F MASS

F.1 *Dynamics of gravity*

Surely this script is not supposed to be an introduction to general relativity with its heavy usage of differential geometry as its mathematical language. For that reason, everything in this chapter is restricted to weak gravity, with perturbations of an otherwise flat Minkowskian spacetime, where the physical picture of fields on top of a Minkowski spacetime is perfectly valid. Weak and strong gravity are quantitative concepts: The curvature of spacetime is defined through second derivatives of a quantity called metric, and as composed of second derivatives the curvature R defines a length scale Δx,

$$\Delta x = \frac{1}{\sqrt{R}} \tag{F.302}$$

For distances larger than Δx, curvature effects are important and gravity is strong, but for distances smaller than Δx, gravity is only a small correction on the Minkowski-metric.

Secondly, the gravitational potential Φ as it appears in the Poisson-equation (already including here the classical cosmological constant λ)

$$\Delta\Phi = 4\pi G\rho + \lambda \tag{F.303}$$

has no dynamics on its own, it changes instantaneously at every point in space if ρ is not stationary. But we've seen that hyperbolic field equations usually show propagation along the light cones and the existence of wave-like solutions, so we would expect this to apply to gravity, too. Table 2 gives an overview over different regimes of gravity in physical systems.

Table 2: regimes of gravity

	static	dynamic
weak	Newton-gravity	gravitational waves
strong	black holes	FLRW-cosmologies

An attempt to make the Poisson-equation relativistic could be the replacement $\partial^i \rightarrow \partial^\mu$, along with $\gamma_{ij} \rightarrow \eta_{\mu\nu}$. And in addition, the kinetic energy in the random motion of particles in a substance, i.e. the pressure p, should contribute along the matter density to the gravitational field, arriving at

$$\Box\frac{\Phi}{c^2} = -\frac{4\pi G}{c^4}(\rho c^2 + 3p) + \Lambda \tag{F.304}$$

with $\Lambda = \lambda/c^2$. This relation is interesting as well because it makes a statement about the dimensionless potential Φ/c^2, so c^2 provides a scale for Φ. Looking ahead at the Schwarzschild-radius r_S one could imagine this argumentation. $\Phi/c^2 = 1$ marks a particular strength of the potential, which could be given by a mass M observed at distance r_S, $GM/c^2 = r_S$, which is correct up to a factor of 2. At the same time you see that the factor G/c^2 has units of length/mass, so it enables us to assign a length scale to a mass.

There were actual observational findings that suggested a new theory of gravity, albeit with a lot of experimental uncertainty. While Newton-gravity predicts the orbits of Planets to be closed ellipses with a fixed ratio between the orbital period and the large semi-axis in form of Kepler's third law, Mercury was found not to obey this. In particular, Mercury's orbit showed a precession of the point of closest proximity to the Sun, which implied a slight deviation $\Phi \propto r^{-(1+\epsilon)}$, $\epsilon > 0$, from the Newtonian potential.

The standard Poisson-equation $\Delta\Phi = 4\pi G\rho$ as the field equation of classical gravity, can be motivated with these arguments: The gravitational acceleration g^i is the field strength of the the gravitational field and appears in an appropriate Gauß-law,

$$\partial_i g^i = -4\pi G\rho \tag{F.305}$$

such that the Poisson-equation is recovered when setting $g^i = -\partial^i\Phi$. Applying the Gauß-integral law and assuming spherical symmetry gives

$$\int_V \mathrm{d}^3 r\, \partial_i g^i = \int_{\partial V} \mathrm{d}S_i\, g^i = g\, 4\pi r^2 = -4\pi G \int_V \mathrm{d}^3 r\, \rho = -4\pi G M \tag{F.306}$$

with the mass M. This implies

$$g = -\frac{GM}{r^2} \quad \text{and consequently,} \quad \Phi = -\frac{GM}{r} \tag{F.307}$$

Effectively, the scaling $g \propto 1/r^2$ and $\Phi \propto 1/r$ is a consequence of the surfaces of spheres in 3-dimensional Euclidean space, where the Gauß-law ensures that the flux $\int \mathrm{d}S_i g^i$ is conserved across every surface $\partial V = S \propto r^2$. Mechanical similarity applied to the $1/r$-potential delivers Kepler's third law $t^2 \propto r^3$, so that the reason for Kepler's law is ultimately geometric, and the origin of Mercury's precession is unclear. Please keep in mind that a Yukawa-type screening modifies Φ at large and not at small distances, so it could not serve as an explanation.

