H COSMIC STRUCTURE FORMATION

H.1 *Structure formation equations*

Structure formation with cold dark matter is driven by self-gravity of cosmic structures that have been seeded by cosmic inflation as inhomogeneities in the density field. At the highest degree of simplification, the dark matter density is subjected to fluid mechanics but without effects of pressure and viscosity (as they would derive from the microscopic interactions between the particles). While the background on which structure formation takes place, is a dynamics spacetime conforming to the FLRW-symmetries, structure formation is well captured in the Newtonian limit, with both Newtonian gravity in the form of a potential Φ, $|\Phi| \ll c^2$ and with non-relativistic velocities $|\upsilon| \ll c$ in the comoving frame.

The formation of cosmic structure is a phenomenon that only involves weak, Newtonian gravitational fields, slowly moving matter and scales much smaller than the Hubble scale. Therefore, we are going to use a Newtonian description of gravity on the relativistic FLRW-background, a nonrelativistic equation of motion and neglect retardation effects due to the finite propagation speed of the gravitational field as well as gravitative effects on moving objects such as gravitomagnetic forces.

As coordinates, we use the conformal time η and comoving coordinates x^i as those coordinates are particularly suited for FLRW-spacetimes, implying that the rate of change of physical coordinate $r = ax$ with physical time gives rise to two contributions in velocity:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \dot{a}x + a\dot{x} = a\mathrm{H}x + a\dot{x} = a\mathrm{H}x + \upsilon \tag{H.388}$$

with the peculiar velocity υ relative to the Hubble flow. Clearly, both terms would contribute to a measurement of redshift. The peculiar velocity υ would likewise be the rate of change of comoving coordinate with conformal time,

$$\upsilon = a\dot{x} = a\underbrace{\frac{\mathrm{d}\eta}{\mathrm{d}t}}_{=1/a}\frac{\mathrm{d}x}{\mathrm{d}\eta} = \frac{\mathrm{d}x}{\mathrm{d}\eta} \tag{H.389}$$

Comoving coordinates have the advantage that the advection of matter due to the Hubble-expansion is absorbed by the coordinates, and we only need to consider relative motion of particles with respect to the comoving coordinate frame. Being a hydrodynamical self-gravitating phenomenon, structure formation is described in the this comoving frame by the system of differential equations composed of (i) the continuity equation

$$\frac{\partial}{\partial\eta}\delta + \mathrm{div}\left[(1+\delta)\boldsymbol{\upsilon}\right] = 0, \tag{H.390}$$

which relates the time-evolution of the density field to the divergence of the matter fluxes $\boldsymbol{\jmath} = (1+\delta)\boldsymbol{\upsilon}$, (ii) the Euler-equation

$$\frac{\partial}{\partial\eta}\boldsymbol{\upsilon} + a\mathrm{H}\boldsymbol{\upsilon} + (\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon} = -\nabla\Phi, \tag{H.391}$$

which describes the evolution of the peculiar velocity field $\boldsymbol{\upsilon}$ from the gradient $\nabla\Phi$

of the peculiar gravitational potential Φ, acting on a fluid element, and finally (*iii*) the comoving Poisson-equation

$$\Delta\Phi = \frac{3}{2}\Omega_m(\eta)\,(a\mathrm{H})^2\delta = \frac{3\mathrm{H}_0^2\Omega_m}{2a}\delta, \tag{H.392}$$

which gives the gravitational potential Φ induced by the matter distribution δ (Newton's constant has been replaced with the definition of the critical density, $\rho_{\text{crit}} = 3\mathrm{H}_0^2/(8\pi\mathrm{G})$ and the density parameter $\Omega_m = \bar{\rho}/\rho_{\text{crit}}$. In the last step, we used the adiabatic relation

$$\frac{\Omega_m(a)}{\Omega_m} = \frac{\mathrm{H}_0^2}{a^{3(1+w)}\mathrm{H}(a)^2} \tag{H.393}$$

while setting $w = 0$ for nonrelativistic matter.

The three equations are sufficient to describe the dynamics of the three relevant fields δ, $\boldsymbol{\upsilon}$ and Φ, because there are no dissipative and pressure forces due to the collisionlessness of dark matter, and it is not necessary to track the energy balance or to introduce and an equation of state parametrising the pressure-density relation.

H.2 *Linearised equations on an expanding background*

Linearisation of the structure formation equations by substituting a perturbative expansion and neglecting all terms involving products of two or more fields. This methods yields the linearised continuity equation,

$$\frac{\partial}{\partial\eta}\delta + \mathrm{div}\boldsymbol{\upsilon} = 0, \tag{H.394}$$

and the linearised Euler-equation,

$$\frac{\partial}{\partial\eta}\boldsymbol{\upsilon} + a\mathrm{H}\boldsymbol{\upsilon} = -\nabla\Phi, \tag{H.395}$$

which are valid as long as the deviation from the mean density is small, $|\delta| \ll 1$. The Newtonian Poisson-equation is always linear, or the superposition principle of classical gravity would not apply.

The three linearised relationships between δ, $\boldsymbol{\upsilon}$ and Φ can be combined into the growth-equation: By taking the divergence of the Euler-equation and the time-derivative of the continuity-equation one can eliminate $\partial\mathrm{div}\boldsymbol{\upsilon}/\partial\eta$ and re-substitute the continuity equation to obtain an expression

$$\frac{\partial^2}{\partial\eta^2}\delta + a\mathrm{H}\frac{\partial}{\partial\eta}\delta = \Delta\Phi \tag{H.396}$$

where, after substitution of the Poisson-equation for $\Delta\Phi$ all spatial derivatives have vanished. This implies that structure growth in the linear regime is homogeneous and can not depend on position. It merely scales all amplitudes in the density field with a factor that only depends on time, $\delta(\mathbf{x},\eta) = \mathrm{D}_+(a)\delta(\mathbf{x},\eta = 0)$, and this factor is commonly referred to as the growth function $\mathrm{D}_+(a)$.

One can continue to replace the time derivatives with respect to conformal time η by derivatives with respect to the scale factor a to obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2}D_+(a) + \frac{1}{a}\left(3 + \frac{\mathrm{d}\ln H}{\mathrm{d}\ln a}\right)\frac{\mathrm{d}}{\mathrm{d}a}D_+(a) = \frac{3}{2a^2}\Omega_m(a)D_+(a). \tag{H.397}$$

where the dependence on the background cosmology is clearer, and reflects the change of the Hubble function, i.e. acceleration or deceleration in the Hubble rate $H(a)$ as well as the change of the background matter density with time. The homogeneity of the growth is the reason why e.g. inflationary models of structure formation can be investigated by observations of the statistical properties of the large-scale structure today: Even though inflation takes place at incredibly high redshifts of $z \simeq 10^{30}$, the cosmic structure is conserving the density field perfectly as long as it is linearly evolving.

