C FLRW-COSMOLOGIES

C.1 *Dynamics of spacetime*

Over hundert years ago, E. Hubble did research on spectral lines of distant galaxies. He discovered that the spectral lines are shifted towards longer wave lengths, which he interpreted as a Doppler shift caused by the motion of galaxies away from us as observers. This recession motion increases proportionally to distance:

$$\upsilon = H_0 r \tag{C.156}$$

with the Hubble-Lemaître constant $H_0 = 10^5 h$ m/s/Mpc, and the Hubble-parameter $h = 0.68 \ldots 0.72$, depending on the measurement method. For a galaxy 10 Mpc away from the Milky Way, the recession velocity would be $\beta = \upsilon/c \simeq 0.003$, which is easily measurable through spectroscopy. While the interpretation of a recession motion is absolutely valid in Newtonian cosmology, general relativity brings in a new concept, namely that the laws of Nature, in particular gravity, are fully covariant, i.e. that coordinate choice does not matter, and that different coordinate choices require different physical interpretations.

If one adopts physical coordinates, consisting of a static coordinate grid, through which the galaxies move isotropically as the Universe expands, one obtains for the relation between velocity and distance

$$\upsilon(r, t) = H(t) r \tag{C.157}$$

which can only depend on time in fulfilment of the cosmological principle: Including any nonlinear dependence of r causes a violation of homogeneity: Starting from the continuity equation for the matter density ρ

$$\dot{\rho} + \partial_i \jmath^i = 0 \tag{C.158}$$

with the momentum density $\jmath^i = \rho\upsilon^i$. If the velocity fulfils the Hubble-law $\upsilon^i = H_0 r^i$ it would imply for the divergence

$$\partial_i \jmath^i \overset{\text{isotropy}}{=} \frac{1}{r^2}\partial_r(r^2\rho\upsilon_r) = \frac{\rho H}{r^2}\partial_r(r^3) = 3H\rho \tag{C.159}$$

if one in addition assumes isotropy such that the velocity has only a radial dependence and using spherical coordinates to formulate the divergence. If ρ does not depend on r, as used in the last step, its time evolution will make sure that it will stay homogeneous. The situation would be fundamentally different if for instance $\upsilon = r^\alpha$, with $\alpha \neq 1$. Then,

$$\partial_i \jmath^i = \frac{1}{r^2}\partial_r(r^2\rho\upsilon_r) = \frac{\rho H}{r^2}\partial_r(r^{2+\alpha}) = (2+\alpha)H\rho r^{\alpha-1} \tag{C.160}$$

such that the time evolution of ρ depends on r, and the continuity equation can not uphold homogeneity, in violation of the cosmological principle.

In comoving coordinates the picture is different: The coordinate grid expands along with the flow of matter, and all particles stay at their comoving coordinate. We therefore differentiate between comoving coordinates x^i and physical coordinates

$r^i = a(t)x^i$, which are related through the scale factor $a(t)$, which itself is only a function of time t. The coordinate change of the physical coordinate with time is given by

$$\frac{\mathrm{d}r^i}{\mathrm{d}t} = \upsilon^i = \dot{a}x^i + a\dot{x}^i = \mathrm{H}(t)r^i + a\upsilon^i_{\mathrm{pec}} \tag{C.161}$$

with two possible contributions of the spectroscopically measured velocity υ^i: The cosmological part due to a nonzero $\mathrm{H}(t) = \dot{a}/a$ and a peculiar motion $\upsilon^i_{\mathrm{pec}}$ relative to the (comoving) coordinate grid: When considering truly fundamental observers and test particles, the peculiar velocity would be zero.

The FLRW-metric of a flat space is usually given in terms of the line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, which reads in comoving coordinates

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = c^2\mathrm{d}t^2 - a^2(t)\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{C.162}$$

so that the spatial part of the metric (here written down in Cartesian and in spherical coordinates) is scaled by the scale factor $a(t)^2$. The choice of comoving coordinates is uniquely suited to the symmetries of a FLRW-spacetime: Neither does the metric depend on position, nor does it single out any particular direction.

This form of the line element, however, is not the most general possible compatible with the cosmological principle: The spatial part of the spacetime can have a constant curvature such that the scaling of surfaces of spheres with their radii differs from the Euclidean prediction. Introducing a curvature parameter k we can write

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{C.163}$$

There is a peculiarity of the FLRW-spacetime that concerns the passage of time: While it is perfectly normal that time passes at a different rate at different locations in a gravitational potential, this is not the case in FLRW-cosmologies. In fact, the line element $\mathrm{d}s$ is perceived as the elapsed proper time $\mathrm{d}\tau$ by an observer,

$$c^2\mathrm{d}\tau^2 = \mathrm{d}s^2 \tag{C.164}$$

such that according to homogeneity, $\mathrm{d}\tau = \mathrm{d}t$ for the FLRW-spacetime: Every observer sees the same passage of time and the coordinate time t is equal to proper time τ. This has profound consequences, as it enables a universal definition of the age of the Universe, which necessarily needs to be equal for every observer.

