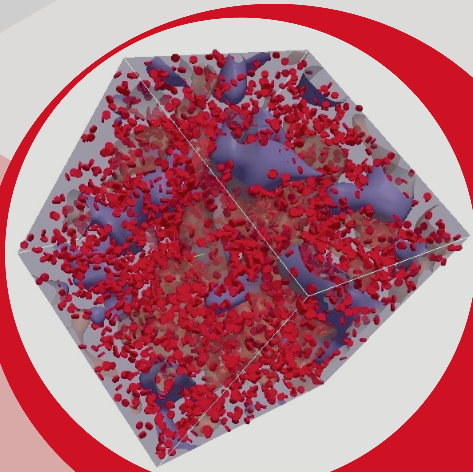
Lecture Notes
Physik

Cosmology

From the Large-Scale Structure of Spacetime to Galaxy Formation

BJÖRN MALTE SCHÄFER



HEIDELBERG
UNIVERSITY PUBLISHING

Cosmology

Lecture Notes Physik

Cosmology

From the Large-Scale Structure of Spacetime to Galaxy Formation

BJÖRN MALTE SCHÄFER

HEIDELBERG
UNIVERSITY PUBLISHING

About the Author

Björn Malte Schäfer works at Heidelberg University on problems in modern cosmology, relativity, statistics, and on theoretical physics in general.

ORCID®

Björn Malte Schäfer https://orcid.org/0000-0002-9453-5772

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.dnb.de.

Published at Heidelberg University Publishing (heiUP), 2022

Heidelberg University/Heidelberg University Library
Heidelberg University Publishing (heiUP)
Grabengasse 1, 69117 Heidelberg, Germany
https://heiup.uni-heidelberg.de

The electronic open access version of this work is permanently available on Heidelberg University Publishing's website: https://heiup.uni-heidelberg.de
urn: urn:nbn:de:bsz:16-heiup-book-1057-6
doi: https://doi.org/10.17885/heiup.1057

ISSN 2566-4816
eISSN 2512-4455

ISBN 978-3-96822-167-0 (Softcover)
ISBN 978-3-96822-168-7 (PDF)

CONTENTS

A COSMOLOGY AND THE DYNAMICS OF THE UNIVERSE

A.1 *Physics on cosmological scales*

Modern cosmology is a highly interdisciplinary subject: It is concerned with the dynamics of spacetime on the largest scales, processes in the early Universe such as cosmic inflation and big bang nucleosynthesis, as well as processes in the late Universe such as galaxy formation and evolution. It has introduced the concepts of dark matter and dark energy to explain new gravitational phenomena, and offers a view on fundamental physics on the largest scales in one of the few exactly solvable cases of the gravitational field equation. There are many links to the physics of elementary particles as possible explanations for dark matter. Cosmology joins seemingly separate areas of general relativity as the theory of gravity with (relativistic) fluid mechanics for the motion of matter and radiation, thermodynamics for systems in which thermal equilibria is established and statistics for a description of fluctuations in the matter distribution. Observables in cosmology are very diverse, ranging from fluctuations in the radiation backgrounds to the large-scale distribution of galaxies, peculiar astronomical objects like supernovæ and shape distortions due to gravitational light deflection.

Typical scales involved in cosmology are defined through the realisation that distant galaxies seem to be in a recession motion away from us as observers in the Milky Way. This recession motion can be measured as a redshift in the spectra of these galaxies and the recession velocity υ increases as a function of distance r, summarised in the Hubble-law:

$$\upsilon = H_0 r \tag{A.1}$$

with the Hubble-Lemaître-constant H_0. Inspecting the units in eqn. A.1 shows that $1/H_0$ is a time scale, so we can define:

- $t_H = 1/H_0 \approx 10^{17}$ s is the Hubble-time, which is a good time scale for the age of the Universe, all known objects are younger than $1/H_0$.
- $\chi_H = c/H_0 \approx 10^{25}$ m is the Hubble-distance, which corresponds to the size of the observable Universe. With the definition of a parsec we get that $\chi_H \simeq 3$ Gpc.
- Together with the gravitational constant G one can define a density scale $\rho_{crit} = 3H_0^2/(8\pi G) \approx 10^{-26}$ kg/m^3. This again is the typical density of matter in the Universe, and corresponds to a galaxy per cubic Mpc or a few atoms per cubic meter.

Astronomers are famous for choosing weird units and for defining everything in counter-intuitive ways, and the Hubble-Lemaître-constant H_0 is no exception: A galaxy at a distance of a Mpc has a recession velocity of about 100 km/s, implying

$$H_0 = \frac{100\ \text{km/s}}{\text{Mpc}} = \frac{10^5\ \text{m/s}}{\text{Mpc}}, \tag{A.2}$$

which in fact is an inverse time scale, with 1 Mpc = 3.0857×10^{22} m. The common basis for cosmology are Friedmann-Lemaître-Robertson-Walker models, in which the dynamics of spacetime and the distribution of matter fulfils the cosmological principle: Homogeneity, as observations from every position would yield the same result, and isotropy, as observations into different directions are equivalent. Specifically this implies, that the the same Hubble-law would be derived from observations with the same Hubble-Lemaître-constant H_0 from any position in the Universe and for every direction (at the current time, observations at earlier or later times might yield a different H_0, depending on the cosmological model). As a consequence, the matter distribution and the spacetime properties on large scales do not show any spatial gradients, neither radially nor tangentially, and changes as a function of time only. This in turn implies, that spherical coordinates should be chosen as an embodiment of isotropy, and that the coordinate origin can be set to any position due to homogeneity, most conveniently though to be coinciding with the Milky Way as the galaxy from which we carry out our observations. As the cosmological principle only holds for the matter distribution and the properties of spacetime on large scales, and as galaxies such as the Milky Way show motion relative to the large-scale averaged matter distribution, the idea of a fundamental FLRW-observer is quite abstract, effectively being at rest relative to the large-scale averaged matter distribution.