Homogeneous structure formation corresponds to independently growing Fourier modes,

$$\delta(\mathbf{x}, a) = D_+(a)\delta(\mathbf{x}, a = 1) \longrightarrow \delta(\boldsymbol{k}, a) = D_+(a)\delta(\boldsymbol{k}, a = 1), \tag{H.398}$$

which conserves every statistical property of the initial conditions, in particular Gaussianity. The Gaussianity of the initial density perturbations is a consequence of inflation, where a large number of uncorrelated quantum fluctuations are superimposed, yielding a Gaussian amplitude distribution due to the central limit theorem. In fact, homogeneous growth in the linear regime is the reason why investigation of inflationary processes in structure is possible by observing the large-scale structure today, even after the cosmic time $1/H_0$ has passed.

A convenient way for approximating the growth function is the γ-parameter, introduced by in the study of peculiar velocities:

$$\frac{\mathrm{d}\ln D_+}{\mathrm{d}\ln a} \simeq \Omega_m(a)^\gamma, \tag{H.399}$$

with $\gamma \simeq 0.6$ in ΛCDM. Solving this equation for the growth function yields

$$D_+(a) = \exp\left(\int_0^a \mathrm{d}\ln a\, \Omega_m(a)^\gamma\right). \tag{H.400}$$

In dynamic dark energy models, γ can be approximated by $\gamma \simeq 0.55 + 0.05(1 + w(z = 1))$ with the dark energy equation of state parameter taken at unit redshift. The effect of adding a fluid with a negative equation of state is a slower growth in the recent cosmic past and a faster growth in the remote past (if the growth function is normalised to unity today). Solutions for $D_+(a)$ for different dark energy cosmologies are compared in Fig. 10.

H.3 *Peculiar velocity field*

Matter streams in the large-scale structure drive structure formation: If they converge, they transport matter into a volume and increase the local density, according to the continuity equation. In order to investigate the properties of the velocity field one can carry out a Helmholtz-decomposition into its curl and gradient components $\theta = \mathrm{div}\boldsymbol{\upsilon}$ and $\boldsymbol{\omega} = \mathrm{rot}\boldsymbol{\upsilon}$. From the Euler-equation one obtains and evolution equation for the divergence of the matter fluxes ,

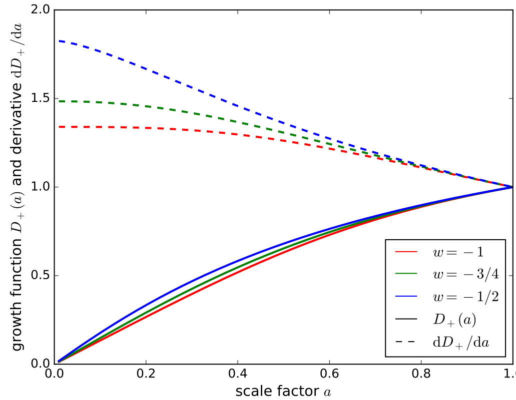

Figure 10: Growth functions $D_+(a)$ *for different dark energy cosmologies, as well as the derivative* dD_+/da

$$\frac{\partial}{\partial \eta}\theta + aH\theta + \frac{3H_0^2\Omega_m}{2a}\delta = 0 \tag{H.401}$$

and the corresponding equation for the vorticity ω,

$$\frac{\partial}{\partial \eta}\omega + aH\omega = 0. \tag{H.402}$$

With the definition of the differential of the conformal time, $da = a^2 H d\eta$, one immediately notices that $d \ln \omega = -d \ln a$, and hence $\omega \propto 1/a$ in the matter dominated phase: Vorticity can not be generated in linear structure formation in collisionless fluids, and the flows are necessarily laminar. The divergence θ can be linked to the evolution of the density field using the continuity equation,

$$\theta = -aH\frac{d \ln D_+}{d \ln a}\delta, \tag{H.403}$$

which underlines the fact that in the linear regime of structure formation, the velocity field is the gradient of a potential. At the same time, eqn. H.403 suggests that a natural scale for the velocity divergence is the comoving Hubble-rate aH.

H.4 *Linear structure formation*

The linear growth equation is given by

$$\frac{d^2D}{da^2} + \frac{1}{a}\left(3 + \frac{d \ln H}{d \ln a}\right)\frac{dD}{da} = \frac{3}{2a^2}\Omega_m(a)D(a) = 0. \tag{H.404}$$

Therefore, linear cosmic structure formation is governed by magnitude and time evolution of two terms: the density of matter as given by $\Omega_m(a)$ and the term $3 + d \ln H/d \ln a$ describing a change in the expansion rate. This latter term is sometimes referred to as Hubble-drag, but although the interpretation as a drag term is formally correct it does not represent the physical picture correctly. In particular it would be wrong to formulate a time-scale for Hubble expansion $1/H(t)$ and compare it to a time scale $t = 1/\sqrt{G\rho}$ because the structure in the overdensity field δ are invariant in shape and amplitude under Hubble-expansion as both densities $\rho(\boldsymbol{x}, a)$ and $\bar{\rho}$ scale identically $\propto a^{-3}$. The relevant physical mechanism is an acceleration of matter relative to the Hubble expansion and a change in the expansion velocity, i.e. an acceleration or deceleration in the cosmological model. This is apparent when writing the growth equation with e.g. the scale factor a as an evolution parameter. In this case, the Hubble-drag term reflects a derivative of the Hubble-expansion with a, and the term $3 + \mathrm{d}\ln H/\mathrm{d}\ln a$ is in fact equal to $2 - q$, with the deceleration parameter $q = -\ddot{a}a/\dot{a}^2$.

Linear structure formation is scale invariant, at a rate determined purely by the FLRW-cosmology through q and H, which determines the evolution of Ω_m and hence of the strength of gravitational fields through the relation

$$\frac{\Omega_m(a)}{\Omega_m} = \frac{H_0^2}{a^3 H(a)^2} \tag{H.405}$$

as a consequence of the continuity equation for normal matter with $w = 0$, which itself is a consequence of conserved 4-momentum $\nabla_\mu T^{\mu\nu} = 0$. As such, it allows the investigation of the the cosmological model through the Hubble function and its derivative if measurements of the amplitude of structures as a function of scale factor or redshift are available. Redshift information is crucial because the same amplitude of cosmic structures is reached in different cosmologies at different times, and this information would be impossible to disentangle without redshift information.

