C FIELDS

C.1 *Lagrange-description of field dynamics*

Relativistic field equations in Nature, for instant for the Maxwell-field A^μ or for the metric $g_{\mu\nu}$ are commonly hyperbolic, second-order partial differential equations, and due to their hyperbolicity there is wave-like propagation of excitations along a light-cone, which is defined by the underlying geometric structure of spacetime. The first notion of a field was Newton's idea of an action at a distance: Somehow gravity from and on Earth needed to extent to the Moon and other celestial bodies. This is really a revolutionary thought as it was the first time in physics where the constituents of a system are not in direct physical contact. The question whether the fields are real or just a convenient way of computing forces between charges that couple to the field, is a bit philosophical but after all, all physical concepts that apply to the "material world" apply to fields in exactly the same way, including the point that the associated energy and momentum content of a field is able to source gravity.

C.2 *Lagrange-description of scalar field dynamics*

Deriving the field equation of a scalar field ϕ is almost like dissipationless continuum mechanics. Let's ignore dynamical evolution for a second and derive the most general linear theory with a second-order partial differential field equation, which would be necessarily elliptical if there's no proper time evolution. As expected one would write down a kinetic and potential term in a suitable Lagrange-density,

$$\mathcal{L}(\phi, \partial^i\phi) = \gamma_{ij}\partial^i\phi\,\partial^j\phi - 8\pi\rho\phi \tag{C.154}$$

and establish Hamilton's principle δS for varying the action S

$$\delta S = \delta\int \mathrm{d}^3x\,\mathcal{L}(\phi, \partial^i\phi) = \int \mathrm{d}^3x\left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial^i\phi}\delta\partial^i\phi\right) = \int \mathrm{d}^3x\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial^i\frac{\partial\mathcal{L}}{\partial\partial^i\phi}\right)\delta\phi \tag{C.155}$$

after writing $\delta\partial^i\phi = \partial^i\delta\phi$ and a successive integration by parts. Substitution of the Lagrange-density eqn. C.154 into the Euler-Lagrange-equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial^i\frac{\partial\mathcal{L}}{\partial\partial^i\phi} = 0 \tag{C.156}$$

which can be isolated from eqn. C.155 yields the Poisson-equation

$$\partial^i\partial_i\phi = \Delta\phi = -4\pi\rho \tag{C.157}$$

by realising that

$$\frac{\partial}{\partial\partial^i\phi}(\gamma_{ab}\partial^a\phi\partial^b\phi) = \gamma_{ab}\left(\frac{\partial\partial^a\phi}{\partial\partial^i\phi}\partial^b\phi + \partial^a\phi\frac{\partial\partial^b\phi}{\partial\partial^i\phi}\right) = \gamma_{ab}\left(\delta^a_i\partial^b\phi + \partial^a\phi\delta^b_i\right) = 2\partial_i\phi \tag{C.158}$$

while the rest of the terms in the Euler-Lagrange-equation is pretty easy.

Repeating the arguments for finding the most general Lagrange-function for a point particle leads to the Lagrange-density

$$\mathcal{L}(\phi, \partial^i\phi) = \frac{1}{2}\gamma_{ij}\partial^i\phi\,\partial^j\phi - 4\pi\rho\phi + \lambda\phi + \frac{m^2}{2}\phi^2 \tag{C.159}$$

with the associated field equation

$$\left(\Delta - m^2\right)\phi = -4\pi\rho + \lambda \tag{C.160}$$

for the most general scalar field equation that is linear and compatible with Ostrogradsky's theorem. If ϕ is the Newtonian gravitational potential Φ and interpreting the generalised Poisson-equation in terms of a gravitational theory we now know that m must be truly small, and that λ is small but certainly nonzero. While all this looks straightforward from an arithmetic point of view, the conceptual interpretation is not so easy: Hamilton's principle $\delta S = \delta\int d^3x\,\mathcal{L} = 0$ looks for a field configuration ϕ which minimises the action, and for a vacuum solution the kinetic term $\partial\mathcal{L}/\partial\partial^i\phi$ would be required to be perpendicular to $\delta\partial^i\phi$, which is perhaps a bit reminiscent of d'Alembert's principle.

