A COSMOLOGY AND THE DYNAMICS OF THE UNIVERSE

A.1 *Physics on cosmological scales*

Modern cosmology is a highly interdisciplinary subject: It is concerned with the dynamics of spacetime on the largest scales, processes in the early Universe such as cosmic inflation and big bang nucleosynthesis, as well as processes in the late Universe such as galaxy formation and evolution. It has introduced the concepts of dark matter and dark energy to explain new gravitational phenomena, and offers a view on fundamental physics on the largest scales in one of the few exactly solvable cases of the gravitational field equation. There are many links to the physics of elementary particles as possible explanations for dark matter. Cosmology joins seemingly separate areas of general relativity as the theory of gravity with (relativistic) fluid mechanics for the motion of matter and radiation, thermodynamics for systems in which thermal equilibria is established and statistics for a description of fluctuations in the matter distribution. Observables in cosmology are very diverse, ranging from fluctuations in the radiation backgrounds to the large-scale distribution of galaxies, peculiar astronomical objects like supernovæ and shape distortions due to gravitational light deflection.

Typical scales involved in cosmology are defined through the realisation that distant galaxies seem to be in a recession motion away from us as observers in the Milky Way. This recession motion can be measured as a redshift in the spectra of these galaxies and the recession velocity υ increases as a function of distance r, summarised in the Hubble-law:

$$\upsilon = \mathrm{H}_0 r \tag{A.1}$$

with the Hubble-Lemaître-constant H_0. Inspecting the units in eqn. A.1 shows that $1/\mathrm{H}_0$ is a time scale, so we can define:

- $t_\mathrm{H} = 1/\mathrm{H}_0 \approx 10^{17}$ s is the Hubble-time, which is a good time scale for the age of the Universe, all known objects are younger than $1/\mathrm{H}_0$.

- $\chi_\mathrm{H} = c/\mathrm{H}_0 \approx 10^{25}$ m is the Hubble-distance, which corresponds to the size of the observable Universe. With the definition of a parsec we get that $\chi_\mathrm{H} \simeq 3$ Gpc.

- Together with the gravitational constant G one can define a density scale $\rho_\mathrm{crit} = 3\mathrm{H}_0^2/(8\pi\mathrm{G}) \approx 10^{-26}$ kg/m^3. This again is the typical density of matter in the Universe, and corresponds to a galaxy per cubic Mpc or a few atoms per cubic meter.

Astronomers are famous for choosing weird units and for defining everything in counter-intuitive ways, and the Hubble-Lemaître-constant H_0 is no exception: A galaxy at a distance of a Mpc has a recession velocity of about 100 km/s, implying

$$\mathrm{H}_0 = \frac{100\ \mathrm{km/s}}{\mathrm{Mpc}} = \frac{10^5\ \mathrm{m/s}}{\mathrm{Mpc}}, \tag{A.2}$$

which in fact is an inverse time scale, with 1 Mpc = 3.0857×10^{22} m. The common basis for cosmology are Friedmann-Lemaître-Robertson-Walker models, in which the dynamics of spacetime and the distribution of matter fulfils the cosmological principle: Homogeneity, as observations from every position would yield the same result, and isotropy, as observations into different directions are equivalent. Specifically this implies, that the the same Hubble-law would be derived from observations with the same Hubble-Lemaître-constant H_0 from any position in the Universe and for every direction (at the current time, observations at earlier or later times might yield a different H_0, depending on the cosmological model). As a consequence, the matter distribution and the spacetime properties on large scales do not show any spatial gradients, neither radially nor tangentially, and changes as a function of time only. This in turn implies, that spherical coordinates should be chosen as an embodiment of isotropy, and that the coordinate origin can be set to any position due to homogeneity, most conveniently though to be coinciding with the Milky Way as the galaxy from which we carry out our observations. As the cosmological principle only holds for the matter distribution and the properties of spacetime on large scales, and as galaxies such as the Milky Way show motion relative to the large-scale averaged matter distribution, the idea of a fundamental FLRW-observer is quite abstract, effectively being at rest relative to the large-scale averaged matter distribution.

