G FLUID MECHANICS

G.1 *fluid mechanics as a continuum theory*

The motion of matter on large scale and for small perturbations can be described by fluid mechanics, such that the evolution of the cosmic density field and the cosmic velocity field is determined through the equations of fluid mechancis, namely the continuity and the Navier-Stokes equation, both with gravity as the driving force of structure formation. For the purpose of this book we restrict ourselves to nonrelativistic fluid mechanics with a Newtonian description of gravity and Galilean relativity. The motion of a fluid is primarily determined by the continuity and the Navier-Stokes equation, which determine the time evolution of the density and the velocity fields, respectively. Fluid mechanics is a continuum theory, because it considers the fluids as continuous media without any microscopic structure, and as such it can only describe fluid elements which are large enough that they contain a large number of particles. The description of collisionless systems under the influence of gravity is conceptually not clear, because (i) individual particles can gain very large velocities in many-body-interactions, such that the particle density might not be sufficient to define a smooth fluid through averaging of particle properties and because (ii) self-gravitating systems produce structures on small scales, which are not wiped out by collisions such that in the averaging process in deriving smooth fields information on the phase-space structure is lost.

It is very important to notice that both the continuity and the Navier-Stokes equations are nonlinear, as both involve products between the density and the velocity, and between the velocity and gradients of the velocity, respectively. In addition, the equation of state $p(\rho)$, if present in the Navier-Stokes equation, can be nonlinear as well and can, in addition depend on other quantities, for instance the entropy density s or temperature T, leading to additional terms in particular in the vorticity equation. Alternative, one can choose to work with the momentum density $\rho\boldsymbol{v}$ instead of the velocity $\boldsymbol{v}$, which would render the continuity equation linear but would make the gravitational force in the Navier-Stokes equation nonlinear.

G.2 *From relativistic to non-relativistic fluid mechanics*

Energy-momentum conservation in the covariant form $\nabla_\mu T^{\mu\nu} = 0$ is equivalent to relativistic fluid mechanics of ideal fluids. In the non-relativistic limit with slow velocities $|\upsilon| \ll c$ on a Minkowski-background with $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu} = 0$. In the non-relativistic limit, $p \ll \rho c^2$ and the motion of the fluid elements proceeds essentially only in $\mathrm{d}t$-direction:

$$\rho c^2 \underbrace{(\partial_t \beta^j + (\beta^i \partial_i)\beta^j)}_{=u^\mu \nabla_\mu u^\nu = u^\mu \partial_\mu u^\nu} = \rho(\partial_t \upsilon^j + \upsilon^i \partial_j \upsilon^j) = -\partial^j p \quad \text{or} \quad \partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} \tag{G.332}$$

which is exactly the non-relativistic Euler-equation. Including gravity requires to use $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$ with a corresponding nonzero Christoffel-symbol. In the weak-field limit $|\Phi| \ll c^2$ on has the line element

$$\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2 \mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right)\mathrm{d}x_i \mathrm{d}x^i \tag{G.333}$$

where only the first term contributes as the displacements in the spatial $\mathrm{d}x^i$-directions are small:

$$g_{tt} = 1 + \frac{2\Phi}{c^2} \tag{G.334}$$

The gravitational acceleration is computed from the Christoffel symbols

$$\Gamma^{\alpha}_{\ \mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu g_{\beta\nu} - \partial_\nu g_{\mu\beta} + \partial_\beta g_{\mu\nu}\right) \tag{G.335}$$

where in the weak-field limit the inverse metric is replaced by the (inverse) Minkowski metric $g^{\alpha\beta} = \eta^{\alpha\beta}$ but of course the gradients $\partial_\beta g_{\mu\nu}$ are kept. In static gravitational fields $\partial_t g_{\alpha\beta} = 0$ and only nonzero spatial derivatives $\partial_i g_{\alpha\beta} = \frac{2}{c^2}\partial_i\Phi\delta_{\alpha\beta}$, from which one would expect gradients $\partial^j\Phi$ to appear:

$$u^\mu(\nabla_\mu u^\nu) = u^\mu(\partial_\mu u^\nu + \Gamma^{\nu}_{\ \mu\alpha}\, u^\alpha) = u^\mu\partial_\mu u^\nu + \Gamma^{\nu}_{\ \mu\alpha}\, u^\mu u^\alpha \tag{G.336}$$

