F EARLY UNIVERSE AND COSMIC INFLATION

F.1 *Need for inflation and scales*

There are indications that the Universe underwent an episode of rapid, accelerated expansion at very early times, commonly referred to as cosmic inflation. Firstly, there is the horizon problem: If we consider thermal equilibrium in the early Universe, the horizon scale for this equilibrium is $c\Delta t$ with the time for equilibration being roughly equal to the travel time of photons. The observed homogeneity of the cosmic microwave background is therefore very surprising, it should be made of patches corresponding to the horizon size as the photons were set free. To make this more quantitative, one can have a look at the comoving horizon at the time when the CMB was generated, which was at a redshift of $z = 10^3$ or equivalently, a scale factor of $a = 10^{-3}$:

$$\chi_{\mathrm{H}} = c \int_0^{10^{-3}} \frac{\mathrm{d}a}{a^2 \mathrm{H}(a)} \approx 100 \text{ Mpc}/h \tag{F.270}$$

The comoving distance to the CMB is ~ 10 Gpc/h for ΛCDM. Taking the ratio of these two scales one arrives at an angular scale of

$$\Delta\Theta \sim \frac{1}{100} \sim 1^\circ. \tag{F.271}$$

This would be an estimate of the patch size for homogeneity on a small scales. This can be changed by including modification to the Hubble-function at early times, in particular by making it very small, such that the horizon scale becomes large as a consequence. Secondly, there is the flatness problem. As we know, the curvature Ω_K is smaller than $\Omega_K \lesssim 0.01$, which is very small, but it grows in matter and radiation dominated phases. One can describe this in FLRW-cosmologies with fluids Ω_w with EOS-parameter w and curvature $\Omega_K = 1 - \Omega_w$.

$$\mathrm{H}^2(a) = \mathrm{H}_0^2 \left(\frac{\Omega_w}{a^{3(1+w)}} + \frac{\Omega_K}{a^2} \right) \tag{F.272}$$

wherein Ω_K's behaviour can be described like a fluid with $w = -\frac{1}{3}$. We can write

$$\frac{\Omega_w(a)}{\Omega_w} = \frac{\mathrm{H}_0^2}{a^{3(1+w)} \mathrm{H}^2(a)} \tag{F.273}$$

derived from

$$\Omega(a) = \frac{\rho(a)}{\rho_{\mathrm{crit}}(a)} \quad \text{with} \quad \rho_{\mathrm{crit}}(a) = \frac{3\mathrm{H}(a)^2}{8\pi \mathrm{G}} \tag{F.274}$$

Therefore we obtain for curvature in adiabatic evolution

$$\frac{\Omega_K(a)}{\Omega_K} = \frac{\mathrm{H}_0^2}{a^2 \mathrm{H}^2(a)} = \frac{1}{\frac{\Omega_w}{a^{3(1+w)-2}} + \Omega_K} \tag{F.275}$$

which indicates directly the evolution of curvature in the presence of another fluid the model universe:

(1) if $3(1+w)-2=0$ then $w=-\frac{1}{3}$ and resulting no changes, as $q=0$, $\ddot{a}=0$ using $3(1+w)=2(1+q)$ for $\Omega=1$ and therefore $\Omega_K=$ const. Effectively, there is another fluid with $w=-1/3$ present and both fluids keep due to their analogous evolution the density parameters fixed at constant values.

(2) if $3(1+w)-2>0$ the resulting $\ddot{a}$ is smaller than 0, thus $q>0$ and in result Ω_K is increasing. An additional fluid with an equation of state more positive than $w=-1/3$ gives rise to a decelerating universe with an associated growth of curvature.

(3) if $3(1+w)-2<0$ the fluid EOS-parameter $w<-\frac{1}{3}$, further $q<0$ and $\ddot{a}>0$, in this configuration Ω_K is decreasing. This case is certainly interesting for us, as this drives Ω_K to small values, as a consequence of the dominating energy density of the additional fluid with an equation of state more negative than $w=-1/3$.