A.2 *Newtonian gravity*

The first section of this script is concerned with Newtonian gravity and Newtonian cosmology, before turning to general relativity in the subsequent sections: It is illustrative and educating to see how far one can actually get with a classical theory of gravity! Newtonian gravity is linear, so the superposition principle holds, and typically one postulates the Poisson-equation

$$\Delta\Phi = 4\pi G\rho \tag{A.3}$$

as the field equation for the potential Φ being sourced by the matter density ρ. One could argue that the reasoning behind the Poisson-equation is the gravitational acceleration $g^i = -\partial^i\Phi$ as the field strength, which follows a Gauß-law similar to electrodynamics: $\partial_i g^i = -4\pi G\rho$. This can be interpreted pictorially by the Gauß-theorem

$$\int_V d^3x\,\partial_i g^i = \int_{\partial V} dS_i\,g^i = 4\pi r^2\,g_r = -4\pi G\int_V d^3x\,\rho = -4\pi GM \quad\rightarrow\quad g_r = -\frac{GM}{r^2} \tag{A.4}$$

with the mass M, such that the field strength in radial direction g_r follows a Coulomb-like law, and the quadratic decrease in acceleration is a direct consequence of the quadratic increase of surfaces of spheres in three dimensions.

A better argument for deriving the Poisson-equation is a variational principle: A good starting point could be a Lagrange-density $\mathcal{L}$ which would depend on the field Φ and its first derivatives $\partial^i\Phi$:

$$\mathcal{L}(\Phi, \partial_i\Phi) = \frac{1}{2}\gamma^{ij}\partial_i\Phi\partial_j\Phi + 4\pi G\rho\Phi \tag{A.5}$$

with the Euclidian metric γ^{ij}: We use it to form a rotationally invariant quantity as a square of first derivatives; as an expression of the rotational invariance of Euclidean space. The integral over the Lagrange-density over the domain where the field is defined defines the action S

$$S = \int d^3x\,\mathcal{L}(\Phi, \partial_i\Phi) \tag{A.6}$$

and the Hamilton principle supposes that $\delta S = 0$ leads to the field equation determining the relation between potential and charge, in our case the matter density:

$$\delta S = \int d^3x \, \frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_i \Phi)} \underbrace{\delta(\partial_i \Phi)}_{\partial_i(\delta\Phi)} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \Phi}\right)\delta\Phi = 0 \tag{A.7}$$

where in the second step an integration by parts has been carried out, with the assumption of vanishing variations on the boundary of the domain. The integral can only be universally zero if the term in the brackets is zero: This is the well-known Euler-Lagrange-equation

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \Phi} = 0 \tag{A.8}$$

From the particular Lagrange-density eqn. A.5 we can first derive

$$\frac{\partial \mathcal{L}}{\partial \Phi} = 4\pi G\rho \tag{A.9}$$

and then take care of the second derivative, where it's always a good idea to rename the indices:

$$\frac{\partial \mathcal{L}}{\partial \partial_i \Phi} = \frac{\partial}{\partial \partial_i \Phi}\left(\gamma^{ab}\partial_a\Phi\partial_b\Phi\right) = \gamma^{ab}\left(\underbrace{\frac{\partial \partial_a \Phi}{\partial \partial_i \Phi}}_{\delta^i_a}\partial_b\Phi + \partial_a\Phi\underbrace{\frac{\partial \partial_b \Phi}{\partial \partial_i \Phi}}_{\delta^i_b}\right) = \gamma^{ab}\left(\delta^i_a\partial_b\Phi + \delta^i_b\partial_a\Phi\right) \tag{A.10}$$

followed by a further differentiation ∂_i as required by the Euler-Lagrange equation A.8:

$$\partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \Phi)} = \frac{1}{2}\gamma^{ab}\left(\partial_i\delta^i_a\partial_b\Phi + \partial_i\delta^i_b\partial_a\Phi\right) = \Delta\Phi \tag{A.11}$$

with $\Delta = \gamma^{ab}\partial_a\partial_b$, leading to the classical Poisson-equation:

$$\Delta\Phi = 4\pi G\rho \tag{A.12}$$

The Lagrange-density is the ideal expression to generalise the theory: If we restrict ourselves to linear theories that can fulfil the superposition principle and have at most field equations of second order, the most general expression would be

$$\mathcal{L}(\Phi, \partial_i\Phi) = \frac{1}{2}\gamma^{ij}\partial_i\Phi\partial_j\Phi + 4\pi G\rho\Phi + \lambda\Phi + \frac{m^2}{2}\Phi^2 \tag{A.13}$$

with m and λ as new constants, neither of the new terms would violate linearity. Variation of the action $S = \int d^3x \, \mathcal{L}$ suggests as the field equation