The three terms naturally correspond to gravitational acceleration in an inhomogeneous field, to the Coriolis- and centrifugal accelerations:

$$\Gamma^{\nu}_{\ \mu\alpha} = \Gamma^{\nu}_{\ tt}\underbrace{u^t}_{c}\underbrace{u^t}_{c} + \underbrace{\Gamma^{\nu}_{\ ti}\, u^t u^i + \Gamma^{\nu}_{\ it}\, u^i u^t}_{2\Gamma^{\nu}_{\ it}\, cv^i} + \Gamma^{\nu}_{\ ij}\underbrace{u^i}_{v^i}\underbrace{u^j}_{v^j} \tag{G.337}$$

The first term is clearly dominating for small velocities

$$u^\mu\nabla_\mu u^\nu = u^\mu\partial_\mu u^\nu + \Gamma^{\nu}_{\ \mu\alpha}\, u^\mu u^\alpha = \partial_t v^j + (v^i\partial_i)v^j - \Gamma^{j}_{\ tt}\, c^2 = -\frac{1}{\rho}\partial^j p \tag{G.338}$$

with the Christoffel-symbol $\Gamma^{j}_{\ tt}$

$$\Gamma^{j}_{\ tt} \simeq \frac{\eta^{jk}}{2}\left(\partial_t g_{tk} + \partial_t g_{kt} - \partial_k g_{tt}\right) = -\partial^j\frac{\Phi}{c^2} \tag{G.339}$$

as only the last term $g_{tt} = 2\Phi/c^2$ contributes and the first two terms vanish, because of the assumption of static gravitational fields. At the same time, the terms $\Gamma^{i}_{\ tj}$ and $\Gamma^{i}_{\ jk}$ offer a natural and consistent way to incorporate other inertial accelerations. So the final result is the non-relativistic Euler-equation with gravity

$$\partial_t v^j + v^i\partial_i v^j = -\frac{1}{\rho}\partial^j p - \partial^j\Phi \quad \text{or} \quad \partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi \tag{G.340}$$

It is quite interesting that the nonlinearities in the fluid-mechanical equations have a relativistic origin, and that one needs empirical reasoning to make sense of them in classical mechanics. The advective term $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ is interpreted as the rate of change of the velocity at a fixed point in the laboratory frame as the flow sweeps new fluid elements to this point which may carry a different velocity (the velocity the fluid element has had upstream an infinitesimal time in the past), while only $\partial_t v$ is the proper rate of change of the flow velocity, measured in terms of coordinate time instead of proper time.

G.3 *Continuity*

The continuity equation is an expression of the conservation of matter. If the density field changes in a volume element at a fixed point it must necessarily be because fluxes have converged and have transported matter into that element:

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{v}) = 0. \tag{G.341}$$

The interpretation of the continuity equation is particularly clear if one applies the Gauss-theorem:

$$\int_V \mathrm{d}V\, \partial_t \rho = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}V\, \rho = \frac{\mathrm{d}}{\mathrm{d}t}M = -\int_V \mathrm{d}V\, \mathrm{div}(\rho \boldsymbol{v}) = -\int_{\partial V} \mathrm{d}\boldsymbol{A}\cdot(\rho \boldsymbol{v}), \tag{G.342}$$

such that the mass M changes if there are fluxes through the surface of the volume element. The continuity equation is nonlinear because the definition of the flux $\rho\boldsymbol{v}$ involves the product of two fields.

G.4 *Navier-Stokes equation*

The Navier-Stokes equation is the equation of motion for fluid elements as a generalisation of Newton's third axiom,

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} - \nabla\Phi + \mu\Delta\boldsymbol{v}, \tag{G.343}$$

as it relates the acceleration of a fluid element with the specific force density. Relevant forces include pressure gradients, gradients in the gravitational potential or viscous forces. The Navier-Stokes-equation seems to have the shape of an evolution equation, but in fact it originates together with the continuity equation from a relativistic conservation equation $\partial_\mu T^{\mu\nu} = 0$ with the energy-momentum-tensor $T^{\mu\nu}$ of the fluid. In a chosen reference frame it is possible to separate the conservation equation in the time-part containing the conservation of mass and a spatial part with the conservation of momentum.