C.2 *Light propagation on a FLRW-spacetime and redshift*

As the coordinate choice is arbitrary rates of change of coordinates should not be assigned any physical meaning, in particular if these velocities are compared to the speed of light c. Whether a cosmological object is visible or not depends on whether a geodesic line between that object and an observer exists or not, specifically for photons this must be a null-geodesic with a normalisation $\mathrm{d}s^2 = g_{\mu\nu}k^\mu k^\nu = 0$ of the wave vector $k^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$.

The null-property of the wave vector ensures that photons propagate dispersion-free in vacuum. In fact, writing k^μ in components

$$k^\mu = \begin{pmatrix} \omega/c \\ k^i \end{pmatrix} \tag{C.165}$$

with the angular frequency ω and the spatial wave vector k^i leads to the norm

$$g_{\mu\nu}k^\mu k^\nu = \left(\frac{\omega}{c}\right)^2 - k_i k^i = 0 \tag{C.166}$$

leads to a linear relation between angular frequency and wave number

$$\omega(k) = \pm ck \tag{C.167}$$

and in consequence to equal phase and group velocities,

$$\upsilon_\mathrm{group} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = c \quad \text{and} \quad \upsilon_\mathrm{phase} = \frac{\omega}{k} = c. \tag{C.168}$$

Dispersion-free propagation of photons $\upsilon_\mathrm{group} = \upsilon_\mathrm{phase}$ is encoded by the fact that their wave vector k^μ is a null-vector.

The null-condition $\mathrm{d}s^2 = 0$ has a very intricate connection to FLRW-spacetimes, as they are conformally flat: The full Riemann-curvature decomposes into two contributions: Weyl-curvature and Ricci-curvature, and the FLRW-symmetries in fact make sure that cosmological solutions are of pure Ricci-curvature, as the Weyl-tensor vanishes identically. Spacetimes, in which this is the case, are conformally flat, as their metric can be written as a rescaled Minkowksi-metric with a conformal factor $\Omega(x^\mu)^2 > 0$, which is strictly positive,

$$g_{\mu\nu} = \Omega(x^\mu)^2\,\eta_{\mu\nu} \tag{C.169}$$

as conformal transformations leave they Weyl-tensor $C_{\alpha\beta\mu\nu}$ invariant and conserve in the FLRW-case its value of zero. Applied to cosmology, we would write for the line element

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = a^2(t)\left(c^2\frac{\mathrm{d}t^2}{a^2}\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) = a^2(t)\left(c^2\mathrm{d}\eta^2 - \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) \tag{C.170}$$

with a new time coordinate $\mathrm{d}\eta$, which is called conformal time:

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a} \tag{C.171}$$

and the scale-factor $a(t)$ is in fact the conformal factor $\Omega(x^\mu)$ which in our case only depends on time and not on the spatial coordinates.

In fact, $\mathrm{d}\eta$ is not uniformly passing unlike $\mathrm{d}t$. Only today with $a = 1$ time intervals in η and t are identical, and as $a < 1$ in the past, intervals in η have been larger than those in t.

This has two interesting consequences: Firstly, light-propagation in a conformally flat spacetime proceeds in a perfectly Minkowskian way as the conformal factor drops out in the null-condition:

$$ds^2 = a^2(t)\left(c^2 d\eta^2 - \gamma_{ij} dx^i dx^j\right) = 0 \tag{C.172}$$

Secondly, the conformal age of the Universe is in fact infinite even if the actual age of the Universe (defined as the physical time passing since the instant $a = 0$) is finite, as the coordinate axes of Minkowski-space stretch out to infinity. Because homogeneity of the FLRW-spacetime allow always to place the origin of the coordinate frame at the observer, all photons are radially moving, so one can write for the line element

$$ds^2 = \left(c^2 d\eta^2 - d\chi^2\right) = (c d\eta + d\chi)(c d\eta - d\chi) = dv dw = 0 \tag{C.173}$$

and define light cone coordinates $dv = c d\eta + d\chi$ and $dw = c d\eta - d\chi$, reminiscent of Kruskal-coordinates.