$$(\Delta - m^2)\Phi = 4\pi G\rho + \lambda. \tag{A.14}$$

λ is the (classical) cosmological constant, even if $\rho = 0$ the potential Φ would be sourced $\Delta\Phi = \lambda$ with the solution $\Phi = \lambda r^2/6$:

$$\Delta\Phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\Phi\right) = \lambda \tag{A.15}$$

such that there are gravitational effects even in empty space, as there is an acceleration $g_r = -\partial_r\Phi = -\lambda r/3$. The parameter m introduces a screening of the gravitational potential at large distances: Φ would fall off more rapidly than $1/r$ as the solution for Φ would be $\Phi \propto \exp(-mr)/r$ (in 3 dimensions). The field equation $(\Delta - m^2)\Phi = 4\pi G\rho$ is the Yukawa-field equation and $1/m$ plays the role of a screening length, keeping Φ from propagating to large distances.

When Albert Einstein worked on general relativity there were only weak indications from experiment that Newtonian gravity was not the correct theory of gravity, for instance the tiny perihelion advance of the planet Mercury, which does not follow an exact closed Kepler-ellipse. Conceptually, one weird issue is that the changes to the gravitational potential would be instantaneous, as the Poisson-equation does not include any dynamical description of the field Φ. But with some intuition about relativity one could make the replacements $\partial_i \rightarrow \partial_\mu$ and $\gamma^{ij} \rightarrow \eta^{\mu\nu}$ such that a dynamical linear gravitational theory would be:

$$\mathcal{L}(\Phi, \partial_\mu\Phi) = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 4\pi G\rho\Phi - \lambda\Phi - \frac{m^2}{2}\Phi^2 \tag{A.16}$$

which, after variation, suggests a wave equation for the potential Φ:

$$(\Box + m^2)\Phi = -4\pi G\rho - \lambda \tag{A.17}$$

with the d'Alembert-operator $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_{ct}^2 - \Delta$ replacing the Laplace-operator $\Delta = \gamma^{ij}\partial_i\partial_j$. The wave equation is of course solved by plane waves,

$$\Phi \sim \exp(\pm i k_\mu x^\mu) = \exp(\pm i \eta_{\mu\nu} k^\mu x^\mu) \quad \rightarrow \quad \Box\Phi = -k_\mu k^\mu \Phi = \left(-\left(\frac{\omega}{c}\right)^2 + k^2\right)\Phi \tag{A.18}$$

with a wave vector $k^\mu = (\omega/c, k^i)^t$, which leads to a vacuum solution (for $\rho = 0$):

$$(\Box + m^2)\Phi = (-k_\mu k^\mu + m^2)\Phi = \lambda\Phi \quad \text{with} \quad \upsilon(k) = \frac{d\omega}{dk} = \frac{\pm ck}{\sqrt{k^2 + m^2 - \lambda}} \tag{A.19}$$

such that $m > 0$ causes the waves to travel at sub-luminal speeds (if $\lambda = 0$ always): This suggests the interpretation of the Yukawa-screening length $1/m$ as a mass! General relativity suggests that the term m is exactly zero already from theoretical arguments (we'll come to that!), and it is in fact measured through the propagation velocity of gravitational waves to be near vanishing.

Jumping ahead to Friedmann-cosmologies, where matter is uniformly distributed throughout space and where the gravitational potential does not change along the spatial coordinates ($\partial_i\Phi = 0$) and only evolves with time ($\partial_{ct}\Phi \neq 0$), one can get very close to the second Friedmann equation, as

$$\partial_{ct}^2\frac{\Phi}{c^2} = -\frac{4\pi G\rho c^2}{c^4} + \frac{\lambda}{c^2} \quad \text{bears similarities to} \quad \frac{\ddot{a}}{a} = -\frac{4\pi G\rho c^2}{c^4} + \frac{\Lambda}{3c^2} \tag{A.20}$$

Table 1: Compilation of the simplest solutions of general relativity together with their symmetries and peculiar physical properties. It should be emphasised that a coordinate choice has been taken which is particularly suited to the symmetry of the respective spacetimes.