F.2 *Inertial accelerations and equivalence*

It is a central tenet in relativity that forces are velocity dependent to conserve the normalisation of velocities, which in turn is needed by causal motion. The prime example are Lorentz-forces,

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{q}{m}F^{\mu\nu}u_\nu = \frac{q}{m}F^{\mu t}u_t + \frac{q}{m}F^{\mu i}u_i \tag{F.308}$$

which can not accelerate a particle with specific charge q/m from timelike velocities $u_\mu u^\mu = c^2 > 0$ to spacelike velocities $u_\mu u^\mu < 0$. The split in the summation over ν shows a contribution that doesn't depend on velocity due to the electric fields $F^{\mu t}$ and a contribution proprotional to the velocities υ due to the magnetic fields $F^{\mu i}$.

Making a giant conceptual leap to gravity we realise that there is no such thing as specific charge: The inertial of a particle and its coupling to a gravitational field are both equal to its mass, so gravity affects all particles in exactly the same way: From this point of view it might be better to speak about gravitational acceleration instead of gravitational force. Gravitational accelerations share this property with inertial accelerations such as the Coriolis- or centrifugal accelerations: This prompted

Einstein to postulate the equivalence principle with a general indistinguishability between gravitational and inertial accelerations.

Looking at inertia it becomes clear very quickly that these accelerations are velocity dependent, and could this be an expression that gravity is relativistic? That is in fact the case, as equation of motion in general relativity for a freely falling particle is the geodesic equation

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -\Gamma^\alpha_{\mu\nu}\,u^\mu u^\nu = -\Gamma^\alpha_{tt}\,u^t u^t - 2\Gamma^\alpha_{it}\,u^t u^i - \Gamma^\alpha_{ij}\,u^i u^j \tag{F.309}$$

with $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau = \gamma(c, u^i)^t$ as always. $\Gamma^\alpha_{\mu\nu}$ is the Christoffel-symbol. In a weakly perturbed Minkowski-spacetime one has $\Gamma^\alpha_{tt} = \partial^\alpha\Phi/c^2$, which would give rise to a Newtonian equation of motion in the slow-motion limit, $\mathrm{d}^2x^i/\mathrm{d}t^2 + \partial^i\Phi = 0$, if the field is static and if $\gamma \simeq 1$ such that $t = \tau$. $2\Gamma^\alpha_{it}\,u^t u^i = 2\Gamma^\alpha_{it}\,cu^i$ would correspond to the Coriolis-acceleration with its proportionality to 2υ, and lastly $\Gamma^\alpha_{ij}\,u^i u^j$ would give rise to the centrifugal acceleration $\propto \upsilon^2$.

F.3 *Classical Raychaudhury-equation and geodesic deviation*

The idea of a test particle is very transparent: It couples through its charge to the corresponding field (without changing the field itself!) and moves according to its equation of motion, indicating the strength and orientation of the field. It is worthwhile noticing that in this way the relativity principle concerning the motion of the test particle is applied to the dynamics and transformation properties of the field, in order to have the two consistent with each other: The transformation of the velocity unter Lorentz-transforms is given by $u^\mu \rightarrow \Lambda^\mu{}_\alpha u^\alpha$, and of the field tensor $F^{\mu\nu} \rightarrow \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta F^{\alpha\beta}$.