Thirdly, there is the scale problem, which arises if one tries to predict typical scales of the Universe from natural constants. In the Planck-system, constants are c, G and $\hbar$, whereas in the Hubble system we use c, G and Λ, and inflation catapults the Universe from a system that is described by the Planck length $l_P=\sqrt{\frac{G\hbar}{c}}=10^{-35}$ meters, the Planck time $t_P=\sqrt{\frac{G\hbar}{c^3}}=10^{-43}$ seconds and the Planck density $\rho_P=\frac{c^5}{G^2\hbar}=10^{96}\frac{kg}{m^3}$ to a state rather described by the Hubble length $l_H=\frac{1}{\sqrt{\Lambda}}=10^{25}$ meters, the Hubble-time $t_H=\frac{1}{c\sqrt{\Lambda}}=10^{17}$ seconds and the Hubble density $\rho_H=\frac{c^3}{\sqrt{\Lambda}G}=10^{-23}\frac{kg}{m^3}$, where we have made convenient use of the fact that the Universe today is flat and dominated by Λ (in fact, a yet unexplained coincidence). Very interestingly, there is a factor of 10^{60} appearing

$$\frac{l_H}{l_P}=10^{60} \quad \text{as well as} \quad \frac{t_H}{t_P}=10^{60} \tag{F.276}$$

suggesting a factor of 10^{120} between ρ_P and ρ_H. Perhaps a better way to phrase the scale problem is to ask why the Universe is so large an empty, and it is clear that accelerated expansion is able to achieve this, by making the Hubble-Lemaître parameter small and, by extension, giving the critical density a small value, too. All in all, these three problems are solved by having an early period of accelerated expansion, called cosmic inflation: it drives the curvature towards small values, shrinks the horizon and makes Universe large.

F.2 *Why is accelerated expansion (and stopping it) so difficult?*

General relativity provides gravity in the form of spacetime curvature for any energy momentum-tensor $T_{\mu\nu}$, which is covariantly conserved, $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu}=0$, and the trace $T=g^{\mu\nu}T_{\mu\nu}$ of the energy momentum tensor is proportional to the Ricci-curvature, as required by the trace of the entire field equation:

$$R_{\mu\nu}-\frac{R}{2}g_{\mu\nu}=-\frac{8\pi G}{c^4}T_{\mu\nu}-\Lambda g_{\mu\nu} \tag{F.277}$$

resulting from $g^{\mu\nu}R_{\mu\nu} = R$ as well as $g^{\mu\nu}g_{\mu\nu} = \delta^{\mu}_{\mu} = 4$. The trace of the energy momentum tensor is surely an invariant quantity but unlike electric charges which can have either of two possible signs, the energy momentum tensor is subjected to energy conditions, making sure that the energy momentum content of spacetime is bounded by zero from below and that gravity is attractive. Working with an ideal fluid

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - p g_{\mu\nu} \tag{F.278}$$

one can define the energy conditions through contractions with $T_{\mu\nu}$ and reexpressing them with density ρ and pressure p.

1. null energy condition ($\rho + p \geq 0$) resulting from $T_{\mu\nu}k^\mu k^\nu \geq 0$ for all fluids, if k^μ is a null-vector $g_{\mu\nu}k^\mu k^\nu = 0$.
2. weak energy condition ($\rho \geq 0$, matter density always positive) resulting from above's $T_{\mu\nu}u^\mu u^\nu \geq 0$, for time-like u^μ with $g_{\mu\nu}u^\mu u^\nu = c^2$ for the tangent $u^\mu = \mathrm{d}x^\mu \mathrm{d}\tau$ to a world line $x^\mu(\tau)$ of an observer.
3. strong energy condition ($\rho + 3p \geq 0$ for an ideal fluid) resulting from scalar $R_{\mu\nu}u^\mu u^\nu \geq 0$ for all fluids.

Therefore gravity is attractive and curves geodesics towards each other. The three conditions are subsets of each other and are related to each other by contraction of $k^\mu k^\nu$ or $u^\mu u^\nu$ with the field equation, similarly to the contraction with $g_{\mu\nu}$, and working best with an ideal fluid for $T_{\mu\nu}$. Thus it is very complicated to generate repulsive gravity, because all together $\rho \geq 0$ (weak), $\rho + p \geq 0$ (null) and $\rho + 3p \geq 0$ (strong) but for repulsion one needs $p < -\frac{1}{3}\rho$ (or $w < -\frac{1}{3}$) resulting in acceleration, $q > 0$.