The time derivative of the velocity, as required by Newton's equation of motion, is computed for a field which depends on time and on position. In components one would write

$$\frac{\mathrm{d}}{\mathrm{d}t}v_i(r_j,t) = \partial_t v_i + \frac{\partial r_j}{\partial t}\frac{\partial v_i}{\partial r_j} \tag{G.344}$$

With the subsitution of the derivative $\partial_t r_j = v_j$ one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}v_i(r_j,t) = \partial_t v_i + v_j\frac{\partial v_i}{\partial r_j}. \tag{G.345}$$

Rewriting this expression yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} = \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}. \tag{G.346}$$

Therefore, the nonlinearity $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ originates purely from the choice of a fixed coordinate frame, relative to which the fluid moves: The derivative $\partial_t \boldsymbol{v}$ would indicate the acceleration or the rate of change of velocity with time of fluid elements which pass in succession through a fixed position $\boldsymbol{x}$ in space, while the so-called convective derivative $\mathrm{D}_t = \partial_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ describes the acceleration of a single fluid element as it moves around, combining the time-derivative ∂_t with the rate of change of velocity with position $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ projected onto the velocity-components themselves. By choosing instead of a fixed Euler-frame a coordinate frame which moves along with the fluid, referred to as the Lagrange-frame, the fluid equation of motion becomes linear, by introducing comoving, Lagrangian coordinates $\boldsymbol{r} = \boldsymbol{r}_0 + \int \mathrm{d}t\ \boldsymbol{v}$ and reexpressing all derivatives.

Both viscosity and pressure originate from collisions between the particles from which the fluid is composed. The viscosity is usually modelled on the Lamé'-viscosity coefficients and is able to dissipate kinetic energy from the fluid by friction if velocity gradients or shear flows $\partial_i v_j$ are present. If there is such a phenomenon, one needs an analogous energy equation to keep track of the evolution of the energy content of the fluid, in particular because the equation of state might show a dependence on e.g. temperature or entropy density. We will only consider ideal fluids without viscosity, because they approximate dark matter well due to its collisionlessness, and cover the phenomenology of baryonic fluids at low densities.

G.5 *Ideal versus viscous fluid mechanics*

In contrast to the kinematical terms in fluid mechanics and in contrast to gravity, effects associated with the microscopic properties of the fluid itself need to have a phenomenological description. In fact, how bulk properties like fluid-mechanical pressure and viscosity would be determined from the microscopic interactions between the particles that the fluid consists of, is yet not fully understood.

The differential change $\mathrm{d}\boldsymbol{v}$ of the velocity in a fluid is to first order proportional to the displacement

$$\mathrm{d}\boldsymbol{v} = (\mathrm{d}\boldsymbol{r}\nabla)\boldsymbol{v} \quad \rightarrow \quad \mathrm{d}v^i = \partial_j v^i \mathrm{d}x^j \tag{G.347}$$

defining the velocity tensor, which is conveniently decomposed into a symmetric part (shear) and the antisymmetric part (vorticity)

$$\partial_j v^i = \underbrace{\frac{1}{2}(\partial_j v^i + \partial_i v^j)}_{\epsilon_j{}^i} + \underbrace{\frac{1}{2}(\partial_j v^i - \partial_i v^j)}_{\omega_j{}^i} \tag{G.348}$$

Again, this idea is very similar to the Raychauduri-equation: The volume change is given by

$$\mathrm{dV} \sim \mathrm{div}\boldsymbol{v} \sim \partial_i v^i = \epsilon^i{}_i = \mathrm{tr}(\epsilon) \tag{G.349}$$

such that the trace of the velocity tensor induces a change in volume of a fluid element. Incompressible flows have the unique property that the divergence of their velocity field is always zero, and hence there can not be any change in the volume of fluid elements.