Exactly the same applies to the motion of particles through the gravitational field, with one peculiarity: If the particle is in a state of free-fall, one has the impression of perfect weightlessness when travelling along with the particle, and Einstein's equivalence principle then stipulates that the metric is locally Minkowskian and that the first derivative of the metric vanishes. So you might wonder where gravity actually is contained! Gravity determines the relative acceleration between freely falling test particles separated by a distance δ^μ:

$$\frac{\mathrm{d}^2\delta^\mu}{\mathrm{d}\tau^2} = -R^\mu{}_{\alpha\beta\nu}u^\alpha u^\beta\delta^\nu \tag{F.310}$$

with the Riemann-curvature $R^\mu{}_{\alpha\beta\nu}$: If spacetime is flat with no curvature, $R^\mu{}_{\alpha\beta\nu} = 0$ and consequently

$$\frac{\mathrm{d}^2\delta^\mu}{\mathrm{d}\tau^2} = 0 \quad \rightarrow \quad \delta^\mu = a^\mu\tau + b^\mu \tag{F.311}$$

with two integration constants a^μ and b^μ, indicating that there is a linear change in the particle's relative distance δ^μ. If the curvature, however, is non-vanishing, test particles get accelerated relative to each other (despite the fact that nobody travelling along with the particles would *feel* this acceleration).

Let's understand this in Newton-gravity: Two particles follow trajectories according to Newton's equation of motion, $\ddot{x}^i + \partial^i\Phi(x) = 0$ and $\ddot{y}^i + \partial^i\Phi(y) = 0$. Their relative distance $\delta^i = y^i - x^i$ follows then the equation of motion

$$\ddot{\delta}^i = \ddot{y}^i - \ddot{x}^i = -\partial^i\Phi(y) + \partial^i\Phi(x) = -\partial^i\partial_j\Phi\,\delta^j \tag{F.312}$$

with the Taylor-expansion $\Phi(y) \simeq \Phi(x) + \partial_j\Phi\,(y-x)^j$. Therefore, the tidal field $\partial^i\partial_j\Phi$ is responsible for the relative acceleration. This is effectively the Newtonian version of the geodesic deviation equation F.310.

It is very illustrative to imagine the following experiment: Let's have a couple of test particles situated at the corners of a cube fall through space(time) and monitor the change in volume or the change in shape of that cube, because intuitively, the volume change should be related to the enclosed mass. For the relative motion of two corners we would write $y^i = x^i + v^i\Delta t$, so that we can observe a shear

$$\frac{\partial y^i}{\partial x^j} = \delta^i_j + \frac{\partial v^i}{\partial x^j}\Delta t \tag{F.313}$$

if there are velocity gradients. Thinking back of the chapter about Lie-symmetries, we might think that these are just the first two terms of a Taylor-expansion of

$$\frac{\partial y^i}{\partial x^j} = \exp\left(\frac{\partial v^i}{\partial x^j}\Delta t\right) \tag{F.314}$$

Volumes transform under this coordinate change according to

$$\mathrm{d}^3y = \det\left(\frac{\partial y^i}{\partial x^j}\right)\mathrm{d}^3x \tag{F.315}$$

with the functional determinant, so that we get

$$\ln\det\left(\frac{\partial y^i}{\partial x^j}\right) = \mathrm{tr}\ln\left(\frac{\partial y^i}{\partial x^j}\right) \simeq \mathrm{tr}\left(\frac{\partial v^i}{\partial x^j}\Delta t\right) = \partial_i v^i\Delta t \tag{F.316}$$

such that the rate of change of the volume is proportional to the divergence of the velocity field, which is immediately apparent and intuitive. We have used the relation $\ln\det A = \mathrm{tr}\ln A$ and the approximation $\ln(1+\epsilon) \simeq \epsilon$ for small ϵ. For a very small time interval, the velocity is

$$v^i = -\partial^i\Phi\,\Delta t \tag{F.317}$$

and consequently

$$\frac{\partial v^i}{\partial x^j} = -\partial^i\partial^j\Phi\,\Delta t \tag{F.318}$$

The tidal field tensor can be decomposed into a trace and a traceless part,

$$\partial^i\partial^j\Phi = \left(\partial^i\partial^j\Phi - \frac{\Delta\Phi}{3}\delta^{ij}\right) + \frac{\Delta\Phi}{3}\delta^{ij} \tag{F.319}$$

where the velocity divergence would only pick up $\Delta\Phi$, which in turn is given by $4\pi G\rho$ through the Poisson-equation:

$$\ln\det\left(\frac{\partial y^i}{\partial x^j}\right) = 4\pi G\rho\,\Delta t^2 \tag{F.320}$$

That means, that the cloud of freely falling test particles changes its volume dynamically in proportion to $\Delta\Phi$ or, equivalently, $4\pi G\rho$. If the cloud falls through empty space, the volume would change linearly with Δt as the corners of the cloud would follow inertial motion, and the traceless part of the tidal shear field can only have an influence on the shape of the cloud but not its volume.