Often you'll see m interpreted as the mass of the field ϕ, or at least as its inertia, even though at this point it's not more than a scale-invariance breaking inverse length scale. If the field ϕ is allowed to have its own dynamics in accordance with special relativity one would make the replacements $\gamma_{ij} \rightarrow \eta_{\mu\nu}$ and $\partial^i \rightarrow \partial^\mu$ to arrive at

$$S = \int d^4x\,\mathcal{L}(\phi, \partial^\mu\phi) \quad \text{with} \quad \mathcal{L}(\phi, \partial^\mu\phi) = \frac{1}{2}\eta_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - \frac{m^2}{2}\phi^2 \tag{C.161}$$

where we omitted the coupling to ρ on purpose because its transformation property is yet unclear, and let's focus on scales small compared to $1/\sqrt{\lambda}$. Variation then gives

$$\left(\Box + m^2\right)\phi = 0 \quad \rightarrow \quad \eta_{\mu\nu}k^\mu k^\nu = m^2 > 0 \quad \text{in Fourier space} \tag{C.162}$$

such that the wave vector k^μ is timelike and points to a location inside the light cone: Excitations of ϕ travel at speeds less than the speed of light which justifies to think of m as a mass. Please watch out for the minus signs here, as $\Box\exp(\pm i\eta_{\alpha\beta}k^\alpha x^\beta) = -\eta_{\mu\nu}k^\mu k^\nu\exp(\pm i\eta_{\alpha\beta}k^\alpha x^\beta)$ from $i^2 = -1$. We need the opposite sign in eqn. C.161 relative to eqn. A.3 as in the "mostly minus" sign convention η_{ij} are negative and η_{tt} is positive.

C.3 *Maxwell-electrodynamics and the gauge-principle*

We should step up the game after this example of scalar field dynamics and turn to the Maxwell-field A^μ: Firstly, it has internal degrees of freedom and transforms like a Lorentz-vector, $A^\mu \rightarrow \Lambda^\mu{}_\alpha A^\alpha$, secondly, it has the charge density $\jmath^\mu$ as a source, likewise a Lorentz vector, $\jmath^\mu \rightarrow \Lambda^\mu{}_\alpha \jmath^\alpha$. Thirdly, the charge density is conserved in the sense that $\partial_\mu \jmath^\mu = \partial_{ct}(c\rho) + \partial_i \jmath^i = 0$, and the field equation itself is linear, $\partial_\mu F^{\mu\nu} = 4\pi/c\ \jmath^\mu$ with the field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ containing the electric and magnetic fields. Clearly, this equation can not contain the entire information about six field components E^i and B^i to be derived from the field tensor which is coupled to just 4 components of charge $\jmath^\mu$. That's the reason why one needs the Bianchi-identity in addition, $\partial_\mu\tilde{F}^{\mu\nu} = 0$, most conveniently written with the dual field tensor $\tilde{F}^{\mu\nu}$,

$$\tilde{F}^{\mu\nu} = +\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \quad \text{and} \quad \tilde{F}_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{C.163}$$

with the 4-dimensional Levi-Civita symbol $\epsilon^{\mu\nu\alpha\beta}$: One needs an object that is antisymmetric at least in every index pair to give a non-vanishing result. $F^{\mu\nu}$ is auto-dual,

$$\tilde{\tilde{F}}^{\mu\nu} = +\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\tilde{F}_{\alpha\beta} = -\frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\alpha\beta\rho\sigma}F^{\rho\sigma} = F^{\mu\nu}. \tag{C.164}$$

That is a lot to digest, in particular the property of the field tensor being antisymmetric, $F^{\nu\mu} = -F^{\mu\nu}$, as well as the existence of the dual field tensor $F^{\mu\nu}$ and its role in the dynamics of the electromagnetic field: First of all, the Maxwell-equations are hyperbolic partial differential equations in A^{μ}, with propagations traveling along the light cone, as the wave vectors are lightlike, $k_{\mu}k^{\mu} = 0$. The source $\jmath^{\mu}$ can be dynamically changing but under conservation, and the transformation properties of $\jmath^{\mu}$ and A^{μ} are identical.

