D SYMMETRY

D.1 *Lie-groups and their generation*

Symmetry plays an incredibly important role in physics, and we would define symmetry as the invariance of a quantity under a transform. Transformations typically form a group as successive transformations can be combined into a single transform, because there is a unit transformation element with no effect and an inverse transform undoing the action of a previous transform. Many transformation groups in physics contain infinitely many elements, such as rotations that are parameterised by an angle α or Lorentz-transform with the rapidity ψ. In contrast to an index n for a finite group (or a countably infinite group), these groups are continuously parameterised and a rotation $R^i{}_j(\alpha)$ or Lorentz-transform $\Lambda^\mu{}_\nu(\psi)$ exists for every possible value of the real-valued parameters α and ψ: These continuously parameterised groups are referred to as Lie-groups.

D.2 *Generating algebras*

One can ask the (very sensible) question if all elements in a continuously parameterised Lie-group can be formed from an infinitesimally small transformation. Let's have a look at rotations in 2 dimensions, where the transformation matrix is given by

$$R = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \quad\rightarrow\quad \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} = \sigma^{(0)} + \alpha\sigma^{(2)} \quad \text{in the limit of small } \alpha \tag{D.202}$$

If one would like to assemble a rotation out of many individual small rotations, ideally n rotations of magnitude α/n with $n \rightarrow \infty$, one should obtain R:

$$R = \lim_{n\rightarrow\infty}\left(\sigma^{(0)} + \frac{\alpha}{n}\sigma^{(2)}\right)^n = \exp\left(\alpha\sigma^{(2)}\right) \tag{D.203}$$

such that one can speculate if $R^i{}_j$ is as well given by a power series,

$$R = \sum_n \frac{(\sigma^{(2)}\alpha)^n}{n!} = \sigma^{(0)}\sum_n \frac{(-1)^n\alpha^{2n}}{(2n)!} + \sigma^{(2)}\sum_n \frac{(-1)^n\alpha^{2n+1}}{(2n+1)!} = \\ \sigma^{(0)}\cos\alpha + \sigma^{(2)}\sin\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{D.204}$$

by splitting up the summation over even and odd indices, because even powers of $\sigma^{(2)}$ are proportional to the unit matrix $\sigma^{(0)}$ while odd powers stay proportional to $\sigma^{(2)}$. Similarly, Lorentz-transforms Λ are generated by $\sigma^{(3)}$,

$$\Lambda = \sum_n \frac{(\sigma^{(3)}\psi)^n}{n!} = \sigma^{(0)}\sum_n \frac{\psi^{2n}}{(2n)!} + \sigma^{(3)}\sum_n \frac{\psi^{2n+1}}{(2n+1)!} = \\ \sigma^{(0)}\cosh\psi + \sigma^{(3)}\sinh\psi = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \tag{D.205}$$

with the rapidity ψ as a parameter.

In summary, there is a generation of a continuously paramterised Lie-group

$A(t) = \exp(Tt)$ with the parameter t and the generator T, which in our two examples of the rotation and the Lorentz-transformation was traceless. Infinitesimal transforms are given by

$$A(\delta t) = \mathrm{id} + \delta t\ T \tag{D.206}$$

with the generator T, which can be assembled to a finite transform

$$A(t) = \lim_{n\to\infty}\left(\mathrm{id} + \frac{t}{n}T\right)^n = \exp(Tt) \tag{D.207}$$

from n successive transforms of magnitude t/n. Then, there should be an exponential series,

$$A(t) = \exp(Tt) = \sum_n \frac{(Tt)^n}{n!} \tag{D.208}$$

Now we should investigate the group structure of $A(t)$: Successive application of transformations $A(t')A(t)$ is captured by a single element of the same group,

$$A(t')A(t) = \exp(t'T)\exp(tT) = \exp((t'+t)T) = A(t'+t) \tag{D.209}$$

such that the parameter is additive. That realisation immediately gives rise to the definition of an inverse,

$$A(-t)A(t) = A(-t+t) = A(0) = \mathrm{id} \quad \text{such that} \quad A^{-1}(t) = A(-t) \tag{D.210}$$

