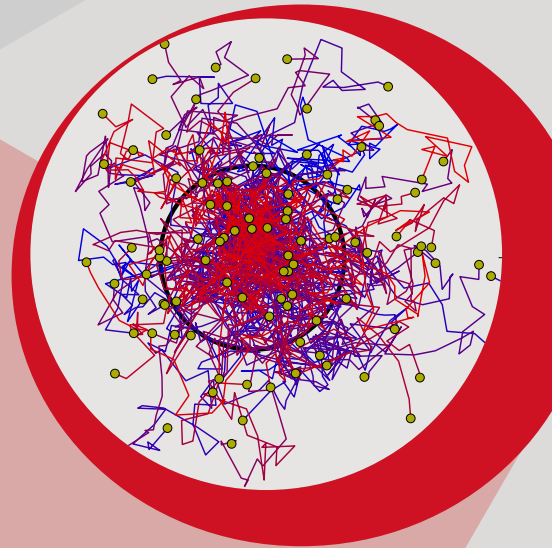


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# Tooltips for Theoretical Physics

Concepts of Modern Theoretical  
Physics, Scales and  
Mathematical Tools

BJÖRN MALTE SCHÄFER





# **Tooltips for Theoretical Physics**

Lecture Notes **Physik**



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Concepts of Modern Theoretical Physics,  
Scales and Mathematical Tools


BJÖRN MALTE SCHÄFER

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### About the Author

Björn Malte Schäfer works at Heidelberg University on problems in modern cosmology, relativity, statistics, and on theoretical physics in general.

### ORCID®

Björn Malte Schäfer  <https://orcid.org/0000-0002-9453-5772>

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## CONTENTS

A	Motion	1
A.1	Scales in physical laws: Poisson vs. Yukawa . . . . .	1
A.2	Buckingham's II-theorem and the Navier-Stokes-equation . . . . .	2
A.3	Constants of Nature and Planck's system of units . . . . .	3
A.4	Classical Lagrange-functions . . . . .	3
A.5	Classical universality and mechanical similarity . . . . .	5
A.6	Total derivatives in the Lagrange-function . . . . .	6
A.7	Virial theorem . . . . .	7
A.8	Galilei-invariance of classical systems . . . . .	9
A.9	Alternatives to the Lagrange-function . . . . .	10
A.10	Beltrami-identity and the conservation of energy . . . . .	11
A.11	Convexity of the Lagrange-function . . . . .	12
A.12	Lorentz- and Galilei-relativity . . . . .	12
A.13	Rapidity . . . . .	15
A.14	Proper time and the relativistic Lagrange-function . . . . .	16
A.15	Geometric view on motion . . . . .	16
A.16	Relativistic energy and momentum . . . . .	18
A.17	Causality and light cones . . . . .	19
B	Waves	21
B.1	Taxonomy of waves . . . . .	21
B.2	Elastic waves and wave equations . . . . .	21
B.3	Partial differential equations: hyperbolic vs. elliptic . . . . .	23
B.4	Relativistic waves and hyperbolicity . . . . .	24
B.5	Causal structure of spacetime . . . . .	25
B.6	Dimensionality of spacetime . . . . .	26
B.7	Huygens-principle . . . . .	28
C	Fields	31
C.1	Lagrange-description of field dynamics . . . . .	31
C.2	Lagrange-description of scalar field dynamics . . . . .	31
C.3	Maxwell-electrodynamics and the gauge-principle . . . . .	32
C.4	Electromagnetic duality and axions . . . . .	34
C.5	Poynting-law and conservation of energy and momentum . . . . .	35
C.6	Covariant electrodynamics in matter . . . . .	37
C.7	Finsler-geometry and Lorentz-forces . . . . .	38
C.8	Light-cone structure beyond metric spacetimes . . . . .	39
D	Symmetry	41
D.1	Lie-groups and their generation . . . . .	41
D.2	Generating algebras . . . . .	41
D.3	Construction of invariants . . . . .	43
D.4	Symplectic structures and canonical time evolution . . . . .	43
D.5	Unitary time-evolution in quantum mechanics . . . . .	44
E	Quanta	47
E.1	Schrödinger-equation as a Helmholtz-differential equation . . . . .	47
E.2	Born's postulate and the conservation of probability . . . . .	47
E.3	Ehrenfest's theorem . . . . .	48

E.4	Dispersion of wave packets and propagation with Green-functions . .	50
E.5	Path integrals . . . . .	51
	E.5.1 Phase space path integral . . . . .	52
	E.5.2 Configuration-space path integral . . . . .	53
E.6	Uncertainty . . . . .	55
E.7	Relativistic quantum mechanics . . . . .	55
E.8	Coupling to fields . . . . .	56
F	Mass . . . . .	59
	F.1 Dynamics of gravity . . . . .	59
	F.2 Inertial accelerations and equivalence . . . . .	60
	F.3 Classical Raychaudhuri-equation and geodesic deviation . . . . .	61
	F.4 Gravitational lensing . . . . .	63
	F.5 Gravitational field equation . . . . .	64
G	Temperature . . . . .	67
	G.1 Thermal energy . . . . .	67
	G.2 Axioms of thermodynamics . . . . .	67
	G.3 Measuring temperature . . . . .	68
	G.4 Carnot-engines . . . . .	69
	G.5 Thermal wavelength and quantum statistics . . . . .	70
	G.6 Boltzmann-factor and the fundamental postulate . . . . .	71
	G.7 Ultra-relativistic Bose gases and the Planck-spectrum . . . . .	73
	G.8 Entropy . . . . .	74
	G.9 Negative absolute temperatures . . . . .	76
	G.10 Reversibility . . . . .	77
	G.11 Information entropies . . . . .	77



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## A MOTION

### A.1 Scales in physical laws: Poisson vs. Yukawa

A good example of a scale-free physical law is the  $1/r$ -potential in electrostatics or in Newton-gravity in 3d dimensions: It follows as a vacuum solution of the Poisson-equation

$$\Delta\Phi = -4\pi\rho \quad (\text{A.1})$$

in the Gauß-system of units. Assuming spherical symmetry for the field away from a point charge one can verify that  $\Phi \propto 1/r$  is in fact a solution to

$$\Delta\Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial\Phi}{\partial r} \right) = 0 \quad (\text{A.2})$$

The solution  $\Phi \propto 1/r$  is perfectly scale free as a power law; increasing the charge can be absorbed in an increased distance. This can be seen directly by the scale transformation  $r \rightarrow \alpha r$ , under which  $\partial_r \rightarrow \alpha^{-1} \partial_r$  and consequently  $\Delta \rightarrow \alpha^{-2} \Delta$ . Then,  $\Phi \rightarrow \alpha^{-1} \Phi$  because  $\rho \rightarrow \alpha^{-3} \rho$ , and two powers of  $\alpha$  cancel, making  $\Phi$  consistent with the scaling of  $r$ .

This scale-invariance expressed by the power law is broken in the Yukawa-equation

$$\left[ \Delta - \lambda^2 \right] \Phi = -4\pi\rho \quad (\text{A.3})$$

with a parameter  $\lambda$ : It has units of inverse length and allows to distinguish between the regimes  $\lambda r \ll 1$  and  $\lambda r \gg 1$ , because despite the fact that the field equation is still linear, scale-invariance is violated. As a solution one finds  $\Phi \propto \exp(-\lambda r)/r$  in 3 dimensions, which behaves  $\Phi \propto 1/r$  for small distances, where  $\exp(-\lambda r) \simeq 1 - \lambda r \pm \dots$ , but at large distances the solution drops faster to zero than  $1/r$ . Therefore, one has constructed a scale-dependent modification of the Poisson-equation. From a physical point of view, Yukawa aimed at a short-range force for explaining the binding of nucleons, and almost at exactly the same time, Debye considered electric fields in electrolytes, where the shielding of ions led to a fast decrease of electric fields around charges.

Please note that much of the arguments are only applicable in 3 dimensions or more. In two dimensions the Poisson-equation reads

$$\Delta\Phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial\Phi}{\partial r} \right) = \frac{1}{r} \frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial r^2} = 0 \quad (\text{A.4})$$

which is solved by  $\Phi \propto \ln r$ : While this is mathematically perfect, there are a couple of issues concerning the physical application. The potential does not vanish for  $r \rightarrow \infty$  and there is no scale-free behaviour of the solution despite the fact that the Poisson-equation is scale free. Adding a Yukawa-type term

$$\left[ \Delta - \lambda^2 \right] \Phi = \frac{1}{r} \frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial r^2} - \lambda^2\Phi = 0 \quad (\text{A.5})$$

gives rise to a differential equation that is known as Emden-Fowler-type and has a (very complicated) solution in terms of Bessel-functions  $J_0$  and  $Y_0$ , where  $\lambda$  appears

as the wavenlength of the oscillation in the Bessel-functions: Surely it plays the role of a scale, but not as clearly as in 3 dimensions.

### A.2 Buckingham's $\Pi$ -theorem and the Navier-Stokes-equation

The example about the Poisson- and Yukawa-equation showed how scales can be introduced in a linear equation, and we should investigate if there can be scale-free behaviour in a nonlinear equation. This is in fact the case, as the example of the dimensionless Navier-Stokes-equation in fluid mechanics shows:

$$\partial_t v^i + (v_j \partial^j) v^i = -\frac{1}{\rho} \partial^i p - \partial^i \Phi + \mu \partial_j \partial^j v^i \quad (\text{A.6})$$

The Navier-Stokes equation describes the acceleration of a fluid with velocity  $v^i$  under the action of forces, for instance gradients in pressure  $p$ , in the gravitational potential  $\Phi$  and viscous forces with the shear viscosity  $\mu$ , all under the condition of incompressible fluids with  $\partial_i v^i = 0$ . Multiplying with  $\rho$  to make things a bit more transparent gives an equation where every term has units of mass/length<sup>2</sup>/time<sup>2</sup>.

If we introduce typical scales, we could reach a form of the Navier-Stokes equation where it would become scale free: It would become an dimensionless equation, and flow patterns of different physical dimension, if they fall back onto the same dimensionless equation, would be scaled versions of each other: Introducing a length scale  $L$  for  $x \rightarrow x^* = x/L$ , a time scale  $T$  for  $t \rightarrow t^* = t/T$ , a velocity scale  $V$  for  $v \rightarrow v^* = v/V$ , a pressure scale  $P$  for  $p \rightarrow p^* = p/P$  and finally a scale for the gravitational acceleration  $G$  for  $g \rightarrow g^* = g/G$  yields

$$\frac{\rho V}{T} \partial_t^* v^{*i} + \frac{\rho V^2}{L} (v^*_j \partial^{*j}) v^{*i} = -\frac{P}{L} \partial^{*i} p^* - \frac{\rho G}{L} \partial^{*i} \Phi^* + \frac{\mu \rho V}{L^2} \partial^*_j \partial^{*j} v^{*i} \quad (\text{A.7})$$

with the dimensionless derivatives

$$\frac{\partial}{\partial x} = \frac{\partial x^*}{\partial x} \frac{\partial}{\partial x^*} = \frac{1}{L} \partial^*_x \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{\partial t^*}{\partial t} \frac{\partial}{\partial t^*} = \frac{1}{T} \partial^*_t \quad (\text{A.8})$$

It should be noted that  $L$  and  $T$  are typical scales on which the flow changes, and that the scale  $V$  is independent of  $L/T$ : You can have a slowly-varying high velocity flow or, vice versa, a rapidly changing low-velocity flow.

In eqn. A.7 one has reached a curious ordering of all terms: The units are concentrated in the prefactors, while all terms involving quantities with a superscript-\* are dimensionless. Dividing the entire formula by the prefactor of the second, nonlinear term then gives rise to:

$$\underbrace{\frac{L}{TV}}_{\text{Strouhal}} \partial_t^* v^{*i} + (v^*_j \partial^{*j}) v^{*i} = -\underbrace{\frac{P}{\rho V^2}}_{\text{Euler}} \partial^{*i} p^* - \underbrace{\frac{G}{V^2}}_{\text{Froude}^{-2}} \partial^{*i} \Phi^* + \underbrace{\frac{\mu}{VL}}_{\text{Reynolds}^{-1}} \partial^*_j \partial^{*j} v^{*i} \quad (\text{A.9})$$

Flows with identical scaling numbers can be mapped onto each other, and the primary application is indeed technical: When designing airplanes, it might be difficult to construct a full-size airplane model and to test it in a wind tunnel at actual velocities. Instead, one can try out a much smaller model at lower air speeds; if the scaling numbers are identical between the two situations, the flow patterns are scaled versions

of each other. In summary, scales might be present in linear laws and there might be scale-free behaviour in nonlinear laws.

### A.3 Constants of Nature and Planck's system of units

There is a clear distinction between classical physics and modern physics: In classical physics, the purpose of constants is to sort out the units and to relate quantities in a phenomenological way: From this point of view there really is not much of a difference between the spring constant  $k$  in Hooke's law

$$F = -kr \quad (\text{A.10})$$

and the gravitational constant  $G$  in Newton's law of gravity

$$F = -G \frac{mM}{r^2} \quad (\text{A.11})$$

Modern physics on the other hand distinguishes between different regimes where Nature behaves classical or shows a markedly new behaviour, for instance at high velocities close to  $c$ , motion at low action close to  $\hbar$ , at low energies comparable to the thermal energy  $k_B T$  and finally at distances close to  $GM/c^2$  at massive objects. In these cases, classical physics gets replaced by special relativity, by quantum mechanics, by statistical physics and finally by general relativity, respectively.

As first noticed by Planck, the four constants  $c$ ,  $\hbar$ ,  $G$  and  $k_B$  can be used to define a *natural* system of units which is universally valid and does not depend on any human concept for length, time, mass or temperature. For instance, a fundamental mass could be constructed by setting

$$m_P = c^\alpha \hbar^\beta G^\gamma = \text{length}^{\alpha+3\beta+2\gamma} \text{time}^{-\alpha-2\beta-\gamma} \text{mass}^{-\beta+\gamma} \quad (\text{A.12})$$

which is solved by  $\alpha = -\beta = \gamma = 1/2$ , defining the Planck-mass  $m_P$ ,

$$m_P = \sqrt{\frac{c\hbar}{G}} \simeq 10^{-8} \text{kg} \simeq 10^{16} \text{GeV}/c^2 \quad (\text{A.13})$$

Similarly, one can define a length-scale  $l_P$ , a time scale  $t_P$  and a temperature scale  $T_P$ ,

$$l_P = \sqrt{\frac{G\hbar}{c^2}} \simeq 10^{-35} \text{m}, \quad t_P = \frac{l_P}{c} \simeq 10^{-43} \text{s}, \quad T_P = \frac{1}{k_B} \sqrt{\frac{c^3\hbar}{G}} \simeq 10^{30} \text{K} \quad (\text{A.14})$$

This beautiful idea is somewhat tainted by the realisation that there are in fact two constants in gravity,  $G$  and the cosmological constant  $\Lambda$ . This second constant makes the construction of a fundamental system of units ambiguous, and what's even more puzzling, starting from  $c$ ,  $G$  and  $\Lambda$  defines a system which very well characterises the Universe today, with a length scale  $1/\sqrt{\Lambda} \simeq 3 \text{Gpc}/h$  and an age of  $1/(\sqrt{\Lambda}c) \simeq 10^{17} \text{s}$ , while even derived quantities like the density scale come out correctly.

### A.4 Classical Lagrange-functions

Classical mechanics describes motion axiomatically with a Lagrange-function  $\mathcal{L}(q^i, \dot{q}^i)$  as a function of the (generalised) coordinates  $q^i$  and the velocities  $\dot{q}^i$ , defined as the

☞ There is a fantastic way of memorising the Reynolds number, which is associated with turbulence:  $VL/\mu$  means, that stirring a coffee fast with a big spoon is making the flow turbulent, but it would not work in honey!

rate of change of the coordinates as the time-parameter evolves. An integration over  $t$  then defines the action  $S$

$$S = \int_{t_i}^{t_f} dt \mathcal{L}(q^i, \dot{q}^i) \quad (\text{A.15})$$

as a functional over the trajectory  $q^i(t)$ . Hamilton's principle

$$\delta S = 0 \quad (\text{A.16})$$

then asserts that the physical motion is the one that extremises the action functional, and incidentally we realise that the linearity of the variation  $\delta S$  induces that the action is affinely invariant.  $S \rightarrow aS + b$  would not change anything in Hamilton's principle, as  $\delta(aS + b) = a\delta S = 0$  shows the irrelevance of  $a$  and  $b$ .

Carrying out the variation is done by writing

$$\delta S = \int_{t_i}^{t_f} dt \left( \frac{\partial \mathcal{L}}{\partial q^i} \delta q^i + \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \delta \dot{q}^i \right) = \int_{t_i}^{t_f} dt \left( \frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) \delta q^i = 0 \quad (\text{A.17})$$

after setting  $\delta \dot{q}^i = d/dt \delta q^i$ , followed by an integration by parts. The boundary term vanishes if the variation on the boundary vanishes,  $\delta q^i(t_i) = \delta q^i(t_f) = 0$ , or at least if their difference is constant. From the last expression we can isolate the Euler-Lagrange-equation,

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{\partial \mathcal{L}}{\partial q^i} \quad (\text{A.18})$$

If one now chooses the Lagrange function to be

$$\mathcal{L} = \frac{m}{2} \gamma_{ab} \dot{q}^a \dot{q}^b - \Phi(q^i) \quad (\text{A.19})$$

with the Euclidean metric  $\gamma_{ab}$  and a potential  $\Phi$ , the Euler-Lagrange-function becomes equivalent to Newton's equation of motion: The gradient of the Lagrange-function with respect to the coordinate yields

$$\frac{\partial \mathcal{L}}{\partial q^i} = -\frac{\partial \Phi}{\partial q^i} \quad (\text{A.20})$$

and the derivative of the kinetic term becomes

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{m}{2} \gamma_{ab} \left( \underbrace{\frac{\partial \dot{q}^a}{\partial \dot{q}^i}}_{\delta_i^a} \dot{q}^b + \dot{q}^a \underbrace{\frac{\partial \dot{q}^b}{\partial \dot{q}^i}}_{\delta_i^b} \right) = m \dot{x}_i \quad (\text{A.21})$$

☞ Please always rename the indices in the kinetic term of the Lagrange-function before substituting it into the Euler-Lagrange equation!

Finally, we arrive at Newton's equation of motion  $m\ddot{q}_i = -\partial_i \Phi$  by differentiation with respect to  $t$ . One might not always have such a convenient separation into a term involving only  $\dot{q}^i$  and only  $q^i$ , for instance, the harmonic oscillator  $\mathcal{L} = \dot{q}^2/2 - \omega^2 q^2/2$  could be rewritten as  $\mathcal{L} = (\dot{q} + \omega q)(\dot{q} - \omega q)/2$ . In these cases, the time-derivative might

act on a function  $\partial\mathcal{L}/\partial\dot{q}$  which is still a function of  $q$ , so one needs to write

$$\frac{d}{dt} \frac{\partial\mathcal{L}}{\partial\dot{q}^i} = \ddot{q}^j \frac{\partial^2\mathcal{L}}{\partial\dot{q}^i\partial\dot{q}^j} + \dot{q}^j \frac{\partial^2\mathcal{L}}{\partial q^j\partial\dot{q}^i} = \frac{\partial\mathcal{L}}{\partial q^i} \quad (\text{A.22})$$

and a solution for  $\ddot{q}^j$  depends on the invertibility of the matrix  $\partial^2\mathcal{L}/\partial\dot{q}^i\partial\dot{q}^j$ :

$$\ddot{q}^j = \left( \frac{\partial^2\mathcal{L}}{\partial\dot{q}^i\partial\dot{q}^j} \right)^{-1} \left( \frac{\partial\mathcal{L}}{\partial q^i} - \dot{q}^j \frac{\partial^2\mathcal{L}}{\partial q^j\partial\dot{q}^i} \right) \quad (\text{A.23})$$

and of course for 1-dimensional motion, it would be enough for  $\partial^2\mathcal{L}/\partial\dot{q}^2$  to be nonzero. Typically,  $\partial^2\mathcal{L}/\partial\dot{q}^2$  is just the mass or inertia of the system, which is strictly positive such that the  $\ddot{q}$ -term can be isolated.

While the Lagrange-formalism seems straightforward as an axiomatic foundation of classical mechanics, there seem to be many issues: There is no fundamental justification for  $\mathcal{L}$  or  $S$ , as they are both not measurable quantities.  $S$  is only determined up to an affine transform, and so must be  $\mathcal{L}$ . At least for motion in a vector space, there is no advantage of using Lagrangian mechanics over the Newton equation of motion, and one might wonder what the relation between Hamilton's principle for the motion of objects and Fermat's principle for the propagation of light might be.

### A.5 Classical universality and mechanical similarity

The Lagrange-function  $\mathcal{L}$  is invariant under affine transformations,

$$\mathcal{L} \rightarrow a\mathcal{L} + b \quad (\text{A.24})$$

with two constants  $a$  and  $b$ , which is no more than a novelty: Clearly, both constants drop out of the Euler-Lagrange equation


$$\frac{d}{dt} \frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = 0 \quad (\text{A.25})$$

$b$ , because it gets lots in the differentiation and  $a$  because both differentiations are linear, so it appears as an irrelevant overall prefactor. But there is a way in which this affine invariance of the Lagrange-function can be used in a sensible way: If one rescales the coordinates  $q \rightarrow \alpha q$  and the time parameter  $t \rightarrow \beta t$ , the kinetic energy  $T$  scales  $T \rightarrow (\alpha/\beta)^2 T$  and the potential energy  $\Phi \rightarrow \alpha^n \Phi$  for a scale-free power-law potential  $\Phi \propto q^n$ . Because the scaling of  $T$  and  $\Phi$  are inherently different, one needs to assume a relation between them, such that the Lagrange-function  $\mathcal{L} = T - \Phi$  just changes by an (irrelevant) overall factor:

$$\frac{\alpha^2}{\beta^2} \propto \alpha^n \quad \text{or, equivalently,} \quad \beta^2 \propto \alpha^{2-n} \quad (\text{A.26})$$

This scaling can be read off from Newton's equation of motion as well (surely it is consistent with the Lagrange-function  $\mathcal{L} = T - \Phi$ ):

$$\ddot{q} = -\frac{\partial\Phi}{\partial q} \rightarrow \frac{\alpha}{\beta^2} \ddot{q} = -\frac{\alpha^n}{\alpha} \frac{\partial\Phi}{\partial q} \quad \text{implying} \quad \beta^2 \propto \alpha^{2-n} \quad (\text{A.27})$$

 Please keep in mind that in classical mechanics the time is just a parameter to describe motion!

for the specific form  $\Phi \propto q^n$ . Therefore, the length and time scales need to be in that particular relation given by the similarity condition  $\beta^2 = \alpha^{2-n}$ , which we can specifically try out for the most common scale-free potentials:

1.  $\Phi \propto q^2$ ,  $n = 2$ : harmonic oscillator

In the case of the harmonic oscillator, similarity implies  $t^2 = \text{const}$ , which indicates that the time scale of e.g. a pendulum is independent of amplitude.

2.  $\Phi \propto q$ ,  $n = 1$ : inclined plane with a constant slope

Here, time and length scale are related by  $t^2 \propto q$ , typical for uniformly accelerated motion.

3.  $\Phi \simeq \text{const}$ ,  $n = 0$ : flat potential

A flat potential is characterised by  $t^2 \propto q^2$ , or equivalently, inertial motion at constant velocity, as no acceleration takes place

4.  $\Phi \propto 1/q$ ,  $n = -1$ : Coulomb-potential

In a Coulomb-potential, Kepler's third law is valid, as  $t^2 \propto q^3$ .

These four examples illustrate the *principle of mechanical similarity* where we can say something profound about motion without performing the variation or solving the actual equation of motion. For instance, we found out that all planetary orbits are scaled version of each other as every orbit needs to fulfil Kepler's law. To formulate this in a very extreme way, for determining the distances of the planets to the Sun one just needs a calendar.

A cute example of mechanical similarity is the motion of astronauts on the surface of the Moon, at a fraction of Earth's gravity: There, everything seems to be happening in slow motion, because accelerations are much lower. Speeding up a movie of astronauts would make everything appear normal again. You might as well have the association that the motion of the astronauts looks as if they were under water: That's sensible, too, because buoyancy reduces the effective gravitational acceleration, leading to the same effect of longer time constants.

### A.6 Total derivatives in the Lagrange-function

The Lagrange-function is only determined up to a total derivative  $dM(q^i, t)/dt$  of a function  $M(q^i, t)$  which may depend on the coordinates  $q^i$  and on the time parameter  $t$ , but *not* on the velocities  $\dot{q}^i$ . In fact, transforming the Lagrange-function

$$\mathcal{L}(q^i, \dot{q}^i) \rightarrow \mathcal{L}(q^i, \dot{q}^i) + \frac{d}{dt}M(q^i, t) \quad (\text{A.28})$$

implies a transformation of the action

$$S = \int dt \mathcal{L}(q^i, \dot{q}^i) \rightarrow S + \int dt \frac{d}{dt}M(q^i, t) \quad (\text{A.29})$$

but Hamilton's principle  $\delta S = 0$  invalidates the new term: Writing the variation with a Euler-Lagrange-operator acting on M

$$\delta S = \delta \int dt \mathcal{L} + \frac{d}{dt} M = \delta \int dt \mathcal{L} + \int dt \left[ \frac{d}{dt} \frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q} \right] \frac{dM}{dt} \quad (\text{A.30})$$

lets us treat each term separately. For the second term, there is

$$\frac{\partial}{\partial q} \frac{dM}{dt} = \frac{\partial}{\partial q} \left( \dot{q} \frac{\partial M}{\partial q} + \frac{\partial M}{\partial t} \right) = \frac{\partial \dot{q}}{\partial q} \frac{\partial M}{\partial q} + \dot{q} \frac{\partial^2 M}{\partial q^2} + \frac{\partial^2 M}{\partial q \partial t} \quad (\text{A.31})$$

because M depends on  $q$  and  $t$ , but not on  $\dot{q}$ . For the first term, we get

$$\frac{\partial}{\partial \dot{q}} \frac{dM}{dt} = \frac{\partial}{\partial \dot{q}} \left( \dot{q} \frac{\partial M}{\partial q} + \frac{\partial M}{\partial t} \right) = \frac{\partial \dot{q}}{\partial \dot{q}} \frac{\partial M}{\partial q} = \frac{\partial M}{\partial q} \quad (\text{A.32})$$

because  $\partial \dot{q} / \partial \dot{q} = 1$ . A successive time derivative yields then

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}} \frac{dM}{dt} = \dot{q} \frac{\partial^2 M}{\partial q^2} + \frac{\partial^2 M}{\partial q \partial t} \quad (\text{A.33})$$

so that all additional terms cancel, because

$$\frac{\partial \dot{q}}{\partial q} = \frac{\partial}{\partial q} \frac{\partial q}{\partial t} = \frac{\partial}{\partial t} \frac{\partial q}{\partial q} = \frac{\partial}{\partial t} 1 = 0 \quad (\text{A.34})$$

where we've use the interchangeability of the second partial derivatives.

Alternatively, one can argue that adding the total derivative changes the action according to

$$S \rightarrow S + \int_{t_i}^{t_f} dt \frac{dM}{dt} = S + M(q(t_f), t_f) - M(q(t_i), t_i) \quad (\text{A.35})$$

The variation  $\delta q$  vanishes at the endpoints  $t_i$  and  $t_f$  by construction, this however does not constrain the value of  $\delta \dot{q}(t)$  at the endpoints. Because  $M(q, t)$  is only a function of  $q$  and not of  $\dot{q}$  we can be sure that  $\delta M$  vanishes for both  $\delta q(t_i)$  and  $\delta q(t_f)$ , cancelling the boundary term.

### A.7 Virial theorem

Lagrangian systems are energy-conserving if  $\mathcal{L}$  does not depend directly on time  $t$ . This can be seen explicitly in Newton's equation of motion

$$m\ddot{q} = -\frac{\partial}{\partial q} \Phi \quad (\text{A.36})$$

if multiplied with  $\dot{q}$ :

$$m\dot{q}\ddot{q} = m \frac{d}{dt} \frac{\dot{q}^2}{2} = -\dot{q} \frac{\partial}{\partial q} \Phi = -\frac{d}{dt} \Phi \quad \rightarrow \quad \frac{d}{dt} \left( \frac{m}{2} \dot{q}^2 + \Phi \right) = 0 \quad (\text{A.37})$$

with the energy  $E = m\dot{q}^2/2 + \Phi$ , because  $\Phi$  depends on  $t$  only through the trajectory  $q(t)$ , and obviously not explicitly. While the total energy is conserved and while the

equation of motion constantly changes kinetic into potential energy and back, one might ask the rather sensible question if the system likes to spend more time in a state of high kinetic energy or in a state of high potential energy.