The influence of the two terms $3 + \mathrm{d}\ln H/\mathrm{d}\ln a$ and $2 - q$ on the growth equation are straightforward to understand in the context of standard cosmologies with two relevant fluids, with dark matter dominating at early and dark energy dominating at late times. In these cosmologies the universe makes a transition from deceleration to acceleration, which is reflected by the growth rate $D(a)$. During matter domination, the Hubble function scales $H \propto a^{-3/2}$ which transitions in the course of cosmic evolution to dark energy domination, where in the extreme case of a cosmological constant $H = \mathrm{const}$. The derivative $3 + \mathrm{d}\ln H/\mathrm{d}\ln a$ would change from 3/2 at early times to 3 at late times, therefore slowing down structure formation. A similar behaviour is found in the matter density, which starts at the value $\Omega_m = 1$ in matter domination and drops to 0 when the dark energy component dominates. In summary, there are now two reasons why structure formation stops at late times under the influence of a cosmological constant: The driving term involving Ω_m, which originates from the Poisson-equation, becomes very small and the damping term $3+\mathrm{d}\ln H/\mathrm{d}\ln a$ assumes the largest possible value.

There are certain cosmologies, where the growth equation has particularly simple solutions. For instance, in a critical FLRW-universe with a constant $\Omega_m = 1$ requires $D(a) = a$. By substitution into the comoving Poisson-equation one immediately sees that the Newtonian potentials Φ scale with D_+/a and are in this particular cosmological model constant in linear structure formation.

Therefore, structures grow proportional to the scale factor. For a general cosmology one can at least infer the asymptotic behaviour by making a power law ansatz for D as a function of scale factor at early times, $D \propto a^\alpha$ and consider solution to the resulting quadratic equation in α, while the exact solution for an arbitrary cosmology defined in terms of $H(a)$ or $q(a)$, or, in terms of the density parameters and their equations of state, is only possible numerically. It is sufficient to formulate the ansatz as a proportionality $D \propto a^\alpha$ because the growth equation is a linear differential equation. Physically, this means that structure growth continues irrespective of the amplitudes of the density field.

To begin, we consider the entire linear growth equation again in the $\Omega_m = 1$-cosmology, which yields as a characteristic polynomial $\alpha^2 + \alpha/2 - 3/2 = 0$, which is solved by $\alpha_+ = 1$ and $\alpha_- = -3/2$: The growth is proportional to the scale factor, as already found by direct substitution, $D_+(a) \propto a$, with a secondary solution $D_-(a) \propto a^{-3/2}$: Due to the fact that the growth equation is of second order in a one expects two solution branches, which need to be combined by linear combination with suitable coefficients such that the boundary condition $D(a) = 1$ at $a = 1$ is met. Usually one neglects the branch $D_-(a)$ because it decreases rapidly.

In addition, it is possible to illustrate the behaviour of the growth equation of individual terms are set to zero and are therefore disfunctional. For instance, the growth in a cosmology with an arbitrary but constant deceleration parameter q, but where gravity in structure formation has been switched off leads with the same ansatz $D(a) \propto a^\alpha$ to a characteristic polynomial $\alpha(\alpha + 1 - q) = 0$ with the two solutions $D_+ = \text{const}$ for $\alpha = 0$ and $D_- \propto a^{q-1}$. Taking this to extremes, the dark energy dominated universe with $q = -1$ and $\Omega_m = 0$ has $\alpha(\alpha + 2) = 0$, implying a constant growing mode $D_+ = \text{const}$ and a fast decaying mode $D_+ \propto a^{-2}$: structure growth is frozen and the amplitudes reached at the point of dark energy domination are conserved from that point on.

Conversely, in an artificial inconsistent universe with a constant expansion rate (vanishing deceleration $q = 0$) and gravitational fields generated by the large-scale structure with $\Omega_m = 1$ one would obtain $\alpha + \alpha - 3/2 = 0$, with the solutions $\alpha = (-1 \pm \sqrt{7})/2$ with a growing $\alpha > 1$ and a decaying solution $\alpha < 1$. Clearly, this is the prototype solution to the differential equation, where the two solutions are modified in any consistent cosmology relative to their actual deceleration and matter density, including their evolution.

H.5 *Nonlinear structure formation*

As long the structure formation is linear, the growth is homogeneous and conserves the Gaussianity of the initial conditions. Nonlinear structure formation implies inhomogeneous growth and the emergence of non-Gaussian features, which is illustrated by a number of arguments: Non-linearity implies inhomogeneity, because if e.g. a void reaches underdensities close to $\delta \simeq -1$ (corresponding to $\rho \simeq 0$), the linearisation fails and the growth has to slow down locally. Inhomogeneity implies non-Gaussianity because the initially Gaussian distribution $p(\delta)\mathrm{d}\delta$ becomes wider with increasing amplitudes δ, but the density δ can not be more negative than -1, requiring the amplitude distribution $p(\delta)\mathrm{d}\delta$ to become asymmetric and to acquire a nonzero skewness. For completing the argument one immediately notices that in inhomogeneous growth, i.e. a position dependence of the growth rate $D_+(\boldsymbol{x}, a)$, the Fourier-modes $\delta(\boldsymbol{k}, a)$ become coupled, violating the central limit theorem such that the superposition of Fourier-modes yields a non-Gaussian amplitude distribution.

- linearity $\leftrightarrow$ homogeneity
    - There are no spatial derivatives in the growth equation, and therefore, the growth must be homogeneous $\delta(\boldsymbol{x}, a) = D_+(a)\delta(\boldsymbol{x})$. Only nonlinear terms would bring in spatial derivatives and make the growth position dependent.

- If the density field is close to $\delta = -1$ somewhere, the growth needs to slow down locally, which leads to different structure formation rates at different positions which eventually breaks homogeneity.

- linearity $\leftrightarrow$ Gaussianity
  - Linear growth introduces a scaling with a function D_+ which itself is a linear transform and therefore preserves statistical properties.
  - Again, if δ approaches -1, the initially Gaussian distribution starts to become asymmetric, as it generates potentially very large positive values for δ but has to be zero for $\delta < -1$.

- homogeneity $\leftrightarrow$ Gaussianity
  - Homogeneous growth $\delta(\boldsymbol{x}, a) = D_+(a)\delta(\boldsymbol{x})$ implies independent growth of all Fourier-modes $\delta(\boldsymbol{k}, a) = D_+(a)\delta(\boldsymbol{k})$, as the Fourier-transform is linear. If a large amount of statistically independent Fourier-modes is superimposed (by inverse Fourier-transform), the resulting δ is a Gaussian distribution.
  - Inhomogeneous growth $\delta(\boldsymbol{x}, a) = D_+(\boldsymbol{x}, a)\delta(\boldsymbol{x})$ results in a convolution in Fourier-space

$$\delta(\boldsymbol{k}, a) = \int \frac{d^3k'}{(2\pi)^3} D_+(\boldsymbol{k} - \boldsymbol{k}', a)\delta(\boldsymbol{k}') \tag{H.406}$$

with a position-dependent growth rate, which breaks the statistical independence by coupling Fourier-modes. Then, the resulting distribution can not be Gaussian anymore.