Deriving the Maxwell-field equation from a variational principle asks the question how the invariance-covariance principle could be incorporated. As a square of first derivatives of A^{μ} as a kinetic term which is invariant under Lorentz-transformations one could use $F^{\mu\nu}F_{\mu\nu}$, such that one can ensure a linear field equation after variation from this particular quadratic invariant. $\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = F^{\mu\nu}F_{\mu\nu} \propto E_iE^i - B_iB^i$, so there is nothing new from using the Frobenius-norm of $\tilde{F}$ instead of F. The other possible quadratic invariant $\tilde{F}^{\mu\nu}F_{\mu\nu} = F^{\mu\nu}\tilde{F}_{\mu\nu} \propto E_iB^i$ would likewise give a linear field equation, but there is an issue because E_iB^i is a scalar product between and axial and a polar vector, and is as such pseudoscalar, i.e. it changes sign under parity inversion and is therefore not a proper scalar. Already at this point one may conjecture that the Lagrange-function is bounded by 0 and that this value corresponds to vacuum solutions: The magnetic and electric field energies of an electromagnetic wave are always exactly equal, such that $E_iE^i - B_iB^i = 0$, and they are necessarily perpendicular to each other, $E_iB^i = 0$. On the side coupling the fields to charges, $A_{\mu}\jmath^{\mu}$ would be perfect in a linear field equation. Collecting these ideas suggests that the Maxwell-action is given by

$$S = \int d^4x \left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{4\pi}{c}A_{\mu}\jmath^{\mu}\right) \tag{C.165}$$

What about terms like $A_{\mu}A^{\mu}$? It would in fact be compatible with a linear field equation with a term proportional to A_{μ}, but it would violate gauge-symmetry as a new symmetry principle. Maxwell's field equation $\partial_{\mu}F^{\mu\nu} = 4\pi/c\ \jmath^{\nu}$ is unchanged under the gauge transform $A^{\mu} \rightarrow A^{\mu} + \partial^{\mu}\chi$ with a scalar field χ, as

$$F^{\mu\nu} \rightarrow \partial^{\mu}(A^{\nu} + \partial^{\nu}\chi) - \partial^{\nu}(A^{\mu} + \partial^{\mu}\chi) = F^{\mu\nu} \tag{C.166}$$

with the interchangeability $\partial^{\mu}\partial^{\nu}\chi = \partial^{\nu}\partial^{\mu}\chi$. And of course, with the invariance of $F^{\mu\nu}$ under gauge transforms one does not possibly observe any change in the observable fields E^i and B^i. The freedom to transform A^{μ} can be used to make the computation of fields easier and to decouple field equations. For instance,

$$\partial_{\mu}F^{\mu\nu} = \partial_{\mu}\partial^{\mu}A^{\nu} - \partial^{\nu}\partial_{\mu}A^{\mu} = \Box A^{\nu} - \partial^{\nu}\partial_{\mu}A^{\mu} = \frac{4\pi}{c}\jmath^{\nu} \tag{C.167}$$

would need to be solved for computing A^{μ} from $\jmath^{\mu}$, such that the fields $F^{\mu\nu}$ are obtained from A^{μ} by successive derivation. There are known Green-functions for solving $\Box A^{\mu} = 4\pi/c\ \jmath^{\mu}$, even index-by-index, but the divergence $\partial_{\mu}A^{\mu}$ couples these four equations together. Under gauge transforms one obtains the transformation

$$\partial_\mu A^\mu \to \partial_\mu A^\mu + \partial_\mu \partial^\mu \chi = 0 \to \Box\chi = -\partial_\mu A^\mu \tag{C.168}$$

implying that one can always find a transform that sets $\partial_\mu A^\mu$ to zero, it is even uniquely defined by the relation $\Box\chi = -\partial_\mu A^\mu$, as the χ needed follows from solving the wave equation with $-\partial_\mu A^\mu$ as a source. $\partial_\mu A^\mu = 0$ is called Lorenz-gauge.