Formally, eqn. D.209 requires the Cauchy-product: In fact,

$$A(t')A(t) = \exp(t'T)\exp(tT) = \sum_n \frac{(Tt')^n}{n!}\sum_m \frac{(Tt)^m}{m!} = \sum_n\sum_m^n \frac{(Tt')^m}{m!}\frac{(Tt)^{n-m}}{(n-m)!} \tag{D.211}$$

where we can proceed by introducing the binomial coefficient

$$A(t')A(t) = \sum_n \frac{T^n}{n!}\sum_m^n \binom{n}{m} t'^m t^{n-m} = \sum_n \frac{(T(t'+t))^n}{n!} = \exp(T(t'+t)) = A(t'+t) \tag{D.212}$$

and use the generalised binomial formula, *if* there is only *a single* generator involved. If one deals with multiple generators T, T′ one needs to employ the Baker-Hausdorff-Campbell-relation,

$$\exp(T)\exp(T') = \exp(T+T')\exp\left(-\frac{1}{2}[T,T']\right) \tag{D.213}$$

where it is apparent that the commutation relations $[T,T'] = TT' - T'T$ determine how the generated group elements get combined. An example that defies (at least my) imagination is the following: Surely the combination of rotations results in a rotation, if the axes are not identical then the result depends on the order as rotations in 3 dimensions are not a commutative group. The combination of Lorentz-transformations into different directions involves a rotation too: In fact, boosting along x followed by a boost along y, and inverting this by first boosting back along x followed by a boost back in y gives you a system with zero relative velocity compared to where you started, but there is an effective rotation.

D.3 *Construction of invariants*

In many cases invariants of Lie-groups can be traced back to the tracelessness of their generators. In fact, my third most favourite formula in theoretical physics implies

$$\ln \det \mathrm{A} = \mathrm{tr} \ln \mathrm{A} \tag{D.214}$$

and as the group elements A typically depend on the generator T through

$$\mathrm{A} = \exp(\mathrm{T}t) = \sum_n \frac{(\mathrm{T}t)^n}{n!} \tag{D.215}$$

one gets

$$\ln \det \mathrm{A} = \mathrm{tr} \ln \mathrm{A} = \mathrm{tr} \ln \exp(t\mathrm{T}) = t \mathrm{tr} \mathrm{T} = 0 \tag{D.216}$$

so that $\det \mathrm{A} = 1$. For a rotation matrix this would be $\cos^2 \alpha + \sin^2 \alpha = 1$ and for a Lorentz-transform $\cosh^2 \psi - \sinh^2 \psi = 1$, but actually one can compute the determinant already from the trace of the generator alone without using properties of the trigonometric or hyperbolic functions!

In the nomenclature of groups you often see a preceding letter S, as in SO(n) for the special orthogonal group in n dimensions, which refers to the property that the determinant of the group elements is equal to 1. From the argument above you understand that this must mean, that the generators are all traceless. If there is a relation like the power series D.215, it would automatically be a solution to a differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{A} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_n \frac{(\mathrm{T}t)^n}{n!} = \sum_n \frac{(\mathrm{T}t)^n}{n!}\mathrm{T} = \mathrm{AT} \tag{D.217}$$

with an index shift due to the differentiation $\mathrm{d}t^n/\mathrm{d}t/n! = t^{n-1}/(n-1)!$. In summary, there are three approaches to the generation of a Lie-group in the exponential form: The infinitesimal transform taken to the nth power, summing of the exponential series and thirdly, the differential equation for the exponential.

D.4 *Symplectic structures and canonical time evolution*

In classical mechanics one encounters a funny property of the symplectic matrix which arises when solving Hamilton's equation of motion for a harmonic oscillator. You'll see that the symplectic matrix is just the Pauli-matrix $\sigma^{(2)}$, so similarities between generating the time evolution of the harmonic oscillator and generating rotations are to be expected! After all, both involve sin and cos, and surely one can transform the coordinates into a rotating coordinate frame in phase space.

The Hamilton-function is $\mathcal{H}(p,q) = p^2/2 + q^2/2$ in a practical choice of units, and therefore

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \quad \text{and} \quad \dot{q} = +\frac{\partial \mathcal{H}}{\partial p} \tag{D.218}$$

which can be combined into a single equation, in particular for the harmonic oscillator where $\partial\mathcal{H}/\partial p = p$ and $\partial\mathcal{H}/\partial q = q$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} \tag{D.219}$$

where with our knowledge of generators we would immediately write

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \exp\left(\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} t\right)\begin{pmatrix} q(0) \\ p(0) \end{pmatrix} \tag{D.220}$$

with a time-evolution operator acting on the initial conditions $q(0)$ and $p(0)$ to give momentum and position of the evolved system. The exponential operator can be evaluated, which amounts to computing powers of the Pauli-matrix $\sigma^{(2)}$,

$$\exp\left(\sigma^{(2)}t\right) = \sum_n \frac{(\sigma^{(2)}t)^n}{n!} \tag{D.221}$$