C.3 *Evolution of the Hubble-expansion with time*

Initially, the Hubble function $H(t)$ was introduced for parameterising the linear relationship between the recessional velocity v and distance r, $v = H(t)r$, and with the definition $H(t) = \dot{a}/a$ we relate it to a Taylor-expansion of $a(t)$ at the current cosmic epoch t_0,

$$a(t) - a(t_0) = \frac{da}{dt}(t - t_0) + \frac{d^2 a}{dt^2}\frac{(t - t_0)^2}{2} \pm \ldots \tag{C.174}$$

which can be rewritten as

$$a(t) = a(t_0)\left(1 + H(t_0)(t - t_0) - q(t_0)H^2(t_0)\frac{(t - t_0)^2}{2}\right) \tag{C.175}$$

by renormalising everything with $a(t_0)$. Taking every function to be evaluated at t_0, $\dot{a}/a$ becomes the Hubble function and $\ddot{a}/a = a\ddot{a}/a^2 \times \dot{a}^2/\dot{a}^2 = -qH^2$ brings in the deceleration parameter, usually defined with a minus-sign:

$$q = -\frac{\ddot{a}a}{\dot{a}^2} \tag{C.176}$$

despite the fact that the Universe is currently accelerating, so $\ddot{a} > 0$ causes q to be negative, as both a and $\dot{a}^2$ are positive; this is in fact a historical remnant. In summary, H determines the current rate at which the scale factor changes as a function of time, and q states by how much that rate changes with time. It is interesting to realise that the Hubble-relation is valid at every instance in time simultaneously for every distance, but of course we do not observe the recession velocity of a distant galaxy at the time that the light was emitted - so as we look out into the distance along the past light cone, we see a record of the recession velocities.

C.4 *Field equation: coupling gravity to matter*

The curvature of spacetime is determined by the energy momentum tensor by means of the field equation

$$\underbrace{R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}}_{G_{\mu\nu}} = -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{C.177}$$

which equates the Einstein-tensor $G_{\mu\nu}$ to the energy momentum tensor $T_{\mu\nu}$ with Newton's gravitational constant G as a coupling constant, but there is an effect of gravity of empty space, too: Even if $T_{\mu\nu} \equiv 0$, the curvature is nonzero due to the presence of the cosmological constant Λ. Actually, this result is not totally surprising as the cosmological constant was already present in the most general linear field theory for a scalar field on a Minkowski-background in the first chapter. The gravitational field equation is unique, as shown by David Lovelock, as the most general (i) second-order partial differential equation in (ii) 4 dimensions, with (iii) covariant energy momentum conservation $\nabla_\mu T^{\mu\nu} = 0$, which establishes a (iv) local relationship between curvature and the source of gravitational field and lastly, if (v) the metric is the only dynamical degree of freedom, from which the curvature is derived. In particular, Lovelock's result makes sure that there are only two tensors, the Einstein-tensor $G_{\mu\nu}$ and the metric $g_{\mu\nu}$, that have vanishing divergences, the first as a consequence of the Bianichi-identity and metric due to metric compatibility. The field equation can be derived by a variation of the Einstein-Hilbert-Lagrange density

$$S = \int d^4x \sqrt{-\det g}(R - 2\Lambda) \tag{C.178}$$

with respect to the (inverse) metric: The choice of this Lagrange-density is unique, again according to Lovelock's theorem.

Within the highly symmetric solutions of general relativity discussed in every textbook cosmology plays a central role: FLRW-spacetimes are, due to the cosmological principle, systems of pure Ricci-curvature (with a vanishing Weyl-tensor); and as such they do not show any propagation effects of gravity. Because of the small value of the cosmological constant, its effect on the dynamics of spacetimes becomes only dominant on scales comparable to the observable Universe.

C.5 *FLRW-spacetimes and their dynamics*

FLRW-cosmologies are a solution to the gravitational field equation with homogeneity and isotropy as symmetries restricting the complexity of the solution, and for ideal fluids as sources. As the only degree of freedom left after imposing the FLRW-symmetries is the scale factor $a(t)$, one effectively ends up at ordinary (albeit nonlinear) differential equations: In fact, the Friedmann equations relate $a(t)$ and its first and second derivatives $\dot{a}$ and $\ddot{a}$ to the properties of the fluid, i.e. density ρ and pressure p, which have to be constant across spactime at a fixed time, in order to fulfil the cosmological principle, too.

Starting from the FLRW-metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ (determined through the defining property $g_{\mu\nu}g^{\nu\alpha} = \delta^\alpha_\mu$) one computes the Christoffel-symbols

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right) \tag{C.179}$$

under the choice of a metric compatible and torsion-free connection. Then, the Riemann-curvature $R^{\alpha}{}_{\beta\mu\nu}$ follows from derivatives and squares of the Christoffel-symbols, and the Ricci-curvature $R_{\beta\nu} = g^{\alpha\mu}R_{\alpha\beta\mu\nu}$

$$R_{tt} = 3\frac{\ddot{a}}{a} \tag{C.180}$$

$$R_{rr} = \frac{-c^2}{1-kr^2}\left(a\ddot{a} + 2\dot{a}^2 + 2c^2k\right) \tag{C.181}$$

$$R_{\theta\theta} = -\frac{c}{r^2}\left(a\ddot{a} + 2\dot{a}^2 + 2c^2k\right) \tag{C.182}$$

$$R_{\phi\phi} = R_{\theta\theta} \cdot \sin^2\theta \tag{C.183}$$

such that contraction $g^{\mu\nu}R_{\mu\nu} = R$ yields the Ricci-scalar,

$$R(t) = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{ck}{a^2}\right]. \tag{C.184}$$

