any force exerted on a body and its consequential acceleration should have anything to do with the ratio between the gravitational force and the body's acceleration in a gravitational field.

The experimentally well-established fact that inertial and gravitational mass are the same at least within our measurement accuracy was raised to a guiding principle by Einstein, the *principle of equivalence*, which can be formulated in several different ways.

Principle of equivalence

The weaker and less precise statement is that *the motion of a test body in a gravitational field is independent of its mass and composition*,

which can be cast into the more precise form that *in an arbitrary gravitational field, no local non-gravitational experiment can distinguish a freely falling, non-rotating system from a uniformly moving system in absence of the gravitational field*.

The latter is *Einstein's Equivalence Principle*, which is the heuristic guiding principle for the construction of general relativity.

It is important to note the following remarkable conceptual advance: Newtonian mechanics starts from Newton's axioms, which introduce the concept of an inertial reference frame, saying that force-free bodies in inertial systems remain at rest or move at constant velocity, and that bodies in inertial systems experience an acceleration which is given by the force acting on them, divided by their mass.

Firstly, inertial systems are a deeply unsatisfactory concept because they cannot be realised in any strict sense. Approximations to inertial systems are possible, but the degree to which a reference frame will approximate an inertial system will depend on the precise circumstances of the experiment or the observation made.

Secondly, Newton's second axiom is, strictly speaking, circular in the sense that it defines forces if one is willing to accept inertial systems, while it defines inertial systems if one is willing to accept the relation between force and acceleration. A satisfactory, non-circular definition of force is not given in Newton's theory. The existence of inertial frames is *postulated*.

Caution Note that Newton assumed the existence of absolute space and time. Strictly speaking, therefore, the problem of inertial frames did not exist when he founded classical mechanics. ◂

Special relativity replaces the rigid Newtonian concept of absolute space and time by a spacetime which carries the peculiar light-cone structure imprinted by the universality of the speed of light demanded by Maxwell's electrodynamics. Newtonian spacetime can be considered as the Cartesian product $\mathbb{R}\times\mathbb{R}^3$. An instant $t\in\mathbb{R}$ in time uniquely identifies the three-dimensional Euclidean space of all simultaneous events.

Of course, it remains possible in special relativity to define simultaneous events, but the three-dimensional hypersurface in four-dimensional Euclidean space $\mathbb{R}^4$ identified in this way depends on the motion of the observer relative to another observer. Independent of their relative motion, however, is the light-cone structure of Minkowskian spacetime. The future light cone encloses events in the future of a point p in spacetime which can be reached by material particles, and its boundary is defined by events which can be reached from p by light signals. The past light cone encloses events in the past of p from which material particles can reach p, and its boundary is defined by events from which light signals can reach p.

Yet, special relativity still makes use of the concept of inertial reference frames. Physical laws are required to be invariant under transformations from the Poincaré group, which translate from one inertial system to another.

Flexible light-cone structure

General relativity keeps the light-cone structure of special relativity, even though its rigidity is given up: the orientation of the light cones can vary across spacetime.

Thus, the relativity of distances in space and time remains within the theory. However, it is one of the great achievements of general relativity that it finally replaces the concept of inertial systems by something else which can be experimentally demonstrated: the principle of equivalence replaces inertial systems by non-rotating, freely-falling frames of reference.

1.3 Consequences of the equivalence principle

Without any specific form of the theory, the equivalence principle immediately allows us to draw conclusions on some of the consequences any theory must have which is built upon it. We discuss two here to illustrate its general power, namely the gravitational redshift and gravitational light deflection.

1.3.1 Gravitational Redshift

We enter an Einstein elevator which is at rest in a gravitational field at $t = 0$. The elevator is assumed to be small enough for the gravitational field to be considered as homogeneous within it, and the (local) gravitational acceleration be g.

According to the equivalence principle, the *downward* gravitational acceleration felt in the elevator cannot locally be distinguished from a constant *upward* acceleration of the elevator with the same acceleration g. Adopting the equivalence principle, we thus assume that the gravitational field is absent and that the elevator is constantly accelerated upward instead.

At $t = 0$, a photon is emitted by a light source at the bottom of the elevator, and received some time Δt later by a detector at the ceiling. The time interval Δt is determined by

$$h + \frac{g}{2}\Delta t^2 = c\Delta t \;, \tag{1.9}$$

where h is the height of the elevator. This equation has the solution

$$\Delta t_\pm = \frac{1}{g}\left[c \pm \sqrt{c^2 - 2gh}\right] = \frac{c}{g}\left[1 - \sqrt{1 - \frac{2gh}{c^2}}\right] \approx \frac{h}{c} \;; \tag{1.10}$$

the other branch makes no physical sense.