H.6 *Eulerian perturbation theory*

The non-linearities in the continuity and Euler-equation make a closed analytical solution impossible. It is possible, however, to obtain a perturbative solution to the structure formation equations, which contains the mode coupling mechanism and describes the generation of non-Gaussianities in nonlinear structure formation. The non-linearities in the continuity- and the Euler-equation translate to convolutions of the density and the velocity fields in Fourier space which couple the individual Fourier modes, violating the central limit theorem and therefore violating Gaussianity. It is worth noting that in the perturbative expansion each field $\delta^{(n)}$ grows homogeneously at the rate $D_+^n(a)$, but the sum does not.

Applying a perturbative solution means to write out perturbation series for δ and Θ in terms of powers of the linear solutions

$$\delta(\boldsymbol{x}, t) = \sum_n \delta^{(n)}(\boldsymbol{x}, t) \quad \text{and} \quad \Theta(\boldsymbol{x}, t) = \sum_n \Theta^{(n)}(\boldsymbol{x}, t) \quad \text{where} \quad \Theta = \frac{\mathrm{div}\boldsymbol{\upsilon}}{aH} \tag{H.407}$$

and substituting them into the fully nonlinear equations:

$$\partial_\tau \delta + \mathrm{div}((1 + \delta)\boldsymbol{\upsilon}) = 0 \quad \text{and} \quad \partial_\tau \boldsymbol{\upsilon} + aH\boldsymbol{\upsilon} + (\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon} = -\nabla\Phi \tag{H.408}$$

where the comoving divergence is computed for the second equation. Differential equations become algebraic in Fourier space, therefore continuity reads

$$\partial_\tau\delta(\boldsymbol{k}) + \Theta(\boldsymbol{k}) = -\int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\int\frac{\mathrm{d}^3k_2}{(2\pi)^3}\Theta(\boldsymbol{k}_1)\delta(k_2)\delta_\mathrm{D}(\boldsymbol{k}-\boldsymbol{k}_{12})\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2) \tag{H.409}$$

and similarly, the Euler-equation becomes

$$\partial_\tau\Theta(\boldsymbol{k})+a\mathrm{H}\Theta(\boldsymbol{k})+\frac{3}{2}\Omega_m(a\mathrm{H})^2\delta(\boldsymbol{k}) = -\int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\int\frac{\mathrm{d}^3k_2}{(2\pi)^3}\Theta(\boldsymbol{k}_1)\Theta(\boldsymbol{k}_2)\delta_\mathrm{D}(\boldsymbol{k}-\boldsymbol{k}_{12})\beta(\boldsymbol{k}_1,\boldsymbol{k}_2) \tag{H.410}$$

keeping in mind that products in real space become convolutions in Fourier-space, here expressed by introducing the Dirac-δ_D function. The derivatives are expressed with

$$\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{1}{k_1^2}\boldsymbol{k}_{12}\boldsymbol{k}_1 \tag{H.411}$$

as well as

$$\beta(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{1}{2k_1^2k_2^2}\boldsymbol{k}_{12}\boldsymbol{k}_1\boldsymbol{k}_2 \tag{H.412}$$

with the abbreviation $\boldsymbol{k}_{12} = \boldsymbol{k}_1 + \boldsymbol{k}_2$. Substitution of the perturbation series yields a recursive relation

$$\delta_n(\boldsymbol{k}) = \int\mathrm{d}^3q_1\ldots\int\mathrm{d}^3q_n\delta_\mathrm{D}(\boldsymbol{k}-\boldsymbol{q}_{1\ldots n})\mathrm{F}_n(\boldsymbol{q}_1\ldots\boldsymbol{q}_n)\delta_1(\boldsymbol{q}_1)\ldots\delta_n(\boldsymbol{q}_n) \tag{H.413}$$

for the density field, as well as

$$\Theta_n(\boldsymbol{k}) = \int\mathrm{d}^3q_1\ldots\int\mathrm{d}^3q_n\delta_\mathrm{D}(\boldsymbol{k}-\boldsymbol{q}_{1\ldots n})\mathrm{G}_n(\boldsymbol{q}_1\ldots\boldsymbol{q}_n)\delta_1(\boldsymbol{q}_1)\ldots\delta_n(\boldsymbol{q}_n) \tag{H.414}$$

for the velocity divergence. Here, F_n is a function of $\mathrm{F}_n(\mathrm{F}_{n-1},\mathrm{G}_{n-1})$ and the same for G_n, all defined inductively starting at $\mathrm{F}_1 = \mathrm{G}_1 = 1$.

The lowest order symmetrised solutions for F_n are $\mathrm{F}_1 = 1$ and

$$\mathrm{F}_2(\boldsymbol{q}_1,\boldsymbol{q}_2) = \frac{5}{7}+\frac{\mu}{2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right)+\frac{2}{7}\mu^2 \quad\text{with}\quad \mu = \frac{\boldsymbol{q}_1\cdot\boldsymbol{q}_2}{q_1q_2} \tag{H.415}$$

being the cosine of the angle between $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Assuming $q_1 = q_2$ for simplicity, the mode coupling function F_2 attains the largest value of $\mathrm{F}_2 = 2$ if the wave vectors are parallel ($\mu = +1$), an intermediate value of $\mathrm{F}_2 = 5/7$ if $\boldsymbol{q}_1 \perp \boldsymbol{q}_2$ ($\mu = 0$) and the smallest value of $\mathrm{F}_2 = 0$ if the the wave vectors are antiparallel ($\mu = -1$). Varying the wave numbers at fixed separation angle μ shows that F_2 is smallest if $q_1 = q_2$, and that the mode coupling increases if the wave numbers are chosen differently. From this point of view, mode-coupling bears resemblance to a resonance phenomenon, where modes with identical direction of propagation experience the strongest coupling. The perturbative solution to the system of equations eqns. (H.390) and (H.391) in terms of a perturbation series in δ and υ is possible due to their renormalisation properties, which hold exactly in the case of SCDM ($\Omega_m = 1$, $\Omega_\varphi = 0$) and approximately for dark energy cosmologies. In these cosmologies, the mode coupling kernels themselves acquire a slow time dependence.

In application to statistics, any correlation function of nonlinear fields can reduced to a higher-order correlation function of the linearly evolving fields, which obey Gaussian statistics, integrated over momentum space with the mode coupling function as a weighting function. While odd n-point correlation functions of Gaussian random fields are equal to zero, even n-point functions can be decomposed into products of two-point functions by virtue of the Wick-theorem,

$$\langle\delta(\boldsymbol{k}_1)\ldots\delta(\boldsymbol{k}_n)\rangle = \sum_{\text{pairs}}\prod_{i,j\in\text{pairs}}\langle\delta(\boldsymbol{k}_i)\delta(\boldsymbol{k}_j)\rangle \tag{H.416}$$

for which a proof can be found in e.g. and which constitutes an extension of the well-known relation $\langle\delta^{2n}\rangle = (2n-1)!!\langle\delta^2\rangle^n$ for the higher moments of a Gaussian random variable δ with $\langle\delta\rangle = 0$.