Substituting the Ricci-tensor and Ricci-scalar into the field equation for an ideal fluid gives the two Friedmann-equations, first from the spatial part of the field equation,

$$H^2(a) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{c^2 a}{a^2} \tag{C.185}$$

as well as from the temporal part,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{p}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{C.186}$$

The combination of Newton's gravitational constant G and the Hubble-constant H_0 provides naturally a density scale

$$\rho_{\mathrm{crit}} = \frac{3H^2}{8\pi G} \tag{C.187}$$

which helps to re-express the densities of all fluids by dimensionless density parameters

$$\Omega_i = \frac{\rho_i}{\rho_{\mathrm{crit}}} \tag{C.188}$$

such that the first Friedmann-equation can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_m}{a^3} + \Omega_\Lambda\right) \tag{C.189}$$

by using $\rho \propto a^{-3}$ for matter. This can easily be extended to further fluids, characterised by their equation of state parameters $w = p/(\rho c^2)$.

We therefore can assign an Ω to k, for consistency

$$1 = \Omega_k + \sum_i \Omega_i \tag{C.190}$$

as otherwise $H(1) \neq H_0$. The curvature Ω_K vanishes if $\sum_i \Omega_i = 1$, in this limit the spatial part of spacetime would be a flat, Euclidean space.

C.6 *Gravitating fluids and their associated dynamics*

By coupling the dynamics of spacetime its energy-momentum content through the field equation, we can predict the time evolution of the scale factor $a(t)$ for a given density and equation of state parameter. While it is obvious that high matter or radiation densities should have a decelerating effect on spacetime, we should have a more detailed look into the effect of the equation of state. Setting up a spatially flat FLRW-cosmology with a single fluid ($\rho = \rho_{crit}$) and a constant equation of state parameter w leads to the realisation that equation of state w and deceleration q are connected by

$$3(1 + w) = 2(1 + q) \tag{C.191}$$

Clearly, a sign change in q takes place at $w = -1/3$: While decelerated universes $q < 0$ need to have equations of state of $w > -1/3$, accelerated universes $q > 0$ are characterised by very negative equations of state $w < -1/3$. Interestingly, a fully curved, empty universe with $q = 0$ has an effective equation of state of $w = -1/3$, in accordance with the a^2-scaling of Ω_K. It expands at a constant $\dot{a}$ as there are no gravitating substances to change the state of motion.

It might be surprising that the deceleration is stronger for photons than for non-relativistic matter, but it is the case that photons on the other hand are more strongly affected by gravitational fields, too: That's the famous factor 2 in gravitational lensing by which the accelerating effect of a gravitational field on a photon is larger compared to a non-relativistic test particle.

Whether the FLRW-spacetime has a finite age depends on whether substances with $w > -1/3$ have been dominating the expansion at early times. Curvature and all substances with more negative equations of state tend to lead to infinitely old universes. As the Universe expands, densities scale proprotional to $a^{-3((1+w)}$, so it is the case that the more negative an equation of state is, the slower the fluid dilutes in the course of the Hubble-expansion, the ultimate example being cosmological constant Λ with $w = -1$, leading to a constant energy density.

C.7 *Redshift and the Hubble-expansion*

We observe spectral lines of distant galaxies, which are shifted towards the red or rather to lower energies. One should not think of the effect as a loss of energy, rather than a transformation effect: Surely, there is a redshifting effect due to the motion of a source relative to the observer already in special relativity, and in addition a geometric effect due to changes in the metric in general relativity. The interpretation of redshifting as a transformation effect can not depend on the choice of coordinates, but of course the prediction has to be independent of a specifically adopted coordinate choice, and in the following derivation we should illustrate this. Due to conformal flatness of FLRW-universes it is best to work in conformal coordinates $(c\eta, \chi)$ which illustrate the Minkowskian causal structure:

$$ds^2 = a^2(t)\left(c^2 d\eta^2 - d\chi^2\right) = 0 \tag{C.192}$$

In these particular coordinates, the metric is the Minkowski-metric, preceeded by $a^2(t)$ as the overall conformal factor,

$$g_{\mu\nu} = a^2(t)\eta_{\mu\nu} = \begin{pmatrix} c^2a^2 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix} \tag{C.193}$$

Motion of photons along the geodesic conserves the normalisation of the wave vector k^μ, so that $g_{\mu\nu}k^\mu k^\nu = 0$ is maintained. A measurement of the frequency of a photon takes place when the photon is intercepted by a timelike observer with a tangent u^μ to her or his world line $x^\mu(\tau)$. The resulting frequency ω is given by the projection

$$\omega = g_{\mu\nu}u^\mu k^\nu \tag{C.194}$$

and is, as a scalar product, a general scalar and invariant under coordinate transforms, as requested for the result of measurement. Clearly, the observed frequency can be affected by the relative orientation of k^μ and u^μ, which is the special relativistic Doppler-effect, but also by a non-Minkowskian scalar product mediated by the metric.