In a phenomenological model one can now relate shears in a fluid to stresses and pressure: In general, the stress tensor σ_{ij} is the ith component of the force acting on a surface element with normal vector into the jth direction: As such, stresses and pressure have the same unit of force normalised by area. One can decompose the stress tensor into the isotropic part $p\delta_{ij}$ and the anisotropic contribution σ'_{ij}

$$\sigma_{ij} = \sigma'_{ij} - p\delta_{ij} \tag{G.350}$$

where

$$\mathrm{tr}(\sigma) = p\ \mathrm{tr}(\delta_{ij}) = -3p. \tag{G.351}$$

so that pressure gets the interpretation of isotropic stress.

Furthermore, the stress tensor is also symmetric $\sigma_{ij} = \sigma_{ji}$. This can be shown by setting up a counter example which turns out to be aphysical: If stresses act on two faces of a cube with volume $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$, one introduces a torque torque M_x if the stresses are unequal, in contradiction to $\sigma_{ij} = \sigma_{ji}$,

$$M_x = \sigma_{zy}(\mathrm{d}x\mathrm{d}z)\mathrm{d}y - \sigma_{yz}(\mathrm{d}x\mathrm{d}y)\mathrm{d}z = (\sigma_{zy} - \sigma_{yz})\mathrm{d}V \tag{G.352}$$

With a Newtonian equation of motion $M = I\ddot{\phi}$ with the inertia $I = (\mathrm{d}y^2 + \mathrm{d}z^2)\mathrm{d}V$ for rotation around the x-axis one would obtain the angular acceleration

$$\ddot{\phi} = \frac{M}{I} \sim V^{-\frac{2}{3}} \tag{G.353}$$

Therefore, for $V \to 0$ the volume term $V^{-\frac{2}{3}}$ diverges, which leads to the conclusion that the angular acceleration $\ddot{\phi}$ diverges, too: Accelerations for the smallest torques would assume arbitrarily high values, which would be aphysical. A way out is the condition $\sigma_{yz} = \sigma_{zy}$ and a symmetric stress tensor σ_{ij}.

G.5.1 Bulk and shear viscosity

With the shear as the differential velocity field into which a fluid is embedded and the stress as the reaction of a fluid element to this external shear it is reasonable to assume a linear relationship between these two symmetric tensors: This is the foundational idea of a Newtonian fluid, if in addition the response of the fluid element is instantaneous to the external shear. The shear tensor ϵ_{ij} and the stress tensor σ'_{ij} are related in Lamé parameterisation by introducing two coefficients η and ξ,

$$\sigma'_{ij} = 2\eta\left(\epsilon_{ij} - \frac{\mathrm{tr}(\epsilon)}{3}\delta_{ij}\right) + \xi\ \mathrm{tr}(\epsilon)\delta_{ij} \tag{G.354}$$

with $\mathrm{tr}(\epsilon) = \partial_i v^i - \mathrm{div}\boldsymbol{v}$ is the divergence of the velocity field. The first term parameterises a reaction of the fluid in form of anisotropic stresses to the traceless shear, which would be realised for instance if there is a shearing motion of fluid layers against each other, motivating the term shear viscosity for η. But there is likewise a reaction of the fluid to changes in volume beyond the effects of pressure mediated by the equation of state: The bulk viscosity ξ parameterises for this case the magnitude of anisotropic stresses.

Again, in flows consisting of purely collisionless dark matter, microscopic stresses and effects of viscosity are not present, but there are, like in the case of pressure, collective effects with emulate these.