A.2 *Newtonian gravity*

The first section of this script is concerned with Newtonian gravity and Newtonian cosmology, before turning to general relativity in the subsequent sections: It is illustrative and educating to see how far one can actually get with a classical theory of gravity! Newtonian gravity is linear, so the superposition principle holds, and typically one postulates the Poisson-equation

$$\Delta\Phi = 4\pi G\rho \tag{A.3}$$

as the field equation for the potential Φ being sourced by the matter density ρ. One could argue that the reasoning behind the Poisson-equation is the gravitational acceleration $g^i = -\partial^i\Phi$ as the field strength, which follows a Gauß-law similar to electrodynamics: $\partial_i g^i = -4\pi G\rho$. This can be interpreted pictorially by the Gauß-theorem

$$\int_V \mathrm{d}^3x\,\partial_i g^i = \int_{\partial V} \mathrm{d}S_i\,g^i = 4\pi r^2\,g_r = -4\pi G\int_V \mathrm{d}^3x\,\rho = -4\pi GM \quad\rightarrow\quad g_r = -\frac{GM}{r^2} \tag{A.4}$$

with the mass M, such that the field strength in radial direction g_r follows a Coulomb-like law, and the quadratic decrease in acceleration is a direct consequence of the quadratic increase of surfaces of spheres in three dimensions.

A better argument for deriving the Poisson-equation is a variational principle: A good starting point could be a Lagrange-density $\mathcal{L}$ which would depend on the field Φ and its first derivatives $\partial^i\Phi$:

$$\mathcal{L}(\Phi,\partial_i\Phi) = \frac{1}{2}\gamma^{ij}\partial_i\Phi\partial_j\Phi + 4\pi G\rho\Phi \tag{A.5}$$

with the Euclidian metric γ^{ij}: We use it to form a rotationally invariant quantity as a square of first derivatives; as an expression of the rotational invariance of Euclidean space. The integral over the Lagrange-density over the domain where the field is defined defines the action S

$$S = \int \mathrm{d}^3x\,\mathcal{L}(\Phi,\partial_i\Phi) \tag{A.6}$$

and the Hamilton principle supposes that $\delta S = 0$ leads to the field equation determining the relation between potential and charge, in our case the matter density:

$$\delta S = \int d^3x \, \frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_i \Phi)} \underbrace{\delta(\partial_i\Phi)}_{\partial_i(\delta\Phi)} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \Phi}\right)\delta\Phi = 0 \qquad \text{(A.7)}$$

where in the second step an integration by parts has been carried out, with the assumption of vanishing variations on the boundary of the domain. The integral can only be universally zero if the term in the brackets is zero: This is the well-known Euler-Lagrange-equation

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \Phi} = 0 \qquad \text{(A.8)}$$

From the particular Lagrange-density eqn. A.5 we can first derive

$$\frac{\partial \mathcal{L}}{\partial \Phi} = 4\pi G\rho \qquad \text{(A.9)}$$

and then take care of the second derivative, where it's always a good idea to rename the indices:

$$\frac{\partial \mathcal{L}}{\partial \partial_i \Phi} = \frac{\partial}{\partial \partial_i \Phi}\left(\gamma^{ab}\partial_a\Phi\partial_b\Phi\right) = \gamma^{ab}\left(\underbrace{\frac{\partial \partial_a \Phi}{\partial \partial_i \Phi}}_{\delta_a^i}\partial_b\Phi + \partial_a\Phi\underbrace{\frac{\partial \partial_b \Phi}{\partial \partial_i \Phi}}_{\delta_b^i}\right) = \gamma^{ab}\left(\delta_a^i\partial_b\Phi + \delta_b^i\partial_a\Phi\right) \qquad \text{(A.10)}$$

followed by a further differentiation ∂_i as required by the Euler-Lagrange equation A.8:

$$\partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \Phi)} = \frac{1}{2}\gamma^{ab}\left(\partial_i\delta_a^i\partial_b\Phi + \partial_i\delta_b^i\partial_a\Phi\right) = \Delta\Phi \qquad \text{(A.11)}$$

with $\Delta = \gamma^{ab}\partial_a\partial_b$, leading to the classical Poisson-equation:

$$\Delta\Phi = 4\pi G\rho \qquad \text{(A.12)}$$

The Lagrange-density is the ideal expression to generalise the theory: If we restrict ourselves to linear theories that can fulfil the superposition principle and have at most field equations of second order, the most general expression would be