The answer to this question is the virial theorem: Multiplying the equation of motion with  $q$  instead of  $\dot{q}$  and averaging over a time interval  $\Delta t$  gives

$$0 = \frac{1}{\Delta t} \int_0^{\Delta t} dt \left( mq\ddot{q} + q \frac{\partial \Phi}{\partial q} \right) = \frac{1}{\Delta t} mq\dot{q}|_0^{\Delta t} - \frac{1}{\Delta t} \int_0^{\Delta t} dt \left( m\dot{q}^2 - q \frac{\partial \Phi}{\partial q} \right) \quad (\text{A.38})$$

after an integration by parts of the first term. The term  $m\dot{q}^2$  gets evaluated at 0 and  $\Delta t$  and can be estimated to be less than the maximum coordinate  $q_{\max}$  times the maximum velocity  $\dot{q}_{\max}$  over the time interval from 0 to  $\Delta t$ , if the motion is bounded:

$$\frac{1}{\Delta t} mq\dot{q}|_0^{\Delta t} \leq \frac{1}{\Delta t} mq_{\max}\dot{q}_{\max} \rightarrow 0 \quad \text{as } \Delta t \rightarrow \infty \quad (\text{A.39})$$

and vanishes then if the average is taken over arbitrarily large time intervals. The term

$$\frac{1}{\Delta t} \int_0^{\Delta t} dt m\dot{q}^2 = 2\langle T \rangle \quad (\text{A.40})$$

becomes twice the average kinetic energy, and for proceeding with the potential term, we need to make an assumption about its functional shape: If it is a homogeneous function of order  $k$ ,  $\Phi \propto q^k$ , we get

$$\frac{1}{\Delta t} \int_0^{\Delta t} dt q \frac{\partial \Phi}{\partial q} = \frac{1}{\Delta t} \int_0^{\Delta t} dt k\Phi = k\langle \Phi \rangle \quad (\text{A.41})$$

because  $q\partial_q \Phi = q\partial_q q^k = kq q^{k-1} = kq^k = k\Phi$ . Therefore, the average energies are related to each other by the virial law

$$2\langle T \rangle = k\langle \Phi \rangle \quad (\text{A.42})$$

A prime example for this is the harmonic oscillator, where both  $T$  and  $\Phi$  are homogeneous functions of order  $k = 2$  in  $\dot{q}$  and  $q$ , respectively, resulting in equal average kinetic and potential energies.

It is perhaps a bit more transparent to derive the virial law from the Euler-Lagrange-equation as the equivalent equation of motion directly. Multiplying with the coordinate  $q$  and averaging gives

$$0 = \frac{1}{\Delta t} \int_0^{\Delta t} dt q \left( \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} \right) = \frac{1}{\Delta t} q \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_0^{\Delta t} - \frac{1}{\Delta t} \int_0^{\Delta t} dt \left( \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} + q \frac{\partial \mathcal{L}}{\partial q} \right) \quad (\text{A.43})$$

which we solve by an integration by parts in the first term. The derivative  $\partial \mathcal{L} / \partial \dot{q}$  is the canonical momentum  $p$  and we can invoke the same argument about bounded systems,



$$\frac{1}{\Delta t} m q p|_0^{\Delta t} \leq \frac{1}{\Delta t} m q_{\max} p_{\max} \rightarrow 0 \quad \text{as } \Delta t \rightarrow 0 \quad (\text{A.44})$$

now in phase space, so that the two remaining averages determine the virial law: Typically, the Lagrange-function is a homogeneous function of order 2 in  $\dot{q}$ ,

$$\frac{1}{\Delta t} \int_0^{\Delta t} dt \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{1}{\Delta t} \int_0^{\Delta t} dt \dot{q} \frac{\partial T}{\partial \dot{q}} = 2\langle T \rangle \quad (\text{A.45})$$

and in the case of power laws a homogeneous function of order  $k$  in  $q$  with an additional minus-sign.

$$\frac{1}{\Delta t} \int_0^{\Delta t} dt q \frac{\partial \mathcal{L}}{\partial q} = -\frac{1}{\Delta t} \int_0^{\Delta t} dt q \frac{\partial \Phi}{\partial q} = -k\langle \Phi \rangle \quad (\text{A.46})$$

and the virial law is established:

$$2\langle T \rangle = k\langle \Phi \rangle \quad (\text{A.47})$$

An illustrative example might be to choose a rather high value of  $k$ : Then, the potential is essentially a box with a flat bottom and high walls, in which the particle zooms from left to right and back in a state of high kinetic energy essentially all the time, and spends little time climbing up the walls and changing its direction of motion. For high  $k$ ,  $\langle T \rangle$  is much higher than  $\langle \Phi \rangle$ . The second example is the impossibility of a gravitationally bound ball of photons: There, the kinetic energy is a homogeneous function of order  $k = 1$  as energy depends linearly on momentum, and for the gravitational potential  $\Phi \propto 1/q$  we have  $k = -1$  as the degree, so the virial law becomes:  $\langle T \rangle = -\langle \Phi \rangle$ , and the total energy  $E = \langle T \rangle + \langle \Phi \rangle = 0$ , but it would need to be negative for a bound system. Lastly, a peculiar case is a harmonic oscillator with  $k = 2$ : Then, the average kinetic and potential energies are exactly equal,  $\langle \dot{x}^2 \rangle = \omega^2 \langle x^2 \rangle$ .

### A.8 Galilei-invariance of classical systems

Classical mechanics uses Galilean relativity, meaning that the equation of motions are identical in every Galilei-frame, which in turn is defined as the class of frames moving at constant relative velocities where inertial forces are absent. Mathematically they are defined as the coordinate transformations  $q \rightarrow q + vt$  with a constant velocity  $v$ , such that  $\dot{q} \rightarrow \dot{q} + v$  and  $\ddot{q} \rightarrow \ddot{q}$ , leaving the Newtonian equation of motion unchained.

On the level of the Lagrange-function there is a change,

$$\mathcal{L} = \frac{m}{2} \dot{q}^2 \rightarrow \frac{m}{2} (\dot{q} + v)^2 = \frac{m}{2} (\dot{q}^2 + 2\dot{q}v + v^2) = \frac{m}{2} \dot{q}^2 + \frac{d}{dt} \left( mqv + \frac{m}{2} v^2 t \right) \quad (\text{A.48})$$

where the additional terms can be absorbed into a the time derivative of a function  $M(q, t)$  which depends on the coordinate  $q$  and  $t$  (please keep in mind that  $v$  is constant!), but not on  $\dot{q}$  directly, so the action  $S = \int dt \mathcal{L}$  is effectively unchanged.

While this looks very convincing there is something fundamental that is being overlooked in Galilean, non-relativistic mechanics. In the process of varying the action, one transitions from an *invariant*, scalar Lagrange-function to a *covariant*

vectorial or tensorial equation of motion with consistent transformation properties. For instance, the Lagrange-function

$$\mathcal{L} = \frac{1}{2} \gamma_{ij} \dot{x}^i \dot{x}^j - \Phi(x^i) \quad (\text{A.49})$$

is rotationally invariant, clearly because of the scalar product  $\gamma_{ij} \dot{x}^i \dot{x}^j$  involving the Euclidean metric  $\gamma_{ij}$ , but also because of the scalar potential  $\Phi$ , which doesn't have any internal degrees of freedom that would be affected by a rotation. After variation, the equation of motion

$$m \ddot{x}_i = -\partial_i \Phi \quad (\text{A.50})$$

puts a linear form  $\ddot{x}_i$  into relation with the gradient  $\partial_i \Phi$ , again written as a linear form, so that the entire formula transforms consistently. Clearly, one could use the inverse Euclidean metric  $\gamma^{ij}$  to write it in vector form,  $m \ddot{x}^i = -\partial^i \Phi$ , with  $\dot{x}^i = \gamma^{ij} \dot{x}_j$  and  $\partial^i \Phi = \gamma^{ij} \partial_j \Phi$ . This property of the variational principles is known as the invariance-covariance principle: You always obtain a covariant equation of motion (or field equation) from an invariant Lagrange-function (or density).

The curiosity is now that actually boosts and rotations form a common group, the proper Lorentz-group, so classical mechanics based on Galilean relativity instead of Lorentzian relativity needs to realise the invariance-covariance principle differently: Time is universal and identical in all frames, and excluded from coordinate transforms. This enables the invariance of the accelerations  $\ddot{q}$  in all frames instead of dealing with a construction a covariant equation of motion.

### A.9 Alternatives to the Lagrange-function

The Lagrange-function  $\mathcal{L} = T(\dot{q}^i) - \Phi(q^i)$  is defined axiomatically in classical mechanics in order to make it consistent with the Newtonian equation of motion. You might want to ask if one could have other terms in the Lagrange-function that would be compatible with a linear, second order equation of motion. As the order of the powers of  $\dot{q}^i$  and  $q^i$  decreases by one through the differentiation in the Euler-Lagrange-equation, there should be at most squares in the Lagrange-function. Higher-order derivatives like  $\ddot{q}^i$  are excluded by the Ostrogradsky-instability (we will come to that!). Therefore, one could imagine a Lagrange-function

$$\mathcal{L} = \gamma_{ij} \dot{q}^i \dot{q}^j - \gamma_{ij} q^i \ddot{q}^j - \Phi + \lambda_i q^i + \mu_i \dot{q}^i + \alpha_{ij} q^i q^j + \beta_{ij} q^i \dot{q}^j + \epsilon + \dots \quad (\text{A.51})$$

and possibly many more terms. But actually, one is quite restricted:  $-\gamma_{ij} q^i \ddot{q}^j$  is just  $\gamma_{ij} \dot{q}^i \dot{q}^j$  after an integration by parts,  $\lambda_i q^i$ ,  $\alpha_{ij} q^i q^j$  and  $\epsilon$  are particular potentials, and  $\beta_{ij} q^i \dot{q}^j$  as well as  $\mu_i \dot{q}^i$  would vanish: After all, they are just total time derivatives of the functions  $\beta_{ij} q^i q^j$  and  $\mu_i q^i$  which just depend on time and position.

It is very interesting to see that any reformulation of the Lagrange-function that can be achieved by integration by parts gives rise to exactly the same equation of motion: That is the case because  $\mathcal{L}$  only ever appears in the action integral  $S = \int dt \mathcal{L}$  with a fixed boundary. But for dealing with a term like  $\gamma_{ij} q^i \ddot{q}^j$  of second order we need a generalisation of the Euler-Lagrange-equation: Performing a variation to second order yields:

$$\delta S = \int dt \left( \frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} + \frac{\partial \mathcal{L}}{\partial \ddot{q}} \delta \ddot{q} \right) = \int dt \left( \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial \mathcal{L}}{\partial \ddot{q}} \right) \delta q = 0 \quad (\text{A.52})$$

with a single integration by parts for the second, and a double integration by parts in the third term. Then, Hamilton's principle defines the generalisation of the Euler-Lagrange-equation to higher orders:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial \mathcal{L}}{\partial \ddot{q}} = 0 \quad (\text{A.53})$$

### A.10 Beltrami-identity and the conservation of energy

The conservation of energy in classical mechanics is realised very differently compared to other conservation laws: In those, one can identify cyclic variables  $q$  defined by the condition  $\partial \mathcal{L} / \partial q = 0$ , so that the Euler-Lagrange-equation makes sure that

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0 \quad \text{and consequently, the canonical momentum} \quad p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \quad (\text{A.54})$$

is conserved,  $dp/dt = 0$ . Time, however, is *not* a coordinate in classical mechanics, so the definition of energy as the canonical momentum  $\partial \mathcal{L} / \partial t$  is *impossible*, it is completely unclear what  $t$  should actually be if not 1.

Instead, one needs the Beltrami-identity: By constructing

$$\frac{d}{dt} \left( \mathcal{L} - \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = \dot{q} \frac{\partial \mathcal{L}}{\partial q} + \ddot{q} \frac{\partial \mathcal{L}}{\partial \ddot{q}} - \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \dot{q} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q} \left( \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = 0 \quad (\text{A.55})$$

because the Euler-Lagrange-equation sets the brackets to zero. Hence, there is a conserved quantity  $\mathcal{H} = \dot{q}p - \mathcal{L}$ , referred to as the Hamilton-function  $\mathcal{H}$ , which depends on the canonical momentum  $p$  and the coordinate  $q$ . This definition already suggests that  $\mathcal{H}(p, q)$  is the Legendre-transform of  $\mathcal{L}(q, \dot{q})$ .

Let's investigate Ostrogradsky's idea that things become unstable if higher derivatives of  $q$  are included and write  $\mathcal{L} = \mathcal{L}(q, \dot{q}, \ddot{q}, \dots)$ , such that

$$\frac{d}{dt} \left( \mathcal{L} - \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = \dot{q} \frac{\partial \mathcal{L}}{\partial q} + \ddot{q} \frac{\partial \mathcal{L}}{\partial \ddot{q}} + \dddot{q} \frac{\partial \mathcal{L}}{\partial \ddot{q}} + \dots - \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \dot{q} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q} \left( \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) + \ddot{q} \frac{\partial \mathcal{L}}{\partial \ddot{q}} + \dots = 0 \quad (\text{A.56})$$

and substituting the general Euler-Lagrange-equation one obtains

$$\frac{d}{dt} \left( \mathcal{L} - \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = \dot{q} \left( -\frac{d^2}{dt^2} \frac{\partial \mathcal{L}}{\partial \ddot{q}} + \frac{d^3}{dt^3} \frac{\partial \mathcal{L}}{\partial \ddot{q}} + \dots \right) + \ddot{q} \frac{\partial \mathcal{L}}{\partial \ddot{q}} + \dots \neq 0 \quad (\text{A.57})$$

which would never work out to be zero.

The Hamilton-function  $\mathcal{H}$  takes on a nice, directly interpretable form for the standard Lagrange-function  $\mathcal{L} = m/2 \dot{q}^2 - \Phi(q)$ : The canonical momentum is  $p = \partial \mathcal{L} / \partial \dot{q} = m\dot{q}$  and therefore,  $\dot{q} = p/m$ , yielding

$$\mathcal{H}(p, q) = p\dot{q} - \mathcal{L}(q, \dot{q}(p)) = \frac{p^2}{m} - \frac{p^2}{2m} + \Phi(q) = \frac{p^2}{2m} + \Phi(q) \quad (\text{A.58})$$

Then,  $d\mathcal{H}/dt = 0$  and  $\mathcal{H}$  is conserved.

### A.11 Convexity of the Lagrange-function

The variational principle relies heavily on the fact that the Lagrange-function  $\mathcal{L}$  is a convex function in  $\dot{q}$ , and that the action  $S$  is a convex functional. Only then, there is a uniquely defined extremum and  $\delta S = 0$  defines the actual equation of motion. Imagine if Hamilton's principle had multiple solutions for  $\delta S = 0$ ! One would clearly end up in an impossible situation where multiple equations of motion would try to determine the evolution of a system.

Furthermore, the Hamilton-function is determined as the Legendre-transform of the Lagrange-function. For that to be feasible one needs the Lagrange function to be convex in  $\dot{q}$ : The canonical momentum  $p = \partial\mathcal{L}/\partial\dot{q}$  is needed for replacing  $\dot{q}$  with  $p$ , and therefore the relation  $p(\dot{q})$  needs to be invertible to give  $\dot{q}(p)$ . Invertibility is given if  $p = \partial\mathcal{L}/\partial\dot{q}$  is monotonically increasing and this is true if  $\mathcal{L}$  is a convex function in  $\dot{q}$ . Then,  $\mathcal{H}$  becomes a convex function in  $p$ , too, and the inverse Legendre-transform back to  $\mathcal{L}$  is well-defined. Convexity (or concavity, any overall sign in  $\mathcal{L}$  is undetermined because of affine invariance  $\mathcal{L} \rightarrow a\mathcal{L} + b$  with  $a = -1$  and  $b = 0$ ) is surely given for a standard form  $\mathcal{L} \propto \dot{q}^2$ , but is there a more fundamental reason for it? The answer to this profound question is relativity:

### A.12 Lorentz- and Galilei-relativity

Lagrange-mechanics is really a bit of physics that was discovered 100 years too early, as many aspect don't make much sense without relativity:  $\mathcal{L} = T - V$  is not measurable and ad-hoc to result in Newton's equation of motion, and the covariance under Galilei-transforms and rotations is realised in very different ways. So, let's approach Lagrange-mechanics through relativity!

In the absence of potentials or curvature, spacetime should be homogeneous as no point or instance in time should play a particular role, and this homogeneity should be reflected in the transformation between different coordinate frames. An observer looking at two coordinate choices could measure the rate at which the coordinates  $x^\mu$  and  $x'^\mu$  containing the collection of spatial coordinates  $x^i$  and time  $t$  are drifting by as a function of her or his proper time  $\tau$ , defining the velocity as the rate of change of the coordinates

$$\frac{dx^\mu}{d\tau} = \text{const.}, \text{ and identically in } S' : \frac{dx'^\mu}{d\tau} = \text{const.} \quad (\text{A.59})$$

which is constant for inertial motion and suitably chosen coordinates, and the corresponding acceleration

$$\frac{d^2x^\mu}{d\tau^2} = 0, \text{ and identically in } S' : \frac{d^2x'^\mu}{d\tau^2} = 0 \quad (\text{A.60})$$

then vanishes in both systems. The relation between the two velocities and accelerations is given by

$$\frac{dx'^\mu}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau} \quad (\text{A.61})$$

$$\frac{d^2x^\mu}{d\tau^2} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau}. \quad (\text{A.62})$$

If the gradient of the Jacobian of the coordinate change vanishes,

$$\frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} = 0 \quad (\text{A.63})$$

vanishing accelerations in one frame imply vanishing accelerations in the other. Then, the rate of change of the coordinate passage  $dx^{\mu}/d\tau$  is constant, and the transitions respects homogeneity. The solution for  $x'^{\mu}(x^{\nu})$  is consequently given by a linear, affine relation,

$$x'^{\mu} = A^{\mu}_{\nu} x^{\nu} + a^{\mu}. \quad (\text{A.64})$$

with two sets of integration constants  $A^{\mu}_{\nu}$  and  $a^{\mu}$ .

Let's construct this transform from the most general transition between two frames, where we align for simplicity the coordinate axes with the direction of relative motion, taken to be the  $x$ -axis. There is an event with coordinates  $x^{\mu}$  in  $S$  and  $x'^{\mu}$  in  $S'$ , and the two frames move with a relative (constant) velocity  $v$ . A linear, affine transform would then be the only one to respect the homogeneity of spacetime (nonlinear transforms would always single out certain spacetime points), so we make the ansatz:

$$x' = ax + bt, \text{ but } x = vt \text{ must imply } x' = 0 \quad (\text{A.65})$$

$$x' = 0 = avt + bt = (av + b)t \Rightarrow b = -av, \text{ and:} \quad (\text{A.66})$$

$$x' = a(x - vt) \quad (\text{A.67})$$

Reversing the roles of  $S$  and  $S'$  implies that

$$x = ax' + bt' \text{ but } x' = -vt' \text{ must imply } x = 0 \quad (\text{A.68})$$

$$x = 0 = -avt' + bt' = (-av + b)t' \Rightarrow b = +av, \text{ and:} \quad (\text{A.69})$$

$$x = a(x' + vt') \quad (\text{A.70})$$

But this relation between  $x$  and  $x'$  is not yet fixed without an additional assumption that determines the value of  $a$ . Here, Nature would have in fact a choice! Either, Nature could work with a universal time coordinate (or rather, a parameter, as it does not participate in transforms unlike the other coordinates). A universal time parameter would require that  $t = t'$ , which is the defining property of Galilei-transforms. Then,

$$x = a(x' + vt) = a(a(x - vt) + vt) = a^2x + (1 - a)vt = x \quad (\text{A.71})$$

which can only be realised if  $a = 1$ . Nature chose instead, for very good reasons, the speed of light to be equal in all frames,  $c = c'$ , which requires Lorentz- instead of Galilei-transforms between frames. In this choice,

$$x' = ct' = a(ct - vt) \quad (\text{A.72})$$

$$x = ct = a(ct' - vt') \quad (\text{A.73})$$

and consequently

$$c^2 tt' = a^2(c - v)(c + v) \cdot tt', \quad (\text{A.74})$$

where the third equation was obtained by multiplying the first two. Dividing by  $tt'$  and solving for  $a$  yields the Lorentz-factor  $\gamma$ ,

$$a = \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \text{with } \beta = \frac{v}{c} \quad (\text{A.75})$$

We should note that Lorentz-transformations, due to their linearity, do not 'mix' the spatial coordinates. The Lorentz-factor  $\gamma$  diverges at  $\beta = 1$  and would indeed become imaginary for values  $\beta > 1$ . Taylor-expanding  $\gamma$  for small velocities  $\beta$  gives the result that

$$\gamma \sim 1 + \left. \frac{\partial^2 \gamma}{\partial \beta^2} \right|_{\beta=0} \cdot \frac{\beta^2}{2} = 1 + \frac{\beta^2}{2}, \quad \text{with } \left. \frac{\partial \gamma}{\partial \beta} \right|_{\beta=0} = 1 \quad (\text{A.76})$$

which is perfectly consistent with the fact that for low velocities  $\beta \ll 1$  and  $\gamma \simeq 1$ , Lorentz- and Galilei-transforms are indistinguishable. Writing  $ct$  and arranging the temporal and spatial coordinates into a vector  $x^\mu = \begin{pmatrix} ct \\ x \end{pmatrix}$  allows to use the standard matrix-form of the Lorentz-transformation:

$$x' = \gamma(x - vt) = \gamma(x - \beta ct) \quad (\text{A.77})$$

$$ct' = \gamma(ct - \beta x), \quad (\text{A.78})$$

so that one arrives at

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \quad (\text{A.79})$$

encapsulating the Lorentz-transform in a matrix  $\Lambda^\mu{}_\nu$ , with  $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ . Therefore, there are only two possible linear transformations, where Nature chooses to conserve the speed  $c$ , and we will see how this is related to the causal structure of spacetime and the hyperbolic evolution of field equations.

Just as rotations leave Euclidean scalar products  $r^2 = \gamma_{ij}x^i x^j$  invariant, quadratic forms  $s^2 = \eta_{\mu\nu}x^\mu x^\nu$  formed with the Minkowski-metric  $\eta_{\mu\nu}$  are untouched by Lorentz-transforms, as one can see by direct computation:

$$\begin{aligned} s'^2 = (ct')^2 - x'^2 &= \gamma^2 \left[ (ct)^2 - 2ct\beta x + \beta^2 x^2 - x^2 + 2x\beta ct - \beta^2 (ct)^2 \right] = \\ &= \underbrace{\gamma^2(1 - \beta^2)}_1 \left( (ct)^2 - x^2 \right) = s^2 \quad (\text{A.80}) \end{aligned}$$

With this realisation, one can define an orthogonality relation:

$$s'^2 = \eta_{\mu\nu}x'^\mu x'^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta = \eta_{\alpha\beta}x^\alpha x^\beta = s^2 \quad (\text{A.81})$$

and therefore

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \quad (\text{A.82})$$

as an orthogonality relation for  $\Lambda^\mu{}_\alpha$ . The physical interpretation of the invariant  $s$  is the proper time  $\tau$  displayed on a comoving clock, i.e. a clock inside the local rest

frame, where  $x = 0$  constantly,

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu = (ct)^2 - x^2 = c^2 \tau^2 \quad (\text{A.83})$$

Differentially, this implies

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2 d\tau^2 = c^2 dt^2 - \gamma_{ij} dx^i dx^j \quad (\text{A.84})$$

and therefore

$$d\tau = \sqrt{1 - \gamma_{ij} \frac{1}{c} \frac{dx^i}{dt} \frac{1}{c} \frac{dx^j}{dt}} dt = \sqrt{1 - \beta^2} dt = \frac{dt}{\gamma} \quad (\text{A.85})$$

with  $\beta^2 = \gamma_{ij} v^i v^j / c^2$ . As  $\gamma \geq 1$  always,  $d\tau < dt$  and one observes a relativistic dilation of proper time relative to the coordinate time.  $\pm 1/\gamma = \pm \sqrt{1 - \beta^2}$  is in fact a semi-circle, so it's a perfectly convex (concave for the other sign choice) function, and would make an excellent candidate for the Lagrange-function. Additionally, its Taylor-expansion

$$\frac{1}{\gamma} = 1 - \frac{\beta^2}{2} \pm \dots \quad (\text{A.86})$$

at low velocities  $\beta \ll 1$  gives a term that is reminiscent of the classical kinetic energy!

### A.13 Rapidity

Lorentz-boosts form a group as their combination always gives a boost, but clearly the velocity is not additive, which can be verified by direct combination of two boosts: The sum of the velocities  $\beta$  always needs to stay below 1. One might wonder then whether there is an additive parameter for the Lorentz-transforms which replaces the velocity  $\beta$ . From the range of values of the terms in the Lorentz-transform  $\Lambda^\mu_\alpha$ , where  $0 \leq \beta\gamma < \infty$  (for positive  $\beta$  with a sign change for negative  $\beta$ !) and  $1 \leq \gamma < \infty$  one could think of a parameterisation  $\cosh \psi = \gamma$  and  $\sinh \psi = \beta\gamma$ , such that

$$\tanh \psi = \beta \quad \rightarrow \quad \psi = \text{artanh} \beta \quad (\text{A.87})$$

with the rapidity  $\psi$  replacing the physical velocity  $\beta$ . Then, the Lorentz-transform can be written as a hyperbolic rotation,

$$\Lambda^\mu_\alpha = \begin{pmatrix} \cosh \psi & \sinh \psi \\ \sinh \psi & \cosh \psi \end{pmatrix} \quad (\text{A.88})$$

where the Lorentz-invariance of  $s^2 = \eta_{\mu\nu} x^\mu x^\nu$  is then re-expressed as the property  $\cosh^2 \psi - \sinh^2 \psi = \gamma^2 - (\beta\gamma)^2 = \gamma^2(1 - \beta^2) = 1$  of the hyperbolic functions. Two successive boosts in this representation then shows that rapidity is additive as a parameter, which we'll elaborate in the chapter about Lie-groups. For the time being, we use the addition theorem of the hyperbolic tangent to get

$$\beta_{\phi+\psi} = \tanh(\phi + \psi) = \frac{\tanh \phi + \tanh \psi}{1 + \tanh \phi \tanh \psi} = \frac{\beta_\phi + \beta_\psi}{1 + \beta_\phi \beta_\psi} \quad (\text{A.89})$$

which falls back onto  $\beta_\phi + \beta_\psi$  for small  $\beta \ll 1$ , where the hyperbolic tangent, at the same time, is approximated well by its argument,  $\beta = \tanh \psi \simeq \psi$ . As  $\tanh \psi$  is bounded by 1 even as  $\psi \rightarrow \infty$ , successive boosts do not exceed  $\beta = 1$  or  $v = c$ .

#### A.14 Proper time and the relativistic Lagrange-function

The rate of change of coordinates  $dx^\mu$  can be measured in units of proper time  $d\tau$ , leading to the definition of 4-velocity,

$$x^\mu = \begin{pmatrix} ct \\ x^i \end{pmatrix} \quad \text{implies that} \quad u^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau} \frac{dx^\mu}{dt} = \gamma \begin{pmatrix} c \\ v^i \end{pmatrix} \quad (\text{A.90})$$

with  $\gamma = dt/d\tau$  and  $v^i = dx^i/dt$ . If the proper time  $\tau$  is used to parameterise the trajectory  $x^\mu(\tau)$  in this way, the tangent vector  $u^\mu = dx^\mu/d\tau$  is normalised to  $c$ ,

$$\eta_{\mu\nu} u^\mu u^\nu = \gamma^2 (c^2 - v^2) = c^2 \gamma^2 (1 - \beta^2) = c^2 \quad (\text{A.91})$$

which is true even for a particle at rest: There, only  $u^t = c$  is nonzero,  $v^i = 0$  and  $\eta_{\mu\nu} u^\mu u^\nu = c^2$ , effectively, the particle drifts along the  $ct$ -axis at velocity  $c$ . Motivated by the idea that  $1/\gamma$  could be a good candidate for the relativistic Lagrange-function, we could imagine that the arc-length  $S = \int ds = c\tau$  through spacetime of a trajectory  $x^\mu(\tau)$  could be extremised, as a very intuitive concept:

$$S = -mc \int ds = -mc^2 \int d\tau = -mc^2 \int dt \sqrt{1 - \beta^2} = -mc^2 \int \frac{dt}{\gamma}, \quad (\text{A.92})$$

therefore  $\mathcal{L} = -mc^2/\gamma$ , with the nonrelativistic limit  $-mc^2(1 - \beta^2)$ , i.e. up to an affine transform the actual kinetic energy  $mv^2/2$ .

