B WAVES

B.1 *Taxonomy of waves*

Waves, i.e. periodic phenomena in x and t are found everywhere in physics and can be differentiated to be in two categories: classical mechanical waves usually rely on the elastic properties of a medium which, due to its internal structure, resists deformation from its equilibrium. The magnitude of the restoring force that the medium provides drives the wave and allows it to travel.

In mechanical waves where the medium could be a fluid described by some type of Navier-Stokes equation, any term on the right hand side could be a suitable restoring force, for instance

$$\partial_t \upsilon^i + (\upsilon_j \partial^j)\upsilon^i = \underbrace{-\rho\partial^i p}_{\text{sound}} - \underbrace{\partial^i \Phi}_{\text{gravity}} + \underbrace{\mu\partial_j\partial^j \upsilon^i}_{\text{elastic}} + \underbrace{2\epsilon^{ijk}\Omega_j \upsilon_k}_{\text{Rossby}} + \ldots \tag{B.111}$$

In sound waves, pressure gradients can accelerate the fluid and if the equation of state provides $\partial p/\partial\rho > 0$, pressure gradients introduce velocities that rarefy the medium, so that it returns to its equilibrium state. Gravity waves are for instance large waves on the surface of water (also called Airy-waves) where the weight of the "mountain" of water is the restoring force. In elastic waves the restoring force is derived from the internal structure of the medium, and even the Coriolis-acceleration can act as a restoring force: This is the case in atmospheric Rossby-waves. Typically, the magnitude of the restoring force is contrasted with the inertia of the medium, and the ratio between the two determine the propagation velocity, which then entirely depends on the material properties of the fluid.

In contrast, relativistic waves are excitations of a field, whose dynamics is described with a field equation, and typically these field equation have a particular mathematical structure allowing for oscillations: Field equations in fundamental physics are hyperbolic partial differential equations which is a natural consequence of the spacetime structure. Personally I find it very interesting, that the same wave equations are found in very different contexts, and that propagation speeds can be determined by relativity on one side and by the internal structure of a medium on the other. When thinking about ideas on the lumiferous aether over a hundred years ago and the measurements of the speed of light that were already available with high precision at that time, it must have been truly daunting to explain the high value of c from the low inertia and the high restoring force of the aether, if light was imagined to be an elastic wave.

B.2 *Elastic waves and wave equations*

Perhaps the most intuitive example of an elastic mechanical wave is that of a string with mass per length ρ under tension σ: Already now one would intuitively think that the ratio between ρ and σ should determine the velocity of elastic waves. In a string instrument, the ratio between velocity and fixed string length gives the frequency ω of a sound, and one observes an increase of frequency with higher string tension and one typically uses thicker strings for lower frequency notes.

The kinetic energy $\mathrm{d}T$ for each differential bit of string is given by the velocity $\partial y/\partial t = \dot{y}$ by which the amplitude y changes along the string with coordinate x,

$$dT = \frac{\rho}{2}\dot{y}^2 dx \quad \rightarrow \quad T = \int dT = \frac{\rho}{2}\int dx\,\dot{y}^2 \tag{B.112}$$

For the corresponding potential energy dW we need to compute by how much the amplitudes $y(x)$ change the overall length of the string: $dl^2 = dx^2 + dy^2$ from Pythagoras' theorem gives $dl = \sqrt{1+y'^2}dx$ with $y' = dy/dx$, and consequently

$$dW = \sigma(dl - dx) = \sigma\left(\sqrt{1+y'^2} - 1\right)dx \simeq \frac{\sigma}{2}y'^2 dx \quad \rightarrow \quad W = \int dW = \frac{\sigma}{2}\int dx\,y'^2 \tag{B.113}$$

Assembling both into a classical Lagrange-function yields

$$\mathcal{L}(\dot{y}, y') = \int dx\,\left(\frac{\rho}{2}\dot{y}^2 - \frac{\sigma}{2}y'^2\right) \tag{B.114}$$

from which we get the action S straight away:

$$S = \int dt\,\int dx\,\left(\frac{\rho}{2}\dot{y}^2 - \frac{\sigma}{2}y'^2\right) \tag{B.115}$$