$$\mathcal{L}(\Phi, \partial_i\Phi) = \frac{1}{2}\gamma^{ij}\partial_i\Phi\partial_j\Phi + 4\pi G\rho\Phi + \lambda\Phi + \frac{m^2}{2}\Phi^2 \qquad \text{(A.13)}$$

with m and λ as new constants, neither of the new terms would violate linearity. Variation of the action $S = \int d^3x \, \mathcal{L}$ suggests as the field equation

$$(\Delta - m^2)\Phi = 4\pi G\rho + \lambda. \qquad \text{(A.14)}$$

λ is the (classical) cosmological constant, even if $\rho = 0$ the potential Φ would be sourced $\Delta\Phi = \lambda$ with the solution $\Phi = \lambda r^2/6$:

$$\Delta\Phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\Phi\right) = \lambda \tag{A.15}$$

such that there are gravitational effects even in empty space, as there is an acceleration $g_r = -\partial_r\Phi = -\lambda r/3$. The parameter m introduces a screening of the gravitational potential at large distances: Φ would fall off more rapidly than $1/r$ as the solution for Φ would be $\Phi \propto \exp(-mr)/r$ (in 3 dimensions). The field equation $(\Delta - m^2)\Phi = 4\pi G\rho$ is the Yukawa-field equation and $1/m$ plays the role of a screening length, keeping Φ from propagating to large distances.

When Albert Einstein worked on general relativity there were only weak indications from experiment that Newtonian gravity was not the correct theory of gravity, for instance the tiny perihelion advance of the planet Mercury, which does not follow an exact closed Kepler-ellipse. Conceptually, one weird issue is that the changes to the gravitational potential would be instantaneous, as the Poisson-equation does not include any dynamical description of the field Φ. But with some intuition about relativity one could make the replacements $\partial_i \rightarrow \partial_\mu$ and $\gamma^{ij} \rightarrow \eta^{\mu\nu}$ such that a dynamical linear gravitational theory would be:

$$\mathcal{L}(\Phi, \partial_\mu\Phi) = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 4\pi G\rho\Phi - \lambda\Phi - \frac{m^2}{2}\Phi^2 \tag{A.16}$$

which, after variation, suggests a wave equation for the potential Φ:

$$(\Box + m^2)\Phi = -4\pi G\rho - \lambda \tag{A.17}$$

with the d'Alembert-operator $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_{ct}^2 - \Delta$ replacing the Laplace-operator $\Delta = \gamma^{ij}\partial_i\partial_j$. The wave equation is of course solved by plane waves,

$$\Phi \sim \exp(\pm \mathrm{i}k_\mu x^\mu) = \exp(\pm\mathrm{i}\eta_{\mu\nu}k^\mu x^\mu) \quad \rightarrow \quad \Box\Phi = -k_\mu k^\mu\Phi = \left(-\left(\frac{\omega}{c}\right)^2 + k^2\right)\Phi \tag{A.18}$$

with a wave vector $k^\mu = (\omega/c, k^i)^t$, which leads to a vacuum solution (for $\rho = 0$):

$$(\Box + m^2)\Phi = (-k_\mu k^\mu + m^2)\Phi = \lambda\Phi \quad \text{with} \quad \upsilon(k) = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\pm ck}{\sqrt{k^2 + m^2 - \lambda}} \tag{A.19}$$

such that $m > 0$ causes the waves to travel at sub-luminal speeds (if $\lambda = 0$ always): This suggests the interpretation of the Yukawa-screening length $1/m$ as a mass! General relativity suggests that the term m is exactly zero already from theoretical arguments (we'll come to that!), and it is in fact measured through the propagation velocity of gravitational waves to be near vanishing.

Jumping ahead to Friedmann-cosmologies, where matter is uniformly distributed throughout space and where the gravitational potential does not change along the spatial coordinates ($\partial_i\Phi = 0$) and only evolves with time ($\partial_{ct}\Phi \neq 0$), one can get very close to the second Friedmann equation, as

$$\partial_{ct}^2\frac{\Phi}{c^2} = -\frac{4\pi G\rho c^2}{c^4} + \frac{\lambda}{c^2} \quad \text{bears similarities to} \quad \frac{\ddot{a}}{a} = -\frac{4\pi G\rho c^2}{c^4} + \frac{\Lambda}{3c^2} \tag{A.20}$$

Table 1: Compilation of the simplest solutions of general relativity together with their symmetries and peculiar physical properties. It should be emphasised that a coordinate choice has been taken which is particularly suited to the symmetry of the respective spacetimes.

	black holes	*grav. waves*	*FLRW-cosmologies*	*white dwarfs*
homogeneous	t	$r \pm ct$	r	t
isotropic	*yes*	*no*	*yes*	*yes*
varies along	r	r,t	t	r
gravity	*strong*	*weak*	*strong*	*weak...strong*
scales	$r_S = \frac{2GM}{c^2}$	*linear physics*	$\rho_{crit} = \frac{3H_0^2}{8\pi G}$	*eqn. of state*
curvature	*Weyl*	*Weyl*	*Ricci*	*Weyl + Ricci*
sources	*vacuum solution*	*vacuum solution*	*p, ρ (ideal fluid)*	*p, ρ (ideal fluid)*

Table 2: Regimes of general relativity and physical systems as examples

	strong	*weak*
static	*black holes*	*Newton gravity*
dynamic	*FLRW-cosmologies*	*gravitational waves*

Hence, we could motivate the second Friedmann-equation by using Newtonian gravity and some aspects of relativity. Clearly, we need to worry about the dynamics of the gravitational field and about the conservation law of the source of the gravitational field.