	black holes	*grav. waves*	*FLRW-cosmologies*	*white dwarfs*
homogeneous	t	$r \pm ct$	r	t
isotropic	*yes*	*no*	*yes*	*yes*
varies along	r	r,t	t	r
gravity	*strong*	*weak*	*strong*	*weak...strong*
scales	$r_S = \frac{2GM}{c^2}$	*linear physics*	$\rho_{crit} = \frac{3H_0^2}{8\pi G}$	*eqn. of state*
curvature	*Weyl*	*Weyl*	*Ricci*	*Weyl + Ricci*
sources	*vacuum solution*	*vacuum solution*	p, ρ *(ideal fluid)*	p, ρ *(ideal fluid)*

Table 2: Regimes of general relativity and physical systems as examples

	strong	*weak*
static	*black holes*	*Newton gravity*
dynamic	*FLRW-cosmologies*	*gravitational waves*

Hence, we could motivate the second Friedmann-equation by using Newtonian gravity and some aspects of relativity. Clearly, we need to worry about the dynamics of the gravitational field and about the conservation law of the source of the gravitational field.

In (nearly) every textbook on general relativity the following four (highly symmetric) solutions of systems with gravity are discussed: black holes, gravitational waves, FLRW-cosmologies and white dwarfs, which are listed in Table A.2.

Sections B and F of this script will illustrate all aspects of relativistic gravity in cosmology, most importantly how gravity can be dynamical, how it can be strong as opposed to Newtonian gravity, and how the equation of state of the gravity-sourcing substances in the Universe matters. The different regimes of gravity are juxtaposed in Table A.2: In cosmology we are dealing with a system of strong, time-varying gravity.

A.3 *Newtonian cosmology*

It is surprising how many features of proper, relativistic cosmology can be recovered and in fact understood on the basis of Newtonian gravity. Imagine two point particles embedded into an infinitely extended homogeneous medium, which changes its density as a function of time as a result of gravity sourced by the medium. The relative motion of the two test particles separated by r follows, by application of Birkhoff's theorem, from the gravitational effect of the matter inside a sphere centered around the first particle, with the second particle residing on the surface of the sphere. The specific total energy E would be given by

$$E = T + V = \frac{\dot{r}^2}{2} - \frac{GM}{r} = \frac{\dot{r}^2}{2} - \frac{4\pi}{3} G\rho r^2 \tag{A.21}$$

where the potential energy is straightforwardly given by a Coulomb-type potential, as all matter inside the sphere acts as if it was concentrated at the centre. Introducing a comoving radius x, which is related to the physical radius by $r(t) = a(t)x$ and which does not depend itself on time, then gives

$$E = \frac{\dot{a}^2 x^2}{2} - \frac{4\pi}{3} G\rho a^2 x^2 \tag{A.22}$$

which would be conserved in the course of time evolution. Solving for the Hubble-function $\dot{a}/a$, defined as the normalised velocity, gives

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{c^2 k}{a^2} \tag{A.23}$$

with the constant $c^2 k = -2E/x^2$. Already in this formula, which will correspond to the first Friedmann-equation, one can see that with the value H_0 for the Hubble-function today one obtains a scale for the density,

$$\rho_{\mathrm{crit}} = \frac{3H_0^2}{8\pi G} \tag{A.24}$$

Differentiation of the first Friedmann-equation with respect to t yields

$$2H\left(\frac{\ddot{a}}{a} - H^2\right) = \frac{8\pi G}{3}\dot{\rho} + 2\frac{c^2 k}{a} H \tag{A.25}$$

but for continuing it would be necessary to know $\dot{\rho}$. The energy density of matter inside the sphere changes as the sphere expands, but depends also on work being performed:

$$dU + p\,dV = T\,dS \tag{A.26}$$

according to the first law of thermodynamics. If there are no heat flows and no heat generation by nuclear or chemical processes, then the expansion is adiabatic with $dS = 0$. If then in addition the medium is pressureless dust, then $p\,dV = 0$ in addition, leaving only the first term. The energy content of the medium would just be the energy associated with rest mass

$$U = \frac{4\pi}{3} a^3 \rho c^2 \quad \rightarrow \quad dU = \dot{U} dt = 0 \quad \text{with} \quad \dot{U} = 4\pi a^2 \dot{a} \rho c^2 + \frac{4\pi}{3} a^3 \dot{\rho} c^2 \tag{A.27}$$

invoking energy conservation. Then,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \quad \text{or} \quad \dot{\rho} + 3H\rho = 0 \tag{A.28}$$

By substitution into eqn. A.25 then yields the second Friedmann-equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho \tag{A.29}$$

implying that there is a gravitational effect on the expansion dynamics of the Universe, as positive matter densities slow down the expansion.