H.7 *Dark matter in astrophysical systems*

With the idea, that all forms of matter, including dark matter, are effected in the same way by gravity as commanded by the equivalence principle of general relativity one would conclude in a range of astrophysical system that the strength of gravitational field can not be explained by luminous matter alone.

H.7.1 Rotation curves of galaxies

Setting up circular orbits for stars in a galactic disk in the gravitational potential of a galaxies would use the condition

$$\frac{\upsilon^2}{r} = \frac{\mathrm{d}\Phi}{\mathrm{d}r} \quad\rightarrow\quad \upsilon^2 = r\frac{\mathrm{d}\Phi}{\mathrm{d}r} \tag{H.417}$$

The gravitational potential Φ would result from solving the Poisson-equation for the total matter density ρ

$$\Delta\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho \tag{H.418}$$

With a matter profile $\rho \propto 1/r^2$ one would obtain, after multiplying with r^2, integrating and multiplying with $\frac{1}{r^2}$ the result

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = 4\pi G\frac{1}{r^2}\int \mathrm{d}r\,\frac{1}{r^2}r^2 = \frac{4\pi G}{r} = \frac{\upsilon^2}{r} \tag{H.419}$$

implying that the rotational velocity υ does not depend on radius r anymore, suggesting the idea that the galactic disc is embedded into a much larger dark matter halo with density $\rho \propto 1/r^2$, which sources the gravitational potential, and which naturally reproduces the observed flat rotation curves.

There are theories that modify dynamical laws in the regimes of really small accelerations which can reproduce galaxy rotation curves even if the gravitational potential is sourced by the visible matter only. The scale at which these MO(dified) N(ewtonian) D(ynamics) theories change the equations of motion is for accelerations close to $a_0 \simeq 10^{-10}\mathrm{m/s}^2$, which corresponds to the acceleration experienced by the Solar System on its orbit around the Milky Way centre. An example of a rotation curve in a low surface-brightness galaxy is provided by Fig. 11.

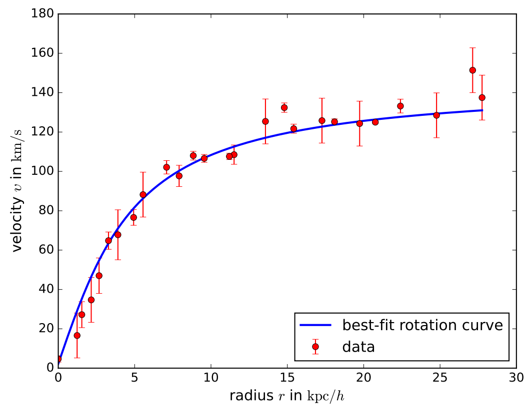

Figure 11: Rotation curve of the galaxy U11616 with an fit to the rotational velocity as a function of radius

H.7.2 Virial equilibria of clusters of galaxies

On the scale of galaxy clusters one again notices similar mismatch: The velocities of the galaxies are too large to be compatible with the gravitational potential if only visible matter should contribute to it. From the positions $\boldsymbol{q}_i$ and the momenta $\boldsymbol{p}_i$ of all galaxies in a cluster one defines the virial G,

$$G = \sum_i \boldsymbol{p}_i \boldsymbol{q}_i \tag{H.420}$$

with the time deriative

$$\frac{dG}{dt} = \sum_i \frac{d\boldsymbol{p}_i}{dt}\boldsymbol{q}_i + \boldsymbol{p}_i\frac{d\boldsymbol{q}_i}{dt} = \sum_i \mathbf{F}_i \boldsymbol{q}_i + m\sum_i \dot{\boldsymbol{q}}_i\dot{\boldsymbol{q}}_i \tag{H.421}$$

where Newton's equation of motion $d\boldsymbol{p}_i/dt = \mathbf{F}_i$ and the definition of momentum $\boldsymbol{p}_i = m\dot{\boldsymbol{q}}_i$ was substituted. Particularly in systems with potentials of power-law shape allow a very compact statement: If Φ is the mutual interaction potential of the particle j onto particle i

$$\Phi(\boldsymbol{q}_i, \boldsymbol{q}_j) \propto |\boldsymbol{q}_i - \boldsymbol{q}_j|^n \tag{H.422}$$

one can find

$$\sum_i \mathbf{F}_i \boldsymbol{q}_i = -\frac{1}{2}\sum_i\sum_j \frac{d\Phi(\boldsymbol{q}_i, \boldsymbol{q}_j)}{dq_{ij}}\frac{|\boldsymbol{q}_i - \boldsymbol{q}_j|^2}{q_{ij}} = -\frac{1}{2}\sum_i\sum_j \frac{d\Phi(\boldsymbol{q}_i, \boldsymbol{q}_j)}{dq_{ij}} q_{ij} \tag{H.423}$$

and in particular for homogeneous potentials of order n that

$$\frac{1}{2}\sum_i\sum_j \frac{d\Phi(\boldsymbol{q}_i, \boldsymbol{q}_j)}{dq_{ij}} q_{ij} = \frac{n}{2}\sum_i\sum_j \Phi(\boldsymbol{q}_i, \boldsymbol{q}_j) = \frac{n}{2}V \tag{H.424}$$

with $\mathrm{V} = \sum_i \sum_j \Phi(\boldsymbol{q}_i, \boldsymbol{q}_j)$ and $\mathrm{T} = m/2 \sum_i \boldsymbol{q}_i^2$. We can therefore conclude that

$$\frac{\mathrm{dG}}{\mathrm{d}t} = 2\mathrm{T} - n\mathrm{V} \tag{H.425}$$

The time averaging yields

$$\left\langle \frac{\mathrm{dG}}{\mathrm{d}t} \right\rangle = \frac{1}{\Delta t} \int_0^{\Delta t} \mathrm{d}t\, \frac{\mathrm{dG}}{\mathrm{d}t} \leq \frac{1}{\Delta t} |\mathrm{G}_{\mathrm{max}} - \mathrm{G}_{\mathrm{min}}| \tag{H.426}$$

if G has a finite range of values, typically realised for systems bounded in the phase space coordinates, the average vanishes in the limit $\Delta t \to \infty$, and therefore the virial theorem applies,

$$2\langle \mathrm{T} \rangle = n \langle \mathrm{V} \rangle \tag{H.427}$$

For Newtonian gravity with a Coulomb-potential we insert $n = -1$ and get

$$\langle \mathrm{T} \rangle = -\frac{1}{2} \langle \mathrm{V} \rangle \tag{H.428}$$

as well as a negative total energy

$$\langle \mathrm{T} \rangle + \langle \mathrm{V} \rangle = \langle \mathrm{T} \rangle - 2\langle \mathrm{T} \rangle = -\langle \mathrm{T} \rangle < 0 \tag{H.429}$$

indicating a bound system. $\langle \mathrm{T} \rangle$ can be measured from the velocity of the galaxies inside the cluster and the potential $\langle \mathrm{V} \rangle$ can be determined from the total mass and size, typically $\langle \mathrm{V} \rangle \sim \mathrm{M}/\mathrm{R}$. Observations, either of the peculiar velocity of galaxies and a mass estimate based on luminosity, or of X-ray temperature and luminosity, show a striking mismatch between data and theory and one would need a n of a few hundred to reconcile $\langle \mathrm{T} \rangle$ with $\langle \mathrm{V} \rangle$, which is clearly at odds with Newtonian gravitational potentials, or alternatively, that there is much more gravitating matter present in these systems compared to luminous matter.