Furthermore, it is clear that in the course of the Hubble expansion, the fluids will dominated in the order of descending value for their equation of state w: Once one has established accelerated expansion with a fluid $w < -1/3$, it is very difficult to return to e.g. matter domination with $w = 0$! Keeping in mind that $3(1 + w) = 2(1 + q)$ for a critical FLRW-universe with density parameter $\Omega = 1$ for a fluid with an arbitrary but constant equation of state w on would get a progression

$$\Omega_r \qquad \Omega_m \qquad \Omega_K \qquad \Omega_\Lambda \tag{F.279}$$

$$w = +\frac{1}{3} \qquad w = 0 \qquad w = -\frac{1}{3} \qquad w = -1 \tag{F.280}$$

$$q = 1 \qquad q = \frac{1}{2} \qquad q = 0 \qquad q = -1 \tag{F.281}$$

To make this explicity, we write down the evolution of the density parameter for a fluid with fixed equation of state w,

$$\frac{\Omega_w(a)}{\Omega_w} = \frac{H_0^2}{a^{3(1+w)}H^2(a)} \tag{F.282}$$

Comparing two such fluids with equations of state w and w' would result in

$$\frac{\Omega_{w'}(a)}{\Omega_w(a)} = \frac{\Omega_{w'}}{\Omega_w} \times a^{-3(w'-w)} \tag{F.283}$$

which increases if $w < w'$ and decreases if $w > w'$. Therefore, the fluid with the most negative equation of state will eventually dominate if the Hubble-function is monotonic: This result is actually very intuitive, as fluids with more negative equation of state parameters tend to have a slower evolution of ρ, such that they eventually dominate. The extreme case of this is Λ with a constant energy density, whose domination will be the natural target of the evolution of the Universe unless the densities of the other fluids are so high that they can halt the Hubble function or make the Universe recollapse.

Therefore, one needs a construction where the Universe is dominated by a fluid with sufficiently negative equation of state $w < -1/3$ such that curvature decreases, but which is able to return eventually back to being dominated by matter with $w = 0$ or radiation with $w = +1/3$, in agreement with observations at redshifts $z > 1$.

F.3 *Quintessence and dynamic dark energy*

Summarising the key results of the last two sections one sees that (i) accelerated expansion can be started with a fluid with a sufficiently negative equation of state but that (ii) it would be difficult to terminate the accelerated expansion and return to radiation- or matter-dominated, decelerated expansion. The solution to this problem is to construct a microscopic model behind the energy momentum tensor consisting of a self-interacting scalar field φ, called quintesence, which follows its own dynamics and which is gravitationally acting on the FLRW-background. Such a system has a dynamically evolving energy density and an equation of state and can terminate accelerated expansion naturally.

The Lagrange-density $\mathcal{L}$ of a scalar field φ on a possibly curved background with a metric $g_{\mu\nu}$ is given by

$$\mathcal{L}(\varphi, \nabla_\mu\varphi) = \frac{1}{2} g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi) \tag{F.284}$$

with a self-interaction potential $V(\varphi)$ including a mass term $V(\varphi) = m^2\varphi^2/2$. The Euler-Lagrange equation follows directly from variation of the action S

$$S = \int d^4x\sqrt{-\det g}\;\mathcal{L}(\varphi, \nabla_\mu\varphi) \tag{F.285}$$

where the covolume $\sqrt{-\det g}$ takes care of non-Cartesian coordinates. Hamilton's principle assumes that $\delta S = 0$ and therefore

$$\delta S = \int d^4x\sqrt{-\det g}\left(\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta(\nabla_\mu\varphi)\right) = \int d^4x\sqrt{-\det g}\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \nabla_\mu\partial\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\right)\delta\varphi \tag{F.286}$$

after an integration by parts, as done with the Gauss-theorem for integrations on manifolds,

$$\int_V d^4x\sqrt{-\det g}\;\nabla_\mu(av^\mu) = \int_{\partial V} dS_\mu\sqrt{|\det\gamma|}(av^\mu) = 0 \tag{F.287}$$

for a vector field υ and a scalar field a, which are assumed to reach values of zero on the integration boundary or at least asymptote towards zero fast enough. Formally, the co-volume $\sqrt{-\det g}$ gives rise to an induced measure $\sqrt{\det\gamma}$ (a co-area, in lack of a better expression) on the boundary ∂V, as γ is the induced metric on ∂V, $\gamma = g(\partial V)$. This leads to