#### A.15 Geometric view on motion

If the relativistic arc-length through spacetime as a candidate for the Lagrange-function were true, force-free motion should proceed along a straight line, as a reflection of the law of inertia: Technically, we replace the variation of the abstract classical action by the much more concrete variation of an arc through spacetime and monitor how the length  $S = \int ds$  would change under a variation,

$$\begin{aligned} \delta S &= -mc^2 \delta \int d\tau = -mc^2 \int \frac{\eta_{\mu\nu}}{2d\tau} [dx^\mu \delta dx^\nu + \delta dx^\mu dx^\nu] = \\ &= -mc^2 \int \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \underbrace{\delta dx^\nu}_{=d\delta x^\nu} = mc^2 \int d\tau \eta_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} \delta x^\nu \end{aligned} \quad (\text{A.93})$$

starting from

$$ds^2 = c^2 d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad \rightarrow \quad cd\tau = \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu} \quad (\text{A.94})$$

using that the Minkowski-metric is symmetric,  $\eta_{\mu\nu} = \eta_{\nu\mu}$  and finally that the differential can be expanded as



$$\frac{d^2x^\mu}{d\tau^2} = \frac{d}{d\tau} \frac{dx^\mu}{d\tau} d\tau = \frac{d^2x^\mu}{d\tau^2} d\tau \quad (\text{A.95})$$

after reshuffling the differentiations by an integration by parts. Hamilton's principle  $\delta S = 0$  therefore implies that

$$\frac{d^2x^\mu}{d\tau^2} = 0 \quad \rightarrow \quad x^\mu(\tau) = a^\mu \tau + b^\mu \quad (\text{A.96})$$

i.e. a straight line through spacetime, with two integration constants  $a^\mu$  and  $b^\mu$ . These trajectories result from minimising the arc-length, which is a convex functional and bounded by  $S = 0$  on the light cone. Affine invariance of the arc-length just means that you're free to measure it as proper time with any unit and from any zero-point.

With the law of inertia explained, could the formalism be adopted to (gravitational) potentials? There, we would indeed expect accelerations  $d^2x^\mu/d\tau^2$  as a consequence of gradients in  $\Phi$ . General relativity is really much beyond the scope of the tooltips-lecture but let's try this idea out with a weakly perturbed Minkowskian spacetime. There, the line element is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1 + 2\frac{\Phi}{c^2}\right) c^2 dt^2 - \left(1 - 2\frac{\Phi}{c^2}\right) \gamma_{ij} dx^i dx^j \quad (\text{A.97})$$

with the metric  $g_{\mu\nu}$  instead of the Minkowski-metric  $\eta_{\mu\nu}$ , and a weak gravitational potential  $\Phi$  with  $|\Phi| \ll c^2$  (Already at this point you can see that without a speed of light we could not say whether the potential is weak or strong!) on top of a Minkowski-vector space (I'm a bit adamant here, because just from  $g_{\mu\nu} \neq \eta_{\mu\nu}$  you can *not* infer the existence of gravitational potentials if there is full freedom in choosing the coordinates.) The first thing we should check if there is an influence of the gravitational potential on the passage of time: After all, proper time should differ from coordinate time which is first of all caused by special relativistic time dilation due to motion. For a stationary object  $dx^i = 0$  because there is no change in coordinate, and we get

$$d\tau = \sqrt{1 + 2\frac{\Phi}{c^2}} dt \simeq \left(1 + \frac{\Phi}{c^2}\right) dt = \left(1 - \frac{GM}{c^2 r}\right) dt \quad (\text{A.98})$$

by substitution of a Newtonian potential  $\Phi = -GM/r$ , where  $2GM/c^2 = r_S$  defines the Schwarzschild-radius: It seems to be the case that  $G/c^2 \simeq 10^{-28} \text{m/kg}$  assigns a length-scale to a mass, and for an object like the Sun with  $M = 10^{30} \text{kg}$  the Schwarzschild-radius comes out with a few hundred meters. As  $d\tau \leq dt$ , we observe a gravitational time dilation of proper time relative to coordinate time, and  $\Phi$  seems to have an influence on the relativistic arc-length just as velocity would. Repeating the above derivation with  $ds = cd\tau = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$  shows

$$S = -mc^2 \int d\tau = -mc \int dt \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} = -mc \int dt \sqrt{\left(1 + 2\frac{\Phi}{c^2}\right) c^2 - \left(1 - 2\frac{\Phi}{c^2}\right) v^2} \quad (\text{A.99})$$

Ignoring  $\Phi v$  as a higher-order term then yields after approximating  $\sqrt{1+x} = 1+x/2$

$$S = -mc^2 \int dt \sqrt{1 + 2\frac{\Phi}{c^2} - \frac{v^2}{c^2}} \simeq \int dt \left( \frac{mv^2}{2} - m\Phi - mc^2 \right) \quad (\text{A.100})$$

i.e. the classical Lagrange-function  $\mathcal{L} = T - \Phi$  with the rest mass as an additional term, which is irrelevant due to affine invariance of  $\mathcal{L}$ . Weirdly enough, motion in classical mechanics proceeds, in the aim to minimise  $S$ , in a way that time dilation is extremised, by being far down in gravitational potentials or by being fast.

### A.16 Relativistic energy and momentum

We have seen in the last chapter that the Lagrange-function is much more a statement of causal motion in spacetime and has little to do with energies: Those appear after Legendre-transform, which is always well defined because the Lagrange function is a convex functional in  $\dot{x}$  - this is, incidentally, the same reason why the variation yields a unique result and finds a unique extremum. It is important to realise that the conservation of the various canonical momenta are ensured by cyclic coordinates, but that energy conservation is a consequence of the Beltrami-identity.

The Legendre-transform of the relativistic Lagrange-function should provide a relativistic dispersion relation, i.e. a relation between energy and momentum. The canonical momentum  $p$  is derived from the Lagrange-function  $\mathcal{L} = -1/\gamma = -\sqrt{c^2 - \dot{x}^2}$ ,

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{c^2 - \dot{x}^2}} \quad (\text{A.101})$$

where this relation can be inverted from  $p(\dot{x})$  to  $\dot{x}(p)$ , as a consequence of the convexity of  $\mathcal{L}$ , and which is needed for computing the Legendre-transform

$$\mathcal{H}(p, x) = p\dot{x}(p) - \mathcal{L}(x, \dot{x}(p)) \quad (\text{A.102})$$

With the inversion

$$p^2 = \frac{v^2}{c^2 - v^2} \quad \rightarrow \quad p^2(c^2 - v^2) = v^2 \quad \rightarrow \quad p^2 c^2 = v^2(1 + p^2) \quad \rightarrow \quad v = \frac{cp}{\sqrt{1 + p^2}} \quad (\text{A.103})$$

with  $v = \dot{x}$  and consequently

$$\mathcal{H}(p, x) = v \underbrace{\frac{v}{\sqrt{c^2 - v^2}}}_p + \underbrace{\sqrt{c^2 - v^2}}_{\frac{v}{p}} = vp + \frac{v}{p} = v \left( p + \frac{1}{p} \right) = \frac{cp}{\sqrt{1 + p^2}} \frac{1 + p^2}{p} = c\sqrt{1 + p^2} \quad (\text{A.104})$$

Including the rest mass  $m$  would yield the relativistic dispersion relation

$$\mathcal{H} = \sqrt{(mc^2)^2 + (cp)^2} \quad (\text{A.105})$$

which is approximated by  $\mathcal{H} = mc^2 + p^2/(2m)$  for  $p \ll mc$  and by  $\mathcal{H} = cp$  for  $p \gg mc$ . The Hamilton-function  $\mathcal{H}$  in turn is again convex and allows an inverse Legendre-transform to recover  $\mathcal{L}$ .

### A.17 Causality and light cones

The Lorentz-invariant  $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$  makes it possible to differentiate between time-like  $ds^2 > 0$ , space-like  $ds^2 < 0$  and light-like  $ds^2 = 0$  separations in spacetime. Weirdly enough, the causal ordering of events, i.e. a statement of  $dt > 0$  or  $dt < 0$  depends on the chosen frame, and it is always possible to change the sign of  $dt$  in space-like separated events from the point of view of a fast enough moving Lorentz-frame.

Only inside the light cone, i.e. for all time-like separated events one observes the same causal order from all Lorentz-frames, which necessarily move at speeds  $\beta < 1$ . Being located inside the light-cone is certainly true for all massive particles: Their 4-velocities  $u^\mu$  are normalised to  $\eta_{\mu\nu} u^\mu u^\nu = c^2 > 0$ , and in fact the light cone is the convex hull of all possible trajectories of massive particles. Approaching  $\beta = 1$  has the Lorentz-factor  $\gamma$  diverge, which is often rewritten as a relativistic mass increase,

$$p^\mu = mu^\mu = \gamma m \begin{pmatrix} c \\ v^i \end{pmatrix} \quad \text{from} \quad u^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau} \frac{dx^\mu}{dt} = \gamma \begin{pmatrix} c \\ v^i \end{pmatrix} \quad (\text{A.106})$$

and therefore, relativistic mass increase is purely a consequence of the dilation of proper time. Asking whether it would be possible to accelerate a (charged) particle with electromagnetic forces will be answered negatively: Starting with the Lorentz-equation

$$\frac{du^\mu}{d\tau} = \frac{q}{m} F^{\mu\nu} u_\nu \quad (\text{A.107})$$

with the field-tensor  $F^{\mu\nu}$  that contains the electric and magnetic fields. Multiplying the Lorentz-equation with  $u_\mu$  gives:

$$u_\mu \frac{du^\mu}{d\tau} = \frac{1}{2} \frac{d}{d\tau} (u_\mu u^\mu) = \frac{q}{m} F^{\mu\nu} u_\mu u_\nu = 0 \quad (\text{A.108})$$

such that the normalisation of  $u^\mu$  is conserved to be  $c^2 > 0$ , and  $u^\mu$  remains time-like and inside the light cone despite being accelerated: The reason is purely geometrical, as the contraction of the antisymmetric tensor  $F^{\mu\nu}$  with the symmetric tensor  $u_\mu u_\nu$  is necessarily zero, making it impossible for  $u_\mu u^\mu$  to change.

At this point, we should start to be careful not to link the Lorentz-geometry to any particular coordinate choice. When considering light cone coordinates,  $du = cdt + dx$  and  $dv = cdt - dx$  the line element is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - dx^2 = (cdt + dx)(cdt - dx) = du \cdot dv, \quad (\text{A.109})$$

and the corresponding Lorentzian metric is represented by the matrix

$$\eta_{\mu\nu} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{A.110})$$

in these coordinates. Surely, the geometry is identical and has not been changed by the new definition of coordinates, and the spectrum of eigenvalues of the new metric is exactly +1 and -1.



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## B WAVES

### B.1 *Taxonomy of waves*

Waves, i.e. periodic phenomena in  $x$  and  $t$  are found everywhere in physics and can be differentiated to be in two categories: classical mechanical waves usually rely on the elastic properties of a medium which, due to its internal structure, resists deformation from its equilibrium. The magnitude of the restoring force that the medium provides drives the wave and allows it to travel.

In mechanical waves where the medium could be a fluid described by some type of Navier-Stokes equation, any term on the right hand side could be a suitable restoring force, for instance

$$\partial_t v^i + (v_j \partial^j) v^i = \underbrace{-\rho \partial^i p}_{\text{sound}} - \underbrace{\partial^i \Phi}_{\text{gravity}} + \underbrace{\mu \partial_j \partial^j v^i}_{\text{elastic}} + \underbrace{2\epsilon^{ijk} \Omega_j v_k}_{\text{Rossby}} + \dots \quad (\text{B.111})$$

In sound waves, pressure gradients can accelerate the fluid and if the equation of state provides  $\partial p / \partial \rho > 0$ , pressure gradients introduce velocities that rarefy the medium, so that it returns to its equilibrium state. Gravity waves are for instance large waves on the surface of water (also called Airy-waves) where the weight of the "mountain" of water is the restoring force. In elastic waves the restoring force is derived from the internal structure of the medium, and even the Coriolis-acceleration can act as a restoring force: This is the case in atmospheric Rossby-waves. Typically, the magnitude of the restoring force is contrasted with the inertia of the medium, and the ratio between the two determine the propagation velocity, which then entirely depends on the material properties of the fluid.

In contrast, relativistic waves are excitations of a field, whose dynamics is described with a field equation, and typically these field equation have a particular mathematical structure allowing for oscillations: Field equations in fundamental physics are hyperbolic partial differential equations which is a natural consequence of the spacetime structure. Personally I find it very interesting, that the same wave equations are found in very different contexts, and that propagation speeds can be determined by relativity on one side and by the internal structure of a medium on the other. When thinking about ideas on the lumiferous aether over a hundred years ago and the measurements of the speed of light that were already available with high precision at that time, it must have been truly daunting to explain the high value of  $c$  from the low inertia and the high restoring force of the aether, if light was imagined to be an elastic wave.

### B.2 *Elastic waves and wave equations*

Perhaps the most intuitive example of an elastic mechanical wave is that of a string with mass per length  $\rho$  under tension  $\sigma$ : Already now one would intuitively think that the ratio between  $\rho$  and  $\sigma$  should determine the velocity of elastic waves. In a string instrument, the ratio between velocity and fixed string length gives the frequency  $\omega$  of a sound, and one observes an increase of frequency with higher string tension and one typically uses thicker strings for lower frequency notes.

The kinetic energy  $dT$  for each differential bit of string is given by the velocity  $\partial y / \partial t = \dot{y}$  by which the amplitude  $y$  changes along the string with coordinate  $x$ ,

$$dT = \frac{\rho}{2} \dot{y}^2 dx \quad \rightarrow \quad T = \int dT = \frac{\rho}{2} \int dx \dot{y}^2 \quad (\text{B.112})$$

For the corresponding potential energy  $dW$  we need to compute by how much the amplitudes  $y(x)$  change the overall length of the string:  $dl^2 = dx^2 + dy^2$  from Pythagoras' theorem gives  $dl = \sqrt{1 + y'^2} dx$  with  $y' = dy/dx$ , and consequently

$$dW = \sigma(dl - dx) = \sigma(\sqrt{1 + y'^2} - 1) dx \approx \frac{\sigma}{2} y'^2 dx \quad \rightarrow \quad W = \int dW = \frac{\sigma}{2} \int dx y'^2 \quad (\text{B.113})$$

Assembling both into a classical Lagrange-function yields

$$\mathcal{L}(\dot{y}, y') = \int dx \left( \frac{\rho}{2} \dot{y}^2 - \frac{\sigma}{2} y'^2 \right) \quad (\text{B.114})$$

from which we get the action  $S$  straight away:

$$S = \int dt \int dx \left( \frac{\rho}{2} \dot{y}^2 - \frac{\sigma}{2} y'^2 \right) \quad (\text{B.115})$$

The Lagrange-function  $\mathcal{L}$  depends on  $\dot{y}$  as well as on  $y'$ , which Hamilton's principle needs to respect. The correct variation of  $S$  would then be

$$\delta S = \int dt \int dx \left( \frac{\partial \mathcal{L}}{\partial y} \delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}} \delta \dot{y} + \frac{\partial \mathcal{L}}{\partial y'} \delta y' \right) \quad (\text{B.116})$$

while the coordinate  $y$  is cyclic and the first term does not play a role, the variations in the second and third term can be rewritten as  $\delta \dot{y} = \partial(\delta y)/\partial t$  and  $\delta y' = \partial(\delta y)/\partial x$  to enable integration by parts, with respect to  $dt$  in the second and with respect to  $dx$  in the third term:

$$\delta S = \int dt \int dx \left( \frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial y'} \right) \delta y = 0 \quad (\text{B.117})$$

where we can isolate the Euler-Lagrange-function

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{y}} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial y'} = \frac{\partial \mathcal{L}}{\partial y} \quad (\text{B.118})$$

Substitution of eqn. B.114 into eqn. B.118 then gives:

$$\rho \ddot{y} - \sigma y'' = 0 \quad \text{or} \quad \left( \frac{\partial^2}{\partial (ct)^2} - \frac{\partial^2}{\partial x^2} \right) y = 0 \quad \text{with} \quad c^2 = \frac{\sigma}{\rho} \quad (\text{B.119})$$

where the speed of propagation of the elastic wave is in fact determined by the ratio of the tension as the restoring force and the inertia of the string. The wave equation can be solved by separating the temporal and spatial dependence with the ansatz  $y(x, t) = \phi(x)\psi(t)$ , such that

$$\frac{1}{\psi(t)} \frac{\partial^2}{\partial (ct)^2} \psi(t) = \frac{1}{\phi(x)} \frac{\partial^2}{\partial x^2} \phi(x) = -k^2 \quad (\text{B.120})$$

after a separation of variables, and because every term depends only on  $t$  or on  $x$ , they need to be independently equal to a constant, which we choose to be negative (for enforcing oscillating solutions). Individually, every term is then solved by a harmonic oscillation, and substitution then shows that

$$y(x, t) \propto \exp(\pm i(kx - \omega t)) \quad (\text{B.121})$$

with the dispersion relation  $\omega = \pm ck$  and the speed of the elastic wave  $c = \sqrt{\sigma/\rho}$ . The sign between  $kx$  and  $\omega t$  follows from requiring whether a plane of constant phase travels into positive or negative  $x$ -direction. Both directions are clearly allowed, as  $\partial_{ct}^2 - \partial_x^2 = (\partial_{ct} - \partial_x)(\partial_{ct} + \partial_x)$  from the binomial formula.

It is important to realise that an elastic wave is able to transport energy even without any transport of the medium on which it travels.

### B.3 Partial differential equations: hyperbolic vs. elliptic

Wave-equations are typically partial differential equations involving second derivatives, for instance for a scalar field

$$\square\phi = \eta_{\mu\nu}\partial^\mu\partial^\nu\phi = \left(\partial_{ct}^2 - \Delta\right)\phi = 0 \quad (\text{B.122})$$

At this point it is well worth to go through the classification of second-order partial differential equations: Comparing  $\square\phi = 0$  as a wave equation with  $\Delta\phi = 0$  as a static field equation shows that the signs of the derivative operators is  $(-, +, +, +)$  in the first case and  $(+, +, +)$  without a change in the second case. This seems to be highly significant, as one obtains oscillatory solutions in the first, and (decreasing, at least in 3 dimensions or more) power-law solutions in the second case.

Before we go through the classification of partial differential equation, we need to introduce some slang, borrowed from the theory of conic sections. Please consider a quadratic form of two coordinates  $x$  and  $y$ ,

$$\begin{pmatrix} x \\ y \end{pmatrix}^t \underbrace{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}}_{\Delta} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + bxy + cy^2 = \text{const.} \quad (\text{B.123})$$

Depending on the structure of eigenvalues of the matrix  $\Delta$ , the quadratic form describes very different curves: If  $b = 0$  (for simplicity) and  $a = c = 1 > 0$  one obtains  $x^2 + y^2 = \text{const}$ , which can be rewritten in a parametric form by setting  $x = \cos t$  and  $y = \sin t$  such that the quadratic form describes a circle as a consequence of  $\cos^2 t + \sin^2 t = 1$ , and if  $a \neq c$  an ellipse. If  $a = 1$  and  $c = -1$ , the quadratic form becomes  $x^2 - y^2 = \text{const}$ , i.e hyperbolae with the hyperbolic functions as parametric forms, using  $\cosh^2 t - \sinh^2 t = 1$ . More generally, the picture arises that  $\det \Delta > 0$  for the elliptical conic section and conversely,  $\det \Delta < 0$  for the hyperbolic conic section.

Applying this idea to the classification of partial differential equations, we start with a homogeneous second-order PDE in two variables,

$$a(x, y) \frac{\partial^2}{\partial x^2} \phi(x, y) + b(x, y) \frac{\partial^2}{\partial x \partial y} \phi(x, y) + c(x, y) \frac{\partial^2}{\partial y^2} \phi(x, y) = \Lambda(x, y) \phi(x, y) \quad (\text{B.124})$$

and assemble the matrix  $\Delta$

$$\Delta = \begin{pmatrix} a(x, y) & \frac{1}{2}b(x, y) \\ \frac{1}{2}b(x, y) & c(x, y) \end{pmatrix} \quad (\text{B.125})$$

The determinant of  $\Delta$  then determines, whether the PDE is elliptical  $\det \Delta > 0$ , parabolic  $\det \Delta = 0$  or hyperbolic  $\det \Delta < 0$ .

Sticking to 2 dimensions and pairs of variables for simplicity, a PDE like the Poisson-equation

$$\Delta\phi = \frac{\partial^2}{\partial x^2}\phi(x, y) + \frac{\partial^2}{\partial y^2}\phi(x, y) = 0 \quad (\text{B.126})$$

would be elliptical, as the determinant of  $\Delta$  would come out positive:  $a = c = 1$  and  $b = 0$ . Elliptical differential equations have only unique solutions after boundary conditions are specified, either of the Dirichlet or Neumann-type. Typical solutions are decreasing (at least in 3 dimensions or higher) with increasing coordinates and parity invariant, as  $(x, y) \rightarrow (-x, -y)$  does not change anything. On the other hand, a wave-equation exhibits a sign change,

$$\square\phi(t, x) = \frac{\partial^2}{\partial(ct)^2}\phi(t, x) - \frac{\partial^2}{\partial x^2}\phi(t, x) = 0 \quad (\text{B.127})$$

with  $a = 1$ ,  $c = -1$  and  $b = 0$  in these coordinates and would be hyperbolic as  $\det \Delta < 0$ . In this case, it is enough to specify initial conditions and the PDE evolves them in a well-defined and unique way into the future. There is clearly the notion of a light-cone and it is actually the case that the metric structure of spacetime with the Minkowski-metric is uniquely suited for hyperbolic PDEs: It is even the fact, that the Lorentzian spacetime as a metric spacetime that allows for hyperbolic evolution is unique! Switching to light-cone coordinates  $\partial_{ct} + \partial_x = \partial_u$  and  $\partial_{ct} - \partial_x = \partial_v$  brings the wave equation into the form

$$\square\phi(u, v) = \frac{\partial^2}{\partial u \partial v}\phi(u, v) = 0 \quad (\text{B.128})$$

this time with  $a = c = 0$  and  $b = 1$ , but the determinant  $\det \Delta < 0$  nonetheless: The wave equation is hyperbolic in light cone coordinates just as well. In the wave equation there is parity invariance and time-reversal invariance. Perhaps it's a very good exercise to go through all iconic PDEs in theoretical physics and classify them as elliptical, parabolic or hyperbolic partial differential equations.

#### B.4 Relativistic waves and hyperbolicity

The dynamics of relativistic fields is described by hyperbolic PDE with their clear notion of a light cone and their time evolution from any field-configuration specified as initial conditions. As an example, we can substitute  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$  into the Maxwell-equation,

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu \quad \rightarrow \quad \partial_\mu \partial^\mu A^\nu - \underbrace{\partial^\nu \partial_\mu A^\mu}_{=0} = \square A^\nu = \frac{4\pi}{c} j^\nu \quad (\text{B.129})$$



which becomes clearly a hyperbolic wave equation. But the Lorenz-gauge  $\partial_\mu A^\mu = 0$  is not required for hyperbolicity, in fact, even without any gauge fixing it would be hyperbolic. As a linear PDE this is most conveniently solved by constructing a Green-function including retardation as the potential for a point charge.

There is a similar wave equation for the field tensor itself: Starting at the Bianchi-identity,

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \quad (\text{B.130})$$

which can immediately be verified by substituting  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ , one can have  $\partial_\lambda$  act on it,

$$\underbrace{\partial_\lambda \partial^\lambda F^{\mu\nu}}_{\square} - \underbrace{\partial^\mu \partial_\lambda F^{\lambda\nu}}_{\frac{4\pi}{c} j^\nu} + \partial^\nu \underbrace{\partial_\lambda F^{\lambda\mu}}_{\frac{4\pi}{c} j^\mu} = 0 \quad (\text{B.131})$$

and arrive at a wave equation with a nicely antisymmetrised source term,

$$\square F^{\mu\nu} = \frac{4\pi}{c} (\partial^\mu j^\nu - \partial^\nu j^\mu) \quad (\text{B.132})$$

The vacuum solutions are  $\square A^\mu = 0$  as well as  $\square F^{\mu\nu} = 0$  are archetypically hyperbolic and solved by plane waves  $\exp(\pm i \eta_{\mu\nu} k^\mu x^\nu)$ , provided that the wave vector  $k^\mu$  is light-like,  $\eta_{\mu\nu} k^\mu k^\nu = 0$ , which has important consequences: Writing  $k^\mu = (\omega/c, k^i)^t$  shows that

$$\omega = \pm k \quad (\text{B.133})$$

from the null-condition  $\eta_{\mu\nu} k^\mu k^\nu = (\omega/c)^2 - k^2 = 0$ , such that there can not be any dispersion:

$$v_{\text{phase}} = \frac{d\omega}{dk} = c = \frac{\omega}{k} = v_{\text{group}} \quad (\text{B.134})$$

as group and phase velocity are identical, and consistent with  $v_{\text{phase}} \times v_{\text{group}} = \omega/k \times d\omega/dk = d\omega^2/dk^2 = c^2$  for a massless particle:  $\omega^2 = c^2 k^2$ , and  $(\omega/c)^2 - k^2 = 0$ . At the same time, it is universally true that relativistic waves are always transverse: The field equation requires  $\partial_\mu F^{\mu\nu} = 0$  in vacuum, so that  $k_\mu F^{\mu\nu} = 0$  and  $k_i E^i = 0$  is always given, and the electric fields are perpendicular to the direction of propagation. Transversality of the magnetic fields is most easily seen with the dual field tensor  $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}/2$ , for which  $\partial_\mu F^{\mu\nu} = 0$  is true: Then,  $k_\mu \tilde{F}^{\mu\nu} = 0$  from which one obtains  $k_i B^i = 0$ .

The analogous statement on the vector potential  $A^\mu$ , however, depends on the gauge: Lorenz-gauge  $\partial_\mu A^\mu = 0$  implies  $k_\mu A^\mu = 0$  for a plane wave, so that  $k_i A^i = \omega A^t/c \neq 0$ , but Coulomb-gauge rather makes sure that  $\partial_i A^i = i k_i A^i = 0$ , such that the potential  $A^i$  is perpendicular to  $k^i$ : That's why it's sometimes called transverse gauge.

## B.5 Causal structure of spacetime

In the last section we have seen that there is a tight connection between hyperbolicity of the wave equation  $\square\phi = 0$  and the lightlike-ness of the wave-vector  $\eta_{\mu\nu} k^\mu k^\nu = 0$ , which is not too surprising because  $\square = \eta_{\mu\nu} \partial^\mu \partial^\nu$ , so the representation of  $\square$  in Fourier-space is  $\eta_{\mu\nu} k^\mu k^\nu$  anyways. The wave equation as a hyperbolic PDE provides a time evolution of initial conditions (and the solution becomes unique if those initial

Table 1: dimensionality required by certain physical phenomena

phenomenon	dimensionality
Huygens' principle	$n$ odd, $n = 1$ or $n = 3$ best
relativistic gravity	$n + 1 \geq 4$
stable planetary systems	$n \leq 3$
Bose-Einstein-condensation	$n \geq 3$
random walks getting lost (Polya's theorem)	$n \geq 3$
as many electric as magnetic fields	$n = 3$
Poisson-solutions vanish at infinity	$n \geq 3$
knots exist	$n \geq 3$

conditions are specified) in a very peculiar way: For the evolution of  $\phi$  at a specific spacetime point  $x^{\mu}$  only the field amplitudes on the past light cone are necessary, clearly as the field excitations can only travel along the light cone. This is perfect, because the light cone structure is Lorentz-invariant, so the field amplitudes that are responsible as initial conditions for  $\phi$  at  $x^{\mu}$  are always the same, despite the fact that  $x^{\mu}$  will get new coordinates.

This idea is truly funny in Galilean relativity: Here,  $c$  is just a velocity and transforms along under Galilei-transforms. Therefore, the two branches of the light cone get velocities  $c + v$  and  $c - v$  formally. Would this be a problem? Well, in the limit  $c \rightarrow 0$  (i.e. as the formal limit of Galilei-relativity from Lorentz-relativity or for everyday, small velocities compared to  $c$ ) the light cone opens up and the field amplitudes on an entire spatial hyperplane set the initial conditions for  $\phi$ . This is consistent with all derivatives  $\partial_{ct}\phi$  becoming small as  $c \rightarrow \infty$ , so that  $\square \rightarrow \Delta$  in this limit: The field equation has lost its dynamics and has become elliptical, such that boundary conditions (possibly on boundaries at  $x^i \rightarrow \pm\infty$ ) need to be specified for uniqueness.

And before you get funny ideas for this: Among all metric spacetimes only the Lorentzian one allows hyperbolic evolution of field equations, but one can construct hyperbolic equations without a metric structure for spacetime! The classic example for this would be covariant electrodynamics in the most general linear model for matter, and we'll come to that in section C.

### B.6 Dimensionality of spacetime

Spacetime has  $n + 1 = 4$  dimensions, 1 temporal and  $n = 3$  spatial, and it is the case that Nature really needs a minimum number of (spatial) dimensions to make certain phenomena possible, a few are summarised in table 1.

The Poisson-equation has the peculiar property that potentials  $\Phi$  only vanish towards infinity in 3 or more dimensions: Looking for vacuum solutions in the spherically symmetric case

$$\Delta\Phi = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial\Phi}{\partial r} \right) = 0 \tag{B.135}$$

is solved when the term in the brackets becomes constant, i.e. when

$$\frac{\partial\Phi}{\partial r} = r^{-(n-1)} \rightarrow \Phi \propto r^{-n+2} \text{ if } n \geq 3, \text{ or } \Phi \propto \ln r \text{ if } n = 2 \tag{B.136}$$

so that one really needs certainly 3 spatial dimensions or more for the potentials to

decrease towards infinity, and one gets logarithmic solutions  $\Phi \propto \ln r$  in 2 dimensions. General relativity as a theory of gravity can only exist in  $n + 1 = 4$  dimensions or more, if gravity as spacetime curvature should be allowed to propagate away from the sources, but this is really beyond the scope of the lecture.