While the wave vector of a photon in conformal coordinates is oblivious to changes in the geometry due to conformal flatness, and the normalisation of the wave vector is conserved in geodesic motion,

$$g_{\mu\nu}k^\mu k^\nu = 0 \tag{C.195}$$

the actual velocities of comoving observers are non-constant: The motion of a galaxy is timelike with the normalisation

$$g_{\mu\nu}u^\mu u^\nu = c^2 > 0 \tag{C.196}$$

and even though the galaxy stays at its comoving coordinate, it moves non-uniformly through spacetime with respect to conformal time! A galaxy at rest in the comoving frame has only a nonzero t-component in its velocity,

$$g_{tt}u^t u^t = c^2 \quad \text{implying} \quad u^t = \frac{c}{\sqrt{g_{tt}}} = \frac{c}{a} \tag{C.197}$$

which is, perhaps a bit surprisingly, changing as $a(t)$ evolves, until it reaches c today: But please keep in mind that in conformal coordinates we're dealing with a non-uniform passing time coordinate. Computing the projection between k^μ and u^μ for the frequency gives

$$\omega' = g_{\mu\nu}u^\mu k^\nu = g_{tt}u^t k^t = a^2 \frac{c}{a}\frac{\omega}{c} = a\omega \tag{C.198}$$

which can be used to derive a relation for the shifted wave length, as $\omega = ck = c\frac{2\pi}{\lambda}$:

$$\lambda' = \frac{1}{a}\lambda \tag{C.199}$$

and therefore define the redshift z according to

$$z = \frac{\lambda' - \lambda}{\lambda} = \frac{1}{a} - 1 \tag{C.200}$$

and vice versa

$$a = \frac{1}{1+z} \tag{C.201}$$

with the convention that $a = 1$ and $z = 0$ for today. Lastly, I'd like to point out that the term redshift is perhaps not ideal: The entire spectrum of a source gets stretched by the scale factor a, and we should think of a shifting of the logarithmic wave length:

$$\ln\lambda = \ln(\lambda' a) = \ln\lambda' + \ln a = \ln\lambda' - \ln(1+z) \tag{C.202}$$

C.8 *Continuity equation and general relativity*

Einstein's field equation is prepared to conserve the energy-momentum-tensor

$$\nabla_\mu T^{\mu\nu} = 0 \tag{C.203}$$

with the energy in the time and the momenta in the spatial components. We arrived (using our covariant derivative) at

$$\partial_t\rho + 3H(t)(1+w)\rho = 0 \tag{C.204}$$

if the equation of state parameter w is constant in time. Pay attention to the fact, the $a(t)$ appears in the continuity equation even if the fluid is at rest in the comoving frame. In this continuity equation, the term $H = \dot{a}/a$ takes care of gravity, which is first of all surprising as there is no effect of Newtonian gravitational potentials on the continuity of classical fluid mechanics, only on the Euler-equation as an accelerating term. Clearly,

$$\partial_t\rho + \partial_i(\rho v^i) = 0 \tag{C.205}$$

does not depend on the gravitational potential Φ. In weak, static gravity on has the line-element

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right)dx_i dx^i \tag{C.206}$$

and see, that there are no gravitational effects whereas there are effects in the FLRW-metric. In the weakly perturbed metric (C.206) there is time-dilatation (which for us is not relevant, since t is the coordinate time). We have

$$\nabla_\mu(\rho u^\mu) + \frac{p}{c^2}\nabla_\mu u^\mu = 0 \tag{C.207}$$

which leaves us with the first term in the non-relativistic limit, as $p \ll \rho c^2$. Computing the covariant divergence then gives

$$\nabla_\mu(\rho u^\mu) = \partial_\mu(\rho u^\mu) + \Gamma^\mu{}_{\mu\alpha}(\rho u^\alpha). \tag{C.208}$$

Wherein the prefactors for the largest component, which is u^t, are 0 by construction, since $\Gamma^{\mu}{}_{\mu t} \sim \partial_t\Phi = 0$ for static fields, showing that there is no first-order influence of static, weak gravitational fields on the continuity equation: This is one instance where gravity really behaves differently than in a classical context.

C.9 *Construction of FLRW-universes for ideal fluids*

Gravity and the dynamical behaviour of the scale-factor $a(t)$ in a FLRW universe is sourced by an ideal fluid, at rest in the comoving frame: With the velocities of the fluid elements given by $u^\mu = (c, 0)^t$, the two only properties of the fluid to be specified are density ρ and pressure p, or equivalently, density ρ and equation of state parameter w. In many cases, w is constant in time and a genuine property of the fluid, such as $w = 0$ for nonrelativistic matter and $w = 1/3$ for photons. If there is just a single fluid with a constant equation of state, the density evolution is determined by the FLRW-background only and one obtains $\rho \propto a^{-3(1+w)}$.