$$\int_V d^4x\sqrt{-\det g}\,\nabla_\mu(a\upsilon^\mu) = \int_V d^4x\sqrt{-\det g}\,(\nabla_\mu a\cdot\upsilon^\mu + a\nabla_\mu\upsilon^\mu) \tag{F.288}$$

as the covariant derivative obeys the Leibnitz-rule, implying that if the surface integral vanishes due to fast enough decaying fields, that

$$\int_V d^4x\sqrt{-\det g}\,\nabla_\mu a\cdot\upsilon^\mu = -\int_V d^4x\sqrt{-\det g}\,a\nabla_\mu\upsilon^\mu \tag{F.289}$$

and everything looks like a straightforward integration by parts.

Deriving now all terms for the Euler-Lagrange equation gives first of all

$$\frac{\partial\mathcal{L}}{\partial\varphi} = -\frac{dV}{d\varphi} \tag{F.290}$$

because the potential V depends only on the field φ, as well as

$$\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)} = \frac{1}{2}\frac{\partial}{\partial(\nabla_\mu\varphi)}\left(g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi\right) = \frac{1}{2}g^{\alpha\beta}\Bigg(\underbrace{\frac{\partial\nabla_\alpha\varphi}{\partial\nabla_\mu\varphi}}_{=\delta^\mu_\alpha}\nabla_\beta\varphi + \nabla_\alpha\varphi\underbrace{\frac{\partial\nabla_\beta\varphi}{\partial\nabla_\mu\varphi}}_{=\delta^\mu_\beta}\Bigg) = g^{\alpha\mu}\nabla_\alpha\varphi \tag{F.291}$$

and further concluding that

$$\nabla_\mu\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)} = \nabla_\mu(g^{\alpha\mu}\nabla_\alpha\varphi) = g^{\alpha\mu}\nabla_\mu\nabla_\alpha\varphi \tag{F.292}$$

using metric compatibility of the covariant derivative. Therefore, the quintessence equation of motion for the field φ looks like a wave equation, or better, a covariant version of the Klein-Gordon equation,

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = -\frac{dV}{d\varphi} \tag{F.293}$$

driven by the self-interaction $V(\varphi)$, which as stated before, may include a mass-term for the field φ. As this will facilitate the treatment later, we can rewrite the term $g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi$ as a covariant divergence for which there is a very practical formula:

$$g^{\mu\nu}\nabla_\mu\underbrace{\nabla_\nu\varphi}_{=\partial_\nu\varphi=\upsilon_\nu} = \nabla_\mu(g^{\mu\nu}\upsilon_\nu) = \nabla_\mu\upsilon^\mu = \frac{1}{\sqrt{-\det g}}\partial_\mu(\sqrt{-\det g}\,\upsilon^\mu) \tag{F.294}$$

making use of metric compatibility again and introducing the determinant g of the metric. For illustrative purposes we have defined the linear form $\upsilon_\mu = \partial_\mu\varphi = \partial_\mu\varphi$ as the field gradient in φ.

Restricting the background now to conform to the FLRW-symmetries on can determine the covolume to be $\sqrt{-\det g} = a^3$ and both spacetime and the field φ only possesses an evolution in the t-direction, such that $\partial_\mu \to \partial_t$. Then, the divergence becomes

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = \frac{1}{a^3}\partial_t(a^3\partial_t\varphi) = 3\frac{\dot{a}}{a}\partial_t\varphi + \partial_t^2\varphi \tag{F.295}$$

leading us finally to

$$\partial_t^2\varphi + 3\mathrm{H}(t)\partial_t\varphi = -\frac{\mathrm{dV}}{\mathrm{d}\varphi} \tag{F.296}$$

which is the Klein-Gordon equation for the field φ with self-interaction V. The FLRW-background manifests itself as the second term in eqn. (F.296), which is proportional to $\mathrm{H} = \dot{a}/a$: For large H it works like a damping term restricting the evolution of the field φ and is aptly named Hubble-drag. But please do keep in mind that there are no dissipative effects implied, the term purely arises because of the dynamic background.