From the scaling of  $\Phi$  in  $n$  dimensions one can derive that planetary systems are not stable if the dimensionality is too high, and the argument would be like that: For the specific energy  $\epsilon = E/m$  of a particle in the potential  $\Phi$  one would write

$$\epsilon = \frac{E}{m} = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\varphi}^2) + \Phi(r) \quad (\text{B.137})$$

in polar coordinates, with  $\Phi(r) = -GM/r^{n-2}$  generated by the central object of mass  $M$ . The motion of planets is restricted to be in a plane, because of angular momentum conservation in a spherically symmetric potential, with the specific angular momentum  $\lambda$

$$\lambda = \frac{L}{m} = r^2 \dot{\varphi} \quad \rightarrow \quad \dot{\varphi} = \frac{\lambda}{r^2} \quad (\text{B.138})$$

which will appear as a repulsive centrifugal potential when replacing  $\dot{\varphi}$ .

$$\epsilon = \frac{1}{2} \left( \dot{r}^2 + \frac{\lambda^2}{r^2} \right) - \frac{GM}{r^{n-2}} \quad (\text{B.139})$$

and counteracts the attractive gravitational potentials. For a stable orbit it is now necessary that the repulsive part of the potential is dominating at small  $r$ , for which  $n$  can not be too large. In fact, in a true Coulomb-potential with  $n = 3$  one gets a long-range attractive  $1/r$  potential superimposed on a short range repulsive  $1/r^2$ -potential, with a nice minimum that harbours the most stable circular orbits. If  $n$  is too high, the roles interchange: There would be a short range attractive gravitational potential superimposed on a long range repulsive potential, with effectively a maximum between the two regimes with unstable orbits. Solving the equation of motion yields

$$\dot{r}^2 = 2 \left( \epsilon + \frac{GM}{r^{n-2}} \right) - \frac{\lambda^2}{r^2} \quad \rightarrow \quad t = \int_0^t dt = \int_{r_{\min}}^{r_{\max}} dr \frac{1}{\sqrt{2 \left( \epsilon + \frac{GM}{r^{n-2}} - \frac{\lambda^2}{2r^2} \right)}} \quad (\text{B.140})$$

by separation of variables: There is an oscillatory motion in the effective potential (if there is a minimum allowing stable orbits), while the planet gets carried around the Sun by the conservation of angular momentum. Bertrand's theorem now states that among all potentials, only two allow for closed orbits: Those are the Keplerian ellipses in  $1/r$ -potentials and the Lissajous-figures in the harmonic  $r^2$ -potential.

As the last point let's investigate the issue that only in  $n = 3$  dimensions there is an equal number of electric and magnetic field components: This becomes most apparent in the field tensor  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ , which is antisymmetric under index exchange,  $F^{\nu\mu} = -F^{\mu\nu}$ . The electric field components are contained in the first row or first column,  $E^i = F^{ti} = -F^{it}$ , and there are  $n$  possible choices, as  $F^{tt} = 0$ . As off-diagonal elements representing the magnetic fields one counts  $n(n-1)/2$ , and  $n(n-1)/2 = n$  is solved by  $n = 3$  (and  $n = 0$ , but this is senseless).

## B.7 Huygens-principle

There is a remarkable peculiarity in the propagation of spherical waves that depends on dimensionality. Writing down a conventional hyperbolic wave equation in  $n$  dimensions

$$\eta_{\mu\nu}\partial^\mu\partial^\nu\psi = \left(\partial_{ct}^2 - \gamma_{ij}\partial^i\partial^j\right)\psi = \left(\partial_{ct}^2 - \sum_{i=1}^n\partial_i^2\right)\psi = 0 \quad (\text{B.141})$$

with an isotropic spatial part, as  $\gamma_{ij}\partial^i\partial^j$  with the Euclidean (inverse) metric  $\gamma_{ij}$  is perfectly invariant under rotations. A spherical wave  $\psi(t, r)$  with  $r^2 = \gamma_{ij}x^ix^j = x_ix^i$  excited at the origin should propagate outwards, and we will try to answer the question whether the wave front is a well-defined shell with radius  $r$  increasing linearly in time,  $r = ct$ . Surprisingly, this is only in 3 dimensions the case. Let's build quickly the derivatives

$$\partial_i r = \frac{x_i}{r} \rightarrow \sum_i (\partial_i r)^2 = \sum_i \left(\frac{x_i}{r}\right)^2 = \frac{1}{r^2} \sum_i x_i^2 = 1 \quad (\text{B.142})$$

and

$$\partial_i^2 r = \frac{r^2 - x_i^2}{r^3} \rightarrow \sum_i \partial_i^2 r = \sum_i \frac{r^2 - x_i^2}{r^3} = \frac{1}{r} \sum_i 1 - \frac{1}{r^3} \sum_i x_i^2 = \frac{n-1}{r} \quad (\text{B.143})$$

from  $r = \sqrt{x_j x^j}$  for later use. When introducing spherical coordinates one would like to replace the  $\partial_i$ -differentiations with respect to Cartesian coordinates by  $\partial_r$  using the chain rule,

$$\partial_i \psi = \partial_i r \cdot \partial_r \psi \quad (\text{B.144})$$

where I put the  $\cdot$  to "stop" the differentiation at this point. For the second derivative one gets

$$\partial_i^2 \psi = \partial_i^2 r \cdot \partial_r \psi + \partial_i r \cdot \partial_i \partial_r \psi \quad (\text{B.145})$$

where the second term can be reshaped

$$\partial_i \partial_r \psi = \partial_r \partial_i \psi = \partial_r (\partial_i r) \partial_r \psi = \partial_i \partial_r r \cdot \partial_r \psi + \partial_i r \partial_r^2 \psi = \partial_i r \partial_r^2 \psi \quad (\text{B.146})$$

with  $\partial_r r = 1$  such that the derivative vanishes. Substitution back into the wave equation gives

$$\partial_i^2 \psi = \partial_i^2 r \partial_r \psi + (\partial_i r)^2 \partial_r^2 \psi \quad (\text{B.147})$$

which, summing over  $i$  and using eqns. B.142 and B.143, leads us to

$$\sum_i \partial_i^2 \psi = \Delta \psi = \frac{n-1}{r} \partial_r \psi + \partial_r^2 \psi \quad (\text{B.148})$$

such that the wave equation for a spherical wave becomes

$$\partial_{ct}^2 \psi = \partial_r^2 \psi + \frac{n-1}{r} \partial_r \psi \quad (\text{B.149})$$

with the additional term  $(n-1)/r \partial_r \psi$  due to spherical symmetry. Of course you can start at

$$\Delta \psi = \frac{1}{r^{n-1}} \partial_r (r^{n-1} \partial_r \psi) = \frac{n-1}{r} \partial_r \psi + \partial_r^2 \psi \quad (\text{B.150})$$

as well to arrive at the same result.

For solving the spherical wave equation, one chooses a separation ansatz  $\psi = r^{-k} \phi$  for factoring out a power-law decrease of the amplitudes. One would expect that the squares of the amplitudes determines the energy flux of the spherical wave, that needs to be conserved over ever increasing surfaces of spherical shells scaling  $\propto r^{n-1}$  in area with radius  $r$ , implying  $k = (n-1)/2$ .

The corresponding derivatives then are

$$\partial_r \psi = -k r^{-(k+1)} \phi + r^{-k} \partial_r \phi \quad (\text{B.151})$$

and

$$\partial_r^2 \psi = k(k+1) r^{-(k+2)} \phi - 2k r^{-(k+1)} \partial_r \phi + r^{-k} \partial_r^2 \phi \quad (\text{B.152})$$

which can be used to reformulate the wave equation in terms of  $\phi$  rather than  $\psi$ :

$$\partial_{ct}^2 \phi - \partial_r^2 \phi - \frac{(n-1)(n-3)}{4r} \partial_r \phi = 0 \quad (\text{B.153})$$

which is a truly remarkable result: Of course, there is no concept of spherical symmetry in 1 dimension, so automatically the wave equation for the amplitude  $\phi$  (which incorporates energy conservation in its suggested scaling with distance, in this case it is constant) is fulfilled. In all other spacetimes with the exception of  $n = 3$  one sees additional terms in the wave equation, which actually slow down the wave relative to  $c$  and fill up the light cone with partial waves, such that (i) neither a spherical wave front would be defined and (ii) there is no clear relation  $r = ct$ : This, however is exactly the case in  $n = 3$  dimensions! In summary,  $n + 1 = 4$  dimensions is the only case where wave propagation of spherical waves is described by a plane wave equation with a relation  $r = ct$  for the radius. If one would decompose an arbitrary wave front into elementary spherical waves according to Huygens' principle, they only propagate with a well-defined wave front defined by  $r = ct$  in 3d to interfere again at a later time.



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## C FIELDS

### C.1 Lagrange-description of field dynamics

Relativistic field equations in Nature, for instant for the Maxwell-field  $A^\mu$  or for the metric  $g_{\mu\nu}$  are commonly hyperbolic, second-order partial differential equations, and due to their hyperbolicity there is wave-like propagation of excitations along a light-cone, which is defined by the underlying geometric structure of spacetime. The first notion of a field was Newton's idea of an action at a distance: Somehow gravity from and on Earth needed to extend to the Moon and other celestial bodies. This is really a revolutionary thought as it was the first time in physics where the constituents of a system are not in direct physical contact. The question whether the fields are real or just a convenient way of computing forces between charges that couple to the field, is a bit philosophical but after all, all physical concepts that apply to the "material world" apply to fields in exactly the same way, including the point that the associated energy and momentum content of a field is able to source gravity.

### C.2 Lagrange-description of scalar field dynamics

Deriving the field equation of a scalar field  $\phi$  is almost like dissipationless continuum mechanics. Let's ignore dynamical evolution for a second and derive the most general linear theory with a second-order partial differential field equation, which would be necessarily elliptical if there's no proper time evolution. As expected one would write down a kinetic and potential term in a suitable Lagrange-density,

$$\mathcal{L}(\phi, \partial^i \phi) = \gamma_{ij} \partial^i \phi \partial^j \phi - 8\pi\rho\phi \quad (\text{C.154})$$

and establish Hamilton's principle  $\delta S$  for varying the action  $S$

$$\delta S = \delta \int d^3x \mathcal{L}(\phi, \partial^i \phi) = \int d^3x \left( \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \partial^i \phi} \delta \partial^i \phi \right) = \int d^3x \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial^i \frac{\partial \mathcal{L}}{\partial \partial^i \phi} \right) \delta \phi \quad (\text{C.155})$$

after writing  $\delta \partial^i \phi = \partial^i \delta \phi$  and a successive integration by parts. Substitution of the Lagrange-density eqn. C.154 into the Euler-Lagrange-equation

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial^i \frac{\partial \mathcal{L}}{\partial \partial^i \phi} = 0 \quad (\text{C.156})$$

which can be isolated from eqn. C.155 yields the Poisson-equation

$$\partial^j \partial_i \phi = \Delta \phi = -4\pi\rho \quad (\text{C.157})$$

by realising that

$$\frac{\partial}{\partial \partial^i \phi} (\gamma_{ab} \partial^a \phi \partial^b \phi) = \gamma_{ab} \left( \frac{\partial \partial^a \phi}{\partial \partial^i \phi} \partial^b \phi + \partial^a \phi \frac{\partial \partial^b \phi}{\partial \partial^i \phi} \right) = \gamma_{ab} (\delta_i^a \partial^b \phi + \partial^a \phi \delta_i^b) = 2\partial_i \phi \quad (\text{C.158})$$

while the rest of the terms in the Euler-Lagrange-equation is pretty easy.

Repeating the arguments for finding the most general Lagrange-function for a point particle leads to the Lagrange-density

$$\mathcal{L}(\phi, \partial^i \phi) = \frac{1}{2} \gamma_{ij} \partial^i \phi \partial^j \phi - 4\pi\rho\phi + \lambda\phi + \frac{m^2}{2} \phi^2 \quad (\text{C.159})$$

with the associated field equation

$$(\Delta - m^2) \phi = -4\pi\rho + \lambda \quad (\text{C.160})$$

for the most general scalar field equation that is linear and compatible with Ostrogradsky's theorem. If  $\phi$  is the Newtonian gravitational potential  $\Phi$  and interpreting the generalised Poisson-equation in terms of a gravitational theory we now know that  $m$  must be truly small, and that  $\lambda$  is small but certainly nonzero. While all this looks straightforward from an arithmetic point of view, the conceptual interpretation is not so easy: Hamilton's principle  $\delta S = \delta \int d^3x \mathcal{L} = 0$  looks for a field configuration  $\phi$  which minimises the action, and for a vacuum solution the kinetic term  $\partial\mathcal{L}/\partial\partial^i\phi$  would be required to be perpendicular to  $\delta\partial^i\phi$ , which is perhaps a bit reminiscent of d'Alembert's principle.

Often you'll see  $m$  interpreted as the mass of the field  $\phi$ , or at least as its inertia, even though at this point it's not more than a scale-invariance breaking inverse length scale. If the field  $\phi$  is allowed to have its own dynamics in accordance with special relativity one would make the replacements  $\gamma_{ij} \rightarrow \eta_{\mu\nu}$  and  $\partial^i \rightarrow \partial^\mu$  to arrive at

$$S = \int d^4x \mathcal{L}(\phi, \partial^\mu \phi) \quad \text{with} \quad \mathcal{L}(\phi, \partial^\mu \phi) = \frac{1}{2} \eta_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - \frac{m^2}{2} \phi^2 \quad (\text{C.161})$$

where we omitted the coupling to  $\rho$  on purpose because its transformation property is yet unclear, and let's focus on scales small compared to  $1/\sqrt{\lambda}$ . Variation then gives

$$(\square + m^2) \phi = 0 \quad \rightarrow \quad \eta_{\mu\nu} k^\mu k^\nu = m^2 > 0 \quad \text{in Fourier space} \quad (\text{C.162})$$

such that the wave vector  $k^\mu$  is timelike and points to a location inside the light cone: Excitations of  $\phi$  travel at speeds less than the speed of light which justifies to think of  $m$  as a mass. Please watch out for the minus signs here, as  $\square \exp(\pm i \eta_{\alpha\beta} k^\alpha x^\beta) = -\eta_{\mu\nu} k^\mu k^\nu \exp(\pm i \eta_{\alpha\beta} k^\alpha x^\beta)$  from  $i^2 = -1$ . We need the opposite sign in eqn. C.161 relative to eqn. A.3 as in the "mostly minus" sign convention  $\eta_{ij}$  are negative and  $\eta_{tt}$  is positive.

### C.3 Maxwell-electrodynamics and the gauge-principle

We should step up the game after this example of scalar field dynamics and turn to the Maxwell-field  $A^\mu$ : Firstly, it has internal degrees of freedom and transforms like a Lorentz-vector,  $A^\mu \rightarrow \Lambda^\mu_\alpha A^\alpha$ , secondly, it has the charge density  $j^\mu$  as a source, likewise a Lorentz vector,  $j^\mu \rightarrow \Lambda^\mu_\alpha j^\alpha$ . Thirdly, the charge density is conserved in the sense that  $\partial_\mu j^\mu = \partial_{ct}(c\rho) + \partial_i j^i = 0$ , and the field equation itself is linear,  $\partial_\mu F^{\mu\nu} = 4\pi/c j^\mu$  with the field tensor  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$  containing the electric and magnetic fields. Clearly, this equation can not contain the entire information about six field components  $E^i$  and  $B^i$  to be derived from the field tensor which is coupled to just 4 components of charge  $j^\mu$ . That's the reason why one needs the Bianchi-identity in addition,  $\partial_\mu \tilde{F}^{\mu\nu} = 0$ , most conveniently written with the dual field tensor  $\tilde{F}^{\mu\nu}$ ,

$$\tilde{F}^{\mu\nu} = +\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \quad \text{and} \quad \tilde{F}_{\mu\nu} = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \quad (\text{C.163})$$



with the 4-dimensional Levi-Civita symbol  $\epsilon^{\mu\nu\alpha\beta}$ : One needs an object that is anti-symmetric at least in every index pair to give a non-vanishing result.  $F^{\mu\nu}$  is auto-dual,

$$\tilde{F}^{\mu\nu} = +\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\tilde{F}_{\alpha\beta} = -\frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\alpha\beta\rho\sigma}F^{\rho\sigma} = F^{\mu\nu}. \quad (\text{C.164})$$

That is a lot to digest, in particular the property of the field tensor being antisymmetric,  $F^{\nu\mu} = -F^{\mu\nu}$ , as well as the existence of the dual field tensor  $F^{\mu\nu}$  and its role in the dynamics of the electromagnetic field: First of all, the Maxwell-equations are hyperbolic partial differential equations in  $A^\mu$ , with propagations traveling along the light cone, as the wave vectors are lightlike,  $k_\mu k^\mu = 0$ . The source  $j^\mu$  can be dynamically changing but under conservation, and the transformation properties of  $j^\mu$  and  $A^\mu$  are identical.

Deriving the Maxwell-field equation from a variational principle asks the question how the invariance-covariance principle could be incorporated. As a square of first derivatives of  $A^\mu$  as a kinetic term which is invariant under Lorentz-transformations one could use  $F^{\mu\nu}F_{\mu\nu}$ , such that one can ensure a linear field equation after variation from this particular quadratic invariant.  $\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = F^{\mu\nu}F_{\mu\nu} \propto E_i E^i - B_i B^i$ , so there is nothing new from using the Frobenius-norm of  $\tilde{F}$  instead of  $F$ . The other possible quadratic invariant  $\tilde{F}^{\mu\nu}F_{\mu\nu} = F^{\mu\nu}\tilde{F}_{\mu\nu} \propto E_i B^i$  would likewise give a linear field equation, but there is an issue because  $E_i B^i$  is a scalar product between an axial and a polar vector, and is as such pseudoscalar, i.e. it changes sign under parity inversion and is therefore not a proper scalar. Already at this point one may conjecture that the Lagrange-function is bounded by 0 and that this value corresponds to vacuum solutions: The magnetic and electric field energies of an electromagnetic wave are always exactly equal, such that  $E_i E^i - B_i B^i = 0$ , and they are necessarily perpendicular to each other,  $E_i B^i = 0$ . On the side coupling the fields to charges,  $A_\mu j^\mu$  would be perfect in a linear field equation. Collecting these ideas suggests that the Maxwell-action is given by

$$S = \int d^4x \left( \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{4\pi}{c}A_\mu j^\mu \right) \quad (\text{C.165})$$

What about terms like  $A_\mu A^\mu$ ? It would in fact be compatible with a linear field equation with a term proportional to  $A_\mu$ , but it would violate gauge-symmetry as a new symmetry principle. Maxwell's field equation  $\partial_\mu F^{\mu\nu} = 4\pi/c j^\nu$  is unchanged under the gauge transform  $A^\mu \rightarrow A^\mu + \partial^\mu \chi$  with a scalar field  $\chi$ , as

$$F^{\mu\nu} \rightarrow \partial^\mu (A^\nu + \partial^\nu \chi) - \partial^\nu (A^\mu + \partial^\mu \chi) = F^{\mu\nu} \quad (\text{C.166})$$

with the interchangeability  $\partial^\mu \partial^\nu \chi = \partial^\nu \partial^\mu \chi$ . And of course, with the invariance of  $F^{\mu\nu}$  under gauge transforms one does not possibly observe any change in the observable fields  $E^i$  and  $B^i$ . The freedom to transform  $A^\mu$  can be used to make the computation of fields easier and to decouple field equations. For instance,

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \square A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} j^\nu \quad (\text{C.167})$$

would need to be solved for computing  $A^\mu$  from  $j^\mu$ , such that the fields  $F^{\mu\nu}$  are obtained from  $A^\mu$  by successive derivation. There are known Green-functions for solving  $\square A^\mu = 4\pi/c j^\mu$ , even index-by-index, but the divergence  $\partial_\mu A^\mu$  couples these four equations together. Under gauge transforms one obtains the transformation

$$\partial_\mu A^\mu \rightarrow \partial_\mu A^\mu + \partial_\mu \partial^\mu \chi = 0 \rightarrow \square \chi = -\partial_\mu A^\mu \quad (\text{C.168})$$

implying that one can always find a transform that sets  $\partial_\mu A^\mu$  to zero, it is even uniquely defined by the relation  $\square \chi = -\partial_\mu A^\mu$ , as the  $\chi$  needed follows from solving the wave equation with  $-\partial_\mu A^\mu$  as a source.  $\partial_\mu A^\mu = 0$  is called Lorenz-gauge.

It is very interesting how gauge-transforms operate on the action or the Lagrange-density:

$$S \rightarrow \int d^4x \left( \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{4\pi}{c} (A_\mu + \partial_\mu \chi) j^\mu \right) = S + \frac{4\pi}{c} \int d^4x (\partial_\mu \chi) j^\mu \quad (\text{C.169})$$

as  $F^{\mu\nu}$  is gauge invariant anyway. The coupling of  $A^\mu$  to  $j^\mu$  can be reformulated using the Leibnitz-theorem,

$$\int d^4x j^\mu \partial_\mu \chi = \int d^4x \partial_\mu (j^\mu \chi) - \int d^4x (\partial_\mu j^\mu) \chi \quad (\text{C.170})$$

where the first term can be reformulated with the Gauß-theorem,

$$\int_V d^4x \partial_\mu (j^\mu \chi) = \int_{\partial V} dQ_\mu j^\mu \chi = 0 \quad (\text{C.171})$$

which can be made to vanish if  $\chi = 0$  on  $\partial V$  by choice. The second term is automatically zero for conserved sources, where  $\partial_\mu j^\mu = 0$ . So effectively, the Lagrange-function is unchanged by the gauge transform if the electric charge density as the source of the Maxwell-field is conserved, which ultimately is the foundation of the knot-rule in electric circuits: That the sum of inflowing and outflowing electric currents at one knot in a circuit cancel exactly if there is not buildup of charge is the consequence of the continuity equation  $\partial_\mu j^\mu = 0$ , which appears consistent with the gauge-freedom of  $A^\mu$ .

#### C.4 Electromagnetic duality and axions

Maxwell-electrodynamics *in vacuum* obeys a peculiar symmetry called electromagnetic duality: In the absence of sources, the field equation  $\partial_\mu F^{\mu\nu} = 0$  and the Bianchi-identity  $\partial_\mu \tilde{F}^{\mu\nu} = 0$  become equal, so the duality transform  $F^{\mu\nu} \leftrightarrow \tilde{F}^{\mu\nu}$  doesn't give rise to any difference in the field dynamics. In terms of fields, the duality transform reads  $E^i \rightarrow B^i$  and  $B^i \rightarrow -E^i$ , which makes perfect sense as  $\partial_\mu F^{\mu\nu} = 0$  contains the two statements  $\partial_i B^i = 0$  and  $\epsilon^{ijk} \partial_j E_k = -\partial_{ct} B^i$ , whereas  $\partial_\mu \tilde{F}^{\mu\nu} = 0$  makes sure that  $\partial_i E^i = 0$  and  $\epsilon^{ijk} \partial_j B_k = +\partial_{ct} E^i$ : Effectively, the two pairs of Maxwell-equations interchange their meaning under the duality transform. Or, to formulate this in a stronger way: Only the presence of charges  $j^\mu$  defines a difference between  $F^{\mu\nu}$  and  $\tilde{F}^{\mu\nu}$ .

In a fantasy world with electric charges  $j^\mu$  and magnetic charges  $i^\mu$  one could set up a perfectly reasonable Maxwell-like theory just by postulating

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu \quad \text{as well as} \quad \partial_\mu \tilde{F}^{\mu\nu} = \frac{4\pi}{c} i^\nu \quad (\text{C.172})$$

provided that both charges are conserved,  $\partial_\mu j^\mu = 0$  and independently  $\partial_\mu i^\mu = 0$ . Both field equations are, due to the antisymmetry of the field tensor, made compatible with

conservation of the respective charge,  $\partial_\mu \partial_\nu \tilde{F}^{\mu\nu} = 4\pi/c \partial_\nu i^\nu = 0$ , as a contraction of the antisymmetric  $\tilde{F}^{\mu\nu}$  with the symmetric  $\partial_\mu \partial_\nu$ , and likewise  $\partial_\mu \partial_\nu F^{\mu\nu} = 4\pi/c \partial_\nu j^\nu = 0$ , for exactly the same reason.

While everything is perfectly well-defined on the basis of the field equations, there is a problem when trying to write down a Lagrange-density: The potential  $A^\mu$  would not exist. In fact,  $A^\mu$  relies on the dual field equation being zero, which can be most easily seen in terms of the components:  $\partial_j B^i = 0$  implies that the magnetic field can be written as  $B^i = \epsilon^{ijk} \partial_j A_k$  derived from a vector potential  $A_k$ , and at the same time  $\epsilon^{ijk} \partial_j E_k = -\partial_{ct} B^i = -\partial_{ct} \epsilon^{ijk} \partial_j A_k$ , such that  $\epsilon^{ijk} \partial_j (E_k + \partial_{ct} A_k) = 0$ . That in turn implies, that the term in brackets can be written as a gradient,  $E_k + \partial_{ct} A_k = -\partial_k \Phi$  with a scalar potential  $\Phi$ . In summary, the components  $A_k$  and  $\Phi$  of the potential  $A^\mu$  rely in their existence on the absence of magnetic charges,  $i^\mu = 0$ .

But one needs  $A^\mu$  for a Lagrange-description of electrodynamics, otherwise the coupling to the sources could not be formulated in the  $A_\mu j^\mu$ -term: Electrodynamics with  $j^\mu \neq 0 \neq i^\mu$  could be defined on the level of the field equations but not with a Lagrange-density.

Let's investigate the second possible quadratic invariant  $\tilde{F}_{\mu\nu} F^{\mu\nu}$  which is expressed in field components  $\propto E_j B^j$  and therefore pseudoscalar: parity inversion  $x^i \rightarrow -x^i$  or inversion of  $ct \rightarrow -ct$  would result in a change in sign and excludes the term from the Lagrange-density as it is not properly scalar. This can be remedied by including a pseudoscalar field  $\theta$  along with its own dynamics

$$S = \int d^4x \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{4\pi}{c} A_\mu j^\mu + \theta F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \partial^\mu \theta \partial^\nu \theta - V(\theta) \right) \quad (C.173)$$

where  $\theta F_{\mu\nu} \tilde{F}^{\mu\nu}$  and  $\eta_{\mu\nu} \partial^\mu \theta \partial^\nu \theta$  are perfectly scalar. The interaction potential  $V(\theta)$  could include a term  $\propto m^2 \theta^2$  which itself is scalar again. Then, the Lagrange-density describes a massive pseudoscalar field  $\theta$ , which in this context is called axion, and variation of eqn. C.173 gives rise to a coupled set of partial differential equations for  $F^{\mu\nu}$  and  $\theta$ .

### C.5 Poynting-law and conservation of energy and momentum

Fields are not only affecting test charges by accelerating them, but they are physically real in their own right: They have their own dynamics, they can transport energy and momentum, and would be ultimately sources of gravity. The energy and momentum content is derived from the independence of the Lagrange-density of position  $x^\mu$ , i.e. the working principle of the fields is supposed to be the same at every location and at every instant in time. One notices how effectively momentum and energy conservation have the same origin now, unlike classical mechanics.

The starting point is to define a shift of the Lagrange-function to a new position in spacetime by a separation  $a_\alpha$ , which can be done by defining the operator  $a_\alpha \partial^\alpha$  and apply it to the Lagrange density,

$$\delta \mathcal{L} = a_\alpha \partial^\alpha \mathcal{L} \quad \text{with the variation being} \quad \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \partial_\mu \phi \quad (C.174)$$

which changes as the fields and their derivatives take one new values as one moves by  $a_\alpha$  across spacetime. Working for simplicity with a scalar field  $\phi$  one gets variations

$$\phi \rightarrow \phi + \underbrace{a_\alpha \partial^\alpha \phi}_{\delta\phi} \quad \text{and} \quad \partial_\mu \phi \rightarrow \partial_\mu \phi + \underbrace{a_\alpha \partial^\alpha \partial_\mu \phi}_{\delta\partial_\mu \phi} \quad (\text{C.175})$$

To deal with the term  $\delta\partial_\mu \phi$  we apply the Leibnitz-rule as follows:

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right) = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\mu \delta\phi}_{\delta\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi} \quad (\text{C.176})$$

such that we can write

$$\delta\mathcal{L} = \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \delta\phi + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right) \quad (\text{C.177})$$

where the first bracket is necessarily zero, as a consequence of the Euler-Lagrange-equations for the field  $\phi$ . Then,

$$\delta\mathcal{L} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right) \quad (\text{C.178})$$

where we can now substitute the displacements by  $a_\alpha$ :

$$a_\alpha \left( \partial^\alpha \mathcal{L} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial^\alpha \phi \right) \right) = 0 \quad (\text{C.179})$$

and by rewriting  $\partial^\alpha = \eta^{\mu\alpha} \partial_\mu$  one can isolate the energy momentum tensor,

$$a_\alpha \underbrace{\partial_\mu \left( \eta^{\mu\alpha} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial^\alpha \phi \right)}_{-\text{T}^{\mu\alpha}} = 0 \quad (\text{C.180})$$

with a corresponding conservation law  $\partial_\mu \text{T}^{\mu\alpha} = 0$  as  $a_\alpha$  is arbitrary: Perhaps it's interesting to note that  $\partial \mathcal{L} / \partial \partial_\mu \phi$  would be the canonical field momentum, so we are actually carrying out a Legendre-transform of  $\mathcal{L}$  to arrive at the energy-momentum tensor.