H.7.3 Gravitational lensing

Substituting non-relativistic particles with relativistic photons for probing gravitational potentials leads to the topic of gravitational lensing. Photons travel along null-geodesics of spacetime, which would be lines with vanishing $\mathrm{d}s^2$ for instance on a Minkowski-background

$$\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right) \mathrm{d}x_i \mathrm{d}x^i = 0 \tag{H.430}$$

slightly curved by a (static) gravitational potential $|\Phi| \ll c^2$. It is sufficient to work with a perturbed Minkowski-metric instead of a FLRW-metric because of conformal flatness of the background: With a suitable choice of conformal time as a coordinate light propagation is impervious to the background dynamics and identical to that in special relativity.

The effective speed of propagation of light is the rate at which the coordinates pass as a function of time,

$$c' = \frac{\mathrm{d}|x^i|}{\mathrm{d}t} = c\sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \simeq c\left(1 - \frac{2\Phi}{c^2}\right) \tag{H.431}$$

where we used the approximation $1/(1+\epsilon) \approx 1 - \epsilon$ for $|\epsilon| \ll 1$. With $c' \neq c$ it is suggestive to define a refractive index

$$n = \frac{c'}{c} \approx 1 - \frac{2\Phi}{c^2} \tag{H.432}$$

The factor of 2 in the effective propagation speed is typical for relativistic particles like photons, on which the effects of gravitational fields is stronger compared to non-relativistic particles. In fact, gravitational time dilation for non-relativistic particles is determined through the interpretation of the line element $\mathrm{d}s$ as the elapsed proper time $\mathrm{d}\tau$

$$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 \tag{H.433}$$

such that

$$\mathrm{d}\tau^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right)\mathrm{d}x_i\mathrm{d}x^i \tag{H.434}$$

If the velocities are small, the displacement in the $\mathrm{d}x^i$-directions are small compared to those into the $\mathrm{d}t$-direction:

$$\mathrm{d}\tau^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 \quad \rightarrow \quad \mathrm{d}\tau \simeq \left(1 + \frac{\Phi}{c^2}\right)\mathrm{d}t \tag{H.435}$$

with the approximation $\sqrt{1+2\epsilon} \simeq 1 + \epsilon$, again for $|\epsilon| \ll 1$. Comparing these two results with Fermat's principle for photons and Hamilton's principle for the motion of massive particles now shows that the effect of gravitational fields on photons is twice as large as that on non-relativistic particles.

Gravitational lensing would be described by the geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}k^\alpha = -\Gamma^\alpha{}_{\mu\nu}k^\mu k^\nu \quad \text{with} \quad k^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{H.436}$$

where the wave vector k^μ is tangent to the trajectory $x^\mu(\lambda)$ and normalised to zero. Using the invariance of geodesics under affine reparameterisation we can choose λ to yield $k^t = 1$, $k_ik^i = -1$, such that $k_\mu k^\mu = (k^t)^2 - k_ik^i = 0$.

The geodesic equation is an implicit relation: one needs to know the trajectory x^μ as the integral curve over k^μ to be able to evaluate the Christoffel-symbol $\Gamma^\alpha{}_{\mu\nu}$ at the right location: In actual numerical application it needs to be solved as a differential equation. To circumvent this, one uses the Born-approximation and assumes that the deflections are small, such that the change of the wave vector δk^α are computed relative to fixed tangents k^μ resulting from a solution of the geodesic equation for the background. Then, only the perturbations $\delta\Gamma^\alpha{}_{\mu\nu}$ determine the deflection:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta k^\alpha = -\delta\Gamma^\alpha{}_{\mu\nu}k^\mu k^\nu \tag{H.437}$$

The changes to the propagation direction can be integrated up directly

$$\delta k^\alpha = -k^\mu k^\nu \int \mathrm{d}\lambda\, \delta\Gamma^\alpha_{\mu\nu}(\lambda) \tag{H.438}$$

The perturbed Christoffel-symbols $\delta\Gamma^\alpha_{\mu\nu}$ are given by the usual relation

$$\delta\Gamma^\alpha_{\mu\nu} = \frac{\eta^{\alpha\beta}}{2}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}\right) \tag{H.439}$$

as derivatives of the weakly perturbed metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{and} \quad g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu} \simeq \eta^{\mu\nu} \tag{H.440}$$

If we assume that the perturbations correspond to Newtonian gravitational potentials, as the perturbed Christoffel-symbols contain gradients of Φ. To evaluate the geodesic equation further we can assume that the unperturbed propagation proceeds into the z-direction and that the gradients in Φ deflect the photons into the perpendicular directions:

$$\delta k^i = -k^\mu k^\nu \int \mathrm{d}\lambda\, \delta\Gamma^i_{\mu\nu} = -\int \mathrm{d}\lambda \left(\delta\Gamma^i_{tt} + 2\delta\Gamma^i_{tz} + \delta\Gamma^i_{zz}\right) \tag{H.441}$$

Inspecting the explicit expressions for the Christoffel-symbol yields

$$\delta\Gamma^i_{tt} = \frac{\eta^{i\beta}}{2}\left(\partial_t h_{\beta t} + \partial_t h_{t\beta} - \partial_\beta h_{tt}\right) = \frac{1}{2}\partial^i h_{tt} = \frac{1}{c^2}\partial^i\Phi \tag{H.442}$$

as well as

$$\delta\Gamma^i_{tz} = \frac{\eta^{i\beta}}{2}\left(\partial_t h_{\beta z} + \partial_z h_{z\beta} - \partial_\beta h_{tz}\right) = 0 \tag{H.443}$$

and

$$\delta\Gamma^i_{zz} = \frac{\eta^{i\beta}}{2}\left(\partial_z h_{\beta z} + \partial_z h_{z\beta} - \partial_\beta h_{zz}\right) = \frac{1}{2}\partial^i h_{zz} = \frac{1}{c^2}\partial^i\Phi \tag{H.444}$$

making heavy use of the diagonal form of the metric and its inverse, and ignoring derivatives along the direction of propagation z. Collecting all results gives for the gravitational light deflection angle $\hat{\alpha} = \delta k^i/k^z \simeq \delta k^i$

$$\delta k^i = -\frac{2}{c^2}\int \mathrm{d}\lambda\, \partial^i\Phi \tag{H.445}$$

In the lensing deflection, the scale of the potential Φ is set by c^2, and the factor 2 originates from the fact that photons as relativistic test particles are more sensitive to gravitational potentials than massive particles. The relevant gradients of Φ are those perpendicular to the line of sight.