F.4 *Gravity of the quintessence filed*

In the previous section we have derived the equation of motion of a scalar field on a FLRW-background and arrived at the Klein-Gordon-equation

$$\partial_t^2\varphi + 3\mathrm{H}(t)\partial_t\varphi = -\frac{\mathrm{dV}}{\mathrm{d}\varphi} \tag{F.297}$$

for the field evolution for a given background dynamics encapsulated in $\mathrm{H}(t)$. The background could be defined by a pre-determined Hubble-function $\mathrm{H}(t)$ with the field φ as a test object, but the more interesting case is certainly where the field φ itself exerts a gravitational effect onto the background, such that one deals with a coupled system of (i) the Klein-Gordon-equation for the evolution of φ and the (ii) Friedmann-equation sourced by the energy momentum-content of φ for the evolution of $\mathrm{H}(t)$.

If $\mathcal{L}$ depends on the field φ and its derivative $\nabla_\mu\varphi$, but not explicitly on the coordinates x^μ, then there is an associated covariant conservation law:

$$g^{\mu\nu}\nabla_\alpha \mathrm{T}_{\mu\nu} = 0 \tag{F.298}$$

Loosely speaking, because the definition of the field dynamics are invariant under shifts on the manifold, energy and momentum are conserved. A counter example would e.g. be a position- or time dependent change in the Lagrange-density of e.g. electrodynamics: Then, the energies of atomic lines would be different in different places of the Universe, and emission processes in the distant Universe would not be compatible with absorption processes in the Milky Way.

As in classical mechanics one can construct the Beltrami-identity

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta(\nabla_\mu\varphi) = \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta\varphi\right) - \nabla_\mu\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi \tag{F.299}$$

where on recognises the Euler-Lagrange equation for φ

$$\delta\mathcal{L} = \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta\varphi\right) + \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \nabla_\mu\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\right)}_{\text{Euler-Lagrange}=0}\delta\varphi \tag{F.300}$$

such that the final result for the variation of $\mathcal{L}$ caused by the field variation $\delta\varphi$ is given by

$$\delta\mathcal{L} = \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta\varphi\right) \tag{F.301}$$

For $\delta\varphi$ we construct an infinitesimal field variation $\delta\varphi$ through a coordinate shift

$$\varphi(x^\mu + \delta x^\mu) = \varphi(x^\mu) + \nabla_\nu\varphi(x^\mu)\delta x^\nu + \ldots \quad \rightarrow \quad \delta\varphi = \varphi(x^\mu + \delta x^\mu) - \varphi(x^\mu) = \nabla_\nu\varphi(x^\mu)\delta x^\nu \tag{F.302}$$

under which the Lagrange-density transforms according to

$$\mathcal{L}(\varphi, \nabla_\mu\varphi) \rightarrow \mathcal{L}(\varphi, \nabla_\mu\varphi) + \nabla_\nu\mathcal{L}\delta x^\nu \quad \rightarrow \quad \delta\mathcal{L} = \nabla_\nu\mathcal{L}\,\delta x^\nu \tag{F.303}$$

Now, we can write the variation $\delta\mathcal{L}$ as resulting from the field variation $\delta\varphi$, as there can not be a variation of the working principle of the field theory with coordinate itself, according to the assumption that the functional shape and therefore the working principle of the field φ is universal and would not depend on the coordinate x^μ:

$$\delta\mathcal{L} = g^{\mu\beta}\nabla_\mu\mathcal{L}\delta x_\beta = \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta\varphi\right) = \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\underbrace{\nabla_\nu\varphi\delta x^\nu}_{=g^{\alpha\beta}\nabla_\alpha\varphi\delta x_\beta}\right) \tag{F.304}$$

resulting in

$$\nabla_\mu\left(g^{\mu\beta}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}g^{\alpha\beta}\nabla_\alpha\varphi\right)\delta x_\beta = 0 \tag{F.305}$$

Identifying the term in the bracket in eqn. F.305 to be the energy-momentum tensor $T^{\mu\beta}$ shows the corresponding covariant conservation law

$$\nabla_\mu T^{\mu\beta} = 0 \tag{F.306}$$

for the energy-momentum tensor $T^{\alpha\beta}$, that results directly from the Lagrange-density $\mathcal{L}$ of the quintessence field φ

$$T^{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi)}g^{\beta\nu}\nabla_\nu\varphi - \mathcal{L}g^{\alpha\beta} \tag{F.307}$$