The same arguments apply to the Maxwell-field  $A^\mu$  but with a small exception as there is gauge-symmetry to be respected in addition. We should not differentiate with respect to the straightforward derivatives  $\partial_\mu A_\nu$  but rather with respect to the anti-symmetrised variant,  $\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$ , which is gauge-invariant. Therefore, the variation of the field  $A^\mu$  under a shift  $a_\alpha$  would be

$$\delta A^\mu = a_\alpha (\partial^\alpha A^\mu - \partial^\mu A^\alpha) = a_\alpha F^{\alpha\mu} \quad (\text{C.181})$$

Therefore, the variation of  $\mathcal{L}$  becomes

$$\delta\mathcal{L} = a_\alpha \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\sigma} F^\alpha_\sigma \right) \quad (\text{C.182})$$

Rewriting  $\delta\mathcal{L} = a_\alpha\partial^\alpha\mathcal{L}$  and  $\partial^\alpha = \eta^{\alpha\mu}\partial_\mu$  then gives

$$a_\alpha\partial_\mu\left(\eta^{\alpha\mu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\partial_\mu A_\sigma}F^\alpha_\sigma\right) = 0 \quad (\text{C.183})$$

with the corresponding energy momentum tensor  $T^{\alpha\mu}$  and its covariant conservation law  $\partial_\mu T^{\alpha\mu} = 0$ . For the Maxwell-Lagrange-density we have

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\sigma} = -\frac{1}{4\pi}F^{\mu\sigma} \quad (\text{C.184})$$

so that

$$T^{\mu\nu} = \frac{1}{4\pi}\left(\eta_{\alpha\beta}F^{\mu\alpha}F^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \quad (\text{C.185})$$

which is naturally symmetric and traceless:

$$4\pi\eta_{\mu\nu}T^{\mu\nu} = \eta_{\mu\nu}\eta_{\alpha\beta}F^{\mu\alpha}F^{\beta\nu} + \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = F_{\alpha\beta}F^{\beta\alpha} + F_{\alpha\beta}F^{\alpha\beta} = 0 \quad (\text{C.186})$$

by switching the index order in one of the terms.

### C.6 Covariant electrodynamics in matter

The Maxwell-equations in matter, written down in an index notation but after choosing an explicit frame, read:

$$\partial_i D^i = 4\pi\rho, \quad \partial_i B^i = 0, \quad \epsilon^{ijk}\partial_j E_k = -\partial_{ct}B^i, \quad \text{and} \quad \epsilon^{ijk}\partial_j H_k = +\partial_{ct}D^i + \frac{4\pi}{c}J^i \quad (\text{C.187})$$

and a peculiar difference between the fields  $D^i$  and  $B^i$  (noted as vectors) and the excitations  $E_i$  and  $H_i$  (written as linear forms) emerges. Of course it's a choice which of the two pairs is written as vectors and which as linear forms, so

$$\partial^i D_i = 4\pi\rho, \quad \partial^i B_i = 0, \quad \epsilon_{ijk}\partial^j E^k = -\partial_{ct}B_i, \quad \text{and} \quad \epsilon_{ijk}\partial^j H^k = +\partial_{ct}D_i + \frac{4\pi}{c}J_i \quad (\text{C.188})$$

is equally valid. Normally, one would need to define tensors to relate the vectors with the linear forms,  $B^i = \mu^{ij}H_j$  with the permeability tensor  $\mu^{ij}$  and  $D^i = \epsilon^{ij}E_j$  with the dielectric tensor  $\epsilon^{ij}$ . Apart from symmetry (which ensures that there is an orthogonal principal axis frame with three real-valued eigenvalues) the two tensors are free and would describe the general linear relationship in a possibly anisotropic medium between the fields and excitations. If the medium is isotropic,  $\mu^{ij} = \mu\delta^{ij}$  and  $\epsilon^{ij} = \epsilon\delta^{ij}$ , so that the usual relation  $B^i = \mu\delta^{ij}H_j = \mu H^i$  and  $D^i = \epsilon\delta^{ij}E_j = \epsilon E^i$  is recovered.

Taking this one step further, one would notice that the two homogeneous Maxwell-equations

$$\partial_i B^i = 0, \quad \epsilon^{ijk}\partial_j E_k = -\partial_{ct}B^i, \quad (\text{C.189})$$

depend on  $B^i$  and  $E_i$ , while the two inhomogeneous Maxwell-equations depend on

the other pair,

$$\partial^i D_i = 4\pi\rho, \quad \epsilon^{ijk} \partial_j H_k = +\partial_{ct} D^i + \frac{4\pi}{c} j^i. \quad (\text{C.190})$$

Because of this separation, one should package  $E^i$  and  $B^i$  into a tensor  $\tilde{F}^{\mu\nu}$  to reproduce the homogeneous equations from  $\partial_\mu \tilde{F}^{\mu\nu} = 0$ . Analogously,  $D^i$  and  $H^i$  should then be part of a tensor  $G^{\mu\nu}$  to generate the inhomogeneous equations from  $\partial_\mu G^{\mu\nu} = 4\pi/c j^\nu$ . Clearly, there is now a second breaking of the duality taking place, because  $G^{\mu\nu} \neq \tilde{F}^{\mu\nu}$ ! That, however is not straightforward: One has  $B^i$  as a vector and  $E_i$  as a linear form for  $\tilde{F}^{\mu\nu}$ , and likewise  $D^i$  as a vector and  $H_i$  as a linear form for  $G^{\mu\nu}$  is given, so one needs to invoke the dielectric and permeability tensors to convert the linear forms  $E_i$  and  $H_i$  to vectors first.

### C.7 Finsler-geometry and Lorentz-forces

A massive test particle tries to minimise proper time as the relativistic generalisation of the action S

$$S = -mc \int ds = -mc^2 \int d\tau \quad (\text{C.191})$$

which is solved in the absence of forces by a straight line,  $d^2 x^\mu/d\tau = 0$ , or equivalently,  $x^\mu(\tau) = a^\mu \tau + b^\mu$  with two integration constants  $a^\mu$  and  $b^\mu$ . If there is a nonzero specific charge  $q/m$  the particle is accelerated by Lorentz-forces

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{q}{m} F^{\mu\nu} \frac{dx_\nu}{d\tau} \quad (\text{C.192})$$

Let's re-derive this equation of motion from a variational principle, because it gives rise to a new geometric structure, called a Finsler-geometry. To cut things short, let's postulate

$$S = -mc^2 \int d\tau + q \int dx^\mu A_\mu \quad (\text{C.193})$$

☞ Can you show that the term  $\int dx^\mu A_\mu$  is gauge-invariant?

with a potential  $A_\mu$ . While the first term is defined by the metric structure of spacetime,  $ds^2 = c^2 d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ , the second term involves the scalar product between  $A_\mu$  and  $dx^\mu$ . If  $A_\mu$  is given directly in terms of a linear form, one actually does not need a metric structure to compute  $dx^\mu A_\mu = \eta_{\mu\nu} dx^\mu A^\nu$ . So effectively, there are two geometric structures at work, the metric structure  $\eta_{\mu\nu}$ , and the structure defined by the scalar product of vectors with the linear form  $A_\mu$ : This is called a Finsler-geometry. The interpretation of the  $A_\mu dx^\mu$ -term is not easy, but perhaps one could imagine  $A_\mu$  as some kind of headwind or tailwind that changes the proper time of the particle depending on in which direction it moves relative to the direction and magnitude of the vector field  $A^\mu$ .

As the values of  $A_\mu$  that the particle sees depends on the trajectory, the variation of the action gives

$$\delta \int A_\mu dx^\mu = \int \delta A_\mu dx^\mu + \int A_\mu \delta dx^\mu = \int \delta A_\mu dx^\mu - \int dA_\mu \delta x^\mu \quad (\text{C.194})$$

with the usual procedure to write  $\delta dx^\mu = d\delta x^\mu$  and a successive integration by parts. Then, we trace back the variation and the differential of  $A_\mu$  to a coordinate shift,

$$\delta A_\mu = \frac{\partial A_\mu}{\partial x^\alpha} \delta x^\alpha \quad \text{and} \quad dA_\mu = \frac{\partial A_\mu}{\partial x^\alpha} dx^\alpha \quad (\text{C.195})$$

such that the variation becomes

$$\delta \int A_\mu dx^\mu = \int \frac{\partial A_\mu}{\partial x^\alpha} \delta x^\alpha dx^\mu - \frac{\partial A_\mu}{\partial x^\alpha} dx^\alpha \delta x^\mu = \int \left( \frac{\partial A_\mu}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\mu} \right) \delta x^\alpha dx^\mu \quad (\text{C.196})$$

after renaming the indices  $\mu \leftrightarrow \alpha$  in the second term (all indices are fully saturated and the terms are both scalar, so it does not matter how the indices are called). Introducing the velocity  $dx^\alpha/d\tau$  and identifying the field tensor brings the integral into the final shape

$$\delta \int A_\mu dx^\mu = \int d\tau F_{\alpha\mu} \frac{dx^\alpha}{d\tau} \delta x^\mu \quad (\text{C.197})$$

which, combined with the variation of  $-mc^2 \int d\tau$  already worked out in eqn. A.93, gives the Lorentz-equation of motion C.192:

$$m \frac{d^2 x^\mu}{d\tau^2} = q F^{\mu\alpha} \frac{dx_\alpha}{d\tau}. \quad (\text{C.198})$$

### C.8 Light-cone structure beyond metric spacetimes

Let's write out the kinetic term of the Lagrange-density of electrodynamics explicitly

$$S = \int d^4x \eta_{\alpha\mu} \eta_{\beta\nu} F^{\alpha\beta} F^{\mu\nu} = \int d^4x \frac{\eta_{\alpha\mu} \eta_{\beta\nu} - \eta_{\alpha\nu} \eta_{\beta\mu}}{2} F^{\alpha\beta} F^{\mu\nu} = \int d^4x G_{\alpha\beta\mu\nu} F^{\alpha\beta} F^{\mu\nu} \quad (\text{C.199})$$

using antisymmetry  $F^{\nu\mu} = -F^{\mu\nu}$  and renaming indices. The quantity  $G_{\alpha\beta\mu\nu}$  is antisymmetric in the first and second index pair and defines a measure of area instead of a measure of length, as a metric  $\eta_{\mu\nu}$  would. In 3 dimensions one determines the area of the parallelogram spanned by two vectors  $a^i$  and  $b^i$  from the norm of  $c_i = \epsilon_{ijk} a^j b^k$ , so effectively through

$$\text{area} = \delta^{il} c_i c_l = \delta^{il} \epsilon_{ijk} \epsilon_{lmn} a^j b^k a^m b^n = [\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}] a^j b^k a^m b^n = a^2 b^2 - (a_i b^i)^2 \quad (\text{C.200})$$

where the square brackets have the same index structure as  $G_{\alpha\beta\mu\nu}$ , so it is justified to speak of  $G_{\alpha\beta\mu\nu}$  as a measure of area. In fact,  $a_i b^i = ab \cos \alpha$  for a standard scalar product, so

$$\text{area} = a^2 b^2 (1 - \cos^2 \alpha) = a^2 b^2 \sin^2 \alpha \quad (\text{C.201})$$

as expected. Perhaps one could imagine that the Maxwell action  $S$  measures the area between the vectors  $\partial^\mu$  and  $A^\nu$  over the spacetime volume.





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## D SYMMETRY

### D.1 *Lie-groups and their generation*

Symmetry plays an incredibly important role in physics, and we would define symmetry as the invariance of a quantity under a transform. Transformations typically form a group as successive transformations can be combined into a single transform, because there is a unit transformation element with no effect and an inverse transform undoing the action of a previous transform. Many transformation groups in physics contain infinitely many elements, such as rotations that are parameterised by an angle  $\alpha$  or Lorentz-transform with the rapidity  $\psi$ . In contrast to an index  $n$  for a finite group (or a countably infinite group), these groups are continuously parameterised and a rotation  $R_j^i(\alpha)$  or Lorentz-transform  $\Lambda_{\nu}^{\mu}(\psi)$  exists for every possible value of the real-valued parameters  $\alpha$  and  $\psi$ . These continuously parameterised groups are referred to as Lie-groups.

### D.2 *Generating algebras*

One can ask the (very sensible) question if all elements in a continuously parameterised Lie-group can be formed from an infinitesimally small transformation. Let's have a look at rotations in 2 dimensions, where the transformation matrix is given by

$$R = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} = \sigma^{(0)} + \alpha \sigma^{(2)} \quad \text{in the limit of small } \alpha \quad (\text{D.202})$$

If one would like to assemble a rotation out of many individual small rotations, ideally  $n$  rotations of magnitude  $\alpha/n$  with  $n \rightarrow \infty$ , one should obtain R:

$$R = \lim_{n \rightarrow \infty} \left( \sigma^{(0)} + \frac{\alpha}{n} \sigma^{(2)} \right)^n = \exp(\alpha \sigma^{(2)}) \quad (\text{D.203})$$

such that one can speculate if  $R_j^i$  is as well given by a power series,

$$R = \sum_n \frac{(\sigma^{(2)} \alpha)^n}{n!} = \sigma^{(0)} \sum_n \frac{(-1)^n \alpha^{2n}}{(2n)!} + \sigma^{(2)} \sum_n \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!} = \sigma^{(0)} \cos \alpha + \sigma^{(2)} \sin \alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \quad (\text{D.204})$$

by splitting up the summation over even and odd indices, because even powers of  $\sigma^{(2)}$  are proportional to the unit matrix  $\sigma^{(0)}$  while odd powers stay proportional to  $\sigma^{(2)}$ . Similarly, Lorentz-transforms  $\Lambda$  are generated by  $\sigma^{(3)}$ ,

$$\Lambda = \sum_n \frac{(\sigma^{(3)} \psi)^n}{n!} = \sigma^{(0)} \sum_n \frac{\psi^{2n}}{(2n)!} + \sigma^{(3)} \sum_n \frac{\psi^{2n+1}}{(2n+1)!} = \sigma^{(0)} \cosh \psi + \sigma^{(3)} \sinh \psi = \begin{pmatrix} \cosh \psi & \sinh \psi \\ \sinh \psi & \cosh \psi \end{pmatrix} \quad (\text{D.205})$$

with the rapidity  $\psi$  as a parameter.

In summary, there is a generation of a continuously parameterised Lie-group

$A(t) = \exp(Tt)$  with the parameter  $t$  and the generator  $T$ , which in our two examples of the rotation and the Lorentz-transformation was traceless. Infinitesimal transforms are given by

$$A(\delta t) = \text{id} + \delta t T \quad (\text{D.206})$$

with the generator  $T$ , which can be assembled to a finite transform

$$A(t) = \lim_{n \rightarrow \infty} \left( \text{id} + \frac{t}{n} T \right)^n = \exp(Tt) \quad (\text{D.207})$$

from  $n$  successive transforms of magnitude  $t/n$ . Then, there should be an exponential series,

$$A(t) = \exp(Tt) = \sum_n \frac{(Tt)^n}{n!} \quad (\text{D.208})$$

Now we should investigate the group structure of  $A(t)$ : Successive application of transformations  $A(t')A(t)$  is captured by a single element of the same group,

$$A(t')A(t) = \exp(t'T) \exp(tT) = \exp((t' + t)T) = A(t' + t) \quad (\text{D.209})$$

such that the parameter is additive. That realisation immediately gives rise to the definition of an inverse,

$$A(-t)A(t) = A(-t + t) = A(0) = \text{id} \quad \text{such that} \quad A^{-1}(t) = A(-t) \quad (\text{D.210})$$

Formally, eqn. D.209 requires the Cauchy-product: In fact,

$$A(t')A(t) = \exp(t'T) \exp(tT) = \sum_n \frac{(Tt')^n}{n!} \sum_m \frac{(Tt)^m}{m!} = \sum_n \sum_m \frac{(Tt')^m (Tt)^{n-m}}{m! (n-m)!} \quad (\text{D.211})$$

where we can proceed by introducing the binomial coefficient

$$A(t')A(t) = \sum_n \frac{T^n}{n!} \sum_m \binom{n}{m} t^m t^{n-m} = \sum_n \frac{(T(t' + t))^n}{n!} = \exp(T(t' + t)) = A(t' + t) \quad (\text{D.212})$$

and use the generalised binomial formula, *if* there is only a *single* generator involved. If one deals with multiple generators  $T, T'$  one needs to employ the Baker-Hausdorff-Campbell-relation,

$$\exp(T) \exp(T') = \exp(T + T') \exp\left(-\frac{1}{2}[T, T']\right) \quad (\text{D.213})$$

where it is apparent that the commutation relations  $[T, T'] = TT' - T'T$  determine how the generated group elements get combined. An example that defies (at least my) imagination is the following: Surely the combination of rotations results in a rotation, if the axes are not identical then the result depends on the order as rotations in 3 dimensions are not a commutative group. The combination of Lorentz-transformations into different directions involves a rotation too: In fact, boosting

along  $x$  followed by a boost along  $y$ , and inverting this by first boosting back along  $x$  followed by a boost back in  $y$  gives you a system with zero relative velocity compared to where you started, but there is an effective rotation.

### D.3 Construction of invariants

In many cases invariants of Lie-groups can be traced back to the tracelessness of their generators. In fact, my third most favourite formula in theoretical physics implies

$$\ln \det A = \text{tr} \ln A \quad (\text{D.214})$$

and as the group elements  $A$  typically depend on the generator  $T$  through

$$A = \exp(Tt) = \sum_n \frac{(Tt)^n}{n!} \quad (\text{D.215})$$

one gets

$$\ln \det A = \text{tr} \ln A = \text{tr} \ln \exp(tT) = t \text{tr} T = 0 \quad (\text{D.216})$$

so that  $\det A = 1$ . For a rotation matrix this would be  $\cos^2 \alpha + \sin^2 \alpha = 1$  and for a Lorentz-transform  $\cosh^2 \psi - \sinh^2 \psi = 1$ , but actually one can compute the determinant already from the trace of the generator alone without using properties of the trigonometric or hyperbolic functions!

In the nomenclature of groups you often see a preceding letter  $S$ , as in  $SO(n)$  for the special orthogonal group in  $n$  dimensions, which refers to the property that the determinant of the group elements is equal to 1. From the argument above you understand that this must mean, that the generators are all traceless. If there is a relation like the power series D.215, it would automatically be a solution to a differential equation of the form

$$\frac{d}{dt} A = \frac{d}{dt} \sum_n \frac{(Tt)^n}{n!} = \sum_n \frac{(Tt)^{n-1}}{(n-1)!} T = AT \quad (\text{D.217})$$

with an index shift due to the differentiation  $dt^n/dt/n! = t^{n-1}/(n-1)!$ . In summary, there are three approaches to the generation of a Lie-group in the exponential form: The infinitesimal transform taken to the  $n$ th power, summing of the exponential series and thirdly, the differential equation for the exponential.

### D.4 Symplectic structures and canonical time evolution

In classical mechanics one encounters a funny property of the symplectic matrix which arises when solving Hamilton's equation of motion for a harmonic oscillator. You'll see that the symplectic matrix is just the Pauli-matrix  $\sigma^{(2)}$ , so similarities between generating the time evolution of the harmonic oscillator and generating rotations are to be expected! After all, both involve  $\sin$  and  $\cos$ , and surely one can transform the coordinates into a rotating coordinate frame in phase space.

The Hamilton-function is  $\mathcal{H}(p, q) = p^2/2 + q^2/2$  in a practical choice of units, and therefore

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \quad \text{and} \quad \dot{q} = +\frac{\partial \mathcal{H}}{\partial p} \quad (\text{D.218})$$

which can be combined into a single equation, in particular for the harmonic oscillator where  $\partial\mathcal{H}/\partial p = p$  and  $\partial\mathcal{H}/\partial q = q$ ,

$$\frac{d}{dt} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \quad (\text{D.219})$$

where with our knowledge of generators we would immediately write

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \exp \left( \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} t \right) \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} \quad (\text{D.220})$$

with a time-evolution operator acting on the initial conditions  $q(0)$  and  $p(0)$  to give momentum and position of the evolved system. The exponential operator can be evaluated, which amounts to computing powers of the Pauli-matrix  $\sigma^{(2)}$ ,

$$\exp(\sigma^{(2)}t) = \sum_n \frac{(\sigma^{(2)}t)^n}{n!} \quad (\text{D.221})$$

Because  $(\sigma^{(2)})^0 = \sigma^{(0)}$ ,  $(\sigma^{(2)})^2 = -\sigma^{(0)}$ ,  $(\sigma^{(2)})^3 = -\sigma^{(2)}$  and  $(\sigma^{(2)})^4 = \sigma^{(0)}$ , continuing cyclically, one only ever obtains terms proportional to  $\sigma^{(0)}$  or  $\sigma^{(2)}$ , with alternating signs:

$$\begin{aligned} \sum_n \frac{(\sigma^{(2)}t)^n}{n!} &= \sigma^{(0)} \sum_n \frac{(-1)^n t^{2n}}{(2n)!} + \sigma^{(2)} \sum_n \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \\ &\sigma^{(0)} \cos t + \sigma^{(2)} \sin t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad (\text{D.222}) \end{aligned}$$

and therefore, the time evolution of the harmonic oscillator is in fact given by sinusoidal oscillations: This is perhaps a bit remarkable; one does not need any intuition about the solution of the Lagrange equation of motion  $\ddot{q} = -q$ , or use a complex exponential  $q \propto \exp(it)$  to transform the differential into an algebraic equation: It can be solved directly with a time-evolution operator that is *constructed*, not guessed.

### D.5 Unitary time-evolution in quantum mechanics

If you look closely at the Schrödinger-equation

$$i\hbar\partial_t\psi = H\psi \quad (\text{D.223})$$

it is perhaps not too dissimilar to eqn. D.219: In fact it would suggest that

$$\psi \propto \exp\left(-\frac{iHt}{\hbar}\right) \quad (\text{D.224})$$

with a time-evolution operator  $\exp(-iHt/\hbar)$  (the minus-sign appear because  $1/i = -i$ ), as if the Hamilton-operator  $H$  is generating the time evolution. The definition of an inverse time evolution operator makes heavy use of the fact that  $H$  is hermitean,  $H^\dagger = H$  and that  $iH$  is anti-hermitean,  $(iH)^\dagger = -iH$ . That's because

$$U(t) = \exp\left(-\frac{iHt}{\hbar}\right) \quad (\text{D.225})$$

is unitary:

$$U^+(t) = \exp\left(-\frac{iHt}{\hbar}\right)^+ = \left(\sum_n \frac{(-iHt/\hbar)^n}{n!}\right)^+ = \sum_n \frac{(iHt/\hbar)^n}{n!} = \exp\left(\frac{iHt}{\hbar}\right) = U(-t) \quad (\text{D.226})$$

Because from additivity we would get  $U(t)U(-t) = \text{id}$  and  $U^{-1}(t) = U(-t)$  we would conclude that  $U^{-1}(t) = U(-t) = U^+(t)$ , and  $U(t)$  is unitary: Its inverse is given by the adjoint. As  $U(t)$  evolves a wave function by  $t$  into the future, the adjoint  $U^+(t) = U(-t)$  evolves it back into the past by  $t$ . In this entire process the normalisation of the wave function is conserved.

There is a shortcut to this result. Complex conjugation of the Schroedinger equation gives

$$i\hbar\partial_t\psi = H\psi \quad \rightarrow \quad -i\hbar\partial_t\psi^* = (H\psi)^* = H^+\psi^* = H\psi^* \quad (\text{D.227})$$

together with the hermiticity  $H^+ = H$  of the Hamilton-operator  $H$ . But the overall sign on the left hand side could be captured by running time backwards,

$$-i\hbar\partial_t\psi^* = i\hbar\partial_{-t}\psi^* \quad (\text{D.228})$$

such that the time-inverted Schrödinger equation becomes

$$i\hbar\partial_{-t}\psi^* = H\psi^* \quad (\text{D.229})$$

Thinking of this as the defining equation of a Lie-time evolution operator gives

$$U(-t) = \exp\left(\frac{iHt}{\hbar}\right) \quad (\text{D.230})$$

for evolving the system backwards in time.

It is amazing to see, how easily the point of time reversal is taken care of in classical, Newtonian mechanics: Neither  $\mathcal{L}(q^i, \dot{q})$  does change, nor the Euler-Lagrange-equation or the resulting equation of motion  $\ddot{q}^i = -\partial^i\Phi$ . Hamilton-mechanics is funny: Both  $\dot{p}$  and  $\dot{q}$  change sign, such that motion proceeds in the opposite direction of the gradients  $\partial\mathcal{H}/\partial p$  and  $\partial\mathcal{H}/\partial q$ , such that one can easily imagine how the motion proceeds backwards in phase space.



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## E QUANTA

### E.1 Schrödinger-equation as a Helmholtz-differential equation

The iconic Schrödinger-equation

$$i\hbar\partial_t\psi = H\psi \quad (\text{E.231})$$

determines the time-evolution of the wave function  $\psi$  with the Hamilton-operator  $H$  of the system. If you'd categorise the Schrödinger equation, it is elliptical rather than hyperbolic, so we need to provide boundary conditions to make the solution for  $\psi$  unique. Please remember the example about particles in an infinitely deep square potential well where  $\psi = 0$  at the edges: This is effectively a Dirichlet boundary condition, likewise, the wave functions for the Coulomb-potential in the hydrogen-problem vanish for  $r \rightarrow \infty$ , in fulfilment of Dirichlet boundaries.

Separating out the time-dependent part by a separation of variables  $\psi(x, t) = \exp(iEt/\hbar)\phi(x)$  recovers the time-independent Schrödinger equation,

$$H\phi(x) = E\phi(x) \quad (\text{E.232})$$

where the phase of the wave function undergoes oscillations with  $\exp(iEt/\hbar)$ . Working in a position representation the Schrödinger equation becomes

$$\left(-\frac{\hbar^2\Delta}{2m} + \Phi(x)\right)\phi = E\phi \quad \rightarrow \quad \left(\Delta + \frac{2m}{\hbar^2}[E - \Phi]\right)\phi = 0 \quad (\text{E.233})$$

which is the archetypical form of a Helmholtz-differential equation  $(\Delta + k^2)\phi = 0$  for constant  $\Phi$ . Weirdly enough, one would arrive at exactly this differential equation starting from a properly hyperbolic wave equation, even though the Schrödinger equation is elliptical, but with the added benefit that because the range of values has been extended to complex numbers,  $k^2$  can be negative and one can switch between oscillatory behaviour for positive energies to exponentially decaying solutions for negative energies.

Non-relativistic quantum mechanics is based on Galilei-invariant classical mechanics, where time is a universal parameter to describe evolution, and as such it is not an observable. Thinking of expectation values  $\langle t \rangle$  is pretty much devoid of meaning, and that's the reason why the energy-time uncertainty is just a different expression of the momentum-position uncertainty. From  $E = p^2/(2m)$  we can conclude that  $\Delta E = dE/dp \Delta p = p/m \Delta p$  and  $x = p/m t$  implies  $\Delta x = dx/dt \Delta t = p/m \Delta t$  such that

$$\Delta E \Delta t = \Delta p \Delta x \geq \frac{\hbar}{2} \quad (\text{E.234})$$

without the need of defining an uncertainty  $\Delta t$  from (non-existent!) expectation values  $\langle t^2 \rangle$  and  $\langle t \rangle$ .

### E.2 Born's postulate and the conservation of probability

Born's postulate gives a probabilistic interpretation to the wave function:  $\rho(x) = \psi^*(x)\psi(x)$  is the probability to find the particle at position  $x$  in a localisation. With this interpretation, the total probability should be conserved in time evolution

$$\frac{d}{dt} \int d^3x \psi^* \psi = 0 \quad (\text{E.235})$$

if the particles are stable and do not decay. From the time derivative  $\partial_t \rho$  of the probability density one can in fact derive a continuity relation:

$$\partial_t \rho = \partial_t (\psi^* \psi) = (\partial_t \psi^*) \psi + \psi^* \partial_t \psi = -\frac{i}{\hbar} (\text{H} \psi^*) \psi + \frac{i}{\hbar} \psi^* (\text{H} \psi) \quad (\text{E.236})$$

by substituting the Schrödinger-equation and its conjugate (keeping in mind that H is hermitean,  $\text{H}^\dagger = \text{H}$ ). For a standard form of the Hamilton-operator in position representation

$$\text{H} = \frac{p^2}{2m} + \Phi = -\frac{\hbar^2 \Delta}{2m} + \Phi \quad (\text{E.237})$$

one can immediately see that the  $\Phi$ -term is not relevant, such that with  $\Delta = \partial_i \partial^i$  one gets

$$\partial_t \rho = \frac{\hbar}{2mi} [(\Delta \psi^*) \psi - \psi^* \Delta \psi] = \frac{\hbar}{2mi} \partial_i [(\partial^i \psi^*) \psi - \psi^* \partial^i \psi] = \partial_i j^i \quad (\text{E.238})$$

with the probability current density  $j^i$

$$j^i = \frac{\hbar}{2mi} [(\partial^i \psi^*) \psi - \psi^* \partial^i \psi] \quad (\text{E.239})$$

Please be careful here: Schrödinger-quantum mechanics is built on Galilean relativity and it's *not possible* to combine the time derivative of  $\rho$  with the divergence of  $j^i$  in to an expression like  $\partial_\mu j^\mu = 0$ .