In (nearly) every textbook on general relativity the following four (highly symmetric) solutions of systems with gravity are discussed: black holes, gravitational waves, FLRW-cosmologies and white dwarfs, which are listed in Table A.2.

Sections B and F of this script will illustrate all aspects of relativistic gravity in cosmology, most importantly how gravity can be dynamical, how it can be strong as opposed to Newtonian gravity, and how the equation of state of the gravity-sourcing substances in the Universe matters. The different regimes of gravity are juxtaposed in Table A.2: In cosmology we are dealing with a system of strong, time-varying gravity.

A.3 *Newtonian cosmology*

It is surprising how many features of proper, relativistic cosmology can be recovered and in fact understood on the basis of Newtonian gravity. Imagine two point particles embedded into an infinitely extended homogeneous medium, which changes its density as a function of time as a result of gravity sourced by the medium. The relative motion of the two test particles separated by r follows, by application of Birkhoff's theorem, from the gravitational effect of the matter inside a sphere centered around the first particle, with the second particle residing on the surface of the sphere. The specific total energy E would be given by

$$E = T + V = \frac{\dot{r}^2}{2} - \frac{GM}{r} = \frac{\dot{r}^2}{2} - \frac{4\pi}{3}G\rho r^2 \tag{A.21}$$

where the potential energy is straightforwardly given by a Coulomb-type potential, as all matter inside the sphere acts as if it was concentrated at the centre. Introducing a comoving radius x, which is related to the physical radius by $r(t) = a(t)x$ and which does not depend itself on time, then gives

$$E = \frac{\dot{a}^2 x^2}{2} - \frac{4\pi}{3} G\rho a^2 x^2 \tag{A.22}$$

which would be conserved in the course of time evolution. Solving for the Hubble-function $\dot{a}/a$, defined as the normalised velocity, gives

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{c^2 k}{a^2} \tag{A.23}$$

with the constant $c^2 k = -2E/x^2$. Already in this formula, which will correspond to the first Friedmann-equation, one can see that with the value H_0 for the Hubble-function today one obtains a scale for the density,

$$\rho_{\mathrm{crit}} = \frac{3H_0^2}{8\pi G} \tag{A.24}$$

Differentiation of the first Friedmann-equation with respect to t yields

$$2H\left(\frac{\ddot{a}}{a} - H^2\right) = \frac{8\pi G}{3}\dot{\rho} + 2\frac{c^2 k}{a}H \tag{A.25}$$

but for continuing it would be necessary to know $\dot{\rho}$. The energy density of matter inside the sphere changes as the sphere expands, but depends also on work being performed:

$$dU + p dV = TdS \tag{A.26}$$

according to the first law of thermodynamics. If there are no heat flows and no heat generation by nuclear or chemical processes, then the expansion is adiabatic with $dS = 0$. If then in addition the medium is pressureless dust, then $pdV = 0$ in addition, leaving only the first term. The energy content of the medium would just be the energy associated with rest mass

$$U = \frac{4\pi}{3}a^3\rho c^2 \quad \rightarrow \quad dU = \dot{U}dt = 0 \quad \text{with} \quad \dot{U} = 4\pi a^2 \dot{a}\rho c^2 + \frac{4\pi}{3}a^3\dot{\rho}c^2 \tag{A.27}$$

invoking energy conservation. Then,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \quad \text{or} \quad \dot{\rho} + 3H\rho = 0 \tag{A.28}$$

By substitution into eqn. A.25 then yields the second Friedmann-equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho \tag{A.29}$$

implying that there is a gravitational effect on the expansion dynamics of the Universe, as positive matter densities slow down the expansion.

B GRAVITY AND CONCEPTS OF RELATIVITY

B.1 *Metric structure of spacetime*

Spacetime is first of all a topological space, where the points are given coordinates by a continuous coordinate mapping (the system of open sets allows specifically to define continuity of a mapping), where the coordinates are arranged in a coordinate tuple, for instance $x^\mu = (ct, x^i)^t$. Unlike in vector spaces, differences between coordinates as distances have no meaning, but one needs a metric tensor to compute the line element $\mathrm{d}s^2$ from an infinitesimal coordinate difference $\mathrm{d}x^\mu$: As the metric tensor can change across the manifold, all definitions are only made in a *local* way.