The Lagrange-function $\mathcal{L}$ depends on $\dot{y}$ as well as on y', which Hamilton's principle needs to respect. The correct variation of S would then be

$$\delta S = \int dt\,\int dx\,\left(\frac{\partial\mathcal{L}}{\partial y}\delta y + \frac{\partial\mathcal{L}}{\partial\dot{y}}\delta\dot{y} + \frac{\partial\mathcal{L}}{\partial y'}\delta y'\right) \tag{B.116}$$

while the coordinate y is cyclic and the first term does not play a role, the variations in the second and third term can be rewritten as $\delta\dot{y} = \partial(\delta y)/\partial t$ and $\delta y' = \partial(\delta y)/\partial x$ to enable integration by parts, with respect to dt in the second and with respect to dx in the third term:

$$\delta S = \int dt\,\int dx\,\left(\frac{\partial\mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{y}} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial y'}\right)\delta y = 0 \tag{B.117}$$

where we can isolate the Euler-Lagrange-function

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{y}} + \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial y'} = \frac{\partial\mathcal{L}}{\partial y} \tag{B.118}$$

Substitution of eqn. B.114 into eqn. B.118 then gives:

$$\rho\ddot{y} - \sigma y'' = 0 \quad \text{or} \quad \left(\frac{\partial^2}{\partial(ct)^2} - \frac{\partial^2}{\partial x^2}\right)y = 0 \quad \text{with} \quad c^2 = \frac{\sigma}{\rho} \tag{B.119}$$

where the speed of propagation of the elastic wave is in fact determined by the ratio of the tension as the restoring force and the inertia of the string. The wave equation can be solved by separating the temporal and spatial dependence with the ansatz $y(x,t) = \phi(x)\psi(t)$, such that

$$\frac{1}{\psi(t)}\frac{\partial^2}{\partial(ct)^2}\psi(t) = \frac{1}{\phi(x)}\frac{\partial^2}{\partial x^2}\phi(x) = -k^2 \tag{B.120}$$

after a separation of variables, and because every term depends only on t or on x, they need to be independently equal to a constant, which we choose to be negative (for enforcing oscillating solutions). Individually, every term is then solved by a harmonic oscillation, and substitution then shows that

$$y(x,t) \propto \exp\left(\pm \mathrm{i}\,(kx - \omega t)\right) \tag{B.121}$$

with the dispersion relation $\omega = \pm ck$ and the speed of the elastic wave $c = \sqrt{\sigma/\rho}$. The sign between kx and ωt follows from requiring whether a plane of constant phase travels into positive or negative x-direction. Both directions are clearly allowed, as $\partial_{ct}^2 - \partial_x^2 = (\partial_{ct} - \partial_x)(\partial_{ct} + \partial_x)$ from the binomial formula.

It is important to realise that an elastic wave is able to transport energy even without any transport of the medium on which it travels.

B.3 *Partial differential equations: hyperbolic vs. elliptic*

Wave-equations are typically partial differential equations involving second derivatives, for instance for a scalar field

$$\Box\phi = \eta_{\mu\nu}\partial^\mu\partial^\nu\phi = \left(\partial_{ct}^2 - \Delta\right)\phi = 0 \tag{B.122}$$

At this point it is well worth to go through the classification of second-order partial differential equations: Comparing $\Box\phi = 0$ as a wave equation with $\Delta\phi = 0$ as a static field equation shows that the signs of the derivative operators is $(-,+,+,+)$ in the first case and $(+,+,+)$ without a change in the second case. This seems to be highly significant, as one obtains oscillatory solutions in the first, and (decreasing, at least in 3 dimensions or more) power-law solutions in the second case.