H.8 *Properties of dark matter*

A number of experiments (rotation curves of galaxies, virial equilibria in galaxy clusters, gravitational lensing, amplitude of CMB temperature fluctuations) suggests the existence of non-baryonic dark matter. Dark matter is significantly more abundant than normal matter, as $\Omega_m/\Omega_b \simeq 7$, and has extreme properties. Apart from exotic models of macroscopic dark matter such as primordial black holes, many cosmologists suspect dark matter to be made up from yet undetected elementary particles, for instance by WIMPs in the TeV-range, or by ultra-light axions. The dark matter particles are required to interaction by the weak force and by gravity, and they are required to have these properties:

- Dark matter is **cold**, meaning that there is little or none thermal motion of the dark matter particles. Therefore, this kind of dark matter is non-relativistic, and as there is no thermal motion of the particles, any structures on small scales seeded by cosmic inflation is preserved: Neither diffusive motion of the dark matter particles themselves nor radiation pressure can break up small-scale structures.
- Dark matter is, well, **dark** and shows no signs of interactions through electromagnetism: There are no annihilation or decay processes of dark matter producing photons, nor are there effects of radiation pressure on dark matter particles.
- In fact, the only appreciable interaction of dark **matter** is gravitational, and its presence manifests itself in rotation curves, virial equilibria, gravitational lensing or in the amplitude of CMB-fluctuations.
- Dark matter is **collisionless**, meaning that there is only a very small cross-section for elastic collisions, as demonstrated e.g. by the bullet cluster: In this system, ob observes a merging of two clusters at high velocity, where the dark matter component as mapped out by lensing is unperturbed in the passage of the two clusters, whereas the baryonic component is not, which clearly indicates differences in the fluid mechanics of the two components: It is not possible to predict fluid properties like pressure and viscosity from the microscopic interaction of particles for dark matter.

H.9 *Spherical collapse of dark matter haloes*

The gravitational dynamics of a homogeneous sphere of matter under its own gravity is, due to its high degree of symmetry, one of the few exactly solvable systems, in Newtonian gravity as well as in general relativity. A spherically symmetric perturbation would initially follow the Hubble-expansion, but its own gravity would slow down the local expansion rate, ultimately stalling the perturbation and decoupling it from the Hubble-flow, before it collapses on itself. During the collapse one can expect that virialisation processes take place such that a stabilised bound state is reached, in which the baryonic component can cool and form stars.

In classical gravity the radius R of a spherical perturbation of mass M follows the Newtonian equation of motion

$$\ddot{R} = -\frac{GM}{R^2} \tag{H.446}$$

The instant t at which the radius stalls, $\dot{R} = 0$, defines the moment of turn-around. With a_a and R_a as scale factor and radius at turn-around, respectively, on defines the dimensionless variables $x = \frac{a}{a_a}$ and $y = \frac{R}{R_a}$.

If we assume for simplicity a flat, matter-dominated FLRW background with $\Omega_m = 1$ the Hubble function is given by

$$H = \frac{\dot{a}}{a} = H_0 a^{-3/2} \tag{H.447}$$

And, due to flatness and matter-domination, $\rho = \rho_{\mathrm{crit}}$, and we can define a dimensionless parameter

$$\tau = H_a t = H(a_a)t = H_0 a_a^{-3/2} t \tag{H.448}$$

which allows to re-express the dynamical equations for x as

$$x' = \frac{dx}{d\tau} = \frac{dt}{d\tau}\frac{dx}{dt} = \frac{1}{H_a}\dot{x} = \frac{1}{H_a}\frac{\dot{a}}{a_a} = \frac{1}{H_a}\frac{a}{a_a}\frac{\dot{a}}{a} = \frac{H}{H_a}x \tag{H.449}$$

substituting the Friedmann-equation in the last step. Similarly,

$$\ddot{R} = -\frac{GM}{R^2} = -\frac{4\pi G}{3}\rho_a R_a^3 \frac{1}{R^2} \tag{H.450}$$

with the background density $\rho_a = \frac{3H_a^2}{8\pi G}\xi$ the density contrast at turn around $\xi > 1$. In a similar way, the dynamical equation for y can be rewritten

$$y'' = -\frac{\xi}{2y^2} \tag{H.451}$$

with the natural initial conditions $y'\big|_{x=1} = 0$ and $y\big|_{x=0} = 0$. The collapse equations are solved analytically through

$$\frac{dx}{d\tau} = x^{-1/2} \quad \rightarrow \quad d\tau = x^{1/2}dx \quad \rightarrow \quad \tau = \frac{2}{3}x^{3/2} + c \tag{H.452}$$

as well as

$$y' = \pm\sqrt{\xi}\sqrt{\frac{1}{y} - 1} \tag{H.453}$$

which can be combined to

$$\tau = \frac{1}{\sqrt{\xi}}\left(\frac{1}{2}\arcsin(2y-1) - \sqrt{y - y^2} + \frac{\pi}{4}\right) \tag{H.454}$$

At turn around $x = y = 1$ one obtains $\tau = \frac{2}{3}$ leading to $\xi = \left(\frac{3\pi}{4}\right)^2$. The density inside the halo results from the ratio

$$\Delta = \left(\frac{x}{y}\right)^3 \approx 1 + \underbrace{\frac{3}{5}y}_{=\delta} \tag{H.455}$$

If we now extrapolate the density to $x = 1$ by $\delta_a = \frac{\delta}{x} = \frac{3y}{5x}$ and use

$$\frac{1}{x} = \left(\frac{3\tau}{2}\right)^{-2/3} \approx \left(\frac{3\pi}{4}\right)^{2/3}\frac{1}{y} \quad \rightarrow \quad \delta_a = \frac{3}{5}\left(\frac{3\tau}{4}\right)^{2/3} \tag{H.456}$$

we receive the time $\tau = \frac{4}{3}$ of the collapse. From this one can deduce a linear growth up to the critical density δ_c

$$\delta_c = 2^{2/3}\delta_a \approx 1.69 \tag{H.457}$$

at which the collapse sets in.

H.10 *Mass function of dark matter haloes*

The central result on spherical collapse was the overdensity of $\delta_c \simeq 1.69$ for a perturbation to collapse in its own gravitational field against the Hubble-expansion of the background. This number can be used to determine the number of objects such as clusters or galaxies per comoving volume that can form from initial conditions with suitably high initial densities. The formalism for achieving this was discovered in three different contexts: Assuming that the noise in an electric circuit is described by a one-dimensional Gaussian random field, the probability for a peak in the voltage exceeding a certain threshold would result from the spectrum of the fluctuations. Similarly, the occurrences of waves on the surface of the ocean above a certain threshold would likewise result from the fluctuation statistics of a Gaussian random field, now in two dimensions. And lastly, objects like galaxies form if the density exceeds the threshold for spherical collapse, and how often this happens in a comoving volume in a Gaussian random field is an application of the same idea in three dimensions.