One could violate the probability conserving continuity equation by adding an anti-hermitean term to the Hamilton-operator, for instance  $\text{H} \rightarrow \text{H} - i\Gamma$ , with  $(i\Gamma)^\dagger = -i\Gamma \neq i\Gamma$  with a real-valued  $\Gamma$ . Then, focusing on this term alone, we would get

$$\partial_t \rho = (\partial_t \psi^*) \psi + \psi^* \partial_t \psi = -\left(\frac{\Gamma}{\hbar} \psi^*\right) \psi - \psi^* \left(\frac{\Gamma}{\hbar} \psi\right) = -2\frac{\Gamma}{\hbar} \psi^* \psi = -2\frac{\Gamma}{\hbar} \rho \quad (\text{E.240})$$

which, depending on the sign of  $\Gamma$ , leads to exponential increase or decrease: The  $\Gamma$ -term would be suitable to describe creation or decay of particles.

### E.3 Ehrenfest's theorem

The transition from quantum mechanics to classical mechanics is conceptually very complicated but needs in some way take care of the fact that in the limit  $\hbar \rightarrow 0$  classical mechanics should be recovered. The transition is gradual as  $\hbar$  provides a scale for the action S, as will be explained in Sect. E.5. But there is a more direct relation between quantum mechanics and classical mechanics in the form of Ehrenfest's theorem: For any hermitean operator A,  $A^\dagger = A$ , one can derive the time evolution of its expectation value  $\langle A \rangle$ ,

$$\frac{d}{dt} \langle A \rangle = \frac{d}{dt} \int d^3x \psi^* A \psi = \int d^3x \partial_t \psi^* A \psi + \psi^* A \partial_t \psi \quad (\text{E.241})$$



if  $A$  is stationary. The time derivatives  $\partial_t \psi$  and  $\partial_t \psi^*$  can be replaced by the Schrödinger-equation and its complex conjugate, keeping in mind that the Hamilton-operator  $H$  is itself hermitean,  $H^\dagger = H$ ,

$$\frac{d}{dt} \langle A \rangle = \int d^3x \frac{H\psi^*}{-i\hbar} A\psi + \psi^* A \frac{H\psi}{i\hbar} = \frac{i}{\hbar} \int d^3x \psi^* [H, A] \psi = \frac{i}{\hbar} \langle [H, A] \rangle \quad (\text{E.242})$$

which is very reminiscent of the Poisson-equation of motion. Let's go through a couple of particular cases: The simplest choice would be the identity operator  $A = \text{id}$ , which commutes with everything,  $[H, \text{id}] = 0$ , so the statement we'd derive would be

$$\frac{d}{dt} \langle \text{id} \rangle = \frac{d}{dt} \int d^3x \psi^* \psi = \frac{d}{dt} \int d^3x \rho = 0 \quad (\text{E.243})$$

such that the normalisation of the wave function is conserved in time evolution and the probability density  $\rho = \psi^* \psi$  integrates up to one at every instant in time. Similarly easy is the choice  $A = H$ , and as  $H$  commutes with itself,  $[H, H] = 0$  and the expectation value of energy is conserved,

$$\frac{d}{dt} \langle H \rangle = 0 \quad (\text{E.244})$$

which is the quantum mechanical version of the Poisson-bracket  $d\mathcal{H}/dt = \{\mathcal{H}, \mathcal{H}\}$  applied to the Hamilton-function  $\mathcal{H}$ . A slightly more interesting case is  $A = x$  with the position operator  $x$ : Then,

$$\frac{d}{dt} \langle x \rangle = \frac{i}{\hbar} \langle [H, x] \rangle \quad (\text{E.245})$$

and evaluating the commutator proceeds like this. The potential  $\Phi$  in  $H$  commutes with  $x$  because it is just a function of position, so  $[p^2, x]$  is left over: Working in the position-representation

$$[p^2, x]\psi = p^2 x \psi - x p^2 \psi = (i\hbar)^2 [\partial_x^2(x\psi) - x \partial_x^2 \psi] = (i\hbar)^2 [\partial_x(\psi + x \partial_x \psi) - x \partial_x^2 \psi] \quad (\text{E.246})$$

which simplifies to

$$[p^2, x]\psi = (i\hbar)^2 [\partial_x \psi + \partial_x \psi + x \partial_x^2 \psi - x \partial_x^2 \psi] = 2(i\hbar)^2 \partial_x \psi = 2i\hbar p \psi \quad (\text{E.247})$$

such that

$$\frac{d}{dt} \langle x \rangle = \frac{\langle p \rangle}{m} \quad (\text{E.248})$$

taking care of the  $-1/(2m)$ -prefactor in the Hamilton-operator. The result implies that the expectation value of position changes in time with the expectation value of momentum,

$$\frac{d}{dt} \langle x \rangle = \frac{\langle p \rangle}{m} \quad \rightarrow \quad m \langle x \rangle = \int dt \langle p \rangle \quad (\text{E.249})$$

as one would expect in classical mechanics. Similarly, the evolution of  $\langle p \rangle$  can be evaluated: This time,  $p$  commutes with the kinetic energy,  $[p, p^2] = 0$  but does not

commute with the potential energy,  $[p, \Phi] \neq 0$  because  $\Phi$  is a function of  $x$ . Using again the position representation of  $p$  shows

$$[p, \Phi]\psi = p\Phi\psi - \Phi p\psi = i\hbar[\partial_x(\Phi\psi) - \Phi\partial_x\psi] = i\hbar[\partial_x\Phi\psi + \Phi\partial_x\psi - \Phi\partial_x\psi] = i\hbar\partial_x\Phi\psi \quad (\text{E.250})$$

such that

$$\frac{d}{dt}\langle p \rangle = -\langle \partial_x \Phi \rangle \quad \rightarrow \quad \langle p \rangle = - \int dt \langle \partial_x \Phi \rangle \quad (\text{E.251})$$

as a Newton-equation of motion for the expectation values. Perhaps it's a nice catch how Ehrenfest's theorem can be derived using unitary time evolution with an operator  $U$  instead of substituting the Schrödinger equation (of course, the two would be absolutely equivalent). Then,  $\psi(t) = U\psi_0$  and  $\psi^*(t) = U^+\psi_0^*$  with initial conditions  $\psi_0$ :

$$\frac{d}{dt}\langle A \rangle = \frac{d}{dt} \int d^3x \psi^* A \psi = \frac{d}{dt} \int d^3x \psi_0^* U^+ A U \psi_0 \quad (\text{E.252})$$

and consequently, the time derivatives only operate on  $U$  and  $U^+$ :

$$\frac{d}{dt}\langle A \rangle = \int d^3x (\psi_0^* (\partial_t U^+) A U \psi_0 + \psi_0^* U^+ A (\partial_t U) \psi_0) \quad (\text{E.253})$$

As  $U = \exp(-iHt/\hbar)$  and  $U^+ = \exp(+iHt/\hbar)$  the differentiation just gives  $\pm iH/\hbar$  as a factor, and because  $H$  and  $U$  commute, one arrives at

$$\frac{d}{dt}\langle A \rangle = \frac{i}{\hbar} \int d^3x \psi_0^* U^+ [H, A] U \psi_0 = \frac{i}{\hbar} \int d^3x \psi^* [H, A] \psi = \frac{i}{\hbar} \langle [H, A] \rangle \quad (\text{E.254})$$

Here, we've used that

$$\partial_t U^+ = \partial_t \exp\left(\frac{iHt}{\hbar}\right) = \partial_t \sum_n \frac{1}{n!} \left(\frac{iHt}{\hbar}\right)^n = \frac{iH}{\hbar} \sum_n \frac{1}{n!} \left(\frac{iHt}{\hbar}\right)^n = \frac{iH}{\hbar} U^+ \quad (\text{E.255})$$

and similarly for  $U$ .

Loosely speaking, the centres of wave packets follow the classical equation of motions: This is the central statement of the Ehrenfest-theorem. Any quantification by how much the wave packets are focused on these locations in  $x$  and  $p$  requires the computation of the uncertainties  $\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2$  as well as  $\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2$

#### E.4 Dispersion of wave packets and propagation with Green-functions

Restarting with unitary time evolution of a wave function and introducing the bracket-notation

$$|\psi(t_f)\rangle = U(t_f, t_i) |\psi(t_i)\rangle \quad \text{with} \quad U(t_f, t_i) = \exp\left(-\frac{iH(t_f - t_i)}{\hbar}\right) \quad (\text{E.256})$$

with a unitary time evolution operator  $U(t_f, t_i)$ , which is true in generality as it solves the time-dependent Schrödinger equation, asks the question whether we can find a position representation in which the Hamilton-operator  $H$  is usually written down,

essentially by setting  $p^2 = -\hbar^2\Delta/(2m)$ . This is in fact possible by projection

$$\psi(x_f, t_f) = \langle x_f | \psi(t_f) \rangle = \langle x_f | U(t_f, t_i) | \psi(t_i) \rangle \quad (\text{E.257})$$

Squeezing in an orthonormal basis set to write the state  $|\psi(t_i)\rangle$  in position representation as well

$$\psi(x_f, t_f) = \int d^3x_i \langle x_f | U(t_f, t_i) | x_i \rangle \langle x_i | \psi(t_i) \rangle \quad (\text{E.258})$$

shows that the final state is related to the initial one in a convolution relation

$$\psi(x_f, t_f) = \int d^3x_i K(x_i, t_i \rightarrow x_f, t_f) \psi(x_i, t_i) \quad (\text{E.259})$$

with the Green-function (or propagator)  $K(x_i, t_i \rightarrow x_f, t_f)$  evolving the wave function by collecting up all amplitudes of the initial state and assembling the final state.

For getting a specific shape of  $K(x_i, t_i \rightarrow x_f, t_f)$ , we would specialise the case to the propagation of a free particle with  $H = p^2/(2m)$  and working in momentum representation, as the eigenfunctions of  $p$  and  $p^2$  are particularly simple:

$$K(x_i, t_i \rightarrow x_f, t_f) = \int d^3p_f \int d^3p_i \langle x_f | p_f \rangle \langle p_f | \exp(-iH(t_f - t_i)/\hbar) | p_i \rangle \langle p_i | x_i \rangle \quad (\text{E.260})$$

as  $\langle x_f | p_f \rangle$  and  $\langle p_i | x_i \rangle$  are only plane waves  $\exp(+ip_f x_f)$  and  $\exp(-ip_i x_i)$ , one can substitute them and integrate the expression, essentially performing a Fourier-transform,

$$K(x_i, t_i \rightarrow x_f, t_f) = \sqrt{\frac{m\hbar}{2\pi i(t_f - t_i)}} \exp\left(-\frac{im}{2\hbar} \frac{(x_f - x_i)^2}{t_f - t_i}\right) \quad (\text{E.261})$$

which can be loosely interpreted as a diffusion kernel with the typical behaviour that the width of an initially  $\delta_D$ -shaped wave function increases  $\propto \sqrt{t}$ .

## E.5 Path integrals

Path integrals are a great view on the relation between the quantum mechanical propagation of probability and the classical variational principles, and they establish  $\hbar$  as a scale for the action  $S$ . But please keep R. MacKenzie's words in mind who said: *"As far as I am aware, path integrals give us no dramatic new results in quantum mechanics of a single particle. Indeed, most, if not all calculations in quantum mechanics which can be done by path integrals can be done with considerable greater ease with the standard formulations"*.

Ignoring MacKenzie's advice for this section, let's introduce an intermediate time  $t_i$  in the propagation of a free particle and see how the propagators need to be linked together, i.e. whether they form a group: Time as the parameter in the unitary time evolution operator is additive, therefore

$$U(t_f - t_i) = U(t_f - t_1 + t_1 - t_i) = U(t_f - t_1)U(t_1 - t_i) \quad (\text{E.262})$$

and therefore the Green-function  $K(x_i, t_i \rightarrow x_f, t_f)$  becomes

$$\begin{aligned} \langle x_f | U(t_f, t_i) | x_i \rangle &= \langle x_f | U(t_f - t_1) U(t_1 - t_i) | x_i \rangle = \\ &= \int d^3 x_1 \langle x_f | U(t_f - t_1) | x_1 \rangle \langle x_1 | U(t_1 - t_i) | x_i \rangle \end{aligned} \quad (\text{E.263})$$

which implies that

$$K(x_i, t_i \rightarrow x_f, t_f) = \int d^3 x_1 K(x_i, t_i \rightarrow x_1, t_1) K(x_1, t_1 \rightarrow x_f, t_f) \quad (\text{E.264})$$

with effectively a marginalisation over all possible stop-over points  $x_1$ . This result can be generalised to more stop-over points such that the time interval  $t_f - t_i$  is subdivided into  $n$  steps,  $\delta = (t_f - t_i)/n$ :

$$\langle x_f | U(t_f, t_i) | x_i \rangle = \langle x_f | (\exp(-iH\delta/\hbar))^n | x_i \rangle = \langle x_f | (\exp(-iH\delta/\hbar) \dots \exp(-iH\delta/\hbar)) | x_i \rangle \quad (\text{E.265})$$

using  $\exp(iH(t_f - t_i)) = \exp(iHn\delta) = \exp(iH\delta)^n$ . This can be decomposed by introducing complete basis sets into all  $n - 1$  gaps between the individual factors of  $\exp(iH\delta)$ :

$$K(x_i, t_i \rightarrow x_f, t_f) = \prod_j^{n-1} \int d^3 x_j K(x_j, t_j \rightarrow x_{j+1}, t_j + \delta) \quad (\text{E.266})$$

with  $x_0 = x_i$  and  $x_n = x_f$ , likewise  $t_0 = t_i$  and  $t_n = t_f = t_i + n\delta$ . Of course you're already suspecting that we should take the limit  $\delta \rightarrow 0$  or  $n \rightarrow \infty$ , while keeping  $t_f - t_i = n\delta$  fixed, and link this to the idea of generating the time evolution  $U$  with  $H$ .

### E.5.1 Phase space path integral

Our view on quantum-mechanical propagation is now very abstract: There are Green-functions that depend on the Hamilton-operator, which collect up the amplitudes of  $\psi$  at  $t_i$  and assemble them at  $t_f$ . Introducing intermediate steps requires to combine the Greens-functions in a convolution relation. The unitary time evolution operator, which is equivalent to the Green-function in position representation, can be replaced by its generator  $H$  in the limit of very small time intervals  $\delta$ :

$$K(x_j, t_j \rightarrow x_{j+1}, t_j + \delta) = \langle x_j | \exp(-iH\delta/\hbar) | x_{j+1} \rangle \simeq \langle x_j | 1 - iH\delta/\hbar | x_{j+1} \rangle \quad (\text{E.267})$$

Let's inspect the two resulting terms separately: Firstly, the term

$$\langle x_j | x_{j+1} \rangle = \int \frac{d^3 p_j}{(2\pi)^3} \exp(ip_j(x_j - x_{j+1})) = \int \frac{d^3 p_j}{(2\pi)^3} \exp(ip_j \dot{x}_j \delta) \quad (\text{E.268})$$

is effectively the  $\delta_D$ -function, which we rewrite as a  $dp$ -integral. A clever step is to extend the term by  $\delta/\delta$  and identify  $(x_i - x_{j+1})/\delta$  as  $\dot{x}_j$ . Secondly, we obtain

$$\langle x_j | H | x_{j+1} \rangle = \int \frac{d^3 p_j}{(2\pi)^3} H \exp(ip_j(x_j - x_{j+1})) \quad (\text{E.269})$$

by substitution of the Hamilton-operator in position representation. Reconstructing the Taylor-series then gives

$$K(x_j, t_j \rightarrow x_{j+1}, t_j + \delta) = \int \frac{d^3 p_j}{(2\pi)^3} \exp\left(\frac{i\delta}{\hbar}(p_j \dot{x}_j - H)\right) \quad (\text{E.270})$$

and collecting all intermediate steps results in

$$K(x_i, t_i \rightarrow x_f, t_f) = \prod_j \int d^3 x_j \int \frac{d^3 p_j}{(2\pi)^3} \exp\left(\frac{i}{\hbar} \delta \sum_j^{n-1} p_j \dot{x}_j - H\right) \quad (\text{E.271})$$

where in the exponential a very interesting term appears:  $p_j \dot{x}_j - H$  is the reverse Legendre-transform of  $H$  which would result in the Lagrange-function  $\mathcal{L}$ , and the summation over  $n - 1$  time steps of size  $\delta$  over  $\mathcal{L}$  would correspond to the action  $S$ , which is being de-dimensionalised by the Planck-constant  $\hbar$ . In the continuum limit one arrives at the phase space path integral,

$$K(x_i, t_i \rightarrow x_f, t_f) = \int \mathcal{D}x \int \mathcal{D}p \exp\left(\frac{i}{\hbar} \int dt \mathcal{L}\right) = \int \mathcal{D}x \int \mathcal{D}p \exp\left(\frac{i}{\hbar} S\right) \quad (\text{E.272})$$

Please be careful that we did in fact carry out the derivation in a simplified case for  $H = p^2/(2m)$  without any potential  $\Phi$ , but we generalised the expression  $\dot{x}p - \mathcal{H}$  to be the Lagrange-function (or rather, operator in this case)  $\mathcal{L}$ .

### E.5.2 Configuration-space path integral

At least half of the integrals in the phase space path integral can be solved for a standard form

$$H = \frac{p^2}{2m} + \Phi \quad (\text{E.273})$$

for the Hamilton operator  $H$ . Realising that for this form of the Hamilton-operator the phase space path integral separates in a factor involving only momenta and a factor involving only coordinates and returning to the discrete representation for a second we can write

$$K = \prod_j \int d^3 x_j \exp\left(\frac{i}{\hbar} \delta \sum_j^{n-1} \Phi(x_j)\right) \times \int \frac{d^3 p_j}{(2\pi)^3} \exp\left(\frac{i}{\hbar} \delta \sum_j^{n-1} p_j \dot{x}_j - \frac{p_j^2}{2m}\right) \quad (\text{E.274})$$

The second factor only involves Gaussian-integrals which can be solved by completion of the square in the exponent, whereas this strategy would only work for the first term for a very specific physical system: the harmonic oscillator. Carrying out all  $d^3 p$ -integrations yields:

$$K = \prod_j \int d^3 x_j \left(\frac{m\hbar}{2\pi i \delta}\right)^{n/2} \exp\left(\frac{i}{\hbar} \delta \sum_j^{n-1} \frac{m}{2} \dot{x}_j^2 - \Phi(x_j)\right) \quad (\text{E.275})$$

where one again recognises the Lagrange-function  $\mathcal{L}$  in the exponent: The squares of  $\dot{x}$  needed for kinetic energy  $m\dot{x}^2$  was provided by the Gaussian integrals over

momentum space at the step of completing the square. Therefore, the final result reads:

$$K(x_i, t_i \rightarrow x_f, t_f) = \int \mathcal{D}x \exp\left(\frac{i}{\hbar} \int dt \mathcal{L}\right) = \int \mathcal{D}x \exp\left(\frac{i}{\hbar} S\right) \quad (\text{E.276})$$

The interpretation of the path-integral for the propagation Green-function is highly interesting, as it joins quantum mechanics with classical mechanics. First of all, all possible paths between  $x_i$  and  $x_f$  can be taken without any energy consideration: Didn't strike you as odd that there was no boundary condition on the classical variational principles if a particle can "afford" a certain path energetically? In quantum mechanics this is not so dramatic as there is tunneling and a penetration of the wave function into energetically disallowed regions. Making the transition from quantum mechanics to classical mechanics in the limit  $\hbar \rightarrow 0$  should collapse the path integral and the "tube" where the wave function propagates, onto an infinitely thin line as the classical trajectory. Introducing a variation

$$x'(t) \rightarrow x(t) + \eta(t) \quad (\text{E.277})$$

gives a corresponding variation of the action

$$\delta S = S[x'] - S[x] = \int dt \left( \frac{\delta S}{\delta x} \eta + \frac{\delta^2 S}{\delta x^2} \frac{\eta^2}{2} + \dots \right) \quad (\text{E.278})$$

This difference in the action gives rise to a variation of the Green-function

$$\delta K = \exp\left(\frac{i}{\hbar} \int dt \frac{\delta S}{\delta x} \eta + \frac{\delta^2 S}{\delta x^2} \frac{\eta^2}{2} + \dots\right) \quad (\text{E.279})$$

The classical path is defined as an extremum of the action, so  $\delta S = 0$  and  $\delta S/\delta x = 0$ , as required by Hamilton's principle, so the decisive term is  $\propto \eta^2$ : Any large deviation from the classical path, no matter if positive or negative, introduces strong oscillations into  $K$  if

$$\frac{\delta^2 S}{\delta x^2} \frac{\eta^2}{2} \gg \hbar \quad (\text{E.280})$$

and in the path-integration  $\mathcal{D}x$  these oscillating terms cancel each other out. In this sense, the Planck-constant  $\hbar$  is a scale for the action  $S$ , differentiating between classical motion and quantum mechanical propagation. With a little overinterpretation one could even imagine that the particle wave duality is the mechanism by which the action is extremised: A particle sends out the probability waves and could propagate along all possible paths, but the classical one is singled out as it is assigned the highest probability by constructive interference. Deviations around the classical path are of order  $\hbar$  in  $S$ , and whether this matters or not depends on the magnitude of  $S/\hbar$ . Therefore, we have identified three key properties in the transition from quantum to classical mechanics:  $\hbar$  as the action  $S$  becomes less and less important, the uncertainty  $\Delta p \Delta x \geq \hbar/2$  becomes irrelevant and the averages needed for Ehrenfest's theorem become perfectly defined without any dispersion.

E.6 *Uncertainty*

All observables are represented by hermitean operators, whose spectrum of eigenvalues (which are necessarily real-valued) are possible outcomes of a measurement. Sometimes the situation arises that two observables are not measurable at arbitrary precision at the same time, and the criterion whether a simultaneous measurement is possible is the value of the commutator  $[A, B] = AB - BA$  of the two operators  $A$  and  $B$ . Defining the expectation values and variances

$$\langle A \rangle = \int d^3x |\psi|^2 A \quad \text{and} \quad \langle A^2 \rangle = \int d^3x |\psi|^2 A^2 \quad (\text{E.281})$$

leads to the uncertainty  $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ , where we'll use the simplification  $\Delta A = \sqrt{\langle A^2 \rangle}$  because setting  $\langle A \rangle = 0$  is always possible by redefining the operator.

The Cauchy-Schwarz-inequality implies that

$$\left( \int d^3x |\psi|^2 AB^* \right)^2 \leq \int d^3x |\psi|^2 A^2 \times \int d^3x |\psi|^2 B^2 \quad (\text{E.282})$$

or short,  $\langle AB \rangle^2 \leq \langle A^2 \rangle \langle B^2 \rangle$ . We can use the Cauchy-Schwarz-inequality as a lower bound:

$$\Delta A \Delta B = \sqrt{\langle A^2 \rangle} \times \sqrt{\langle B^2 \rangle} \geq \left| \frac{1}{2i} (\langle AB \rangle - \langle AB \rangle^*) \right| = \frac{1}{2} |\langle AB \rangle - \langle BA \rangle| = \frac{1}{2} |\langle [A, B] \rangle| \quad (\text{E.283})$$

where we've used that the modulus of a complex number is always larger than its imaginary part, that the scalar product is hermitean,

$$\langle AB \rangle^* = \langle B^+ A^+ \rangle = \langle BA \rangle \quad (\text{E.284})$$

and that the operators themselves are hermitean. Fundamental commutators between coordinates and their canonical momenta are for instance  $[p, x] = i\hbar$ :

$$[p, x]\psi = (px - xp)\psi = i\hbar [\partial_x(x\psi) - x\partial_x\psi] = i\hbar [\psi + x\partial_x\psi - x\partial_x\psi] = i\hbar\psi. \quad (\text{E.285})$$

E.7 *Relativistic quantum mechanics*

The rationale behind the Schrödinger-equation is a classical dispersion relation,  $E = p_i p^i / (2m)$  with the canonical displacements  $E \rightarrow i\hbar \partial_t$  and  $p_i \rightarrow i\hbar \partial_i$ . But we know already that the classical dispersion relation is only valid for small  $cp \ll mc^2$  compared to the rest mass  $m$ .

One possibility to generalise the Schrödinger-equation is to use the fully relativistic dispersion relation

$$E^2 = (cp)^2 + (mc^2)^2 \quad (\text{E.286})$$

and perform canonical replacement on this equation: This procedure leads to the Klein-Gordon-equation,

$$\partial_{ct}^2 \psi - \partial_i \partial^i \psi + \left( \frac{mc}{\hbar} \right)^2 \psi = 0 \quad (\text{E.287})$$

or shorthand

$$\left(\square + \lambda^{-2}\right)\psi = 0 \quad \text{with} \quad \lambda = \frac{\hbar}{mc} \quad (\text{E.288})$$

with a fully Lorentz-covariant  $\square = \eta_{\mu\nu}\partial^\mu\partial^\nu$ . The constant term is rather interesting, as it defines the de Broglie-wavelength,  $\lambda = \hbar/(mc)$  pertaining to the momentum  $mc$ . And for static situations, where  $\partial_{ct}\psi = 0$  and consequently  $\square \rightarrow \Delta$ , we recover the Yukawa-field equation A.3

$$\left(\Delta - \lambda^{-2}\right)\psi = 0 \quad (\text{E.289})$$

and now the de Broglie-scale  $\lambda = \hbar/(mc)$  gets the interpretation of a screening length, which effectively truncates the Coulomb-potential at  $\lambda r \simeq 1$ . We now understand perfectly Yukawa's reasoning: By introducing a mass-term into the field equation one can deviate from  $\Phi \propto 1/r$  and make the potential finite-ranged. The range is controlled by the mass of the particle that is described by the Klein-Gordon-equation, so choosing about 100 MeV as a mass gives a restriction to nuclear dimensions, exactly what has been found in the  $\pi$ -mesons.

Effectively, we could even take this idea one step further and define directly a canonical substitution to the relativistic momentum  $p^\mu$ :

$$p^\mu = \begin{pmatrix} E/c \\ p^i \end{pmatrix} \rightarrow i\hbar \begin{pmatrix} \partial_{ct} \\ -\partial^i \end{pmatrix} = i\hbar\partial^\mu \quad (\text{E.290})$$

and obtain  $p^\mu p_\mu = -\hbar^2\square = (mc)^2$  directly. There is a second path which leads to the Dirac-equation and the inclusion of spin-degrees of freedom of the wave functions (so-called spinors), but this is beyond the scope of this lecture.

### E.8 Coupling to fields

Coupling of wave functions  $\psi$  to fields  $F^{\mu\nu}$  is rather subtle as it proceeds over the potential  $A^\mu$  with all kinds of conceptual difficulties involving the Aharonov-Bohm-experiment. Establishing a relation between the momentum  $p^\mu$  and the potential  $A^\mu$  is done by minimal coupling,

$$p^\mu \rightarrow p^\mu - \frac{q}{c}A^\mu \quad (\text{E.291})$$

If one tries this with the Klein-Gordon equation E.287:

$$p^\mu p_\mu \psi = \left(i\hbar\partial^\mu - \frac{q}{c}A^\mu\right)\left(i\hbar\partial_\mu - \frac{q}{c}A_\mu\right)\psi = (mc)^2\psi \quad (\text{E.292})$$

which becomes under the assumption of Lorenz-gauge  $\partial_\mu A^\mu = 0$

$$\square\psi + \left(\frac{q}{c\hbar}\right)^2 A_\mu A^\mu \psi = \left(\frac{mc}{\hbar}\right)^2 \psi \quad (\text{E.293})$$

which is in this form difficult to interpret, in particular the  $A_\mu A^\mu$ -term looks weird. Let's therefore try this again in a fixed frame, where  $A^\mu = (\Phi, A^i)^t$ :

$$p^\mu p_\mu \psi = \left(i\hbar\partial_{ct} - \frac{q}{c}\Phi\right)^2 \psi - \left(i\hbar\partial^i - \frac{q}{c}A^i\right)\left(i\hbar\partial_i - \frac{q}{c}A_i\right)\psi = (mc)^2\psi \quad (\text{E.294})$$



With a very similar calculation, assuming stationary electric potentials  $\partial_{ct}\Phi = 0$  and Coulomb-gauge  $\partial_i A^i = 0$  for simplicity,

$$\square\psi - \left(\frac{q}{\hbar c}\right)^2 (\Phi^2 + A_i A^i) \psi = \left(\frac{mc}{\hbar}\right)^2 \psi \quad (\text{E.295})$$

where clearly the  $A_\mu A^\mu$ - and  $A_i A^i$ -terms are not gauge invariant. There seems to be an issue, which gets resolved by considering the action of gauging on the wave function  $\psi$  itself:

The gauge principle states that one can change the potentials  $A^\mu \rightarrow A^\mu + \partial^\mu \chi$  with a gauge function  $\chi$ , without changing the physical fields  $F^{\mu\nu}$ . The purpose of gauging is to simplify and decouple the field equations, for instance by enforcing  $\partial_\mu A^\mu = 0$ . Up to this point, we've shown as well, that the Lagrange-function of electrodynamics is gauge invariant *if* charges are conserved, so we've got reason to assume that gauge invariance is the symmetry principle behind charge conservation. Charge conservation is a property of the matter, though, it needs to function in a way that the charge density can only change locally if there are currents converging on that point and accumulate charge. In summary, we would want to have a gauge invariant wave equation for the matter fields, and understand why gauge-invariance implies charge conservation.