Before we go through the classification of partial differential equation, we need to introduce some slang, borrowed from the theory of conic sections. Please consider a quadratic form of two coordinates x and y,

$$\begin{pmatrix} x \\ y \end{pmatrix}^t \underbrace{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}}_{\Delta} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + bxy + cy^2 = \text{const.} \tag{B.123}$$

Depending on the structure of eigenvalues of the matrix Δ, the quadratic form describes very different curves: If $b = 0$ (for simplicity) and $a = c = 1 > 0$ one obtains $x^2 + y^2 = \text{const}$, which can be rewritten in a parametric form by setting $x = \cos t$ and $y = \sin t$ such that the quadratic form describes a circle as a consequence of $\cos^2 t + \sin^2 t = 1$, and if $a \neq c$ an ellipse. If $a = 1$ and $c = -1$, the quadratic form becomes $x^2 - y^2 = \text{const}$, i.e hyperbolae with the hyperbolic functions as parametric forms, using $\cosh^2 t - \sinh^2 t = 1$. More generally, the picture arises that $\det\Delta > 0$ for the elliptical conic section and conversely, $\det\Delta < 0$ for the hyperbolic conic section.

Applying this idea to the classification of partial differential equations, we start with a homogeneous second-order PDE in two variables,

$$a(x,y)\frac{\partial^2}{\partial x^2}\phi(x,y) + b(x,y)\frac{\partial^2}{\partial x\partial y}\phi(x,y) + c(x,y)\frac{\partial^2}{\partial y^2}\phi(x,y) = \mathrm{A}(x,y)\phi(x,y) \tag{B.124}$$

and assemble the matrix Δ

$$\Delta = \begin{pmatrix} a(x,y) & \frac{1}{2}b(x,y) \\ \frac{1}{2}b(x,y) & c(x,y) \end{pmatrix} \tag{B.125}$$

The determinant of Δ then determines, whether the PDE is elliptical $\det\Delta > 0$, parabolic $\det\Delta = 0$ or hyperbolic $\det\Delta < 0$.

Sticking to 2 dimensions and pairs of variables for simplicity, a PDE like the Poisson-equation

$$\Delta\phi = \frac{\partial^2}{\partial x^2}\phi(x,y) + \frac{\partial^2}{\partial y^2}\phi(x,y) = 0 \tag{B.126}$$

would be elliptical, as the determinant of Δ would come out positive: $a = c = 1$ and $b = 0$. Elliptical differential equations have only unique solutions after boundary conditions are specified, either of the Dirichlet or Neumann-type. Typical solutions are decreasing (at least in 3 dimensions or higher) with increasing coordinates and parity invariant, as $(x,y) \to (-x,-y)$ does not change anything. On the other hand, a wave-equation exhibits a sign change,

$$\Box\phi(t,x) = \frac{\partial^2}{\partial(ct)^2}\phi(t,x) - \frac{\partial^2}{\partial x^2}\phi(t,x) = 0 \tag{B.127}$$

with $a = 1$, $c = -1$ and $b = 0$ in these coordinates and would be hyperbolic as $\det\Delta < 0$. In this case, it is enough to specify initial conditions and the PDE evolves them in a well-defined and unique way into the future. There is clearly the notion of a light-cone and it is actually the case that the metric structure of spacetime with the Minkowski-metric is uniquely suited for hyperbolic PDEs: It is even the fact, that the Lorentzian spacetime as a metric spacetime that allows for hyperbolic evolution is unique! Switching to light-cone coordinates $\partial_{ct} + \partial_x = \partial_u$ and $\partial_{ct} - \partial_x = \partial_v$ brings the wave equation into the form

$$\Box\phi(u,v) = \frac{\partial^2}{\partial u \partial v}\phi(u,v) = 0 \tag{B.128}$$

this time with $a = c = 0$ and $b = 1$, but the determinant $\det\Delta < 0$ nonetheless: The wave equation is hyperbolic in light cone coordinates just as well. In the wave equation there is parity invariance and time-reversal invariance. Perhaps it's a very good exercise to go through all iconic PDEs in theoretical physics and classify them as elliptical, parabolic or hyperbolic partial differential equations.