When the photon is received at the ceiling, the ceiling moves with the velocity

$$\Delta v = g\Delta t \approx \frac{gh}{c} \tag{1.11}$$

compared to the floor when the photon was emitted. The photon is thus Doppler shifted with respect to its emission, and is received with the longer wavelength

$$\lambda' \approx \left(1 + \frac{\Delta v}{c}\right)\lambda \approx \left(1 + \frac{gh}{c^2}\right)\lambda\,. \tag{1.12}$$

The gravitational acceleration is given by the gravitational potential Φ through

$$g = |\vec{\nabla}\Phi| \quad \Rightarrow \quad gh \approx \Delta\Phi\,, \tag{1.13}$$

where $\Delta\Phi \approx |\vec{\nabla}\Phi|\,h$ is the change in Φ from the floor to the ceiling of the elevator.

Gravitational redshift

The equivalence principle demands a gravitational redshift of

$$z \equiv \frac{\lambda' - \lambda}{\lambda} \approx \frac{\Delta\Phi}{c^2} \tag{1.14}$$

of a light ray passing the potential difference $\Delta\Phi$.

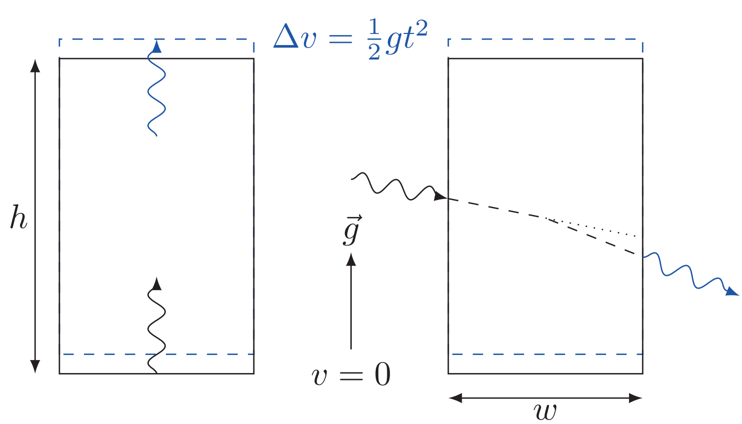

Figure 1.3 Two Einstein elevators, both outside a gravitational field and accelerated upwards with acceleration $\vec{g}$. When the photon reaches the top of the elevator (left) or while the light ray crosses it (right), the elevator is accelerated to the velocity $\Delta v = gt^2/2$. This leads to redshift (left) and aberration (right).

1.3.2 Gravitational Light Deflection

Similarly, it can be concluded from the equivalence principle that light rays should be curved in gravitational fields. To see this, consider again the Einstein elevator from above which is at rest in a gravitational field $g = |\vec{\nabla}\Phi|$ at $t = 0$.

As before, the equivalence principle asserts that we can consider the elevator as being accelerated upwards with the acceleration g.

Suppose now that a horizontal light ray enters the elevator at $t = 0$ from the left and leaves it at a time $\Delta t = w/c$ to the right, if w is the horizontal width of the elevator.

As the light ray leaves the elevator, the elevator's velocity has increased to

$$\Delta v = g\Delta t = \frac{|\vec{\nabla}\Phi|w}{c} \tag{1.15}$$

such that, in the rest frame of the elevator, it leaves at an angle

$$\Delta\alpha = \frac{\Delta v}{c} = \frac{|\vec{\nabla}\Phi|w}{c^2} \tag{1.16}$$

downward from the horizontal because of the aberration due to the finite light speed.

Light deflection by gravitational fields

Since the *upward* accelerated elevator corresponds to an elevator at rest in a *downward* gravitational field, this leads to the expectation that light will be deflected towards gravitational fields.

Although it is possible to construct theories of gravity which obey the equivalence principle and do not lead to gravitational light deflection, the bending of light in gravitational fields is by now a well-established experimental fact.

1.4 Futile attempts

1.4.1 Gravitational Redshift

We have seen before that the equivalence principle implies a gravitational redshift, which has been demonstrated experimentally. We must thus require from a theory of gravity that it does lead to gravitational redshift.

Suppose we wish to construct a theory of gravity which retains the Minkowski metric $\eta_{\mu\nu}$. In such a theory, how ever it may look in detail, the proper time measured by observers moving along a world line $x^\mu(\lambda)$ from λ_1 to λ_2 is

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} , \tag{1.17}$$

where the minus sign under the square root appears because we choose the signature of $\eta_{\mu\nu}$ to be $(-1, 1, 1, 1)$.

Now, let a light ray propagate from the floor to the ceiling of the elevator in which we have measured gravitational redshift before. Specifically, let the light source shine between coordinates times t_1 and t_2. The emitted photons will propagate to the receiver at the ceiling along world lines which may be curved, but must be *parallel* because the metric is constant. The time interval within which the photons arrive at the receiver must thus equal the time interval $t_2 - t_1$ within which they left the emitter. Thus there cannot be gravitational redshift in a theory of gravity in flat spacetime.