Starting with a Euclidean manifold with a metric γ^{ij} one would write down for the line element

$$\mathrm{d}s^2 = \mathrm{d}x_i \mathrm{d}x^i = \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{B.30}$$

where the last equality is true for Cartesian coordinates as a particular coordinate choice, where $\gamma_{ij} = \delta_{ij}$. Euclidian space is a flat space with no curvature, and there is invariance of $\mathrm{d}s^2$ under rotations. Generalising to Minkowskian space, we get the line-element

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{B.31}$$

again with the last equality being applicable if Cartesian coordinates have been chosen. Minkowskian space is flat, too, there is no curvature and it is invariant under Lorentz-transformations. In opposite to the Euclidian line-element, the line-element is no longer positive definite, which means that there can be negative distances. In practice, this is never an actual issue, as only events with positive distances $\mathrm{d}s^2 > 0$ are causally related to each other. At the same time, $\mathrm{d}s^2 = 0$ defines a light cone structure for the manifold. Both examples are (pseudo-)Riemannian manifolds where the line element is given by a quadratic form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{B.32}$$

with a general metric tensor $g_{\mu\nu}$. In 4 dimensions there are 10 independent entries of $g_{\mu\nu}$ due to the symmetry $g_{\mu\nu} = g_{\nu\mu}$: Any anti-symmetric part would not be able to influence the value of $\mathrm{d}s^2$ as $\mathrm{d}x^\mu \mathrm{d}x^\nu$ is fully symmetric. On a manifold we will establish invariance of line elements as general scalars under *arbitrary* coordinate transforms, generalising the idea of the invariance of the Euclidean line element under rotations and the invariance of the Minkowski-line element under Lorentz-transforms. To make this specific, we have for an *invertible* and *differentiable* coordinate change (a so-called diffeomorphism):

$$x'^\rho = x'^\rho(x^\mu) \quad \rightarrow \quad \mathrm{d}x'^\rho = \frac{\partial x'^\rho}{\partial x^\mu}\mathrm{d}x^\mu \tag{B.33}$$

as well as the inverse

$$x^\mu = x^\mu(x'^\rho) \quad \rightarrow \quad \mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial x'^\rho}\mathrm{d}x'^\rho \tag{B.34}$$

If the line element is to be invariant as a scalar, the metric $g_{\mu\nu}$ needs to transform inversely to $\mathrm{d}x^\mu$:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = g_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}\mathrm{d}x'^\rho\mathrm{d}x'^\sigma = g'_{\rho\sigma}\mathrm{d}x'^\rho\mathrm{d}x'^\sigma \tag{B.35}$$

i.e.

$$g_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} = g'_{\rho\sigma} \tag{B.36}$$

and the general picture emerges that contravariant (superscript) indices transform with Jacobians $\frac{\partial x'^\rho}{\partial x^\mu}$ whereas covariant (subscript) indices transform with inverse Jacobians $\frac{\partial x^\mu}{\partial x'^\rho}$.

B.2 *Metric and inner products*

Picking up this idea lets us write for a vector υ^μ with contravariant indices

$$\upsilon^\mu \to \upsilon'^\rho = \frac{\partial x'^\rho}{\partial x^\mu}\upsilon^\mu \tag{B.37}$$

with a Jacobian $\frac{\partial x'^\rho}{\partial x^\mu}$ and for a linear form w_μ with covariant indices

$$w_\mu \to w'_\rho = \frac{\partial x^\mu}{\partial x'^\rho}w_\mu \tag{B.38}$$

with an inverse Jacobian $\frac{\partial x^\mu}{\partial x'^\rho}$, such that inner products stay invariant:

$$w_\mu\upsilon^\mu = g_{\mu\nu}w^\mu\upsilon^\nu \quad\to\quad w'_\mu\upsilon'^\mu = g'_{\mu\nu}w'^\mu\upsilon'^\nu = \underbrace{\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\sigma}}_{\delta^\rho_\mu} w_\rho\upsilon^\sigma = w_\sigma\upsilon^\sigma \tag{B.39}$$

as Jacobian cancels with the inverse Jacobian and simply a renaming of the indices is taking place.

The index shift carried out by the metric $\upsilon_\mu = g_{\mu\rho}\upsilon^\rho$ is undone by the inverse metric $\upsilon^\sigma = g^{\sigma\mu}\upsilon_\mu = g^{\sigma\mu}g_{\mu\rho}\upsilon^\rho = \delta^\sigma_\rho\upsilon^\rho$, such that the inverse metric fulfils

$$g^{\sigma\mu}g_{\mu\rho} = \delta^\sigma_\rho \tag{B.40}$$

Please keep in mind that

$$g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = 4 \tag{B.41}$$

in 4 dimensions, and *not equal* to 2, as on might (naively) think.

B.3 *Vectors and covariant derivatives*

Considering a curve $x^\mu(\lambda)$ with parameter λ cutting through a field $\varphi(x^\mu)$: How would φ change along the curve as λ changes? The chain rule suggests that

$$\frac{d\varphi}{d\lambda} = \frac{d}{d\lambda}\varphi(x^\mu(\lambda)) = \underbrace{\frac{dx^\mu}{d\lambda}}_{\text{tangent}}\frac{\partial\varphi}{\partial x^\mu} = u^\mu\frac{\partial\varphi}{\partial x^\mu} \tag{B.42}$$

with the tangent $u^\mu = dx^\mu/d\lambda$, such that the rate of change of φ along the curve $x^\mu(\lambda)$ is given as a projection of the gradient field $\partial\varphi/\partial x^\mu = \partial_\mu\varphi$ onto the tangent u^μ. From this we recognise that u^μ as well as dx^μ are vectors, and $\partial_\mu\varphi$ is a linear form. It is possible to run curves through a point A in all possible directions and construct vectors dx^μ tangent to them, and the minimal collection of dx^μ would constitute the basis of a tangent space T_AM of the manifold M at A, relative to which all tensor of vector fields can be expressed in components. Most sensibly, one would run these curves through A by changing a single coordinate at a time: But this implies that the construction of the basis for T_AM would depend on the coordinate choice and could be different at another point B! That has in fact profound implications when considering changes to a vector or tensor field across the manifold: The components of the field can become different because the basis has changed going from T_AM to T_BM, or there could be a *genuine* change in the field, and the two cases would need to be distinguished.