B.4 *Relativistic waves and hyperbolicity*

The dynamics of relativistic fields is described by hyperbolic PDE with their clear notion of a light cone and their time evolution from any field-configuration specified as initial conditions. As an example, we can substitute $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ into the Maxwell-equation,

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}\jmath^\nu \quad \to \quad \partial_\mu\partial^\mu A^\nu - \partial^\nu \underbrace{\partial_\mu A^\mu}_{=0} = \Box A^\nu = \frac{4\pi}{c}\jmath^\nu \tag{B.129}$$

which becomes clearly a hyperbolic wave equation. But the Lorenz-gauge $\partial_\mu A^\mu = 0$ is not required for hyperbolicity, in fact, even without any gauge fixing it would be hyperbolic. As a linear PDE this is most conveniently solved by constructing a Green-function including retardation as the potential for a point charge.

There is a similar wave equation for the field tensor itself: Starting at the Bianchi-identity,

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \tag{B.130}$$

which can immediately be verified by substituting $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, one can have ∂_λ act on it,

$$\underbrace{\partial_\lambda\partial^\lambda}_{\square} F^{\mu\nu} - \partial^\mu \underbrace{\partial_\lambda F^{\lambda\nu}}_{\frac{4\pi}{c}\jmath^\nu} + \partial^\nu \underbrace{\partial_\lambda F^{\lambda\mu}}_{\frac{4\pi}{c}\jmath^\mu} = 0 \tag{B.131}$$

and arrive at a wave equation with a nicely antisymmetrised source term,

$$\square F^{\mu\nu} = \frac{4\pi}{c}\left(\partial^\mu \jmath^\nu - \partial^\nu \jmath^\mu\right) \tag{B.132}$$

The vacuum solutions are $\square A^\mu = 0$ as well as $\square F^{\mu\nu} = 0$ are archetypically hyperbolic and solved by plane waves $\exp(\pm i\eta_{\mu\nu}k^\mu x^\nu)$, provided that the wave vector k^μ is light-like, $\eta_{\mu\nu}k^\mu k^\nu = 0$, which has important consequences: Writing $k^\mu = (\omega/c, k^i)^t$ shows that

$$\omega = \pm k \tag{B.133}$$

from the null-condition $\eta_{\mu\nu}k^\mu k^\nu = (\omega/c)^2 - k^2 = 0$, such that there can not be any dispersion:

$$\upsilon_{\text{phase}} = \frac{d\omega}{dk} = c = \frac{\omega}{k} = \upsilon_{\text{group}} \tag{B.134}$$

as group and phase velocity are identical, and consistent with $\upsilon_{\text{phase}} \times \upsilon_{\text{group}} = \omega/k \times d\omega/dk = d\omega^2/dk^2 = c^2$ for a massless particle: $\omega^2 = c^2k^2$, and $(\omega/c)^2 - k^2 = 0$. At the same time, it is universally true that relativistic waves are always transverse: The field equation requires $\partial_\mu F^{\mu\nu} = 0$ in vacuum, so that $k_\mu F^{\mu\nu} = 0$ and $k_i E^i = 0$ is always given, and the electric fields are perpendicular to the direction of propagation. Transversality of the magnetic fields is most easily seen with the dual field tensor $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}/2$, for which $\partial_\mu F^{\mu\nu} = 0$ is true: Then, $k_\mu \tilde{F}^{\mu\nu} = 0$ from which one obtains $k_i B^i = 0$.

The analogous statement on the vector potential A^μ, however, depends on the gauge: Lorenz-gauge $\partial_\mu A^\mu = 0$ implies $k_\mu A^\mu = 0$ for a plane wave, so that $k_i A^i = \omega A^t/c \neq 0$, but Coulomb-gauge rather makes sure that $\partial_i A^i = ik_i A^i = 0$, such that the potential A^i is perpendicular to k^i: That's why it's sometimes called transverse gauge.