G.5.2 Viscous fluid mechanics

The effects of pressure and viscosity can obviously change the state of motion of a fluid element, as expressed by the momentum density $\rho\boldsymbol{\upsilon}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}V\,\mathrm{div}(\rho\boldsymbol{\upsilon}) = -\int_V \mathrm{d}V\,\rho\nabla\Phi + \int_{\partial V}\mathrm{d}\mathbf{A}\,\sigma \tag{G.355}$$

such that apart from bulk forces $\rho\nabla\Phi$ acting on the fluid element as a whole there are stresses as surface forces σ. The first term in the momentum equation can be reformulated as a surface integral, too, yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}V\,\mathrm{div}(\rho\boldsymbol{\upsilon}) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial V}\mathrm{d}\mathbf{A}\,\rho\boldsymbol{\upsilon} = \int_{\partial V}\mathrm{d}\mathbf{A}\,\rho\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} \tag{G.356}$$

in a Lagrangian frame that moves along with the flow: Following the fluid element in this way tracks the momentum evolution as forces are acting on its surface, and because there is no exchange of matter with the environment of a fluid element, the time derivative only acts on the velocity. The stresses acting on the surface of the volume element are given by

$$\left(\int_{\partial V}\mathrm{d}\mathbf{A}\,\sigma\right)_i \overset{(\mathrm{d}\mathbf{A})_i=\mathrm{d}An_i}{=} \int_{\partial V}\mathrm{d}A\,\sigma_{ij}n_j = \int_V \mathrm{d}V\,\frac{\partial}{\partial x^j}\sigma_{ij} = \left(\int_V \mathrm{d}V\,\nabla\sigma\right)_i \tag{G.357}$$

Substituting back gives

$$\rho\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} = \rho\left(\frac{\partial\boldsymbol{\upsilon}}{\partial t} + (\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon}\right) = -\rho\nabla\Phi + \nabla\sigma \tag{G.358}$$

Introducing viscosity and pressure

$$(\nabla\sigma)_i = (\nabla\sigma')_i - \frac{\partial}{\partial x^j}(p\delta_{ij}) = (\nabla\sigma' - \nabla p)_i \tag{G.359}$$

leads to the expression

$$\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} = \frac{\partial}{\partial t}\boldsymbol{\upsilon} + (\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon} = -\nabla\Phi - \frac{\nabla p}{\rho} + \frac{1}{\rho}\nabla\sigma' \tag{G.360}$$

If now viscosity is parameterised by the Lamé-coefficients η and ξ

$$(\nabla\sigma')_i = \frac{\partial}{\partial x_j}\sigma'_{ij} = \eta\frac{\partial^2 \upsilon_i}{\partial x_j^2} + \left(\xi - \frac{\eta}{3}\right)\frac{\partial}{\partial x_i}\underbrace{\frac{\partial \upsilon_k}{\partial x_k}}_{=\mathrm{div}\,\boldsymbol{\upsilon}} \tag{G.361}$$

and if the fluid is incompressible with the condition $\mathrm{div}\boldsymbol{\upsilon} = 0$, the bulk viscosity is irrelevant and one arrives at the Navier-Stokes equation

$$\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\nabla\Phi - \frac{1}{\rho}\nabla p + \mu\Delta\boldsymbol{v} \tag{G.362}$$

with the kinematic viscosity $\mu = \eta/\rho$.

G.6 *Fluid mechanical similarity and scaling relations*

Since nobody has found a general solution to the Navier-Stokes-equation, one wants to use some properties of mechnical similarity to bring the Navier-Stokes-equation to an already solved case. One might argue at this point, that classical fluid mechanics is scale-free from fundamental theory, but scales can enter through macroscopic properties of the fluid. Therefore we have some 'typical' behaviour of flows and can use corresponding scale symmetries. For this we first need to look for a dimensionless form of the Navier-Stokes-equation. To do so, we rescale

$$x \rightarrow x^* = \frac{x}{L} \qquad t \rightarrow t^* = \frac{t}{T} \tag{G.363}$$

as well as

$$\upsilon \rightarrow \upsilon^* = \frac{\upsilon}{V} \quad p \rightarrow p^* = \frac{p}{P} \quad g \rightarrow g^* = \frac{g}{G} = \frac{\nabla\Phi}{G} \tag{G.364}$$

It'd be important to realise that the scaling with L and T is relevant for derivatives in the fluid mechanical equations, but that V as a scale for the velocity is not automatically L/T: There can be high-velocity flows that vary only slowly with time or position, and vice versa.