But before we investigate that in detail, we should try out a remapping of the coordinates in equation B.42. Writing $x^\mu(x'^\alpha)$ we can introduce a "one" $\delta^\nu_\mu = \partial x^\nu/\partial x^\mu$ in the form of two mutually annihilating Jacobians,

$$\frac{d\varphi}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial\varphi}{\partial x^\mu} = \frac{dx^\mu}{d\lambda}\delta^\nu_\mu\frac{\partial\varphi}{\partial x^\nu} = \frac{dx^\mu}{d\lambda}\underbrace{\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\alpha}}_{\delta^\nu_\mu}\frac{\partial\varphi}{\partial x^\nu} = \frac{dx'^\alpha}{d\lambda}\frac{\partial\varphi}{\partial x'^\alpha} \tag{B.43}$$

suggesting that vectors such as u^μ transform with the Jacobian $\frac{\partial x'^\alpha}{\partial x^\mu}$ while linear forms like $\partial_\mu\varphi$ transform with the inverse Jacobian $\frac{\partial x^\nu}{\partial x'^\alpha}$.

We need the concept of a parallel transport to quantify changes in a vector field υ^μ across the manifold M. The components of the vector are given in terms of a local coordinate frame which is the basis of T_AM, and which might differ from the frame at T_BM, implying that the same abstract vector υ could have different components at A and B: We need to disentangle changes of the tangent space from genuine changes in the vector field! For this purpose, one introduces parallel transport, which moves a vector perfectly from A to B and tracks only the change in tangent space. If the two points are separated by δx, the parallel-transported, perfect copy $\upsilon^\mu_\parallel$ at the point B with coordinates $x+\delta x$ of the original vector υ^μ at point A with coordinates x is given at linear order

$$\upsilon^\mu_\parallel(x+\delta x) = \upsilon^\mu(x) - \Gamma^\mu{}_{\alpha\beta}\,\upsilon^\alpha(x)\delta x^\beta + \ldots \tag{B.44}$$

where the minus-sign is chosen by convention. The coefficients $\Gamma^\mu{}_{\alpha\beta}$ form the Christoffel-symbol. A vector field would now change genuinely if it differs at position $x+\delta x$ from the parallel-transported vector field. We are now only comparing two vector fields $\upsilon^\mu(x+\delta x)$ and $\upsilon^\mu_\parallel(x+\delta x)$ at the same point within the same tangent space

T_BM, as opposed to a direct comparison of $\upsilon^\mu(x+\delta x)$ with $\upsilon^\mu(x)$ which is senseless as the tangent spaces T_AM and T_BM are in general different and the component expansion of υ exists in two different bases:

$$\lim_{\delta x^\beta \to 0} \frac{\upsilon^\mu(x+\delta x) - \upsilon^\mu_\parallel(x+\delta x)}{\delta x^\beta} = \lim_{\delta x^\beta \to 0} \frac{\upsilon^\mu(x+\delta x) - \upsilon^\mu(x)}{\delta x^\beta} + \Gamma^\mu{}_{\alpha\beta}\, \upsilon^\alpha(x)\frac{\delta x^\beta}{\delta x^\beta} = \partial_\beta \upsilon^\mu + \Gamma^\mu{}_{\alpha\beta}\, \upsilon^\alpha \equiv \nabla_\beta \upsilon^\alpha \quad \text{(B.45)}$$

Here, we have identified a straightforward index-by-index change of the vector field over the shift δx as the partial differentiation $\partial_\beta \upsilon^\mu$, which gets corrected by the Christoffel-symbol tracking the change of the tangent spaces.

It is important to realise that the covariant differentiation becomes only relevant for fields that have internal degrees of freedom, whose decomposition in components depend on the change in tangent space moving from T_AM to T_BM. Scalar fields are oblivious to these changes, and therefore the covariant differentiation falls back on the conventional partial differentiation:

$$\nabla_\beta \varphi = \partial_\beta \varphi \quad \text{(B.46)}$$

For higher-order tensorial fields one needs a Christoffel-symbol for each index: You can imagine that the basis for such an object is the Cartesian product, and that the differentiation fulfils a Leibnitz-rule, such that we get

$$\nabla_\beta T^{\mu\nu} = \partial_\beta T^{\mu\nu} + \Gamma^\mu{}_{\beta\alpha}\, T^{\alpha\nu} + \Gamma^\nu{}_{\beta\alpha}\, T^{\mu\alpha} \quad \text{(B.47)}$$