B.5 *Causal structure of spacetime*

In the last section we have seen that there is a tight connection between hyperbolicity of the wave equation $\square\phi = 0$ and the lightlike-ness of the wave-vector $\eta_{\mu\nu}k^\mu k^\nu = 0$, which is not too surprising because $\square = \eta_{\mu\nu}\partial^\mu\partial^\nu$, so the representation of $\square$ in Fourier-space is $\eta_{\mu\nu}k^\mu k^\nu$ anyways. The wave equation as a hyperbolic PDE provides a time evolution of initial conditions (and the solution becomes unique if those initial

Table 1: dimensionality required by certain physical phenomna

phenomenon	dimensionality
Huygens' principle	n odd, $n = 1$ or $n = 3$ best
relativistic gravity	$n + 1 \geq 4$
stable planetary systems	$n \leq 3$
Bose-Einstein-condensation	$n \geq 3$
random walks getting lost (Polya's theorem)	$n \geq 3$
as many electric as magnetic fields	$n = 3$
Poisson-solutions vanish at infinity	$n \geq 3$
knots exist	$n \geq 3$

conditions are specified) in a very peculiar way: For the evolution of ϕ at a specific spacetime point x^μ only the field amplitudes on the past light cone are necessary, clearly as the field excitations can only travel along the light cone. This is perfect, because the light cone structure is Lorentz-invariant, so the field amplitudes that are responsible as initial conditions for ϕ at x^μ are always the same, despite the fact that x^μ will get new coordinates.

This idea is truly funny in Galilean relativity: Here, c is just a velocity and transforms along under Galilei-transforms. Therefore, the two branches of the light cone get velocities $c + \upsilon$ and $c - \upsilon$ formally. Would this be a problem? Well, in the limit $c \to 0$ (i.e. as the formal limit of Galilei-relativity from Lorentz-relativity or for everyday, small velocities compared to c) the light cone opens up and the field amplitudes on an entire spatial hyperplane set the initial conditions for ϕ. This is consistent with all derivatives $\partial_{ct}\phi$ becoming small as $c \to \infty$, so that $\Box \to \Delta$ in this limit: The field equation has lost its dynamics and has become elliptical, such that boundary conditions (possibly on boundaries at $x^i \to \pm\infty$) need to be specified for uniqueness.

And before you get funny ideas for this: Among all metric spacetimes only the Lorentzian one allows hyperbolic evolution of field equations, but one can construct hyperbolic equations without a metric structure for spacetime! The classic example for this would be covariant electrodynamics in the most general linear model for matter, and we'll come to that in section C.

B.6 *Dimensionality of spacetime*

Spacetime has $n + 1 = 4$ dimensions, 1 temporal and $n = 3$ spatial, and it is the case that Nature really needs a minimum number of (spatial) dimensions to make certain phenomena possible, a few are summarised in table 1.

The Poisson-equation has the peculiar property that potentials Φ only vanish towards infinity in 3 or more dimensions: Looking for vacuum solutions in the spherically symmetric case

$$\Delta\Phi = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial\Phi}{\partial r}\right) = 0 \tag{B.135}$$

is solved when the term in the brackets becomes constant, i.e. when

$$\frac{\partial\Phi}{\partial r} = r^{-(n-1)} \quad \to \quad \Phi \propto r^{-n+2} \text{ if } n \geq 3, \text{ or } \quad \Phi \propto \ln r \text{ if } n = 2 \tag{B.136}$$

so that one really needs certainly 3 spatial dimensions or more for the potentials to decrease towards infinity, and one gets logarithmic solutions $\Phi \propto \ln r$ in 2 dimensions. General relativity as a theory of gravity can only exist in $n + 1 = 4$ dimensions or more, if gravity as spacetime curvature should be allowed to propagate away from the sources, but this is really beyond the scope of the lecture.