1.4.2 A Scalar Theory of Gravity and the Perihelion Shift

Let us now try and construct a scalar theory of gravity starting from the field equation

$$\Box\phi = -4\pi\mathcal{G}T , \tag{1.18}$$

where ϕ is the gravitational potential and $T = T^\mu_{\ \mu}$ is the trace of the energy-momentum tensor. Note that ϕ is made dimensionless here by dividing the Newtonian gravitational potential Φ by c^2.

In the limit of weak fields and non-relativistic matter, this reduces to Poisson's equation

$$\vec{\nabla}^2\Phi = 4\pi\mathcal{G}\rho , \tag{1.19}$$

since then the time derivatives in d'Alembert's operator and the pressure contributions to T can be neglected.

Let us further adopt the Lagrangian

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = -mc\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,(1+\phi) , \tag{1.20}$$

which is the ordinary Lagrangian of a free particle in special relativity, multiplied by the factor $(1 + \phi)$. This is the *only* possible Lagrangian that yields the right weak-field (Newtonian) limit.

We can write the square root in (1.20) as

$$\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = \sqrt{c^2 - \vec{v}^2} = c\sqrt{1 - \vec{\beta}^2} , \tag{1.21}$$

where $\vec{\beta} = \vec{v}/c$ is the velocity in units of c. The weak-field limit of (1.20) for non-relativistic particles is thus

$$\mathcal{L}(x^\mu, \dot{x}^\mu) \approx -mc^2\left(1 - \frac{\vec{v}^2}{2c^2}\right)(1+\phi) \approx -mc^2 + \frac{m}{2}\vec{v}^2 - mc^2\phi\ , \quad (1.22)$$

which is the right expression in Newtonian gravity.

The equations of motion can now be calculated inserting (1.20) into Euler's equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha} = \frac{\partial\mathcal{L}}{\partial x^\alpha}\ . \quad (1.23)$$

On the right-hand side, we find

$$\frac{\partial\mathcal{L}}{\partial x^\alpha} = -mc^2\sqrt{1-\beta^2}\,\frac{\partial\phi}{\partial x^\alpha}\ . \quad (1.24)$$

On the left-hand side, we first have

$$\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha} = \frac{1}{c}\frac{\partial\mathcal{L}}{\partial\beta^\alpha} = \frac{mc\beta^\alpha}{\sqrt{1-\beta^2}}(1+\phi)\ , \quad (1.25)$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha} = mc(1+\phi)\left(\frac{\dot{\beta}^\alpha}{\sqrt{1-\beta^2}} + \frac{\beta^\alpha\vec{\beta}\cdot\dot{\vec{\beta}}}{(1-\beta^2)^{3/2}}\right) + \frac{mc\beta^\alpha}{\sqrt{1-\beta^2}}\,\dot{\phi}\ . \quad (1.26)$$

We shall now simplify these equations assuming that the potential is static, $\dot{\phi} = 0$, and that the motion is non-relativistic, $\beta \ll 1$. Then, (1.26) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\vec{v}} \approx mc(1+\phi)\dot{\vec{\beta}}\left(1 + \frac{\beta^2}{2}\right) \approx m(1+\phi)\ddot{\vec{x}}\ , \quad (1.27)$$

and (1.24) turns into

$$\frac{\partial\mathcal{L}}{\partial\vec{x}} \approx -mc^2\left(1 - \frac{\beta^2}{2}\right)\vec{\nabla}\phi \approx -mc^2\vec{\nabla}\phi\ . \quad (1.28)$$

The equation of motion thus reads, in this approximation,

$$(1+\phi)\ddot{\vec{x}} = -c^2\vec{\nabla}\phi \quad (1.29)$$

or

$$\ddot{\vec{x}} = -c^2\vec{\nabla}\phi(1-\phi) = -c^2\vec{\nabla}\left(\phi - \frac{\phi^2}{2}\right)\ . \quad (1.30)$$

Compared to the equation of motion in Newtonian gravity, therefore, the potential is augmented by a quadratic perturbation.

For a static potential and non-relativistic matter, the potential is given by Poisson's equation.