A spherical perturbation of radius R encloses the mass M

$$M = \frac{4\pi}{3}R^3\Omega_m\rho_{\mathrm{crit}} \quad \rightarrow \quad R = \sqrt[3]{\frac{3M}{4\pi\Omega_m\rho_{\mathrm{crit}}}} \tag{H.458}$$

with the ambient density $\Omega_m\rho_{\mathrm{crit}}$, $\rho_{\mathrm{crit}} = 3H_0^2/(8\pi G)$, such that each mass M corresponds to a length scale R(M). If we now filter the density field δ by convolution with a filter W_R of spatial size R(M)

$$\bar{\delta}(\boldsymbol{x}) = \int d^3x' W_{R(M)}(|\boldsymbol{x} - \boldsymbol{x}'|)\delta(\boldsymbol{x}'), \tag{H.459}$$

then the convolved density field $\bar{\delta}$ consists of fluctuations that are massive enough that they can form objects of mass M by spherical collapse. In Fourier-space, the convolution relation reads

$$\bar{\delta}(\boldsymbol{k}) = W_R(\boldsymbol{k})\delta(\boldsymbol{k}) \tag{H.460}$$

with the Fourier-transform $W_R(\boldsymbol{k})$ of the filter function. The convolution as a linear operation does not change fundamentally the distribution of the density field amplitude, but changes the variance. Working with a Gaussian distribution

$$p(\bar{\delta}, a) = \frac{1}{\sqrt{2\pi\sigma_R^2(a)}} \exp\left(-\frac{1}{2}\left(\frac{\bar{\delta}}{\sigma_R(a)}\right)^2\right) \tag{H.461}$$

where the variance is growing in linear structure formation according to with the relation

$$\sigma_R^2(a) = \sigma_R^2(\text{today})D_+^2(a) \tag{H.462}$$

With this distribution we can ask how often in a fixed comoving volume the smoothed density field reaches amplitudes sufficient for spherical collapse, i.e. where the condition $\bar{\delta} > \delta_c$ is fulfilled. The probability of finding those is equal to the volume fraction filled with halos of mass M,

$$F(M, a) = \int_{\delta_c}^{\infty} d\bar{\delta}\, p(\bar{\delta}, a) = \frac{1}{2}\mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_R(a)}\right) \tag{H.463}$$

with the complementary error function erfc(). One determines the halo-distribution by differentiation

$$\frac{\partial F}{\partial M} = \frac{dR}{dM}\frac{dF}{dR}\frac{\delta_c}{\sigma_R D_+}\frac{d\ln\sigma_R}{dM}\exp\left(-\frac{1}{2}\left(\frac{\delta_c}{\sigma_R D_+}\right)^2\right) \tag{H.464}$$

because $\frac{d}{dx}\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\exp(-x^2)$. To obtain the comoving number density we divide the by halos occupied volume fraction by the halo-volume $\frac{4\pi}{3}R^3$ and get

$$n(M, a) = \frac{\rho_0}{\sqrt{2\pi}}\frac{\delta_c}{\sigma_R D_+}\frac{d\ln\sigma_R}{d\ln M}\exp\left(-\frac{1}{2}\left(\frac{\delta_c}{\sigma_R D_+}\right)^2\right)\frac{1}{M} \tag{H.465}$$

The mass function or Press-Schechter function $n(M, a)$ is a valuable source of cosmological information as it is sensitive to the shape of the CDM-spectrum $P(k)$ through the variance σ_R^2 and its derivative $d\sigma^2/dM$. Practical numbers to remember are about 100 clusters of galaxies above $5 \times 10^{13} M_\odot/h$ in a volume of $(100\mathrm{Mpc}/h)^3$, and about 10^4 galaxies with masses between $10^{11} M_\odot/h$ and $10^{12} M_\odot/h$ in the same volume. An important caveat is that the number of haloes per comoving volume is not observable, and neither would be comoving distance, but redshift is straightforwardly observable. Then, a cosmological probe could be the number of haloes observed within a fixed solid angle $\Delta\Omega$ between two redshifts $z_{\min}$ and $z_{\max}$

$$N = \frac{\Delta\Omega}{4\pi}\int_{z_{\min}}^{z_{\max}} dz\, \frac{dV}{dz}\int_{M_{\min}(z)}^{\infty} dM\, n(M, a(z)) \tag{H.466}$$

where the minimal mass $M_{\min}(z)$ for an object to be detectable is commonly determined by the observational technique. But in almost all cases, the magnitude of

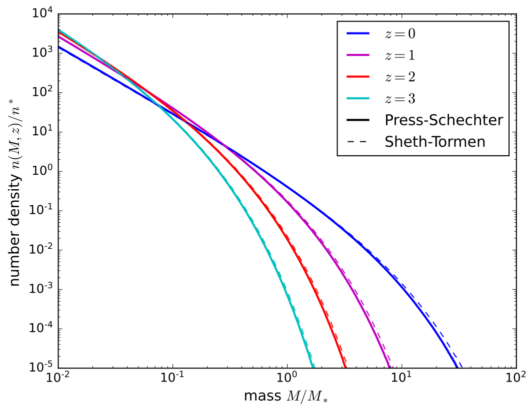

Figure 12: Halo mass function $n(\mathrm{M}, z)$ at different redshifts

observable properties of haloes, like luminosity or temperature, scale with halo mass. The comoving volume evolves with redshift z according to

$$\mathrm{V} = \frac{4\pi}{3}\chi^3(a(z)) \quad \rightarrow \quad \frac{\mathrm{dV}}{\mathrm{d}z} = \frac{\mathrm{d}a}{\mathrm{d}z}\frac{\mathrm{d}\chi}{\mathrm{d}a}\frac{\mathrm{dV}}{\mathrm{d}\chi} \tag{H.467}$$

Due to $\chi = c\int \mathrm{d}a/(a^2\mathrm{H}(a))$ and $a = 1/(1+z)$ this expression becomes

$$\frac{\mathrm{d}\chi}{\mathrm{d}a} = \frac{c}{a^2\mathrm{H}} \quad \text{as well as} \quad \frac{\mathrm{d}a}{\mathrm{d}z} = \frac{1}{(1+z)^2} = a^2 \quad \text{and therefore,} \quad \frac{\mathrm{dV}}{\mathrm{d}z} = \frac{c}{\mathrm{H}}4\pi\chi^2 \tag{H.468}$$

The halo mass function $n(\mathrm{M}, z)$ is shown in Fig. 12 for a ΛCDM-cosmology, in two different parameterisations. Clearly, the most massive halos only appear at late times in the Universe.