F.4 *Gravitational lensing*

We should spend a couple of minutes on the issue of gravitational light deflection to clear up misconceptions about how light could be at all influenced by gravity or curvature. There is a perfectly valid set of Maxwell's equations on a curved background which allow for wave-like solutions, but here we should see how null-lines defined by $\mathrm{d}s^2 = 0$ as photon trajectories notice gravity.

A good starting point is a weakly perturbed Minkowski line element,

$$\mathrm{d}s^2 = \left(1 + 2\frac{\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1 - 2\frac{\Phi}{c^2}\right)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{F.321}$$

valid with a Cartesian coordinate choice and if $|\Phi| \ll c^2$. γ_{ij} is the Euclidean metric.

A conventional, non-relativistic particle experiences the line element as the passage of proper time, $\mathrm{d}s^2 = c^2\mathrm{d}\tau^2$, and if the particle is non-relativistic, it moves essentially only in the $\mathrm{d}t$-direction and doesn't change its spatial coordinates by a large amount, $\mathrm{d}x^i = 0$. Then there will be a gravitational dilation of proper time relative to coordinate time

$$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 = \left(1 + 2\frac{\Phi}{c^2}\right)c^2\mathrm{d}t^2 \quad\rightarrow\quad \tau = \sqrt{1 + 2\frac{\Phi}{c^2}}\mathrm{d}t \simeq \left(1 + \frac{\Phi}{c^2}\right)\mathrm{d}t \tag{F.322}$$

caused by the gravitational potential Φ, which is negative as $\Phi = -GM/r$, such that $\mathrm{d}\tau < \mathrm{d}t$, with the approximation $\sqrt{1+\epsilon} \simeq 1 + \epsilon/2$.

A photon, however, traces out a trajectory characterised by $\mathrm{d}s^2 = 0$ and proper time is not sensibly defined. The effective speed of propagation of the photon is the rate at which the coordinates $\mathrm{d}x$ pass by in units of coordinate time $\mathrm{d}t$, leading to

$$c' = \frac{\mathrm{d}x}{\mathrm{d}t} = \pm\sqrt{\frac{1 + 2\frac{\Phi}{c^2}}{1 - 2\frac{\Phi}{c^2}}}\,c \simeq \pm\left(1 + 2\frac{\Phi}{c^2}\right)c \tag{F.323}$$

with the approximation $1/(1-\epsilon) \simeq 1 + \epsilon$ for small ϵ. That is a surprising result, as the effect of a gravitational field on a relativistic particle is twice as strong as on a non-relativistic particle. If again $\Phi = -GM/r$, the effective speed of propagation c' becomes zero at $2GM/c^2 = r_S$, which is known as the Schwarzschild radius. You see, it's not a matter of energy or of time of flight when a photon can not escape from a black hole; in these coordinates it's the case that the effective speed of propagation reaches zero at r_S, so the photon does not make any headway (in either direction!).

It's a good idea to follow this thought a bit further: For a radially moving photon in the potential $\Phi = -GM/r$, we have

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\left(1 - \frac{2GM}{c^2 r}\right)c \quad \rightarrow \quad \frac{\mathrm{d}r}{1 - r_S/r} = \pm c\mathrm{d}t \tag{F.324}$$

which is solved by $r_S \ln(r - r_S) + r = \pm ct$ up to an integration constant, let's call it p for the + branch and q for the − branch. This integration constant can be made the new radial coordinate,

$$p = r_S \ln(r - r_S) + r + ct \quad \text{and} \quad q = r_S \ln(r - r_S) + r - ct \tag{F.325}$$

or differentially with $\alpha = (1 - r_S/r)^{-1}$:

$$\mathrm{d}p = c\mathrm{d}t + \alpha\mathrm{d}r \quad \text{and} \quad \mathrm{d}q = c\mathrm{d}t - \alpha\mathrm{d}r \tag{F.326}$$