Defining dimensionless derivatives is possible by writing

$$\frac{\partial}{\partial t} = \frac{\partial t^*}{\partial t}\frac{\partial}{\partial t^*} = \frac{1}{T}\frac{\partial}{\partial t^*} \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{\partial x^*}{\partial x}\frac{\partial}{\partial x^*} = \frac{1}{L}\frac{\partial}{\partial x^*} \tag{G.365}$$

Rewriting the entire Navier-Stokes equation for incompressible flows in terms of dimensionless variables and dimensionless derivatives gives

$$\frac{\rho V}{T}\frac{\partial}{\partial t^*}\boldsymbol{v}^* + \rho\frac{V^2}{L}(\boldsymbol{v}^*\nabla)\boldsymbol{v}^* = -\frac{P}{L}\nabla^* p^* - \rho G\underbrace{\nabla^*\Phi^*}_{=g^*} + \frac{\eta V}{L^2}\Delta^*\boldsymbol{v}^* \tag{G.366}$$

As all prefactors are equal in their units to $\frac{\rho V^2}{L}$ one can divide this factor out and arrive at

$$\underbrace{\frac{L}{TV}}_{St}\frac{\partial}{\partial t^*}\boldsymbol{v}^* + (\boldsymbol{v}^*\nabla)\boldsymbol{v}^* = -\underbrace{\frac{p}{\rho V^2}}_{Eu}\nabla^* p^* - \underbrace{\frac{GL}{V^2}}_{Fr^{-2}}\nabla^*\Phi^* + \underbrace{\frac{\eta}{\rho VL}}_{Re^{-1}}\Delta^*\boldsymbol{v}^* \tag{G.367}$$

which defines the scaling numbers:

- Strouhal-number $St = \frac{L}{TV}$ - proper acceleration
- Euler-number $Eu = \frac{p}{\rho V^2}$ - pressure vs. kinetic energy density
- Froude-number $Fr = \sqrt{\frac{V}{GT}}$ - potential vs. kinetic energy density

- Reynolds-number $\mathrm{Re} = \frac{\mu}{\mathrm{VL}}$ - magnitude of viscous forces

Working with the dimensionless form of the Navier-Stokes equation implies that the information about the actual physical properties of the system is replaced with the four scaling numbers. If two flows on physically different scales have the same scaling numbers, one must be able to map them onto each other by a similarity or scaling transform. This implies that there should be a classification of fluid mechanical problems into categories according to the dominating scaling numbers. Again, dark matter poses the conceptual problem how the Euler- and Reynolds-numbers should be defined, with the absence of microscopic interactions between the particles there is no pressure and no viscosity.

G.7 *Gravity and the Poisson-equation*

The gravitational force in the fluid-mechanical equations

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\nabla\Phi \tag{G.368}$$

could be determined through the Poisson equation,

$$\Delta\Phi = 4\pi G\rho, \tag{G.369}$$

and describes gravity in the weak field limit and at distances smaller than the Hubble-distance such that retardation effects do not play a role. In addition, all additional gravitational effects on and by moving objects are neglected: In summary, the equation is valid for $|\Phi| \ll c^2$, $|\upsilon| \ll c$ and on scales $\ll c/H_0$.

Due to the fact that it is the same density field ρ which is driven in its evolution by gradients $\nabla\Phi$ in the gravitational potential and which is at the same time sourcing the gravitational potential through the Poisson equation speaks of cosmic structure formation as a self gravitating phenomenon: Heuristically, a perturbation in the matter distribution generates a potential, which attracts matter from the surrounding of the perturbation, making it stronger. Then, the potential becomes deeper and the fields amplify, such that more matter is falling towards the perturbation, making it grow rapidly and at an exponential rate with time, if the influence of the background cosmology is neglected.

G.8 *Wave-type solutions and the Jeans-scale*

Pressure gradients have an influence on the evolution of the velocity field, and they typically lead to wave-type solutions: Compressing the medium builds up pressure, causing the medium to re-expand:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} \tag{G.370}$$

In order to construct a determined system of differential equations one would need to specify a relation between pressure p and density ρ, i.e. an equation of state, which accompanies the equation of continuity. Then, there are three relations (Euler, continuity and equation of state) for three fields ρ, $\boldsymbol{v}$ and p. For collisionless dark matter, though, pressure would not exist.