Because $(\sigma^{(2)})^0 = \sigma^{(0)}$, $(\sigma^{(2)})^2 = -\sigma^{(0)}$, $(\sigma^{(2)})^3 = -\sigma^{(2)}$ and $(\sigma^{(2)})^4 = \sigma^{(0)}$, continuing cyclically, one only ever obtains terms proportional to $\sigma^{(0)}$ or $\sigma^{(2)}$, with alternating signs:

$$\sum_n \frac{(\sigma^{(2)}t)^n}{n!} = \sigma^{(0)}\sum_n \frac{(-1)^n t^{2n}}{(2n)!} + \sigma^{(2)}\sum_n \frac{(-1)^n t^{2n+1}}{(2n+1)!} =$$

$$\sigma^{(0)}\cos t + \sigma^{(2)}\sin t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \tag{D.222}$$

and therefore, the time evolution of the harmonic oscillator is in fact given by sinusoidal oscillations: This is perhaps a bit remarkable; one does not need any intuition about the solution of the Lagrange equation of motion $\ddot{q} = -q$, or use a complex exponential $q \propto \exp(\mathrm{i}t)$ to transform the differential into an algebraic equation: It can be solved directly with a time-evolution operator that is *constructed*, not guessed.

D.5 *Unitary time-evolution in quantum mechanics*

If you look closely at the Schrödinger-equation

$$\mathrm{i}\hbar\partial_t\psi = \mathrm{H}\psi \tag{D.223}$$

it is perhaps not too dissimilar to eqn. D.219: In fact it would suggest that

$$\psi \propto \exp\left(-\frac{\mathrm{iH}t}{\hbar}\right) \tag{D.224}$$

with a time-evolution operator $\exp(-\mathrm{iH}t/\hbar)$ (the minus-sign appear because $1/\mathrm{i} = -\mathrm{i}$), as if the Hamilton-operator H is generating the time evolution. The definition of an inverse time evolution operator makes heavy use of the fact that H is hermitean, $\mathrm{H}^+ = \mathrm{H}$ and that iH is anti-hermitean, $(\mathrm{iH})^+ = -\mathrm{iH}$. That's because

$$\mathrm{U}(t) = \exp\left(-\frac{\mathrm{iH}t}{\hbar}\right) \tag{D.225}$$

is unitary:

$$\mathrm{U}^+(t) = \exp\left(-\frac{\mathrm{iH}t}{\hbar}\right)^+ = \left(\sum_n \frac{(-\mathrm{iH}t/\hbar)^n}{n!}\right)^+ = \sum_n \frac{(\mathrm{iH}t/\hbar)^n}{n!} = \exp\left(\frac{\mathrm{iH}t}{\hbar}\right) = \mathrm{U}(-t) \tag{D.226}$$

Because from additivity we would get $\mathrm{U}(t)\mathrm{U}(-t) = \mathrm{id}$ and $\mathrm{U}^{-1}(t) = \mathrm{U}(-t)$ we would conclude that $\mathrm{U}^{-1}(t) = \mathrm{U}(-t) = \mathrm{U}^+(t)$, and $\mathrm{U}(t)$ is unitary: Its inverse is given by the adjoint. As $\mathrm{U}(t)$ evolves a wave function by t into the future, the adjoint $\mathrm{U}^+(t) = \mathrm{U}(-t)$ evolves it back into the past by t. In this entire process the normalisation of the wave function is conserved.

There is a shortcut to this result. Complex conjugation of the Schroedinger equation gives

$$\mathrm{i}\hbar\partial_t\psi = \mathrm{H}\psi \quad \rightarrow \quad -\mathrm{i}\hbar\partial_t\psi^* = (\mathrm{H}\psi)^* = \mathrm{H}^+\psi^* = \mathrm{H}\psi^* \tag{D.227}$$

together with the hermiticity $\mathrm{H}^+ = \mathrm{H}$ of the Hamilton-operator H. But the overall sign on the left hand side could be captured by running time backwards,

$$-\mathrm{i}\hbar\partial_t\psi^* = \mathrm{i}\hbar\partial_{-t}\psi^* \tag{D.228}$$

such that the time-inverted Schrödinger equation becomes

$$\mathrm{i}\hbar\partial_{-t}\psi^* = \mathrm{H}\psi^* \tag{D.229}$$

Thinking of this as the defining equation of a Lie-time evolution operator gives

$$\mathrm{U}(-t) = \exp\left(\frac{\mathrm{iH}t}{\hbar}\right) \tag{D.230}$$

for evolving the system backwards in time.

It is amazing to see, how easily the point of time reversal is taken care of in classical, Newtonian mechanics: Neither $\mathcal{L}(q^i, \dot{q})$ does change, nor the Euler-Lagrange-equation or the resulting equation of motion $\ddot{q}^i = -\partial^i\Phi$. Hamilton-mechanics is funny: Both $\dot{p}$ and $\dot{q}$ change sign, such that motion proceeds in the opposite direction of the gradients $\partial\mathcal{H}/\partial p$ and $\partial\mathcal{H}/\partial q$, such that one can easily imagine how the motion proceeds backwards in phase space.