What change could one apply to a wave function without changing any of the physical observables? The answer is clearly a phase transformation,

$$\psi \rightarrow \psi \exp(+i\alpha(x)) \quad \psi^* \rightarrow \psi^* \exp(-i\alpha(x)) \quad (\text{E.296})$$

with a real-valued field  $\alpha(x)$ , and it is obvious that for instance the probability density  $\rho = \psi^* \psi$  is invariant under these phase transformations. Derivatives of the wave function pick up an additional term,

$$\partial_\mu \psi \rightarrow \partial_\mu (\psi \exp(i\alpha)) = (\partial_\mu \psi) \exp(i\alpha) + i\psi \exp(i\alpha) \partial_\mu \alpha \quad (\text{E.297})$$

so terms like  $\psi^* \partial_\mu \psi$  or  $\partial_\mu \psi^* \partial^\mu \psi$  are not gauge invariant. Let's try out a new derivative  $D_\mu$  with this property:

$$D_\mu \psi \rightarrow \exp(i\alpha) D_\mu \psi \quad (\text{E.298})$$

where for instance  $D_\mu \psi^* D^\mu \psi$  would be perfectly invariant.

These gauge-covariant derivatives do not commute

$$\begin{aligned} [D_\mu, D_\nu] \psi &= \left[ \partial_\mu - i\frac{q}{c} A_\mu, \partial_\nu - i\frac{q}{c} A_\nu \right] \psi = \\ &= \left( \partial_\mu - i\frac{q}{c} A_\mu \right) \left( \partial_\nu - i\frac{q}{c} A_\nu \right) \psi - \left( \partial_\nu - i\frac{q}{c} A_\nu \right) \left( \partial_\mu - i\frac{q}{c} A_\mu \right) \psi \end{aligned} \quad (\text{E.299})$$

where a tedious but straightforward calculation shows that

$$[D_\mu, D_\nu] \psi = \left[ \partial_\mu - i\frac{q}{c} A_\mu, \partial_\nu - i\frac{q}{c} A_\nu \right] \psi = -i\frac{q}{c} (\partial_\mu A_\nu - \partial_\nu A_\mu) \psi = -i\frac{q}{c} F_{\mu\nu} \psi \quad (\text{E.300})$$

with the gauge-invariant field tensor  $F_{\mu\nu}$  appearing. This idea, that the second (gauge)-covariant differentiations do not interchange, is central to general relativity and define

the Riemann-curvature  $R^{\alpha}_{\beta\mu\nu}$ :

$$[\nabla_{\mu}\nabla_{\nu}]v^{\alpha} = -R^{\alpha}_{\beta\mu\nu}v^{\beta} \quad (\text{E.301})$$

through the non-interchangeability of the second covariant derivatives acting on the vector  $v^{\alpha}$ .

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F MASS

F.1 *Dynamics of gravity*

Surely this script is not supposed to be an introduction to general relativity with its heavy usage of differential geometry as its mathematical language. For that reason, everything in this chapter is restricted to weak gravity, with perturbations of an otherwise flat Minkowskian spacetime, where the physical picture of fields on top of a Minkowski spacetime is perfectly valid. Weak and strong gravity are quantitative concepts: The curvature of spacetime is defined through second derivatives of a quantity called metric, and as composed of second derivatives the curvature  $R$  defines a length scale  $\Delta x$ ,

$$\Delta x = \frac{1}{\sqrt{R}} \tag{F.302}$$

For distances larger than  $\Delta x$ , curvature effects are important and gravity is strong, but for distances smaller than  $\Delta x$ , gravity is only a small correction on the Minkowski-metric.

Secondly, the gravitational potential  $\Phi$  as it appears in the Poisson-equation (already including here the classical cosmological constant  $\lambda$ )

$$\Delta\Phi = 4\pi G\rho + \lambda \tag{F.303}$$

has no dynamics on its own, it changes instantaneously at every point in space if  $\rho$  is not stationary. But we've seen that hyperbolic field equations usually show propagation along the light cones and the existence of wave-like solutions, so we would expect this to apply to gravity, too. Table 2 gives an overview over different regimes of gravity in physical systems.

*Table 2: regimes of gravity*

	static	dynamic
weak	Newton-gravity	gravitational waves
strong	black holes	FLRW-cosmologies

An attempt to make the Poisson-equation relativistic could be the replacement  $\partial^i \rightarrow \partial^\mu$ , along with  $\gamma_{ij} \rightarrow \eta_{\mu\nu}$ . And in addition, the kinetic energy in the random motion of particles in a substance, i.e. the pressure  $p$ , should contribute along the matter density to the gravitational field, arriving at

$$\square \frac{\Phi}{c^2} = -\frac{4\pi G}{c^4}(\rho c^2 + 3p) + \Lambda \tag{F.304}$$

with  $\Lambda = \lambda/c^2$ . This relation is interesting as well because it makes a statement about the dimensionless potential  $\Phi/c^2$ , so  $c^2$  provides a scale for  $\Phi$ . Looking ahead at the Schwarzschild-radius  $r_S$  one could imagine this argumentation.  $\Phi/c^2 = 1$  marks a particular strength of the potential, which could be given by a mass  $M$  observed at distance  $r_S$ ,  $GM/c^2 = r_S$ , which is correct up to a factor of 2. At the same time you see that the factor  $G/c^2$  has units of length/mass, so it enables us to assign a length scale to a mass.

There were actual observational findings that suggested a new theory of gravity,

albeit with a lot of experimental uncertainty. While Newton-gravity predicts the orbits of Planets to be closed ellipses with a fixed ratio between the orbital period and the large semi-axis in form of Kepler's third law, Mercury was found not to obey this. In particular, Mercury's orbit showed a precession of the point of closest proximity to the Sun, which implied a slight deviation  $\Phi \propto r^{-(1+\epsilon)}$ ,  $\epsilon > 0$ , from the Newtonian potential.

The standard Poisson-equation  $\Delta\Phi = 4\pi G\rho$  as the field equation of classical gravity, can be motivated with these arguments: The gravitational acceleration  $g^i$  is the field strength of the the gravitational field and appears in an appropriate Gauß-law,

$$\partial_i g^i = -4\pi G\rho \quad (\text{E.305})$$

such that the Poisson-equation is recovered when setting  $g^i = -\partial^i\Phi$ . Applying the Gauß-integral law and assuming spherical symmetry gives

$$\int_V d^3r \partial_i g^i = \int_{\partial V} dS_i g^i = g \ 4\pi r^2 = -4\pi G \int_V d^3r \rho = -4\pi GM \quad (\text{E.306})$$

with the mass M. This implies

$$g = -\frac{GM}{r^2} \quad \text{and consequently,} \quad \Phi = -\frac{GM}{r} \quad (\text{E.307})$$

Effectively, the scaling  $g \propto 1/r^2$  and  $\Phi \propto 1/r$  is a consequence of the surfaces of spheres in 3-dimensional Euclidean space, where the Gauß-law ensures that the flux  $\int dS_i g^i$  is conserved across every surface  $\partial V = S \propto r^2$ . Mechanical similarity applied to the  $1/r$ -potential delivers Kepler's third law  $t^2 \propto r^3$ , so that the reason for Kepler's law is ultimately geometric, and the origin of Mercury's precession is unclear. Please keep in mind that a Yukawa-type screening modifies  $\Phi$  at large and not at small distances, so it could not serve as an explanation.

## F.2 Inertial accelerations and equivalence

It is a central tenet in relativity that forces are velocity dependent to conserve the normalisation of velocities, which in turn is needed by causal motion. The prime example are Lorentz-forces,

$$\frac{du^\mu}{d\tau} = \frac{q}{m} F^{\mu\nu} u_\nu = \frac{q}{m} F^{\mu t} u_t + \frac{q}{m} F^{\mu i} u_i \quad (\text{E.308})$$

which can not accelerate a particle with specific charge  $q/m$  from timelike velocities  $u_\mu u^\mu = c^2 > 0$  to spacelike velocities  $u_\mu u^\mu < 0$ . The split in the summation over  $\nu$  shows a contribution that doesn't depend on velocity due to the electric fields  $F^{\mu t}$  and a contribution proportional to the velocities  $\nu$  due to the magnetic fields  $F^{\mu i}$ .

Making a giant conceptual leap to gravity we realise that there is no such thing as specific charge: The inertial of a particle and its coupling to a gravitational field are both equal to its mass, so gravity affects all particles in exactly the same way: From this point of view it might be better to speak about gravitational acceleration instead of gravitational force. Gravitational accelerations share this property with inertial accelerations such as the Coriolis- or centrifugal accelerations: This prompted

Einstein to postulate the equivalence principle with a general indistinguishability between gravitational and inertial accelerations.

Looking at inertia it becomes clear very quickly that these accelerations are velocity dependent, and could this be an expression that gravity is relativistic? That is in fact the case, as equation of motion in general relativity for a freely falling particle is the geodesic equation

$$\frac{du^\alpha}{d\tau} = -\Gamma_{\mu\nu}^\alpha u^\mu u^\nu = -\Gamma_{tt}^\alpha u^t u^t - 2\Gamma_{it}^\alpha u^t u^i - \Gamma_{ij}^\alpha u^i u^j \quad (\text{F.309})$$

with  $u^\mu = dx^\mu/d\tau = \gamma(c, u^i)^t$  as always.  $\Gamma_{\mu\nu}^\alpha$  is the Christoffel-symbol. In a weakly perturbed Minkowski-spacetime one has  $\Gamma_{tt}^\alpha = \partial^\alpha\Phi/c^2$ , which would give rise to a Newtonian equation of motion in the slow-motion limit,  $d^2x^i/dt^2 + \partial^i\Phi = 0$ , if the field is static and if  $\gamma \approx 1$  such that  $t = \tau$ .  $2\Gamma_{it}^\alpha u^t u^i = 2\Gamma_{it}^\alpha c u^i$  would correspond to the Coriolis-acceleration with its proportionality to  $2v$ , and lastly  $\Gamma_{ij}^\alpha u^i u^j$  would give rise to the centrifugal acceleration  $\propto v^2$ .

### F.3 Classical Raychaudhury-equation and geodesic deviation

The idea of a test particle is very transparent: It couples through its charge to the corresponding field (without changing the field itself!) and moves according to its equation of motion, indicating the strength and orientation of the field. It is worthwhile noticing that in this way the relativity principle concerning the motion of the test particle is applied to the dynamics and transformation properties of the field, in order to have the two consistent with each other: The transformation of the velocity under Lorentz-transforms is given by  $u^\mu \rightarrow \Lambda^\mu_\alpha u^\alpha$ , and of the field tensor  $F^{\mu\nu} \rightarrow \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}$ .

Exactly the same applies to the motion of particles through the gravitational field, with one peculiarity: If the particle is in a state of free-fall, one has the impression of perfect weightlessness when travelling along with the particle, and Einstein's equivalence principle then stipulates that the metric is locally Minkowskian and that the first derivative of the metric vanishes. So you might wonder where gravity actually is contained! Gravity determines the relative acceleration between freely falling test particles separated by a distance  $\delta^\mu$ :

$$\frac{d^2\delta^\mu}{d\tau^2} = -R^\mu_{\alpha\beta\nu} u^\alpha u^\beta \delta^\nu \quad (\text{F.310})$$

with the Riemann-curvature  $R^\mu_{\alpha\beta\nu}$ : If spacetime is flat with no curvature,  $R^\mu_{\alpha\beta\nu} = 0$  and consequently

$$\frac{d^2\delta^\mu}{d\tau^2} = 0 \quad \rightarrow \quad \delta^\mu = a^\mu \tau + b^\mu \quad (\text{F.311})$$

with two integration constants  $a^\mu$  and  $b^\mu$ , indicating that there is a linear change in the particle's relative distance  $\delta^\mu$ . If the curvature, however, is non-vanishing, test particles get accelerated relative to each other (despite the fact that nobody travelling along with the particles would *feel* this acceleration).

Let's understand this in Newton-gravity: Two particles follow trajectories according to Newton's equation of motion,  $\ddot{x}^i + \partial^i\Phi(x) = 0$  and  $\ddot{y}^i + \partial^i\Phi(y) = 0$ . Their relative distance  $\delta^i = y^i - x^i$  follows then the equation of motion

$$\ddot{\delta}^i = \dot{y}^i - \dot{x}^i = -\partial^i \Phi(y) + \partial^i \Phi(x) = -\partial^i \partial_j \Phi \delta^j \quad (\text{E.312})$$

with the Taylor-expansion  $\Phi(y) \simeq \Phi(x) + \partial_j \Phi (y - x)^j$ . Therefore, the tidal field  $\partial^i \partial_j \Phi$  is responsible for the relative acceleration. This is effectively the Newtonian version of the geodesic deviation equation F.310.

It is very illustrative to imagine the following experiment: Let's have a couple of test particles situated at the corners of a cube fall through space(time) and monitor the change in volume or the change in shape of that cube, because intuitively, the volume change should be related to the enclosed mass. For the relative motion of two corners we would write  $y^i = x^i + v^i \Delta t$ , so that we can observe a shear

$$\frac{\partial y^i}{\partial x^j} = \delta_j^i + \frac{\partial v^i}{\partial x^j} \Delta t \quad (\text{E.313})$$

if there are velocity gradients. Thinking back of the chapter about Lie-symmetries, we might think that these are just the first two terms of a Taylor-expansion of

$$\frac{\partial y^i}{\partial x^j} = \exp\left(\frac{\partial v^i}{\partial x^j} \Delta t\right) \quad (\text{E.314})$$

Volumes transform under this coordinate change according to

$$d^3 y = \det\left(\frac{\partial y^i}{\partial x^j}\right) d^3 x \quad (\text{E.315})$$

with the functional determinant, so that we get

$$\ln \det\left(\frac{\partial y^i}{\partial x^j}\right) = \text{tr} \ln\left(\frac{\partial y^i}{\partial x^j}\right) \simeq \text{tr}\left(\frac{\partial v^i}{\partial x^j} \Delta t\right) = \partial_i v^i \Delta t \quad (\text{E.316})$$

such that the rate of change of the volume is proportional to the divergence of the velocity field, which is immediately apparent and intuitive. We have used the relation  $\ln \det A = \text{tr} \ln A$  and the approximation  $\ln(1 + \epsilon) \simeq \epsilon$  for small  $\epsilon$ . For a very small time interval, the velocity is

$$v^i = -\partial^i \Phi \Delta t \quad (\text{E.317})$$

and consequently

$$\frac{\partial v^i}{\partial x^j} = -\partial^i \partial^j \Phi \Delta t \quad (\text{E.318})$$

The tidal field tensor can be decomposed into a trace and a traceless part,

$$\partial^i \partial^j \Phi = \left(\partial^i \partial^j \Phi - \frac{\Delta \Phi}{3} \delta^{ij}\right) + \frac{\Delta \Phi}{3} \delta^{ij} \quad (\text{E.319})$$

where the velocity divergence would only pick up  $\Delta \Phi$ , which in turn is given by  $4\pi G\rho$  through the Poisson-equation:

$$\ln \det \left( \frac{\partial y^i}{\partial x^j} \right) = 4\pi G\rho \Delta t^2 \quad (\text{F.320})$$

That means, that the cloud of freely falling test particles changes its volume dynamically in proportion to  $\Delta\Phi$  or, equivalently,  $4\pi G\rho$ . If the cloud falls through empty space, the volume would change linearly with  $\Delta t$  as the corners of the cloud would follow inertial motion, and the traceless part of the tidal shear field can only have an influence on the shape of the cloud but not its volume.

#### F.4 Gravitational lensing

We should spend a couple of minutes on the issue of gravitational light deflection to clear up misconceptions about how light could be at all influenced by gravity or curvature. There is a perfectly valid set of Maxwell's equations on a curved background which allow for wave-like solutions, but here we should see how null-lines defined by  $ds^2 = 0$  as photon trajectories notice gravity.

A good starting point is a weakly perturbed Minkowski line element,

$$ds^2 = \left(1 + 2\frac{\Phi}{c^2}\right) c^2 dt^2 - \left(1 - 2\frac{\Phi}{c^2}\right) \gamma_{ij} dx^i dx^j \quad (\text{F.321})$$

valid with a Cartesian coordinate choice and if  $|\Phi| \ll c^2$ .  $\gamma_{ij}$  is the Euclidean metric.

A conventional, non-relativistic particle experiences the line element as the passage of proper time,  $ds^2 = c^2 d\tau^2$ , and if the particle is non-relativistic, it moves essentially only in the  $dt$ -direction and doesn't change its spatial coordinates by a large amount,  $dx^i = 0$ . Then there will be a gravitational dilation of proper time relative to coordinate time

$$ds^2 = c^2 d\tau^2 = \left(1 + 2\frac{\Phi}{c^2}\right) c^2 dt^2 \quad \rightarrow \quad \tau = \sqrt{1 + 2\frac{\Phi}{c^2}} dt \simeq \left(1 + \frac{\Phi}{c^2}\right) dt \quad (\text{F.322})$$

caused by the gravitational potential  $\Phi$ , which is negative as  $\Phi = -GM/r$ , such that  $d\tau < dt$ , with the approximation  $\sqrt{1 + \epsilon} \simeq 1 + \epsilon/2$ .

A photon, however, traces out a trajectory characterised by  $ds^2 = 0$  and proper time is not sensibly defined. The effective speed of propagation of the photon is the rate at which the coordinates  $dx$  pass by in units of coordinate time  $dt$ , leading to

$$c' = \frac{dx}{dt} = \pm \sqrt{\frac{1 + 2\frac{\Phi}{c^2}}{1 - 2\frac{\Phi}{c^2}}} c \simeq \pm \left(1 + 2\frac{\Phi}{c^2}\right) c \quad (\text{F.323})$$

with the approximation  $1/(1 - \epsilon) \simeq 1 + \epsilon$  for small  $\epsilon$ . That is a surprising result, as the effect of a gravitational field on a relativistic particle is twice as strong as on a non-relativistic particle. If again  $\Phi = -GM/r$ , the effective speed of propagation  $c'$  becomes zero at  $2GM/c^2 = r_S$ , which is known as the Schwarzschild radius. You see, it's not a matter of energy or of time of flight when a photon can not escape from a black hole; in these coordinates it's the case that the effective speed of propagation reaches zero at  $r_S$ , so the photon does not make any headway (in either direction!).

It's a good idea to follow this thought a bit further: For a radially moving photon in the potential  $\Phi = -GM/r$ , we have

$$\frac{dr}{dt} = \pm \left(1 - \frac{2GM}{c^2 r}\right) c \quad \rightarrow \quad \frac{dr}{1 - r_S/r} = \pm c dt \quad (\text{E.324})$$

which is solved by  $r_S \ln(r - r_S) + r = \pm ct$  up to an integration constant, let's call it  $p$  for the + branch and  $q$  for the - branch. This integration constant can be made the new radial coordinate,

$$p = r_S \ln(r - r_S) + r + ct \quad \text{and} \quad q = r_S \ln(r - r_S) + r - ct \quad (\text{E.325})$$

or differentially with  $\alpha = (1 - r_S/r)^{-1}$ :

$$dp = c dt + \alpha dr \quad \text{and} \quad dq = c dt - \alpha dr \quad (\text{E.326})$$

with these new coordinates, the line element becomes

$$\begin{aligned} ds^2 &\simeq \alpha^{-1} c^2 dt^2 - \alpha dr^2 = \alpha^{-1} (dp - \alpha dr)(dq + \alpha dr) - \alpha dr^2 = \\ &\alpha^{-1} (dp dq + \alpha (dp - dq) dr - \alpha dr^2) - \alpha dr^2 \end{aligned} \quad (\text{E.327})$$

which becomes by using  $dp - dq = 2\alpha dr$  simply

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dp dq \quad (\text{E.328})$$

The line element is effectively given now in terms of light cone coordinates, with a so-called conformal factor in front: This conformal factor doesn't change light propagation as  $ds^2 = 0$  and the factor is never zero, so  $dp dq = 0$  already characterises the trajectory of a photon: We have absorbed the action of the gravitational field in a redefinition of the coordinates.

### F.5 Gravitational field equation

From what we've learned the gravitational field equation should be a second-order hyperbolic field equation which is at least covariant under Lorentz-transforms. A first guess could be that gravity is some kind of electrodynamics for masses, so we could write

$$\square A^\mu = -\frac{4\pi G}{c} j^\mu \quad (\text{E.329})$$

with  $A^t$  being the gravitational potential  $\Phi$  and  $j^t$  the matter density  $\rho$ , the idea being that momentum density along with rest mass sources the gravitational field. Already now it might be a bit weird that  $j^t$  is *not* the rest mass energy density.

Surely, in the case of static field one would fall back onto the Poisson-equation, but for instance the incorporation of the cosmological constant  $\lambda$  would be unclear, as the equation is vectorial and not scalar as our intuitive rewriting of the Poisson equation

$$\square \frac{\Phi}{c^2} = -\frac{4\pi G}{c^4} (c^2 \rho + 3p) + \frac{\lambda}{c^2} \quad (\text{E.330})$$

But there is a more fundamental problem: The rest mass energy density  $c^2 \rho$  transforms differently than the electric charge density. If you imagine a cloud of



electrical charges viewed from a moving system, one perceives that cloud Lorentz-contracted by a factor of  $\gamma$  such that the charge density is higher by that factor, in agreement with the transformation property of a vector  $j^\mu \rightarrow \Lambda^\mu_\alpha j^\alpha$ . A cloud of matter viewed from another Lorentz-system has the same effect of Lorentz-contraction of the volume along the direction of motion, but also a relativistic mass increase by another factor of  $\gamma$  (indirectly, as a consequence of time dilation: one assigns a higher amount of inertia to the system). To get two powers of  $\gamma$  in the transformation,  $c^2 \rho$  must be the  $tt$ -component of a tensor, in this case the energy momentum tensor  $T^{\mu\nu}$ . The transformation property would be  $T^{\mu\nu} \rightarrow \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}$ , and with the proportionality of  $\Lambda^\mu_\alpha \propto \gamma$  this actually works out. In summary, the gravitational field equation would need to be at least tensorial, in the form

$$\square h^{\mu\nu} = -\frac{4\pi G}{c^2} T^{\mu\nu} \quad (\text{F.331})$$

A second large conceptual difference is the nonlinearity in energy-momentum conservation, expressed by the innocently looking conservation law  $\partial_\mu T^{\mu\nu} = 0$ , with typical nonlinear terms arising in the equations of relativistic fluid mechanics. This in turn implies that eqn. F.331 can only be valid in a linearised limit.

The solution to these problems is much more complex and requires differential geometry: Gravity is thought to be equivalent to spacetime geometry, where curvature is sourced by the energy-momentum content. If that relationship is to be (i) a second-order, hyperbolic relation, which (ii) respects energy-momentum conservation, if (iii) spacetime is 4-dimensional and if the (iv) metric of spacetime is linked to the energy-momentum tensor in a (v) local way, then general relativity is uniquely defined, as stated by [Lovelock's theorem](#).



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## G TEMPERATURE

### G.1 *Thermal energy*

Thermodynamics is a very abstract physical theory which defines state variables and establishes relationships between them. Most importantly, it provides a definition of temperature for systems in a state of thermal equilibrium. But thermodynamics does not make any assumption about the internal structure of systems and how they would give rise to relations between their state variables: that particular relation is provided by statistical mechanics, i.e. the mechanical theory of systems with many degrees of freedom. One can think of thermodynamics as an effective theory of these systems (typically, the number of degrees of freedom is of the order of Avogadro's number or higher) in thermal equilibrium, where on average each mode carries the same share of the total thermal energy: this is meant by equipartition.

It is only sensible to speak about thermal energy in thermal equilibrium: While microscopically there is a continuous reshuffling of energy between all degrees of freedom of a system, macroscopically there is no discernible dynamics at all. The amount  $Q$  of thermal energy that an object of mass  $m$  stores at temperature  $T$  is given by

$$Q = c(T)mT \tag{G.332}$$

with the specific energy  $c(T)$ , possibly a function of  $T$ . It summarises how much thermal energy can be stored in a system, effectively by counting the degrees of freedom the substance provides if the total mass is  $m$ .

### G.2 *Axioms of thermodynamics*

Thermodynamics is a sensibly defined theory *even if* one does not have the slightest clue about the internal structure of matter. But we can turn this around to our advantage: There are sensible thermodynamical properties and definitions of temperature for almost every physical system, even if we do not know a priori how it works internally. For instance, it is quite reasonable to consider the electrodynamic field in thermal equilibrium or to imagine thermodynamical properties of black holes, i.e. of gravity.

Thermodynamics is defined axiomatically, and it's really a good place to appreciate the abstractness of it:

0. Heat flows from hot to cold until a thermal equilibrium is established, characterised by a common temperature  $T$ .
1. The energy content  $U$  of a system can be changed by  $dU$ , either by performing mechanical work  $\delta W$  on it or by changing its thermal energy by  $\delta Q$ : Effectively, the law of energy conservation encompasses thermal energy, too:  $dU = \delta W + \delta Q$
2. A system can not perform mechanical work out of thermal equilibrium; you need a non-equilibrium to generate work out of a heat flow, and that is done imperfectly.
3. Entropy approaches zero at absolute zero in temperature.

With reference to the first law I'd like to clarify that changing the internal energy content can be done in very different ways:

$$dU = TdS - pdV + \mu dN + \Phi dQ + \mathbf{B} \cdot d\mathbf{M} + \dots \tag{G.333}$$

where one sees typically the combination of an extensive (i.e. proportional to the size of the system or the amount of substance) state variable with an intensive (independent of the size of the system) one: work can be performed by a change  $dV$  in volume against pressure  $p$ , or by changing the charge  $dQ$  against an external electric potential  $\Phi$ , by changing the magnetisation  $d\mathbf{M}$  against an external magnetic field  $\mathbf{B}$ , by changing the number of particles in the system  $dN$  against the chemical potential  $\mu$  and lastly, by changing the entropy  $dS$  against temperature  $T$ .

The separation between intensive and extensive state variables is sometimes washed out, and please be very careful in these cases of "non-extensive thermodynamics": If the potential  $\Phi$  is in fact sourced by  $Q$  itself, it would become extensive.

### G.3 Measuring temperature

Temperature is perhaps the most abstract concept in theoretical physics, despite the fact that everyone has the feeling of intuitiveness about temperature: This is because the fundamental definition of temperature  $T$  is

$$\frac{\partial S}{\partial E} = \frac{1}{k_B T} \quad (\text{G.334})$$

so one needs to have an intuition about entropy  $S$  first and how it depends on  $E$ , before thinking about temperature  $T$ . When measuring temperature, one can go about in very different ways: Firstly, one can use some empirical relationship between an easily observable quantity, for instance the length of a column of mercury as it is determined by the [thermal expansion coefficient](#) or the volume of a gas as an expression of [Gay-Lussac's law](#) to estimate  $T$ . Specifically, for an ideal gas we have a constant  $V/T$  at constant pressure  $p$ , so

$$\frac{V_1}{T_1} = \frac{V_2}{T_2} \quad \rightarrow \quad T_2 = \frac{V_2}{V_1} T_1 \quad (\text{G.335})$$

with a reference  $V_1$ ,  $T_1$  and a measurement of  $V_2$ , where for instance, the reference could be defined through the [molar volume](#) of 22.41396954 litres at [standard conditions](#)  $p = 101.325$  kPa and  $T = 273.15$  Kelvin.

But conceptually, one would like to measure temperature mechanically: By conversion of thermal energy to mechanical energy, the world of thermodynamics is linked to the world of mechanics, and mechanical energies can be measured unambiguously and in accordance with the laws of Galilean or Lorentzian relativity, for instance by accelerating an object of known mass. Surely, this can not be achieved in thermal equilibrium, as clarified by the second law of thermodynamics, but if there is a disequilibrium one can employ a Carnot-engine to convert thermal energy  $\delta Q$  to mechanical energy  $\delta W$

$$\delta W = \eta \delta Q \quad (\text{G.336})$$

at a known and unique efficiency  $\eta$ , which is only a function of temperatures

$$\eta = 1 - \frac{T_2}{T_1} < 1 \quad (\text{G.337})$$

with the temperature  $T_1$  of the hot reservoir and  $T_2$  fo the cold reservoir: With a Carnot-engine one can determine at least temperature differences relative to a

reference temperature that needs to be fixed. For this one chooses the [triple point of water](#) at  $T = 273.1600$  Kelvin at  $p = 611.657$  Pa. At the triple point, the phases of ice, water and vapour exist simultaneously which is easy to observe.