It is very interesting how gauge-transforms operate on the action or the Lagrange-density:

$$S \to \int d^4x \left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{4\pi}{c}(A_\mu + \partial_\mu\chi)\jmath^\mu\right) = S + \frac{4\pi}{c}\int d^4x\,(\partial_\mu\chi)\jmath^\mu \tag{C.169}$$

as $F^{\mu\nu}$ is gauge invariant anyway. The coupling of A^μ to $\jmath^\mu$ can be reformulated using the Leibnitz-theorem,

$$\int d^4x\,\jmath^\mu\partial_\mu\chi = \int d^4x\,\partial_\mu(\jmath^\mu\chi) - \int d^4x\,(\partial_\mu \jmath^\mu)\chi \tag{C.170}$$

where the first term can be reformulated with the Gauß-theorem,

$$\int_V d^4x\,\partial_\mu(\jmath^\mu\chi) = \int_{\partial V} dQ_\mu\,\jmath^\mu\chi = 0 \tag{C.171}$$

which can be made to vanish if $\chi = 0$ on ∂V by choice. The second term is automatically zero for conserved sources, where $\partial_\mu \jmath^\mu = 0$. So effectively, the Lagrange-function is unchanged by the gauge transform if the electric charge density as the source of the Maxwell-field is conserved, which ultimately is the foundation of the knot-rule in electric circuits: That the sum of inflowing and outflowing electric currents at one knot in a circuit cancel exactly if there is not builtup of charge is the consequence of the continuity equation $\partial_\mu \jmath^\mu = 0$, which appears consistent with the gauge-freedom of A^μ.

C.4 *Electromagnetic duality and axions*

Maxwell-electrodynamics *in vacuum* obeys a peculiar symmetry called electromagnetic duality: In the absence of sources, the field equation $\partial_\mu F^{\mu\nu} = 0$ and the Bianchi-identity $\partial_\mu \tilde{F}^{\mu\nu} = 0$ become equal, so the duality transform $F^{\mu\nu} \leftrightarrow \tilde{F}^{\mu\nu}$ doesn't give rise to any difference in the field dynamics. In terms of fields, the duality transform reads $E^i \to B^i$ and $B^i \to -E^i$, which makes perfect sense as $\partial_\mu F^{\mu\nu} = 0$ contains the two statements $\partial_i B^i = 0$ and $\epsilon^{ijk}\partial_j E_k = -\partial_{ct} B^i$, whereas $\partial_\mu \tilde{F}^{\mu\nu} = 0$ makes sure that $\partial_i E^i = 0$ and $\epsilon^{ijk}\partial_j B_k = +\partial_{ct} E^i$: Effectively, the two pairs of Maxwell-equations interchange their meaning under the duality transform. Or, to formulate this in a stronger way: Only the presence of charges $\jmath^\mu$ defines a difference between $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$.

In a fantasy world with electric charges $\jmath^\mu$ and magnetic charges $\imath^\mu$ one could set up a perfectly reasonable Maxwell-like theory just by postulating

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}\jmath^\nu \quad \text{as well as} \quad \partial_\mu \tilde{F}^{\mu\nu} = \frac{4\pi}{c}\imath^\nu \tag{C.172}$$

provided that both charges are conserved, $\partial_\mu \jmath^\mu = 0$ and independently $\partial_\mu \imath^\mu = 0$. Both field equations are, due to the antisymmetry of the field tensor, made compatible with conservation of the respective charge, $\partial_\mu\partial_\nu\tilde{F}^{\mu\nu} = 4\pi/c\,\partial_\nu\imath^\nu = 0$, as a contraction of the antisymmetric $\tilde{F}^{\mu\nu}$ with the symmetric $\partial_\mu\partial_\nu$, and likewise $\partial_\mu\partial_\nu F^{\mu\nu} = 4\pi/c\,\partial_\nu \jmath^\nu = 0$, for exactly the same reason.