The field equation reduces to the two Friedmann-equations under the assumption of the FLRW-symmetries, and as the field equation itself already respects covariant energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$, is is automatically fulfilled. This implies that of the two Friedmann-equations and the continuity equation only two relations are truly independent. Commonly, such as in the ΛCDM-class of cosmological models one assumes that (i) all fluids are independent (i.e. there is no direct coupling or transition of energy from one fluid to another) and that (ii) the equation of state parameter is fixed through the properties of the fluid (we will encounter different examples later, such as quintessence) and governs the adiabatic, energy-momentum conserving behaviour of the fluid. Then, the Hubble function can be assembled by writing

$$H(a) = H_0\sqrt{\sum_i \frac{\Omega_i}{a^{3(1+w_i)}} + \frac{\Omega_K}{a^2}} \tag{C.209}$$

with the sum over the individual densities fixing the global curvature,

$$\sum_i \Omega_i = 1 - \Omega_K. \tag{C.210}$$

Statements on acceleration as done by the second Friedmann-equation can be computed by taking the derivative of $H(a)$, leading to the deceleration parameter q. The time evolution of the density parameters is determined from $\rho(a)$ of the respective fluids with their equation of state, and the time evolving critical density $\rho_{\mathrm{crit}}(a)$, determined through the Hubble-function $H(a)$. Then,

$$\frac{\Omega_w(a)}{\Omega_w} = \frac{H_0^2}{a^{3(1+w)}H^2(a)} \tag{C.211}$$

which is illustrated for Ω_m $(w = 0)$, Ω_γ $(w = +1/3)$ and Ω_Λ $(w = -1)$ in Figure 1, clearly indicating phases, where the FLRW-dynamics is dominated by a single fluid, in order of descending w.

Auxiliary to the last argument, we can compute $\Omega_m(a)$ in its time evolution and compare it to the Hubble-function $H(a)$ for a range of dark energy models with differing w. It is very practical for this type of plot to scale out the behaviour of H in the matter-dominated phase, where it is $\propto a^{-3/2}$ and consider $a^{3/2}H(a)$. The result

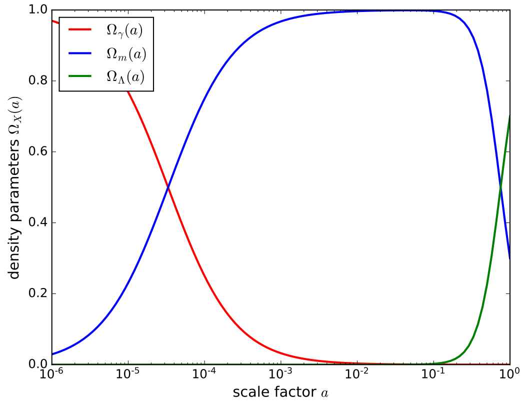

Figure 1: Density parameters $\Omega(a)$ for radiation, matter and the cosmological constant

is shown in Fig. 2, where the double logarithmic derivative $\mathrm{d}\ln \mathrm{H}/\mathrm{d}\ln a$ shows the effective power law behaviour of H.

C.10 *Cosmological distance measures*

Coordinate differences between objects are irrelevant, as the coordinate choice is completely arbitrary: For defining actual distances on needs to go through the metric which maps infinitesimal coordinate differences $\mathrm{d}x^\mu$ onto spacetime distances $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. The result of this operation is only differential, so any macroscopic distance measure involves an integration, and it would make a difference whether the coordinate differentials $\mathrm{d}x^\mu$ are part of a timelike or lightlike geodesic, so one would need to describe an actual experiment that defines the measurement of a distance on a metric manifold.

Perhaps most intuitive is the *proper distance* p, where one derives the distance from the light travel time, given infinitesimally by

$$\mathrm{d}p = c\mathrm{d}t \quad \text{with} \quad \mathrm{d}t = \frac{\mathrm{d}a}{a\mathrm{H}(a)} \tag{C.212}$$

so that p can be determined by integration as

$$p = c\int_a^1 \mathrm{d}a\, \frac{1}{a\mathrm{H}(a)} \tag{C.213}$$

and is naturally related to the amount of time passing between a and 1. Next, we define the *comoving distance* χ, which must never be confused with comoving coordinates! The null-condition for FLRW-universes reads

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2\mathrm{d}\chi^2 = 0 \tag{C.214}$$