#### G.4 Carnot-engines

Carnot-engines are thermal engines: They can convert thermal energy back to mechanical energy (or pump thermal energy from the cold to the hot reservoir against the natural tendency of heat to flow from hot to cold). The conversion from thermal to mechanical energy is not ideal but happens at an ideal efficiency  $\eta = 1 - T_2/T_1$ , which is identical for all Carnot-engines irrespective of how they are built: All that matters is perfect reversibility in their working principle: If the temperatures are changed, the efficiency changes without any delay. And Carnot-engines work in a cyclic fashion: There is no energy stored internally after one sequence is completed.

A traditional construction is the steam-engine-type, where one proceeds in four phases through (i) isothermal expansion at  $T_1$ , sucking in the heat  $Q_1$ , followed by (ii) adiabatic expansion to bring the temperature from  $T_1$  down to  $T_2$  with no heat exchange, (iii) isothermal compression at  $T_2$ , squeezing out the heat  $Q_2$ , and completed by (iv) adiabatic compression to get the temperature from  $T_2$  up to  $T_1$  again. The amounts of work gained in step (ii) is equal to the work to be invested in step (iv), so all that matters are the energies in step (i) and (iii):

$$Q_1 = \int dS T = T_1 \Delta S > 0 \quad \text{and} \quad Q_2 = \int dS T = T_2 \Delta S < 0 \quad (\text{G.338})$$

After one complete cycle of the engine there is no change  $dU$  in internal energy as it works cyclically, so one can conclude

$$dU = \delta W + \delta Q = 0 \quad \rightarrow \quad W = -Q = -(Q_1 + Q_2) = (T_1 - T_2)\Delta S \quad (\text{G.339})$$

so we can find for the Carnot-efficiency  $\eta$  the iconic result

$$\eta = \frac{W}{Q_1} = 1 - \frac{T_2}{T_1} \quad (\text{G.340})$$

But actually this is only one way of constructing a Carnot-engine. A completely different engine would be a propeller on an axis submerged in a gas at temperature  $T_1$  on which there is a ratched and pawl-mechanism that only allows the engine to turn into one direction, lifting a weight in the process and performing mechanical work. In forward motion, one needs an energy  $\epsilon$  to disengage the ratched such that an amount  $W$  of work can be gained, and this happens at a rate  $\exp(-(\epsilon + W)/(k_B T_1))$ . Motion in the opposite direction frees work  $W$ , but nevertheless needs an activation energy  $\epsilon$  of the ratched.  $W$  is lost, or more exactly, transferred to the gas and dissipated there, while a random thermal fluctuation provides  $\epsilon$  to the ratched, at a rate  $\exp(-\epsilon/(k_B T_2))$ . In a reversible engine, the two rates are equal,

$$\exp\left(-\frac{\epsilon + W}{k_B T_1}\right) = \exp\left(-\frac{\epsilon}{k_B T_2}\right) \quad (\text{G.341})$$

from which we can define the efficiency  $\eta$  as

$$\eta = \frac{W}{\epsilon} = 1 - \frac{T_2}{T_1} \quad (\text{G.342})$$

as before! It is perhaps a funny image to combine the steam engine with the ratchet and pawl-machine, one operating as a thermal engine and the other one as a heat pump: They would exactly cancel each other out. And in fulfilment of the second law of thermodynamics: In thermal equilibrium,  $T_1 = T_2$ , the efficiency drops to zero  $\eta = 0$  and no mechanical work can be performed.

### G.5 Thermal wavelength and quantum statistics

We can take this idea of determining temperature with a mechanical measurement one step further, specifically by bringing in quantum mechanics and focusing on kinetic systems, i.e. systems where the thermal energy is present in the form of kinetic energy in the motion of the particles: The de Broglie-wavelength  $\lambda$  of a particle at momentum  $p$  is given by the relation

$$p = \frac{h}{\lambda} \quad (\text{G.343})$$

with Planck's constant  $h$ . On the other hand, the particle's typical energy would be

$$E = k_B T \quad (\text{G.344})$$

as a consequence of equipartition. The two ideas are linked through the dispersion relation

$$E = \sqrt{(cp)^2 + (mc^2)^2} = \begin{cases} cp, & \text{for large momenta } cp \gg mc^2 \\ \frac{p^2}{2m} & \text{for small momenta } cp \ll mc^2 \end{cases} \quad (\text{G.345})$$

Then, we can define the thermal wavelength for relativistic particles,

$$E = k_B T = cp = \frac{ch}{\lambda} \quad \rightarrow \quad \lambda = \frac{ch}{k_B T} \quad (\text{G.346})$$

and analogously for non-relativistic particles,

$$E = k_B T = \frac{p^2}{2m} = \frac{1}{2m} \left( \frac{h}{\lambda} \right)^2 \quad \rightarrow \quad \lambda = \frac{h}{\sqrt{2mk_B T}} \quad (\text{G.347})$$

In both cases, the thermal wavelength becomes shorter with increasing temperature, as a reflection of the particle's higher momenta. Measuring  $\lambda$  spectroscopically by means of a diffraction grid is a perfectly valid determination of temperature: From the fact that the light of the Sun is visible and yellow in colour we can conclude that the surface temperature of the Sun must be around 6000 Kelvin.

Thermal wavelength as a scale matters physically as it is intricately linked to the particle's being indistinguishable: If the separation of two particles is small compared to the thermal wavelength, their wave functions, which have a typical extension of the order  $\lambda$  overlap heavily and a localisation of the particles is not able to determine which particle is which!

Additionally, thermal wavelength  $\lambda$  and the associated volume  $\lambda^3$  of a wave packet provide a scale for the volume of the system, as can be seen for instance in the canonical partition  $Z(T, V, N)$  for an ideal, non-relativistic gas:

$$Z(T, V, N) = \frac{1}{N!} \int \prod_i \frac{d^3 p_i d^3 q_i}{h^3} \exp\left(-\frac{1}{k_B T} \sum_i \frac{p_i^2}{2m}\right) \quad (\text{G.348})$$

which, due to the non-interaction of particles, separates:

$$Z(T, V, N) = Z(T, V, 1)^N \quad (\text{G.349})$$

Evaluating the partition sum shows that  $\int d^3 q$  is just the volume  $V$  of the system and substituting the thermal wavelength  $\lambda$  gives

$$Z(T, V, N) = \frac{1}{N!} \left(\frac{V}{\lambda^3}\right)^N \quad (\text{G.350})$$

Funnily, exactly the same expression is valid for a relativistic ideal gas! Writing

$$Z(T, V, N) = \frac{1}{N!} \int \prod_i \frac{d^3 p_i d^3 q_i}{h^3} \exp\left(-\frac{1}{k_B T} \sum_i c p_i\right) \quad (\text{G.351})$$

by substitution of the definition of  $\lambda$  for the relativistic case. Clearly,  $\lambda^3$  is a scale for  $V$  and we would expect some kind of quantum-mechanical interference effect when the wave functions are extended and become comparable to the size of the system,  $V \approx N\lambda^3$ . And the example of these two classical gases show that  $h$  plays a role even in apparently non-quantum mechanical systems.

## G.6 Boltzmann-factor and the fundamental postulate

To imagine a system in thermal equilibrium is not straightforward: On the macroscopic level, nothing at all is happening as the system does not evolve mechanically, and there are no heat fluxes in or out either. But on the microscopic level, there is a lot going on! All degrees of freedom follow their dynamics defined in Hamiltonian mechanics and are continuously reshuffling energy, but maintaining equipartition on average, with a typical energy of  $k_B T$  present in every degree of freedom. In addition, there is the fundamental postulate to be fulfilled, that finding the system in any of the microstates is equally probable, and that observing a degree of freedom acquiring an amount of energy  $\epsilon$  by a thermal fluctuation is given by the Boltzmann-probability  $\exp(-\epsilon/(k_B T))$ .

There are various ways which suggest the Boltzmann-factor convincingly, but it is clearly one way to enforce transitivity: The probability  $p(\epsilon, T)$  to find a fluctuation of  $\epsilon$  at temperature  $T$  should be a function of energy difference, i.e.

$$\frac{p(\epsilon_2, T)}{p(\epsilon_1, T)} = g(\epsilon_2 - \epsilon_1, T) \quad (\text{G.352})$$

where we can introduce an intermediate step,

$$g(\epsilon_3 - \epsilon_1, T) = \frac{p(\epsilon_3, T)}{p(\epsilon_1, T)} = \frac{p(\epsilon_3, T)}{p(\epsilon_2, T)} \frac{p(\epsilon_2, T)}{p(\epsilon_1, T)} = g(\epsilon_3 - \epsilon_2, T)g(\epsilon_2 - \epsilon_1, T) \quad (\text{G.353})$$

This suggests a functional equation for the unknown function  $g(\epsilon, T)$  which is uniquely solved by  $g(\epsilon, T) = \exp(-\beta(T)\epsilon)$ . The dependence of  $\beta$  on temperature  $T$  is heuristically given by

$$\beta = \frac{1}{k_B T} \quad (\text{G.354})$$

with the Boltzmann-constant  $k_B$ . Heuristically, this is very sensible, as higher temperatures make large thermal fluctuations more likely, and the minus-sign is a reflection of stability if the system is energetically bounded from below and if the temperature is positive. This would be the opposite in systems with negative absolute temperature: Please refer to Sect. G.9 for this.

Let's see whether thermal fluctuations are real and how they would enter in a discrete picture of matter versus a continuum picture. A molecule in the atmosphere experiences continuous collisions with the other molecules maintaining thermal equilibrium, and by interactions with more than one particle the energy  $\epsilon$  fluctuates. The particle can invest this energy to rise up in the gravitational field of the Earth to the height  $h$  determined by the potential energy,  $\epsilon = mgh$ . This process would take place at the Boltzmann-probability

$$p = \exp\left(-\frac{E}{k_B T}\right) = \exp\left(-\frac{mgh}{k_B T}\right) \quad (\text{G.355})$$

such that the fraction of molecules that can reach the height  $h$ , i.e. the density  $\rho$  becomes proportional to  $\exp(-h)$ : This is just the barometric formula.

In a continuum picture the same result has to be explained by this: The atmosphere as a continuum is described by density  $\rho$ , velocity  $v^i$  and pressure  $p$  by the Euler-equation of ideal fluid mechanics,

$$\partial_t v^i + (v^j \partial_j) v^i = -\frac{\partial^i p}{\rho} - \partial^i \Phi \quad (\text{G.356})$$

from which we derive the hydrostatic equation

$$\frac{\partial^i p}{\rho} = -\partial^i \Phi \quad (\text{G.357})$$

if the velocities vanish,  $v^i = 0$  and if there are no accelerations  $\partial_t v^i = 0$ . To continue, we need to assume a relationship between pressure and density, for instance that  $p \propto \rho$  at fixed temperature as predicted by the law by Boyle and Mariotte for ideal gases. Then,

$$\frac{\partial^i p}{\rho} = \partial^i \ln \rho \propto -\partial^i \Phi \quad \rightarrow \quad \rho \propto \exp(-\Phi) \quad (\text{G.358})$$

with the scaling  $\rho \propto \exp(-h)$  for a homogeneous gravitational potential, again leading to the barometric formula. You see that these two pictures have almost nothing in common yet lead to the same result, and that the equation of state  $p \propto \rho$  at fixed  $T$



brushes over a lot of physics but establishes the equivalence between the two pictures. It is clear that on the level of molecules a fluctuation of  $k_B T \approx 10^{-21}$  Joules matters a lot ( $k_B = 1.380649 \times 10^{-23}$  J/K and room temperature is about 300 Kelvin), but that it is completely irrelevant for macroscopic objects.

### G.7 Ultra-relativistic Bose gases and the Planck-spectrum

The Planck-spectrum of a thermal gas of photons was one of the decisive systems which established quantum mechanics along with the hydrogen atom: As abstract as it may seem, the electromagnetic field can be in a state of thermal equilibrium! As Maxwell-electrodynamics is perfectly linear, the field can not reach equilibrium by itself, it can only do so through the interaction with matter. This is important because otherwise the superposition principle would apply to the modes of the field, and it would not be possible to transfer energy from one mode to another. In Planck's original works he conjures up the picture of a container with perfectly mirrored walls to contain the electromagnetic field and a grain of coal as a means of interaction and thermal equilibration. The grain of coal is able to absorb energy from the field and re-emit it in another mode, called photon, and there is really no resistance of the system to change the photon number, expressed by the chemical potential  $\mu = 0$ . The situation would be different if we're dealing with particles with a finite rest mass. Then, one would need to invest at least  $mc^2$  to change the particle number and the chemical potential would be consequently nonzero.

As the particle number is not fixed but controlled by the chemical potential  $\mu$  with the specific value  $\mu = 0$ , corresponding to the fugacity  $z = \exp(\beta\mu) = 1$ , along with a fixed volume  $V$  and a temperature  $T$ , we have to work with a macrocanonical partition  $\mathcal{Z}(T, V, \mu)$ ,

$$\ln \mathcal{Z}(T, V, \mu) = -\frac{4\pi V}{(hc)^3} \int_0^\infty \epsilon^2 d\epsilon \ln [1 - \exp(\beta\epsilon)] = \frac{4\pi V}{(hc)^3} \frac{\beta}{3} \int_0^\infty d\epsilon \frac{\epsilon^3}{\exp(\beta\epsilon) - 1} \quad (\text{G.359})$$

where we use the linear dispersion  $\epsilon = cp$  valid for photons and  $\beta = 1/(k_B T)$ . Integrals like the one in eqn. G.359 involving a monomial  $\epsilon^n$  and the Bose-factor are typical for calculations with bosons: Substituting  $x = \beta\epsilon$  and  $dx = \beta d\epsilon$  gives

$$\ln \mathcal{Z}(T, V, \mu) = \frac{4\pi V}{3(hc)^3} \frac{1}{\beta^3} \int_0^\infty dx \frac{x^3}{\exp(x) - 1} = \frac{4\pi^5 V}{90(hc)^3} (k_B T)^3 \quad (\text{G.360})$$

using

$$\int_0^\infty dx \frac{x^{n-1}}{\exp(x) - 1} = \zeta(n)\Gamma(n) = \zeta(n)(n-1)! \quad (\text{G.361})$$

with Riemann's  $\zeta$ -function and the  $\Gamma$ -function as a generalisation of the factorial. Here, we need the specific value  $\zeta(4)3! = \pi^4/90$ . With these results, one finds for the macrocanonical potential  $J(T, V, \mu)$  the expression

$$J(T, V, \mu) = -k_B T \ln \mathcal{Z}(T, V, \mu) = \frac{8\pi^5 V}{90(hc)^3} (k_B T)^4 \quad (\text{G.362})$$

where we include an additional pre-factor of 2 to take care of the two possible spin states of photons. From this result derives many properties of the Planck-spectrum automatically: Entropy  $S$  and photon number  $N$  are both proportional to  $T^3$ , pressure is proportional to  $T^4$  as well as total energy, as an expression of the Stefan-Boltzmann-law.

One of the decisive properties is the appearance of the Bose factor

$$\frac{1}{\exp(\beta\epsilon) - 1} \rightarrow \exp(-\beta\epsilon) \quad (\text{G.363})$$

which falls back on the familiar Boltzmann-factor for  $\beta\epsilon \gg 1$ . Primarily the consequences are slight numerical differences to a classical computation involving the Boltzmann-factor only, as carried out by Wien originally, who found puzzling pre-factors that he could not make much sense of. If we isolate the spectral energy density  $S(\omega)$  from eqn.G.360 and rewrite it in terms of frequency  $\omega$ ,  $x = \bar{\omega}/(k_B T)$  we get

$$S(\omega) = \frac{\hbar}{4\pi^2 c^2} \frac{\omega^3}{\exp(\beta\hbar\omega) - 1} \quad (\text{G.364})$$

which transitions for high frequencies  $\hbar\omega \gg k_B T$  into the classical Wien-limit as the Bose-factor can be replaced by the Boltzmann-factor

$$S(\omega) = \frac{\hbar}{4\pi^2 c^2} \omega^3 \exp(-\beta\hbar\omega) \quad (\text{G.365})$$

while for small frequencies  $\hbar\omega \ll k_B T$  one recovers the Rayleigh-Jeans limit,

$$S(\omega) = \frac{\omega^2}{4\pi^2 c^2} k_B T \quad (\text{G.366})$$

which of course does not yield a finite result when integrating over all frequencies: That's the ultraviolet catastrophe. The key result in this context is that at high energies, the system behaves classically and at low energies quantum mechanically with an overabundance of photons at low energies. The two regimes are separated roughly by the peak of the Planck-spectrum as the Wien displacement law shows:

$$\frac{dS(\omega)}{d\omega} = 0 \rightarrow \hbar\omega_{\max} \simeq 2.8k_B T \quad (\text{G.367})$$

This overabundance of photons at low energies leads to a super-Poissonian counting statistic experimentally verified by the Hanbury-Brown and Twiss experiment.

### G.8 Entropy

The concept of entropy is mysterious and perhaps as complex to understand as temperature, so let's go through different aspects of entropy:

- Weirdly, the most straightforward view on entropy is a system of ultra-relativistic bosons. As we've shown in Sect. G.7, entropy  $S \propto T^3$  as well as particle number  $N \propto T^3$ , so that both  $S$  and  $N$  increase in proportion with increasing temperature  $T$ , as the system generates new photons. Therefore, entropy is just the number of photons in the system and is perfectly extensive, and clearly one

does not have independent control over  $S$  and  $N$ : As a macrocanonical system, the ultra-relativistic photon gas has  $T$ ,  $V$  and  $\mu$  as independent state variables, with the particular property  $\mu = 0$  as a reflection of the masslessness of photons.

- In a classical ideal gas the particle number is fixed (and the state variable in the canonical ensemble are  $T$ ,  $V$  and  $N$ ): Entropy is rather a reflection of the volume of the energetically allowed phase space.
- The Carnot-engine offers a completely different view on entropy: The Carnot-engine absorbs  $Q_1$  from one reservoir at  $T_1$  and dumps  $Q_2 = T_2/T_1 Q_1$  onto the reservoir at  $T_2$ , because  $Q_i/T_i = S_i$ , which is equal: Entropy controls the conversion from thermal to mechanical energy.  $W/Q_1 = \eta = 1 - T_2/T_1$  is then the efficiency.
- The first law of thermodynamics states that  $dU = TdS - pdV \pm \dots$ : Entropy is an extensive quantity. If one changes it, one performs work against temperature, similarly to changes in volume perform work against pressure. Vice versa,  $\partial S/\partial E = 1/T$  is the formal definition of temperature as a derived quantity.

A nice example of entropy in a discrete system a polymer chain: Please assume that a polymer string is made from  $N$  monomers, which can be built into the chain in the long configuration with length  $a$  and the short configuration with length  $b < a$ . From the outside, one controls temperature  $T$  (through a heat bath), string tension  $\sigma$  (effectively as an analogue to pressure  $p$ ) while  $N$  is fixed; With  $T$ ,  $\sigma$  and  $N$  the suitable state function is the enthalpy  $G(T, \sigma, N)$ , and as  $N$  is fixed, we'll use a canonical description, with a replacement of  $l$  (or  $V$ ) by  $\sigma$  (or  $p$ ): The canonical partition is given by

$$Z_G(T, \sigma, N) = \sum_i \binom{N}{i} \exp\left(-\frac{\sigma l(i)}{k_B T}\right) \quad (G.368)$$

summing over all possible states weighted by the Boltzmann-factor, with the chain length  $l(i)$

$$l(i) = ai + (n - i)b \quad (G.369)$$

The canonical partition can be summed out to yield

$$Z_G(T, \sigma, N) = \left( \exp\left(-\frac{\sigma a}{k_B T}\right) + \exp\left(-\frac{\sigma b}{k_B T}\right) \right)^N \quad (G.370)$$

which factorises, which is typical for non-interacting states,  $Z_G(T, \sigma, N) = Z_G(T, \sigma, 1)^N$ . The enthalpy  $G$  is given by

$$G(T, \sigma, N) = -k_B T \ln Z(T, \sigma, N) = -TS + \sigma l + \mu N \quad (G.371)$$

along with the differential  $dG$

$$dG = -SdT + l d\sigma + \mu dN \quad (G.372)$$

such that we get

$$S = -\frac{\partial G}{\partial T} = Nk_B \ln Z_G(T, \sigma, 1) + \frac{N}{Z(T, \sigma, 1)} \left( a \exp\left(-\frac{\sigma a}{k_B T}\right) + b \exp\left(-\frac{\sigma b}{k_B T}\right) \right) \frac{\sigma}{T} \quad (\text{G.373})$$

as well as

$$l = \frac{\partial G}{\partial \sigma} = N \times \frac{a \exp\left(-\frac{\sigma a}{k_B T}\right) + b \exp\left(-\frac{\sigma b}{k_B T}\right)}{\exp\left(-\frac{\sigma a}{k_B T}\right) + \exp\left(-\frac{\sigma b}{k_B T}\right)} \quad (\text{G.374})$$

The equation of state  $l(T, \sigma, N)$  has a curious property, as the length of the chain decreases with increasing temperature: Higher temperatures enable the system to transition from elements in the long configuration to the energetically disfavoured short configuration by providing thermal fluctuations more easily. This unusual behaviour is an example of an entropic force, as the shortening of the chain comes with an increase in entropy. In addition, the proportionality of  $S$  with the number  $N$  of chain elements underlines the extensivity of  $S$ .

### G.9 Negative absolute temperatures

To make things even weirder, it's perfectly valid to construct systems of negative absolute temperature! Imagine a system that is energetically bounded from above, with fewer and fewer possibilities to realise states of increasing energy. Then, the derivative  $\partial S/\partial E$  that defines temperature, would be negative, and consequently,  $T$  would be smaller than zero as well, according to

$$\frac{\partial S}{\partial E} = \frac{1}{k_B T} \quad (\text{G.375})$$

Of course, this could never be realised in a gas-dynamical system! There, the energy is bounded from below, and  $S$  increases as a function of  $E$ , as there is a larger phase space at higher energies, and  $T$  is necessarily positive.

It gets even weirder when powering a Carnot-engine with two reservoirs, one at positive and one at negative  $T$ : Then, the Carnot-efficiency  $\eta$  becomes larger than one! In some sense, thermal energy is in this case the more useful form of energy compared to mechanical energy and while the first law of thermodynamics formulating energy conservation for mechanical and thermal energy combined is of course valid, it becomes more attractive to store energy in these systems combining a reservoir with  $T > 0$  with one where  $T' < 0$ . By powering the Carnot-engine with mechanical energy one makes  $T$  more positive and  $T'$  more negative, and using the Carnot-engine as a thermal engine one decreases  $T$  and make  $T'$  less negative, and gets mechanical energy back at the efficiency:

$$\eta = 1 - \frac{T'}{T} > 1 \quad (\text{G.376})$$

so in some sense, thermal energy is the more useful energy form compared to mechanical energy.

The efficiency  $\eta$  would be strictly lower than one for thermal engines operating between only positive absolute temperatures or only negative absolute temperatures. Examples of systems with negative absolute temperature are for instance spin-systems, where the state of highest energy (all spins aligned) corresponds to a state of low entropy because of the high degree of order, and in approaching the state of highest energy would find fewer and fewer realisations compatible with the energy,

such that  $\partial S/\partial E$  becomes negative: Imagine a grid of spins, where the state of highest alignment (all spins pointing into the same direction) would be the state of highest energy. Then, having one spin point into the other direction could be realised in  $n$  different ways, having two spins point into the other direction already by  $n(n-1)$  possibilities, so the number of possible realisations increases with decreasing energy, or decreases with increasing energy: That would imply a negative temperature.

### G.10 Reversibility

It seems that statistics of systems with many degrees of freedom brings in something new: While all fundamental laws of Nature are perfectly time reversible (due to the second derivatives in the field equations or the equations of motion), we see that there are irreversible processes like mixing of liquids accompanied by entropy increase.

But this difference is quantitative and not fundamental: Imagine a ball pit filled with balls; 1000 balls of each of 10 different colours, and a group of children mixing the balls continuously. The state where the ball pit is unmixed is only a single one out of  $10^{1000}$  different realisations! (Unmixed means that each ball is situated in one corner of the ball pit assigned to its colour.) If the children playing in the ball pit stir the balls continuously and if there is a new realisation every second, one can expect a spontaneous unmixing in  $10^{1000}$  seconds. As the Universe is "only"  $10^{17}$  seconds old, one would need to wait  $10^{983}$  times the [age of the Universe](#) for this to occur. And a system of  $10^4$  particles is really nothing compared to [Avogadro's number](#)  $N_A = 6.02214076 \times 10^{23}/\text{mol}$ !

### G.11 Information entropies

The fundamental postulate of statistical mechanics, namely that in thermal equilibrium all states of a given energy are equally probable to be assumed by the system and that states of different energy are weighted relative to each other with the Boltzmann-factor  $\exp(-\beta(\epsilon_2 - \epsilon_1))$  opened the way to a fundamental microscopic theory behind thermodynamics. We can ask the question whether the fundamental postulate can be motivated. You might have already guessed that this is the case: Thermal equilibrium could be characterised by making the least assumption about the system in the sense that the random process that distributes the system among its possible states is as random and non-committal as possible.

For a given discrete random distribution  $p_i$  one can define the Shannon-entropy  $S$

$$S = - \sum_i p_i \ln p_i \quad (\text{G.377})$$

as a measure of randomness. Shannon's entropy  $S$  has these properties:

- $S \geq 0$ , because the overall minus-sign takes care of the fact that  $p_i \leq 1$ , and the entropy is bounded from below.
- $S = 0$  if one of the  $p_i = 1$ . Because  $\sum_i p_i = 1$ , the other  $p_i$  need to be zero, and there is no randomness involved.
- For equally probable outcomes,  $p_i = 1/n$  and consequently  $S = \ln n$ , such that the entropy increases with the number of possible outcomes.

- S is additive for statistically independent events,  $p_{ij} = p_i p_j$  implies

$$S = - \sum_{ij} p_{ij} \ln p_{ij} = - \sum_i \sum_j p_i p_j \ln(p_i p_j) = - \sum_j p_j \sum_i p_i \ln p_i - \sum_i p_i \sum_j p_j \ln p_j = S_i + S_j \quad (\text{G.378})$$

- Shannon’s entropy is maximal for equally probable outcomes. Think of S as a functional dependent on the set of probabilities  $p_i$ , so that variation of S would determine  $p_i$ , under the condition that  $\sum_i p_i = 1$  is met, which can be incorporated with a Lagrange multiplier  $\lambda$ :

$$S = - \sum_i p_i \ln p_i - \lambda \left( \sum_i p_i - 1 \right) \quad (\text{G.379})$$

such that the variation  $\delta S$  with respect to  $p_i$  becomes

$$\delta S = - \sum_i \delta p_i \ln p_i + p_i \frac{1}{p_i} \delta p_i - \lambda \sum_i \delta p_i = - \sum_i (\ln p_i + 1 + \lambda) \delta p_i = 0 \quad (\text{G.380})$$

such that  $p_i$  is constant with the value  $\exp(-(1 + \lambda))$ , where  $\lambda$  can be fixed with the boundary condition  $\sum_i p_i = 1$ .

In particular the last point suggests already now a uniform distribution in maximisation of the information entropy S.

The generalisation of Shannon’s entropy to a continuum of outcomes is very interesting but contains a few dangerous spots: For a probability density  $p(x)dx$  one can define

$$S = - \int dx p(x) \ln p(x) \quad (\text{G.381})$$

which shares with the definition eqn. G.377 for the discrete case the value  $S = 0$  for the certain outcome, is additive for independent events and is proportional to the logarithm of the interval length  $b - a$  for a uniform distribution, which incidentally maximises S, too. But it is not bounded from below by zero, which can be seen in the example of the uniform distribution:  $S = \ln(b - a)$  can assume arbitrarily negative values if the interval size  $b - a$  is small enough.

Perhaps even more importantly, S changes under transforms of the random variable:  $p(x)dx = p(y)dy$  as a transformation law is suggested by integration by substitution,

$$\int dx p(x) = \int dy \frac{dx}{dy} p(x(y)) \quad (\text{G.382})$$

so that  $p(x)dx$  is invariant, but  $\ln p(x)$  is not as it becomes  $\ln p(y) + \ln dx/dy$ . This, however, does not play a role in relative information entropies like the Kullback-Leibler-divergence

$$\Delta S = - \int dx p(x) \ln \frac{p(x)}{q(x)} \quad (\text{G.383})$$

which quantify the relative amount of randomness between the distributions  $p(x)$  and  $q(x)$ : The transformation Jacobian drops out in the ratio  $p(x)/q(x)$ .





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## INDEX

age of the Universe, 77

Avogadro's number, 77

Gay-Lussac's law, 68

Lovelock's theorem, 65

molar volume, 68

standard conditions, 68

thermal expansion coefficient, 68

triple point of water, 69

Theoretical physics is commonly taught in separate lectures, illustrating the physics behind the great constants of Nature: Electrodynamics and the speed of light, quantum mechanics and Planck's constant, thermodynamics and Boltzmann's constant, and finally relativity with Newton's constant as well as the cosmological constant. In these lecture notes, the concepts of theoretical physics are illustrated with their commonalities, and phenomena are traced back to their origin in fundamental concepts.

### **About the Author**

Björn Malte Schäfer works at Heidelberg University on problems in modern cosmology, relativity, statistics, and on theoretical physics in general.



**UNIVERSITÄT  
HEIDELBERG**  
ZUKUNFT  
SEIT 1386