with these new coordinates, the line element becomes

$$\mathrm{d}s^2 \simeq \alpha^{-1}c^2\mathrm{d}t^2 - \alpha\mathrm{d}r^2 = \alpha^{-1}(\mathrm{d}p - \alpha\mathrm{d}r)(\mathrm{d}q + \alpha\mathrm{d}r) - \alpha\mathrm{d}r^2 = \alpha^{-1}\left(\mathrm{d}p\mathrm{d}q + \alpha(\mathrm{d}p - \mathrm{d}q)\mathrm{d}r - \alpha\mathrm{d}r^2\right) - \alpha\mathrm{d}r^2 \tag{F.327}$$

which becomes by using $\mathrm{d}p - \mathrm{d}q = 2\alpha\mathrm{d}r$ simply

$$\mathrm{d}s^2 = \left(1 - \frac{r_S}{r}\right)\mathrm{d}p\mathrm{d}q \tag{F.328}$$

The line element is effectively given now in terms of light cone coordinates, with a so-called conformal factor in front: This conformal factor doesn't change light propagation as $\mathrm{d}s^2 = 0$ and the factor is never zero, so $\mathrm{d}p\mathrm{d}q = 0$ already characterises the trajectory of a photon: We have absorbed the action of the gravitational field in a redefinition of the coordinates.

F.5 *Gravitational field equation*

From what we've learned the gravitational field equation should be a second-order hyperbolic field equation which is at least covariant under Lorentz-transforms. A first guess could be that gravity is some kind of electrodynamics for masses, so we could write

$$\Box A^\mu = -\frac{4\pi G}{c}\jmath^\mu \tag{F.329}$$

with A^t being the gravitational potential Φ and $\jmath^t$ the matter density ρ, the idea being that momentum density along with rest mass sources the gravitational field. Already now it might be a bit weird that $\jmath^t$ is *not* the rest mass energy density.

Surely, in the case of static field one would fall back onto the Poisson-equation, but for instance the incorporation of the cosmological constant λ would be unclear, as the equation is vectorial and not scalar as our intuitive rewriting of the Poisson equation

$$\Box\frac{\Phi}{c^2} = -\frac{4\pi G}{c^4}(c^2\rho + 3p) + \frac{\lambda}{c^2} \tag{F.330}$$

But there is a more fundamental problem: The rest mass energy density $c^2\rho$ transforms differently than the electric charge density. If you imagine a cloud of electrical charges viewed from a moving system, one perceives that cloud Lorentz-contracted by a factor of γ such that the charge density is higher by that factor, in agreement with the transformation property of a vector $\jmath^\mu \rightarrow \Lambda^\mu{}_\alpha \jmath^\alpha$. A cloud of matter viewed from another Lorentz-system has the same effect of Lorentz-contraction of the volume along the direction of motion, but also a relativistic mass increase by another factor of γ (indirectly, as a consequence of time dilation: one assigns a higher amount of inertia to the system). To get two powers of γ in the transformation, $c^2\rho$ must be the tt-component of a tensor, in this case the energy momentum tensor $T^{\mu\nu}$. The transformation property would be $T^{\mu\nu} \rightarrow \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}$, and with the proportionality of $\Lambda^\mu{}_\alpha \propto \gamma$ this actually works out. In summary, the gravitational field equation would need to be at least tensorial, in the form

$$\Box h^{\mu\nu} = -\frac{4\pi G}{c^2} T^{\mu\nu} \tag{F.331}$$

A second large conceptual difference is the nonlinearity in energy-momentum conservation, expressed by the innocently looking conservation law $\partial_\mu T^{\mu\nu} = 0$, with typical nonlinear terms arising in the equations of relativistic fluid mechanics. This in turn implies that eqn. F.331 can only be valid in a linearised limit.

The solution to these problems is much more complex and requires differential geometry: Gravity is thought to be equivalent to spacetime geometry, where curvature is sourced by the energy-momentum content. If that relationship is to be (i) a second-order, hyperbolic relation, which (ii) respects energy-momentum conservation, if (iii) spacetime is 4-dimensional and if the (iv) metric of spacetime is linked to the energy-momentum tensor in a (v) local way, then general relativity is uniquely defined, as stated by Lovelock's theorem.