Let's now have a look at the differentiation of a covariant vector or, equivalently, a linear form. A contraction between the vector υ^μ and the linear form w_μ is scalar, so the covariant differentiation falls back onto the partial one:

$$\nabla_\beta(\upsilon^\mu w_\mu) = \partial_\beta(\upsilon^\mu w_\mu) = \partial_\beta \upsilon^\mu \cdot w_\mu + \upsilon^\mu \partial w_\mu \quad \text{(B.48)}$$

If, on the other side, the covariant differentiation comes with a Leibnitz-rule for dealing with products we would write

$$\nabla_\beta(\upsilon^\mu w_\mu) = \nabla_\beta \upsilon^\mu \cdot w_\mu + \upsilon^\mu \nabla w_\mu = \underbrace{(\partial_\beta \upsilon^\mu + \Gamma^\mu{}_{\alpha\beta}\, \upsilon^\alpha)}_{\partial_\beta \upsilon^\mu \text{ from above}} w_\mu + \upsilon^\mu \underbrace{\nabla_\beta w_\mu}_{\text{isolate this term}} \quad \text{(B.49)}$$

Isolating the covariant derivative $\nabla_\beta w_\mu$ of the linear form w_μ we get:

$$\upsilon^\mu \nabla_\beta w_\mu = \upsilon^\mu \partial_\beta w^\mu - \Gamma^\mu{}_{\alpha\beta}\, \upsilon^\alpha w_\mu = \upsilon^\mu \partial_\beta w^\mu - \Gamma^\alpha{}_{\mu\beta}\, \upsilon^\mu w_\alpha \quad \text{(B.50)}$$

after renaming indices and finally

$$\nabla_\beta w_\mu = \partial_\beta w_\mu - \Gamma^\alpha{}_{\mu\beta}\, w_\alpha \quad \text{(B.51)}$$

implying that a linear form needs a negative Christoffel-symbol, as opposed to a vector with a positive Christoffel-term.

With this definition the covariant derivative depends completely on the choice of the connection coefficients $\Gamma^{\alpha}{}_{\mu\nu}$, but we should be guided by the idea that the two structures that exist on the manifold, (i) the metric structure which allows the measurements of angles between vectors and determinations of their lengths, and (ii) the differential structure which quantifies rates of change of vectors, should be compatible with each other. Specifically, if two vectors are parallel-transported, their length and relative orientation should not change, and as a consequence their scalar product should be unaffected. With the covariant derivative

$$\nabla_\beta \upsilon^\mu = \lim_{\delta x^\beta \to 0} \frac{\upsilon^\mu(x+\delta x) - \upsilon^\mu_{||}(x+\delta x)}{\delta x^\beta} \tag{B.52}$$

based on the parallel transport

$$\upsilon^\mu_{||}(x+\delta x) = \upsilon^\mu(x) + \Gamma^{\mu}{}_{\alpha\beta}\, \upsilon^\alpha \delta x^\beta \tag{B.53}$$

we can reformulate parallel transport in an operator notation: The vector $\upsilon^\mu(x+\delta x)$ must be equal to $\upsilon^\mu_{||}(x+\delta x) + \delta x^\beta \nabla_\beta \upsilon^\mu$. Perfect parallel transport means that the vector $\upsilon^\mu(x+\delta x)$ at $\mathrm{T_B M}$ and $\upsilon^\mu_{||}(x+\delta x)$ transported from $\mathrm{T_A M}$ to $\mathrm{T_B M}$ by the shift δx are now identical, and in this case $\delta x^\beta \nabla_\beta \upsilon^\mu$ must be zero. This shifting operator $\delta x^\beta \nabla_\beta$ can be applied to scalar quantities as well, such as in particular the scalar product $g_{\mu\nu}\upsilon^\mu w^\nu$:

$$\delta x^\beta \nabla_\beta (g_{\mu\nu}\upsilon^\mu w^\nu) = \delta x^\beta (\nabla_\beta g_{\mu\nu} \cdot \upsilon^\mu w^\nu + g_{\mu\nu} \underbrace{\nabla_\beta \upsilon^\mu}_{=0} \cdot w^\nu + g_{\mu\nu}\upsilon^\mu \underbrace{\nabla_\beta w^\nu}_{=0}) = \delta x^\beta \nabla_\beta g_{\mu\nu} \cdot \upsilon^\mu w^\nu = 0 \tag{B.54}$$

as a consequence of the Leibnitz rule, with a single term remaining:

$$\nabla_\beta g_{\mu\nu} = 0 \tag{B.55}$$

which is referred to as the metric compatibility condition: If it is true, the scalar product over perfectly parallel transported vectors does not change across the manifold. As the metric itself is a tensor with covariant indices, the covariant derivative is computed as

$$\nabla_\beta g_{\mu\nu} = \partial_\beta g_{\mu\nu} - \Gamma^{\alpha}{}_{\beta\mu}\, g_{\alpha\nu} - \Gamma^{\alpha}{}_{\beta\nu}\, g_{\mu\alpha} \tag{B.56}$$