The explicit result $T^{\alpha\beta}$ for the scalar field φ by substituting its Lagrange-density

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi - V(\varphi) \tag{F.308}$$

into eqn. F.307, making use of

$$\frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi)} = \frac{1}{2}g^{\mu\nu}\left(\underbrace{\frac{\partial\nabla_\mu\varphi}{\partial\nabla_\alpha\varphi}}_{\delta^\alpha_\mu}\nabla_\nu\varphi + \nabla_\mu\varphi\underbrace{\frac{\partial\nabla_\nu\varphi}{\partial\nabla_\alpha\varphi}}_{\delta^\alpha_\nu}\right) = g^{\mu\alpha}\nabla_\mu\varphi \tag{F.309}$$

such that one arrives at an expression for the energy-momentum tensor as it is determined from the gradients $\nabla_\mu\varphi$ of the field and the strength $V(\varphi)$ of the field's self-interaction:

$$T^{\alpha\beta} = g^{\mu\alpha}g^{\beta\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\nabla_\mu\varphi\nabla_\nu\varphi + V(\varphi)g^{\alpha\beta} \tag{F.310}$$

It is instructive to interpret this result for the the energy-momentum tensor with that of an ideal fluid

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)u^\alpha u^\beta - pg^{\alpha\beta} \tag{F.311}$$

and possibly derive ρ and p from the terms ∇_φ and $V(\varphi)$: In particular for a FLRW-spactime with spatial homogeneity one should then be able to derive ρ and p, as they would result dynamically from solving the Klein-Gordon-equation and compute the evolution of the scale factor a from the Friedmann-equations, such that one has constructed a coupled dynamical system for φ and a, possibly with a dynamical relation between p and ρ, or, equivalently, a dynamically evolving equation of state $w = p/(\rho c^2)$.

Parameterising a FRLW-spacetime with comoving coordinates x^μ yields for the velocities $u^\mu = \mathrm{d}x^\mu/\mathrm{d}t = \mathrm{d}x^\mu/\mathrm{d}t = (c,0)^t$ as tangents to the world lines of fluid elements simplifies the energy-momentum tensor tremendously: It will be diagonal (as the inverse metric $g^{\mu\nu}$ is diagonal is the FLRW-case) and have the tt-component

$$T^{tt} = \left(\rho + \frac{p}{c^2}\right)u^t u^t - pg^{tt} = \rho c^2 \tag{F.312}$$

with $g^{tt} = 1$, and the spatial ii-components

$$T^{ii} = \left(\rho c^2 + \frac{p}{c^2}\right)u^i u^i - pg^{ii} = 3\frac{p}{a^2} \tag{F.313}$$

as the spatial part of the inverse metric is $g^{ii} = -a^{-2}$ and $u^i = 0$ for comoving fluid elements.

Isolating these two components from the energy-momentum tensor for the field φ is straightforward in particular under the assumption of the FLRW-symmetries, where all spatial derivatives are zero and because the field φ is scalar, implying that $\nabla_\mu\varphi = \partial_\mu\varphi$ of which only $\partial_t\varphi$ is nonzero. Therefore, the density ρ must be

$$\rho c^2 = T^{tt} = g^{t\alpha}g^{t\beta}\nabla_\alpha\varphi\nabla_\beta\varphi - \frac{1}{2}g^{tt}g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi + V(\varphi)g^{tt} = \frac{1}{2}(\partial_t\varphi)^2 + V(\varphi) \tag{F.314}$$

and similarly for the spatial part yielding pressure p

$$3\frac{p}{a^2} = T^{ii} = g^{i\alpha}g^{i\beta}\nabla_\alpha\varphi\nabla_\beta\varphi - \frac{1}{2}g^{ii}g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi + V(\varphi)g^{ii} = 3a^{-2}\frac{1}{2}(\partial_t\varphi)^2 - 3a^{-2}V(\varphi) \tag{F.315}$$