Wave-like phenomena are, because they fulfil the superposition principle, obtained as solutions to the linearised Navier-Stokes equation. Linearisations involve perturbing the dynamical fields away from their averages $\rho = \rho_0 + \delta\rho$, $p = p_0 + \delta p$ and $\boldsymbol{v} = \delta\boldsymbol{v}$. Therefore,

$$\partial_t \delta\rho + \rho_0 \mathrm{div}\,\delta\boldsymbol{v} = 0 \quad \text{as well as} \quad \partial_t \delta\boldsymbol{v} + \frac{1}{\rho_0}\nabla\delta p = 0 \tag{G.371}$$

Taking the time-derivative of the continuity equation and the divergence of the Navier-Stokes euqation defines the wave equation

$$\partial_t^2 \delta\rho - \underbrace{\left.\frac{\partial p}{\partial\rho}\right|_{\rho_0}}_{=c_s^2} \Delta\delta\rho = 0 \tag{G.372}$$

if one introduces an equation of state

$$\delta p = \left.\frac{\partial p}{\partial\rho}\right|_{\rho_0} \delta\rho \tag{G.373}$$

The derivative $c_s^2 = \partial p/\partial\rho$ defines sound speed inside the medium and depends typically on the thermodynamic change of state, e.g. isothermal and adiabatic.

Combining both gravity and pressure leads to an interesting concept: the Jeans-scale. If a system of size R and density ρ collapses under its own gravity, we can associate a free-fall time scale with the collapse, estimated to be

$$\tau_{ff} = \frac{1}{\sqrt{G\rho}} \tag{G.374}$$

and it can provide pressure support on the time scale of the sound-crossing time

$$\tau_s = \frac{R}{c_s} \tag{G.375}$$

Now, comparison between the two time scales suggests that if $\tau_{ff} \ll \tau_s$, the system collapses as pressure support can not be established fast enough, and if $\tau_{ff} \ll \tau_s$, the system is stabilised by pressure against gravity. Re-expressing the time scale as a length scale lets us define the Jeans-length $R_J = c_s\tau_{ff}$, and the associated Jeans-mass

$$M_J = \frac{4\pi}{3}\rho R_J^3 = \frac{4\pi}{3}\frac{c_s^3}{\sqrt{G^3\rho}} \tag{G.376}$$

In systems with masses exceeding M_J defined for a given c_s and ρ gravity is dominant over pressure and the system collapses, vice versa, in low-mass systems below M_J, pressure is able to provide support against gravity. Again, these concepts are irrelevant for systems consisting of dark matter only, due to its collisionlessness and the absence of pressure terms from the fluid mechanical equations.

G.9 *Vorticity equation*

The vorticity tensor is the antisymmetric part of velocity tensor $\partial v_j/\partial x_i$

$$\omega_{jk} = \frac{1}{2}(\partial_j v_k - \partial_k v_j) \tag{G.377}$$

and the vorticity-vector ω_j can be written as

$$\omega^i = \epsilon^{ijk}\partial_j v_k = \epsilon^{ijk}\omega_{jk} \tag{G.378}$$

or as $\boldsymbol{\omega} = \mathrm{rot}\boldsymbol{v}$.

The vorticity evolution can be deduced from the Navier-Stokes equation

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} - \nabla\Phi + \mu\Delta\boldsymbol{v} \tag{G.379}$$

by application of the operation rot to the equation and by using

$$(\boldsymbol{v}\nabla)\boldsymbol{v} = \nabla\frac{\boldsymbol{v}^2}{2} - \boldsymbol{v}\times\underbrace{\nabla\times\boldsymbol{v}}_{=\omega} \tag{G.380}$$

arriving at

$$\partial_t\boldsymbol{\omega} - \mathrm{rot}(\boldsymbol{v}\times\boldsymbol{\omega}) = \mu\mathrm{rot}(\Delta\boldsymbol{v}) \tag{G.381}$$