From the scaling of Φ in n dimensions one can derive that planetary systems are not stable if the dimensionality is too high, and the argument would be like that: For the specific energy $\epsilon = E/m$ of a particle in the potential Φ one would write

$$\epsilon = \frac{E}{m} = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \Phi(r) \tag{B.137}$$

in polar coordinates, with $\Phi(r) = -GM/r^{n-2}$ generated by the central object of mass M. The motion of planets is restricted to be in a plane, because of angular momentum conservation in a spherically symmetric potential, with the specific angular momentum λ

$$\lambda = \frac{L}{m} = r^2\dot{\varphi} \quad \rightarrow \quad \dot{\varphi} = \frac{\lambda}{r^2} \tag{B.138}$$

which will appear as a repulsive centrifugal potential when replacing $\dot{\varphi}$.

$$\epsilon = \frac{1}{2}\left(\dot{r}^2 + \frac{\lambda^2}{r^2}\right) - \frac{GM}{r^{n-2}} \tag{B.139}$$

and counteracts the attractive gravitational potentials. For a stable orbit it is now necessary that the repulsive part of the potential is dominating at small r, for which n can not be too large. In fact, in a true Coulomb-potential with $n = 3$ one gets a long-range attractive $1/r$ potential superimposed on a short range repulsive $1/r^2$-potential, with a nice minimum that harbours the most stable circular orbits. If n is too high, the roles interchange: There would be a short range attractive gravitational potential superimposed on a long range repulsive potential, with effectively a maximum between the two regimes with unstable orbits. Solving the equation of motion yields

$$\dot{r}^2 = 2\left(\epsilon + \frac{GM}{r^{n-2}}\right) - \frac{\lambda^2}{r^2} \quad \rightarrow \quad t = \int_0^t \mathrm{d}t = \int_{r_\mathrm{min}}^{r_\mathrm{max}} \mathrm{d}r\, \frac{1}{\sqrt{2\left(\epsilon + \frac{GM}{r^{n-2}} - \frac{\lambda^2}{2r^2}\right)}} \tag{B.140}$$

by separation of variables: There is an oscillatory motion in the effective potential (if there is a minimum allowing stable orbits), while the planet gets carried around the Sun by the conservation of angular momentum. Bertrand's theorem now states that among all potentials, only two allow for closed orbits: Those are the Keplerian ellipses in $1/r$-potentials and the Lissajous-figures in the harmonic r^2-potential.

As the last point let's investigate the issue that only in $n = 3$ dimensions there is an equal number of electric and magnetic field components: This becomes most apparent in the field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, which is antisymmetric under index exchange, $F^{\nu\mu} = -F^{\mu\nu}$. The electric field components are contained in the first row or first column, $E^i = F^{ti} = -F^{it}$, and there are n possible choices, as $F^{tt} = 0$. As off-diagonal elements representing the magnetic fields one counts $n(n-1)/2$, and $n(n-1)/2 = n$ is solved by $n = 3$ (and $n = 0$, but this is senseless).

B.7 *Huygens-principle*

There is a remarkable peculiarity in the propagation of spherical waves that depends on dimensionality. Writing down a conventional hyperbolic wave equation in n dimensions

$$\eta_{\mu\nu}\partial^\mu\partial^\nu\psi = \left(\partial_{ct}^2 - \gamma_{ij}\partial^i\partial^j\right)\psi = \left(\partial_{ct}^2 - \sum_{i=1}^{n}\partial_i^2\right)\psi = 0 \tag{B.141}$$

with an isotropic spatial part, as $\gamma_{ij}\partial^i\partial^j$ with the Euclidean (inverse) metric γ_{ij} is perfectly invariant under rotations. A spherical wave $\psi(t,r)$ with $r^2 = \gamma_{ij}x^ix^j = x_ix^i$ excited at the origin should propagate outwards, and we will try to answer the question whether the wave front is a well-defined shell with radius r increasing linearly in time, $r = ct$. Surprisingly, this is only in 3 dimensions the case. Let's build quickly the derivatives

$$\partial_i r = \frac{x_i}{r} \rightarrow \sum_i(\partial_i r)^2 = \sum_i\left(\frac{x_i}{r}\right)^2 = \frac{1}{r^2}\sum_i x_i^2 = 1 \tag{B.142}$$