which can be simplified to

$$p = \frac{1}{2}(\partial_t\varphi)^2 - V(\varphi) \tag{F.316}$$

Combining both results is a construction of the equation of state w

$$w = \frac{p}{\rho c^2} = \frac{\frac{1}{2}(\partial_t\varphi)^2 - V(\varphi)}{\frac{1}{2}(\partial_t\varphi)^2 + V(\varphi)} \tag{F.317}$$

which gives a direct indication of the gravitational effect of the field φ, as both ρ and p enter the gravitational field equation. In particular, if the evolution of the field is slow and therefore the kinetic term $\frac{1}{2}(\partial_t\varphi)^2$ is much less than the potential term $V(\varphi)$, one obtains for the equation of state is $w \sim -1$. Then, the gravitational effect of φ is identical to that of the cosmological constant Λ and the FLRW-spacetime is accelerating at $q = -1$, leading to exponential expansion.

In the course of time evolution with the Klein-Gordon equation

$$\partial_t^2\varphi + 3H(t)\partial_t\varphi = -\frac{dV}{d\varphi} \tag{F.318}$$

one would expect that $(\partial_t\varphi)^2$ increases at the expense of $V(\varphi)$, and that the equation of state evolves away from the value $w = -1$, as the slow-roll condition

$$\frac{1}{2}(\partial_t\varphi) \ll V(\varphi) \tag{F.319}$$

is violated. For instance, when $\frac{1}{2}(\partial_t\varphi)^2 \sim V(\varphi)$ is reached, the equation of state becomes $w = 0$, corresponding to a decelerated universe with $q = \frac{1}{2}$, as if it was filled with matter. Clearly, the quintessence field shows a variable gravitational effect on the FLRW-background, and in particular does it provide a mechanism of driving accelerated expansion to solve the flatness-, horizon- and scale-problems, and a natural way of stopping inflation and returning to normal expansion dominated by fluids with less negative equations of state.

F.5 *Slow-roll approximation*

Cosmic inflation as driven by the scalar field φ, if it should solve the horizon and flatness problems, has to provide accelerated expansion through a negative enough equation of state and take care that this period of accelerated expansion lasts long enough. These two conditions are ultimately requirements on the potential $V(\varphi)$, usually formulated in terms of the two slow-roll parameters ϵ and η:

$$\epsilon = \frac{1}{8\pi G}\left(\frac{d\ln V}{d\varphi}\right)^2 \quad \text{and} \quad \eta = \frac{1}{24\pi G}\left(\frac{1}{V}\frac{d^2V}{d\varphi^2}\right) \tag{F.320}$$

which are essentially logarithmic derivatives of the quintessence potential $V(\varphi)$. If ϵ and η are small, the potential has a small slope and is weakly curved, implying that the time evolution of φ is weak, slow-roll is maintained for a long time, and exponential, accelerated expansion is maintained, such that a low spatial curvature can be realised and the horizon becomes large enough.

A sufficiently negative equation of state parameter w for accelerated expansion is generated by the slow-roll condition itself, $\frac{1}{2}(\partial_t\varphi)^2 \ll V(\varphi)$. This condition implies directly for the first Friedmann-equation that

$$H^2(t) = \frac{8\pi G}{3}\rho c^2 = \frac{8\pi G}{3}\left(\frac{1}{2}(\partial_t\varphi)^2 + V(\varphi)\right) \quad\rightarrow\quad H^2(t) = \frac{8\pi G}{3}V(\varphi) \tag{F.321}$$

where we used the slow-roll in the last step. The acceleration $\ddot{a}$ can be derived from the latter equation by differentiating it with respect to t, yielding

$$2H\partial_t H = \frac{8\pi G}{3}\partial_t\varphi\frac{dV}{d\varphi} \tag{F.322}$$

by application of the chain rule to $\partial_t V(\varphi(t))$. The slow-roll approximated Klein-Gordon equation F.321 for the FRLW-background

$$3H\partial_t\varphi = -\frac{dV}{d\varphi} \tag{F.323}$$

implies the condition

$$\partial_t H = -4\pi G(\partial_t\varphi)^2 \ll 4\pi G V(\varphi) \tag{F.324}$$

as an expression for slow roll, on the basis of the potential and constrains the evolution of the Hubble-function H. This allows now to formulate the slow-roll parameters ϵ and η defined in eqn. F.320 in their dependence on the potential $V(\varphi)$.