χ would be the comoving coordinate differential, and integrating this up along a

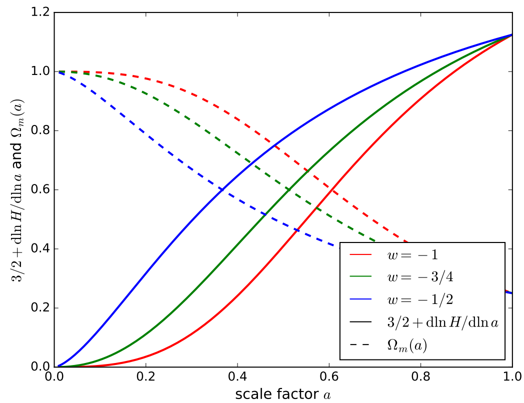

Figure 2: Matter density parameter $\Omega_m(a)$ and the logarithmic slope of the Hubble-function

null-geodesic yields

$$\chi = \int d\chi = c \int dt \, \frac{1}{a} = c \int_a^1 da \, \frac{1}{a^2 H(a)} \tag{C.215}$$

An actually measurable distance indicator is the *angular diameter distance* d_A, as it incorporates an actual experimental setup: If one places an object of a known physical size dA at the distance d_A, it would subtend a (measurable) solid angle $d\Omega$: In a spatially flat universe the two can be related by writing

$$d\Omega = \frac{dA}{d_A^2} = \frac{dQ}{\chi^2} \tag{C.216}$$

As physical size dA and comoving size dQ must be related by a factor of a^2, so must be d_A and χ: For consistency we get $d_a = a\chi$ in a flat, Euclidean universe. With a similar physical idea in mind, one can relate the apparent brightness of a source with its intrinsic luminosity: Spreading out the luminosity L of an object over a sphere with the *luminosity distance* d_L as a radius defines the flux f,

$$f = \frac{L}{4\pi d_L^2} = \frac{L}{4\pi d_A^2} a^4 \tag{C.217}$$

In metric spacetimes there is a general result between the angular sizes of objects and their surface brightnesses, called the Etherington-relation,

$$d_L = \frac{d_A}{a^2} \tag{C.218}$$

which helps to reformulate the apparent flux from a source in terms of comoving or angular diameter distance: The flux is distributed over a sphere with angular

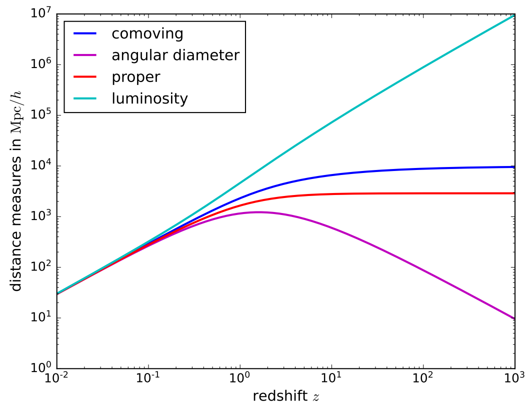

Figure 3: Luminosity distance, comoving distance, proper distance and angular diameter distance for a ΛCDM-cosmology

diameter distance d_A, but as we need to center this sphere on the source and not the observer, the quantity determining the area needs to incorporate a factor of a^2, as the Universe has become larger by a. Additionally, the arrival time of photons is stretched by a as well as their energies redshifted by the same factor. The distance measures are compared to each other for a vanilla ΛCDM-cosmology in Fig. 3.

C.11 *Age of FLRW-universes*

It is only sensible to speak about the age of the Universe, defined as the elapsed time between the instances $a = 0$ (possibly in the mathematical limit) and $a = 1$ (today), if this time interval is identical for all fundamental observers: This is in fact made sure by the FLRW-symmetries. Elapsed proper time τ of a fundamental observer who stays at her or his comoving coordinate with $\mathrm{d}\chi = 0$ is given by $\tau = \int \mathrm{d}\tau = \int \mathrm{d}s/c = \int \mathrm{d}t = t$, and therefore equal to the universally equal coordinate time. With the definition of the Hubble function $H = \dot{a}/a$, which implies that $\mathrm{d}t = \mathrm{d}a/(aH)$ one can compute this time as

$$t = \int \mathrm{d}t = \int_0^1 \mathrm{d}a\, \frac{1}{aH} \tag{C.219}$$

with $1/H_0$ setting the scale of the integral to be about $1/H_0 \simeq 10^{17}$ seconds. The exact number, and whether the integral itself is finite or not, depends on the cosmological model, i.e. the values of the density parameters Ω_i and of the gravitating fluid's equation of state parameters w_i. Let's go through a couple of specific examples with a single dominating fluid: A flat cosmology with only a cosmological constant $\Omega_\Lambda = 1$ and $w = -1$ has a constant Hubble-function, and consequently

$$t = \int_0^1 \mathrm{d}a\, \frac{1}{aH} = \frac{1}{H_0} \int_0^1 \mathrm{d}\ln a = \frac{1}{H_0} \ln a \Big|_0^1 \to \infty \tag{C.220}$$

which is sensible as $a(t) \propto \exp(H_0 t)$ is finite for all finite times and the instant $a = 0$ is never reached. A completely, fully hyperbolically curved universe with $\Omega_K = 1$ and $w = -\frac{1}{3}$ has a Hubble function $H(a) = H_0/a$ and from that we obtain