For an equation of state where pressure only depends on density, $p = p(\rho)$, the pressure term assumes the shape

$$\mathrm{rot}\left(\frac{\nabla p}{\rho}\right) = \frac{\mathrm{rot}\nabla p}{\rho} - \frac{1}{\rho^2}\nabla p\times\nabla\rho = 0 \tag{G.382}$$

making use of the chain rule in $\nabla p(\rho) = \frac{\partial p}{\partial\rho}\nabla\rho$. The Leibnitz-rule applied to $\boldsymbol{v}\times\boldsymbol{\omega}$ suggests

$$\mathrm{rot}(\boldsymbol{v}\times\boldsymbol{\omega}) = (\boldsymbol{\omega}\nabla)\boldsymbol{v} - (\boldsymbol{v}\nabla)\boldsymbol{\omega} + \boldsymbol{\omega}\underbrace{\mathrm{div}\ \boldsymbol{v}}_{=0} + \boldsymbol{v}\underbrace{\mathrm{div}\ \boldsymbol{\omega}}_{=0} \tag{G.383}$$

for incompressible fluids where $\mathrm{div}\boldsymbol{v} = \partial_i v^i = 0$, and because $\mathrm{div}\boldsymbol{\omega} = \epsilon_{ijk}\partial^i\partial^j v^k = 0$ always. Then, making use of

$$\mathrm{rot}(\Delta\boldsymbol{v}) = \mathrm{rot}(\nabla\underbrace{\mathrm{div}\boldsymbol{v}}_{=0} - \mathrm{rot\ rot}\boldsymbol{v}) = -\mathrm{rot\ rot\ rot}\boldsymbol{v} = \mathrm{rot\ rot}\boldsymbol{\omega} = \Delta\boldsymbol{\omega} - \nabla\underbrace{\mathrm{div}\ \boldsymbol{\omega}}_{=0} = \Delta\boldsymbol{\omega} \tag{G.384}$$

one arrives at a relation featuring again an advective derivative

$$\partial_t\boldsymbol{\omega} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} + \mu\Delta\boldsymbol{\omega} \tag{G.385}$$

Combining all results gives the vorticity equation, as a dynamical equation for the vorticity field ω:

$$\partial_t \omega + (\boldsymbol{\upsilon}\nabla)\omega = -\omega \mathrm{div}\boldsymbol{\upsilon} + \frac{\nabla\rho \times \nabla p}{\rho^2} + \mu\Delta\omega \tag{G.386}$$

which has the form of a convection-diffusion equation. The vorticity equation has a convective derivative of the form $\partial_t \boldsymbol{\omega} + (\boldsymbol{\upsilon}\cdot\nabla)\omega$, implying that the vorticity is advected in its own velocity field which is given by inverting the definition $\omega = \mathrm{rot}\boldsymbol{\upsilon}$ by means of the law of Biot-Savart,

$$\boldsymbol{\upsilon}(\boldsymbol{r}) = \int \mathrm{d}^3 r' \,\omega(\boldsymbol{r}') \times \nabla \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{G.387}$$

illustrating that the vorticity field needs to be known in the entire volume for converting back to the velocity field, as an expression of the nonlocal properties of this term. Secondly, the sourcing of the vorticity field can take place through the baroclinic term $\nabla p \times \nabla\rho$, if the density gradient and the pressure gradient are not parallel. Gravity alone is not able to source vorticity because as a scalar field, it can not decide about the orientation of the vorticity vector: $\mathrm{rot}\nabla\Phi = 0$, which immediately suggests the question why spiral galaxies should be rotating, if their dynamics is dominated by gravity. Lastly, the term $\mu\Delta\boldsymbol{\upsilon}$ causes in conjunction with the term $\partial_t\omega$ a diffusion of vorticity with the viscosity μ as the diffusion coefficient.

G.10 *Effective processes in collisionless systems*

Even though dark matter does not show elastic collisions between the particles and even though there is no microscopic origin of pressure and viscosity, there can be collective processes of groups of dark matter particles, emulating pressure and viscosity. After all, we observe that dark matter dominated objects are stable against their own gravity, due to the random motion of the particles, which acts as an effective pressure term in a hdyrostatic equilibrium. Similarly, we observe how systems like galaxies slow down if they enter a high density environment, by a process called dynamical friction.