The square of the approximate Klein-Gordon equation,

$$(3H\partial_t\varphi)^2 = \left(\frac{dV}{d\varphi}\right)^2 \tag{F.325}$$

together with the Friedmann-equation for H^2

$$3^2\frac{8\pi G}{3}V(\varphi)^2(\partial_t\varphi)^2 = \left(\frac{dV}{d\varphi}\right)^2 \tag{F.326}$$

shows that

$$(\partial_t\varphi)^2 = \frac{1}{24\pi G}\left(\frac{1}{V}\frac{dV}{d\varphi}\right)^2 = \frac{1}{24\pi G}\left(\frac{d\ln V}{d\varphi}\right)^2 \equiv \epsilon \ll 1 \tag{F.327}$$

where the slow-roll parameter $\epsilon \ll$ ensures that the kinetic term $(\partial_t\varphi)^2/2$ stays small.

Differentiating the approximate Klein-Gordon equation with respect to t yields

$$3(\partial_t H\partial_t\varphi + H\partial_t^2\varphi) \simeq 3\partial_H\partial_t\varphi = \frac{d^2V}{d\varphi^2}\partial_t\varphi \tag{F.328}$$

where we neglect $H\partial_t^2\varphi$ over $\partial_t H\partial_t\varphi$ and which we divide with $\partial_t\varphi$ for

$$3\partial_t H = -\frac{d^2V}{d\varphi^2} \tag{F.329}$$

But because of the fact that $\partial_t H = -4\pi G(\partial_t\varphi)^2 \ll 4\pi G V(\varphi)$ as derived above, one can conclude that the second slow-roll parameter η,

$$(\partial_t\varphi)^2 = \frac{1}{12\pi G V}\frac{d^2V}{d\varphi^2} \equiv \eta \ll 1 \tag{F.330}$$

must be small compared to one as well.

F.6 *Accelerated expansion in the late Universe*

To what limit the accelerated expansion at the current time is related to quintessence at early times is unclear, but the mechanism works in both cases: at early times, as cosmic inflation and at late times as dark energy. Whether inflation in the early Universe is initiated by randomly setting the right initial conditions for the field φ (the exact mechanism of this is still unclear), achieving domination of φ in the late Universe at redshifts below unity in a natural way is equally difficult. Many dark energy models link accelerated expansion to other physical processes, for instance, the acquisition of mass in neutrinos.

F.7 *Seeding of cosmic structures in inflation*

Apart from solving the flatness and horizon problems, cosmic inflation provides a mechanism for seed fluctuations from which the cosmic large-scale can grow: The exact mechanism is quite technical, but the fundamental idea is that the comoving horizon $c/(aH)$ shrinks during the accelerated expanding phase. Fluctuations in the metric with a fixed comoving wave length are initialised at the instant when they leave the (shrinking) horizon, at an amplitude that is given by the so-called Bunch-Davies vacuum, which corresponds to the ground state amplitude of the field φ.

The amplitude of these perturbation in φ and the associated fluctuations in the metric $\delta\Phi$ are roughly given by $\sqrt{\langle\delta\Phi^2\rangle} \simeq H^2/V(\varphi)$, which is roughly constant while the expansion is exponential. One can now relate fluctuations in the potential Φ to fluctuations in the density field by invoking the Poisson-equation which reads in Four-space $k^2\Phi(k) = -\delta(k)$.

Then, the relation

$$|\delta(k)|^2 \propto k^4\,|\delta\Phi|^2 \propto k^3 P(k) \tag{F.331}$$

for the variance of the density field fluctuations in Fourier-space applies, which is related to the variance in the potential fluctuations. If $|\delta\Phi|^2$ is constant as predicted by the constant Hubble-function, the spectrum $P(k)$ must be $\propto k$ to give a consistent scaling.

In reality, there are tiny deviations from perfect exponential expansion, of the order of the slow-roll parameters ϵ and η. As a consequence, there is a minute evolution of the Hubble-function and the amplitude $\sqrt{\langle\delta\Phi^2\rangle}$ becomes a function of time. As the comoving horizon evolves, that time-dependence can be converted into a scale dependence, which effectively makes $P(k) \propto k^{n_s}$ with $n_s \simeq 0.96$, deviating slightly from unity, by a quantity of the order of the slow-roll parameters.