$$t = \int_0^1 da \frac{1}{aH} = \frac{1}{H_0} \int_0^1 da = \frac{1}{H_0} \tag{C.221}$$

and therefore a finite age! You can easily convince yourself that $w = -1/3$ is the boundary for the age of the Universe to be finite: Lower equation of state parameters make the integral diverge, and higher equation of state parameters cause the integral to converge. In fact, a flat, matter-filled universe with $\Omega_m = 1$ and $w = 0$ would have a Hubble-function with $H(a) = H_0 a^{-3/2}$ and therefore

$$t = \int_0^1 da \frac{1}{aH} = \frac{1}{H_0} \int_0^1 da \frac{1}{a^{-1/2}} = \frac{1}{H_0} \frac{2}{3} a^{3/2} \Big|_1^0 = \frac{2}{3} \frac{1}{H_0}, \tag{C.222}$$

again with a finite age.

C.12 *Causal structure of FLRW-spacetimes and cosmological horizons*

It is immediately obvious that a flat FLRW-spacetime stretches infinitely into the spatial directions but that, depending on the density parameters and the associated equation of state, could have existed only for a finite time, which implies that light from distant regions of the Universe could not yet have arrived at the location of an observer.

The particle horizon is the limit of the past light cone, caused by a finite time since $a = 0$. Working in conformal coordinates we compute the comoving distance as

$$\chi_{PH} = c \int_{-\infty}^{\eta_0} d\eta = c \int_0^{t_0} dt \, \frac{1}{a}, \tag{C.223}$$

which is the maximum comoving distance from which a light signal could have reached us over the finite age of the Universe. Similarly, the future light cone has possibly a limit, corresponding to the maximum distance out to which we can send a light signal in the future: This is called the event horizon, whose comoving distance is given by

$$\chi_{EH} = c \int_{\eta_0}^{+\infty} d\eta = \int_{t_0}^{t_{max}} dt \, \frac{1}{a} \tag{C.224}$$

where the physical age of the Universe is finite in certain cosmological models.

Neither particle nor event horizon should be confused with the Hubble-sphere, which is defined by the physical distance r_{Hubble} at which the recession velocity v reaches the speed of light,

$$c = H r_{Hubble} \quad \rightarrow \quad r_{Hubble} = \frac{c}{H} \tag{C.225}$$

which has today the value $c/H_0 \simeq 3$ Gpc/h. We can perfectly see objects from beyond the Hubble-radius; for instance the cosmic microwave background: All that matters for the visibility of a cosmological object is whether a null-geodesic between the object and observer can be drawn; uninterrupted by a horizon.

Of course, the integrals for particle and event horizon can be reformulated in terms of the scale-factor a, which might be more intuitive and which allows an easier judgement if the integrals converge or not. The expression for the Hubble-function $H(a)$

$$H(a) = H_0\sqrt{\frac{\Omega_\gamma}{a^4} + \frac{\Omega_m}{a^3} + \frac{\Omega_K}{a^2} + \Omega_\Lambda} = H_0\sqrt{\sum_i \frac{\Omega_i}{a^{3(1+w_i)}}} \tag{C.226}$$

suggest that, with the assumption of a monotonically increasing scale factor $\dot{a} > 0$ that the densities $\rho \sim a^{-3(1+w)}$ decrease if $w \geq -1$ and stay constant with $w = -1$. Therefore, the Universe goes typically through all fluids in decreasing order in the value of w in its evolution:

$$\Omega_\gamma \quad \rightarrow \quad \Omega_m \quad \rightarrow \quad \Omega_K \quad \rightarrow \quad \Omega_\varphi \quad \rightarrow \quad \Omega_\Lambda \tag{C.227}$$

Ω_γ and Ω_m are dominant at early times, resulting in decelerating expansion with $q > 0$, whereas in later times dark energy with Ω_φ and the cosmological constant Ω_Λ are dominant, which leads to accelerating expansion with $q < 0$. For any constant equation of state and a single dominating fluid at the critical density we would obtain

$$H = H_0 a^{-\frac{3(1+w)}{2}} = \frac{\dot{a}}{a} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \tag{C.228}$$

such that the integrand for the event- or particle horizon would become

$$\frac{\mathrm{d}t}{a} = \frac{\mathrm{d}a}{a^2} a^{\frac{3(1+w)}{2}} \tag{C.229}$$

and the integral would naturally depend on the equation of state as

$$\int \frac{\mathrm{d}t}{a} = \int \mathrm{d}a\, a^{\frac{3(1+w)}{2}-2} = \int \mathrm{d}a\, a^{\frac{3(w-1)}{2}} \sim a^{\frac{3(w+1)}{2}} \tag{C.230}$$

with a convergent solution at early times for $w > -1/3$ and at late times for $w < -1/3$. Particular problems would occur if the equation of state is more negative than -1: Then, a diverging scale factor $a \rightarrow +\infty$ is reached after a finite physical time. This event is called big rip.