While everything is perfectly well-defined on the basis of the field equations, there is a problem when trying to write down a Lagrange-density: The potential A^μ would not exist. In fact, A^μ relies on the dual field equation being zero, which can be most easily seen in terms of the components: $\partial_i B^i = 0$ implies that the magnetic field can be written as $B^i = \epsilon^{ijk}\partial_j A_k$ derived from a vector potential A_k, and at the same time $\epsilon^{ijk}\partial_j E_k = -\partial_{ct}B^i = -\partial_{ct}\epsilon^{ijk}\partial_j A_k$, such that $\epsilon^{ijk}\partial_j(E_k + \partial_{ct}A_k) = 0$. That in turn implies, that the term in brackets can be written as a gradient, $E_k + \partial_{ct}A_k = -\partial_k\Phi$ with a scalar potential Φ. In summary, the components A_k and Φ of the potential A^μ rely in their existence on the absence of magnetic charges, $\imath^\mu = 0$.

But one needs A^μ for a Lagrange-description of electrodynamics, otherwise the coupling to the sources could not be formulated in the $A_\mu \jmath^\mu$-term: Electrodynamics with $\jmath^\mu \neq 0 \neq \imath^\mu$ could be defined on the level of the field equations but not with a Lagrange-density.

Let's investigate the second possible quadratic invariant $\tilde{F}_{\mu\nu}F^{\mu\nu}$ which is expressed in field components $\propto E_i B^i$ and therefore pseudoscalar: parity inversion $x^i \to -x^i$ or inversion of $ct \to -ct$ would result in a change in sign and excludes the term from the Lagrange-density as it is not properly scalar. This can be remedied by including a pseudoscalar field θ along with its own dynamics

$$S = \int d^4x \left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{4\pi}{c}A_\mu \jmath^\mu + \theta F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\partial^\mu\theta\partial^\nu\theta - V(\theta)\right) \tag{C.173}$$

where $\theta F_{\mu\nu}\tilde{F}^{\mu\nu}$ and $\eta_{\mu\nu}\partial^\mu\theta\partial^\nu\theta$ are perfectly scalar. The interaction potential $V(\theta)$ could include a term $\propto m^2\theta^2$ which itself is scalar again. Then, the Lagrange-density describes a massive pseudoscalar field θ, which in this context is called axion, and variation of eqn. C.173 gives rise to a coupled set of partial differential equations for $F^{\mu\nu}$ and θ.

C.5 *Poynting-law and conservation of energy and momentum*

Fields are not only affecting test charges by accelerating them, but they are physically real in their own right: They have their own dynamics, they can transport energy and momentum, and would be ultimately sources of gravity. The energy and momentum content is derived from the independence of the Lagrange-density of position x^μ, i.e. the working principle of the fields is supposed to be the same at every location and at every instant in time. One notices how effectively momentum and energy conservation have the same origin now, unlike classical mechanics.

The starting point is to define a shift of the Lagrange-function to a new position in spacetime by a separation a_α, which can be done by defining the operator $a_\alpha\partial^\alpha$ and apply it to the Lagrange density,

$$\delta\mathcal{L} = a_\alpha\partial^\alpha\mathcal{L} \quad \text{with the variation being} \quad \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\partial_\mu\phi \tag{C.174}$$

which changes as the fields and their deriatives take one new values as one moves by a_α across spacetime. Working for simplicity with a scalar field ϕ one gets variations

$$\phi \to \phi + \underbrace{a_\alpha \partial^\alpha \phi}_{\delta\phi} \quad \text{and} \quad \partial_\mu \phi \to \partial_\mu \phi + \underbrace{a_\alpha \partial^\alpha \partial_\mu \phi}_{\delta \partial_\mu \phi} \tag{C.175}$$