We now proceed to work out the perihelion shift expected for planetary orbits around the Sun in such a theory of gravity. As we know from the discussion of Kepler's problem in classical mechanics, the radius r and the polar angle φ of such orbits are characterised by

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{mr^2}{L}\sqrt{\frac{2}{m}(E - V_\mathrm{L})}\,, \tag{1.31}$$

where V_L is the effective potential energy

$$V_\mathrm{L} = V + \frac{L^2}{2mr^2}\,, \tag{1.32}$$

and L is the (orbital) angular momentum. Thus,

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{r^2}{L}\sqrt{2m(E - V) - \frac{L^2}{r^2}}\,. \tag{1.33}$$

Caution Recall that the equation of motion (1.31) follows from the conservation laws of angular-momentum,

$$\dot\varphi = \frac{L}{mr^2}\,,$$

and energy,

$$\dot r^2 = \frac{2}{m}\left(E - V_L(r)\right)\,.$$

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Confirm (1.35) by carrying out the calculation yourself.

The perihelion shift is the change in φ upon integrating once around the orbit, or integrating twice from the perihelion radius r_0 to the aphelion radius r_1,

$$\Delta\varphi = 2\int_{r_0}^{r_1}\mathrm{d}r\,\frac{\mathrm{d}\varphi}{\mathrm{d}r}\,. \tag{1.34}$$

Inverting (1.33), it is easily seen that (1.34) can be written as

$$\Delta\varphi = -2\frac{\partial}{\partial L}\int_{r_0}^{r_1}\mathrm{d}r\,\sqrt{2m(E - V) - \frac{L^2}{r^2}}\,. \tag{1.35}$$

Now, we split the potential energy V into the Newtonian contribution V_0 and a perturbation $\delta V \ll V$ and expand the integrand to lowest order in δV, which yields

$$\Delta\varphi \approx -2\frac{\partial}{\partial L}\int_{r_0}^{r_1}\mathrm{d}r\,\sqrt{A_0}\left(1 - \frac{m\delta V}{A_0}\right) \tag{1.36}$$

where the abbreviation

$$A_0 \equiv 2m(E - V_0) - \frac{L^2}{r^2} \tag{1.37}$$

was inserted for convenience.

We know that orbits in Newtonian gravity are closed, so that the first term in the integrand of (1.36) must vanish. Thus, we can write

$$\Delta\varphi \approx 2\frac{\partial}{\partial L}\int_{r_0}^{r_1}\mathrm{d}r\,\frac{m\delta V}{\sqrt{A_0}}\,. \tag{1.38}$$

Next, we transform the integration variable from r to φ, using that

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} \approx \frac{r^2}{L}\sqrt{A_0} \tag{1.39}$$

to leading order in δV, according to (1.31). Thus, (1.38) can be written as

$$\Delta\varphi \approx \frac{\partial}{\partial L}\frac{2m}{L}\int_0^\pi \mathrm{d}\varphi\, r^2 \delta V\ . \tag{1.40}$$

Finally, we specialise the potential energy. Since Poisson's equation for the gravitational potential remains valid, we have

$$V_0 = mc^2\phi = -\frac{\mathcal{G}M_\odot m}{r}\ , \tag{1.41}$$

where $M_\odot$ is the Sun's mass, and following (1.30), the potential perturbation is

$$\delta V = mc^2\frac{\phi^2}{2} = \frac{mc^2}{2}\frac{V_0^2}{m^2c^4} = \frac{\mathcal{G}^2M_\odot^2 m}{2c^2r^2}\ . \tag{1.42}$$

Inserting this into (1.40) yields the perihelion shift

$$\Delta\varphi = \frac{\partial}{\partial L}\frac{m}{L}\frac{\pi\mathcal{G}^2M_\odot^2 m}{c^2} = -\frac{\pi\mathcal{G}^2M_\odot^2 m^2}{c^2L^2}\ . \tag{1.43}$$

The angular momentum L can be expressed by the semi-major axis a and the eccentricity e of the orbit,

$$L^2 = \mathcal{G}M_\odot m^2 a(1-e^2)\ , \tag{1.44}$$

which allows us to write (1.43) in the form

$$\Delta\varphi = -\frac{\pi\mathcal{G}M_\odot}{ac^2(1-e^2)}\ . \tag{1.45}$$

For the Sun, $M_\odot = 2\cdot 10^{33}$ g, thus

$$\frac{\mathcal{G}M_\odot}{c^2} = 1.5\cdot 10^5\,\mathrm{cm}\ . \tag{1.46}$$

For Mercury, $a = 5.8\cdot 10^{12}$ cm and the eccentricity $e = 0.2$ can be neglected because it appears quadratic in (1.45). Thus, we find

$$\Delta\varphi = -8.1\cdot 10^{-8}\,\mathrm{radian} = -0.017'' \tag{1.47}$$

per orbit. Mercury's orbital time is 88 d, i.e. it completes about 415 orbits per century, so that the perihelion shift predicted by the scalar theory of gravity is

$$\Delta\varphi_{100} = -7'' \tag{1.48}$$

per century.

This turns out to be wrong: Mercury's perihelion shift is six times as large, and not even the sign is right. Therefore, our scalar theory of gravity fails in its first comparison with observations, showing that we have to walk along a different route.