and

$$\partial_i^2 r = \frac{r^2 - x_i^2}{r^3} \rightarrow \sum_i \partial_i^2 r = \sum_i \frac{r^2 - x_i^2}{r^3} = \frac{1}{r}\sum_i 1 - \frac{1}{r^3}\sum_i x_i^2 = \frac{n-1}{r} \tag{B.143}$$

from $r = \sqrt{x_jx^j}$ for later use. When introducing spherical coordinates one would like to replace the ∂_i-differentiations with respect to Cartesian coordinates by ∂_r using the chain rule,

$$\partial_i\psi = \partial_i r \cdot \partial_r\psi \tag{B.144}$$

where I put the $\cdot$ to "stop" the differentiation at this point. For the second derivative one gets

$$\partial_i^2\psi = \partial_i^2 r \cdot \partial_r\psi + \partial_i r \cdot \partial_i\partial_r\psi \tag{B.145}$$

where the second term can be reshaped

$$\partial_i\partial_r\psi = \partial_r\partial_i\psi = \partial_r(\partial_i r)\partial_r\psi = \partial_i\partial_r r \cdot \partial_r\psi + \partial_i r\partial_r^2\psi = \partial_i r\partial_r^2\psi \tag{B.146}$$

with $\partial_r r = 1$ such that the derivative vanishes. Subsitution back into the wave equation gives

$$\partial_i^2\psi = \partial_i^2 r\partial_r\psi + (\partial_i r)^2\partial_r^2\psi \tag{B.147}$$

which, summing over i and using eqns. B.142 and B.143, leads us to

$$\sum_i \partial_i^2\psi = \Delta\psi = \frac{n-1}{r}\partial_r\psi + \partial_r^2\psi \tag{B.148}$$

such that the wave equation for a spherical wave becomes

$$\partial_{ct}^2 \psi = \partial_r^2 \psi + \frac{n-1}{r} \partial_r \psi \tag{B.149}$$

with the additional term $(n-1)/r\ \partial_r \psi$ due to spherical symmetry. Of course you can start at

$$\Delta\psi = \frac{1}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r \psi \right) = \frac{n-1}{r} \partial_r \psi + \partial_r^2 \psi \tag{B.150}$$

as well to arrive at the same result.

For solving the spherical wave equation, one chooses a separation ansatz $\psi = r^{-k}\phi$ for factoring out a power-law decrease of the amplitudes. One would expect that the squares of the amplitudes determines the energy flux of the spherical wave, that needs to be conserved over ever increasing surfaces of spherical shells scaling $\propto r^{n-1}$ in area with radius r, implying $k = (n-1)/2$.

The corresponding derivatives then are

$$\partial_r \psi = -k r^{-(k+1)} \phi + r^{-k} \partial_r \phi \tag{B.151}$$

and

$$\partial_r^2 \psi = k(k+1) r^{-(k+2)} \phi - 2k r^{-(k+1)} \partial_r \phi + r^{-k} \partial_r^2 \phi \tag{B.152}$$

which can be used to reformulate the wave equation in terms of ϕ rather than ψ:

$$\partial_{ct}^2 \phi - \partial_r^2 \phi - \frac{(n-1)(n-3)}{4r} \partial_r \phi = 0 \tag{B.153}$$

which is a truly remarkable result: Of course, there is no concept of spherical symmetry in 1 dimension, so automatically the wave equation for the amplitude ϕ (which incorporates energy conservation in its suggested scaling with distance, in this case it is constant) is fulfilled. In all other spacetimes with the exception of $n = 3$ one sees additional terms in the wave equation, which actually slow down the wave relative to c and fill up the light cone with partial waves, such that (i) neither a spherical wave front would be defined and (ii) there is no clear relation $r = ct$: This, however is exactly the case in $n = 3$ dimensions! In summary, $n + 1 = 4$ dimensions is the only case where wave propagation of spherical waves is described by a plane wave equation with a relation $r = ct$ for the radius. If one would decompose an arbitrary wave front into elementary spherical waves according to Huygens' principle, they only propagate with a well-defined wave front defined by $r = ct$ in 3d to interfere again at a later time.