To deal with the term $\delta\partial_\mu\phi$ we apply the Leibnitz-rule as follows:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right) = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\mu \delta\phi}_{\delta\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi} \tag{C.176}$$

such that we can write

$$\delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right) \tag{C.177}$$

where the first bracket is necessarily zero, as a consequence of the Euler-Lagrange-equations for the field ϕ. Then,

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right) \tag{C.178}$$

where we can now substitute the displacements by a_α:

$$a_\alpha \left(\partial^\alpha \mathcal{L} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial^\alpha \phi \right) \right) = 0 \tag{C.179}$$

and by rewriting $\partial^\alpha = \eta^{\mu\alpha}\partial_\mu$ one can isolate the energy momentum tensor,

$$a_\alpha \partial_\mu \underbrace{\left(\eta^{\mu\alpha} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial^\alpha \phi \right)}_{-T^{\mu\alpha}} = 0 \tag{C.180}$$

with a corresponding conservation law $\partial_\mu T^{\mu\alpha} = 0$ as a_α is arbitrary: Perhaps it's interesting to note that $\partial\mathcal{L}/\partial\partial_\mu\phi$ would be the canonical field momentum, so we are actually carrying out a Legendre-transform of $\mathcal{L}$ to arrive at the energy-momentum tensor.

The same arguments apply to the Maxwell-field A^μ but with a small exception as there is gauge-symmetry to be respected in addition. We should not differentiate with respect to the straightforward derivatives $\partial_\mu A_\nu$ but rather with respect to the anti-symmetrised variant, $\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$, which is gauge-invariant. Therefore, the variation of the field A^μ under a shift a_α would be

$$\delta A^\mu = a_\alpha (\partial^\alpha A^\mu - \partial^\mu A^\alpha) = a_\alpha F^{\alpha\mu} \tag{C.181}$$

Therefore, the variation of $\mathcal{L}$ becomes

$$\delta\mathcal{L} = a_\alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\sigma} F^\alpha{}_\sigma \right) \tag{C.182}$$

Rewriting $\delta\mathcal{L} = a_\alpha \partial^\alpha \mathcal{L}$ and $\partial^\alpha = \eta^{\alpha\mu}\partial_\mu$ then gives

$$a_\alpha \partial_\mu \left(\eta^{\alpha\mu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\partial_\mu A_\sigma} F^\alpha{}_\sigma\right) = 0 \tag{C.183}$$

with the corresponding energy momentum tensor $T^{\alpha\mu}$ and its covariant conservation law $\partial_\mu T^{\alpha\mu} = 0$. For the Maxwell-Lagrange-density we have

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\sigma} = -\frac{1}{4\pi} F^{\mu\sigma} \tag{C.184}$$

so that

$$T^{\mu\nu} = \frac{1}{4\pi}\left(\eta_{\alpha\beta} F^{\mu\alpha} F^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right) \tag{C.185}$$

which is naturally symmetric and traceless:

$$4\pi \eta_{\mu\nu} T^{\mu\nu} = \eta_{\mu\nu}\eta_{\alpha\beta} F^{\mu\alpha} F^{\beta\nu} + \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} = F_{\alpha\beta} F^{\beta\alpha} + F_{\alpha\beta} F^{\alpha\beta} = 0 \tag{C.186}$$

by switching the index order in one of the terms.

C.6 *Covariant electrodynamics in matter*

The Maxwell-equations in matter, written down in an index notation but after choosing an explicit frame, read:

$$\partial_i D^i = 4\pi\rho, \quad \partial_i B^i = 0, \quad \epsilon^{ijk}\partial_j E_k = -\partial_{ct} B^i, \text{ and } \quad \epsilon^{ijk}\partial_j H_k = +\partial_{ct} D^i + \frac{4\pi}{c} j^i \tag{C.187}$$

and a peculiar difference between the fields D^i and B^i (noted as vectors) and the excitations E_i and H_i (written as linear forms) emerges. Of course it's a choice which of the two pairs is written as vectors and which as linear forms, so

$$\partial^i D_i = 4\pi\rho, \quad \partial^i B_i = 0, \quad \epsilon_{ijk}\partial^j E^k = -\partial_{ct} B_i, \text{ and } \quad \epsilon_{ijk}\partial^j H^k = +\partial_{ct} D_i + \frac{4\pi}{c} j_i \tag{C.188}$$

is equally valid. Normally, one would need to define tensors to relate the vectors with the linear forms, $B^i = \mu^{ij} H_j$ with the permeability tensor μ^{ij} and $D^i = \epsilon^{ij} E_j$ with the dielectric tensor ϵ^{ij}. Apart from symmetry (which ensures that there is an orthogonal principal axis frame with three real-valued eigenvalues) the two tensors are free and would describe the general linear relationship in a possibly anisotropic medium between the fields and excitations. If the medium is isotropic, $\mu^{ij} = \mu\delta^{ij}$ and $\epsilon^{ij} = \epsilon\delta^{ij}$, so that the usual relation $B^i = \mu\delta^{ij} H_j = \mu H^i$ and $D^i = \epsilon\delta^{ij} E_j = \epsilon E^i$ is recovered.

Taking this one step further, one would notice that the two homogeneous Maxwell-equations

$$\partial_i B^i = 0, \quad \epsilon^{ijk}\partial_j E_k = -\partial_{ct} B^i, \tag{C.189}$$

depend on B^i and E_i, while the two inhomogeneous Maxwell-equations depend on the other pair,

$$\partial^i D_i = 4\pi\rho, \quad \epsilon^{ijk}\partial_j H_k = +\partial_{ct} D^i + \frac{4\pi}{c} \jmath^i. \tag{C.190}$$

Because of this separation, one should package E^i and B^i into a tensor $\tilde{F}^{\mu\nu}$ to reproduce the homogeneous equations from $\partial_\mu \tilde{F}^{\mu\nu} = 0$. Analogously, D^i and H^i should then be part of a tensor $G^{\mu\nu}$ to generate the inhomogeneous equations from $\partial_\mu G^{\mu\nu} = 4\pi/c\, \jmath^\nu$. Clearly, there is now a second breaking of the duality taking place, because $G^{\mu\nu} \neq \tilde{F}^{\mu\nu}$! That, however is not straightforward: One has B^i as a vector and E_i as a linear form for $\tilde{F}^{\mu\nu}$, and likewise D^i as a vector and H_i as a linear form for $G^{\mu\nu}$ is given, so one needs to invoke the dielectric and permeability tensors to convert the linear forms E_i and H_i to vectors first.

C.7 *Finsler-geometry and Lorentz-forces*

A massive test particle tries to minimise proper time as the relativistic generalisation of the action S

$$S = -mc \int ds = -mc^2 \int d\tau \tag{C.191}$$

which is solved in the absence of forces by a straight line, $d^2x^\mu/d\tau = 0$, or equivalently, $x^\mu(\tau) = a^\mu \tau + b^\mu$ with two integration constants a^μ and b^μ. If there is a nonzero specific charge q/m the particle is accelerated by Lorentz-forces

$$\frac{d^2x^\mu}{d\tau^2} = \frac{q}{m} F^{\mu\nu} \frac{dx_\nu}{d\tau} \tag{C.192}$$

Let's re-derive this equation of motion from a variational principle, because it gives rise to a new geometric structure, called a Finsler-geometry. To cut things short, let's postulate

$$S = -mc^2 \int d\tau + q \int dx^\mu A_\mu \tag{C.193}$$

☠ *Can you show that the term $\int dx^\mu A_\mu$ is gauge-invariant?*

with a potential A_μ . While the first term is defined by the metric structure of spacetime, $ds^2 = c^2 d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, the second term involves the scalar product between A_μ and dx^μ. If A_μ is given directly in terms of a linear form, one actually does not need a metric structure to compute $dx^\mu A_\mu = \eta_{\mu\nu} dx^\mu A^\nu$. So effectively, there are two geometric structures at work, the metric structure $\eta_{\mu\nu}$ and the structure defined by the scalar product of vectors with the linear form A_μ: This is called a Finsler-geometry. The interpretation of the $A_\mu dx^\mu$-term is not easy, but perhaps one could imagine A_μ as some kind of headwind or tailwind that changes the proper time of the particle depending on in which direction it moves relative to the direction and magnitude of the vector field A^μ.

As the values of A_μ that the particle sees depends on the trajectory, the variation of the action gives

$$\delta \int A_\mu dx^\mu = \int \delta A_\mu dx^\mu + \int A_\mu \delta dx^\mu = \int \delta A_\mu dx^\mu - \int dA_\mu \delta x^\mu \tag{C.194}$$

with the usual procedure to write $\delta dx^\mu = d\delta x^\mu$ and a successive integration by parts. Then, we trace back the variation and the differential of A_μ to a coordinate shift,

$$\delta A_\mu = \frac{\partial A_\mu}{\partial x^\alpha}\delta x^\alpha \quad \text{and} \quad dA_\mu = \frac{\partial A_\mu}{\partial x^\alpha}dx^\alpha \tag{C.195}$$

such that the variation becomes

$$\delta\int A_\mu dx^\mu = \int \frac{\partial A_\mu}{\partial x^\alpha}\delta x^\alpha dx^\mu - \frac{\partial A_\mu}{\partial x^\alpha}dx^\alpha \delta x^\mu = \int\left(\frac{\partial A_\mu}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\mu}\right)\delta x^\alpha dx^\mu \tag{C.196}$$

after renaming the indices $\mu \leftrightarrow \alpha$ in the second term (all indices are fully saturated and the terms are both scalar, so it does not matter how the indices are called). Introducing the velocity $dx^\alpha/d\tau$ and identifying the field tensor brings the integral into the final shape

$$\delta\int A_\mu dx^\mu = \int d\tau\, F_{\alpha\mu}\frac{dx^\alpha}{d\tau}\delta x^\mu \tag{C.197}$$

which, combined with the variation of $-mc^2\int d\tau$ already worked out in eqn. A.93, gives the Lorentz-equation of motion C.192:

$$m\frac{d^2x^\mu}{d\tau^2} = qF^{\mu\alpha}\frac{dx_\alpha}{d\tau}. \tag{C.198}$$

C.8 *Light-cone structure beyond metric spacetimes*

Let's write out the kinetic term of the Lagrange-density of electrodynamics explicitly

$$S = \int d^4x\, \eta_{\alpha\mu}\eta_{\beta\nu}F^{\alpha\beta}F^{\mu\nu} = \int d^4x\, \frac{\eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\alpha\nu}\eta_{\beta\mu}}{2}F^{\alpha\beta}F^{\mu\nu} = \int d^4x\, G_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu} \tag{C.199}$$

using antisymmetry $F^{\nu\mu} = -F^{\mu\nu}$ and renaming indices. The quantity $G_{\alpha\beta\mu\nu}$ is antisymmetric in the first and second index pair and defines a measure of area instead of a measure of length, as a metric $\eta_{\mu\nu}$ would. In 3 dimensions one determines the area of the parallelogram spanned by two vectors a^i and b^i from the norm of $c_i = \epsilon_{ijk}a^j b^k$, so effectively through

$$\text{area} = \delta^{il}c_i c_l = \delta^{il}\epsilon_{ijk}\epsilon_{lmn}a^j b^k a^m b^n = \left[\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right]a^j b^k a^m b^n = a^2b^2 - (a_i b^i)^2 \tag{C.200}$$

where the square brackets have the same index structure as $G_{\alpha\beta\mu\nu}$, so it is justified to speak of $G_{\alpha\beta\mu\nu}$ as a measure of area. In fact, $a_i b^j = ab\cos\alpha$ for a standard scalar product, so

$$\text{area} = a^2b^2(1-\cos^2\alpha) = a^2b^2\sin^2\alpha \tag{C.201}$$

as expected. Perhaps one could imagine that the Maxwell action S measures the area between the vectors ∂^μ and A^ν over the spacetime volume.