If in addition we assume that the parallel transport is torsion-free the Christoffel-symbol is symmetric in the lower two indices,

$$\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}\,. \tag{B.57}$$

This implies that we write out the combination $\nabla_\mu g_{\beta\nu} + \nabla_\nu g_{\mu\beta} - \nabla_\beta g_{\mu\nu} = 0$ (metric compatibility ensures that the terms vanish already individually!) and solve for the Christoffel-symbol $\Gamma^{\alpha}{}_{\mu\nu}$, which comes out as

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}) \tag{B.58}$$

A connection $\Gamma^{\alpha}{}_{\mu\nu}$ which is metric-compatible ($\nabla_\alpha g_{\mu\nu} = 0$) and torsion free ($\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}$) is called a Levi-Civita connection; it is uniquely compatible with the metric structure on the manifold, as the connection can be computed from the metric and its derivatives alone. A metric manifold with a Levi-Civita connection and the corresponding covariant derivative defines Riemannian geometry.

At this point, a beautiful conceptual picture emerges: Spacetime is a manifold with, first of all, a topological structure, which allows a continuous mapping of coordinates onto spacetime. Then, there is in addition a metric structure, which allows measurements of lengths and angles in vector fields on the manifold: As other fields, the metric tensor may vary across the manifold. We've introduced a differentiable structure on the manifold, in addition, by defining parallel transport and the covariant derivative. This differentiable structure has to be compatible with the metric structure, which is made sure by metric compatibility. Later in this course, we'll see that there is a second notion of derivation, called a Lie-derivative, which is needed to describe symmetries: Those are made compatible with covariant derivative by the requirement of torsion-free connections, giving further support to Levi-Civita connections. A physical motivation for choosing torsion-free connections is the compatibility of covariant derivatives with Lie-derivatives which are used for characterising symmetries of spacetimes.

B.4 *Geodesics and autoparallelity*

A particle drifting through spacetime follows a trajectory $x^\mu(\lambda)$ in a given coordinate choice, parameterised by the affine parameter λ. Then, the rate of change of the coordinates with λ would be the velocity u^μ,

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \dot{x}^\mu \tag{B.59}$$

or equivalently the tangent to the trajectory $x^\mu(\lambda)$. The velocity u^μ and the coordinate differential $\mathrm{d}x^\mu$ are vectors, in contrast to the coordinate tuple x^μ itself. With the idea the operator for parallel transport we might construct a curve whose tangent $u^\mu = \dot{x}^\mu$ stays parallel to itself, exactly through the autoparallelity condition

$$\dot{x}^\mu \nabla_\mu \dot{x}^\nu = 0 \tag{B.60}$$

i.e. $u^\mu = \dot{x}^\mu$ is always a parallel-transported version of itself. It is suggestive to imagine that these curves describe inertial motion through spacetime, as no accelerations are felt, because the velocity $\boldsymbol{u}$ as an abstract vector does not change, only its components u^μ can be different as there can be different tangent spaces along the curve. Taking this thought a little further leads us to the realisation that there is actually no difference between inertial motion and freely falling motion, as both cases are characterised by the absence of physical accelerations.

Surely, $\mathrm{d}u^\mu/\mathrm{d}\lambda$ can be nonzero, but the abstract vector $\boldsymbol{u}$ is conserved.

$$\dot{x}^\mu \nabla_\mu \dot{x}^\nu = \dot{x}^\mu \partial_\mu \dot{x}^\nu + \Gamma^\nu{}_{\mu\alpha}\,\dot{x}^\mu \dot{x}^\alpha = \ddot{x}^\nu + \Gamma^\nu{}_{\alpha\mu}\,\dot{x}^\mu \dot{x}^\alpha = 0 \tag{B.61}$$

using

$$\ddot{x}^\nu = \frac{\mathrm{d}\dot{x}^\nu}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\partial \dot{x}^\nu}{\partial x^\mu} = \dot{x}^\mu \partial_\mu \dot{x}^\nu \tag{B.62}$$

to obtain the second derivative $\ddot{x}^\nu$. The result is the geodesic equation, reading

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\lambda} + \Gamma^\alpha{}_{\mu\nu}\,u^\mu u^\nu = 0, \quad \text{or} \quad \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0. \tag{B.63}$$

if formulated in terms of the tangent vector u^μ. It is a fun realisation that the tangent vector of the Earth's orbit *in 4 dimensions* is autoparallel, in a spacetime which is non-Minkowskian with a very slight curvature introduced by the Sun.

It is possible to tease out the geodesic equation from Newton's equation of motion. In fact,

$$\ddot{x}^i + \partial^i \Phi = 0 \tag{B.64}$$

describes the freely falling motion of a test particle in the gravitational potential Φ. It follows a force-free trajectory, which is straight according to the inertial law formulated by Newton. Surely, we don't make a mistake by writing

$$\ddot{x}^i + \partial^i \frac{\Phi}{c^2}\cdot c \cdot c = 0 \tag{B.65}$$

where now c^2 provides a scale for the potential Φ: Because c has no particular relevance for Galilean physics one would think that the division by c^2 just makes the potential dimensionless. In the slow-motion limit of relativity particles follow trajectories with $\dot{x}^t = c$, so the formula becomes

$$\ddot{x}^i + \partial^i \frac{\Phi}{c^2}\dot{x}^t \dot{x}^t = 0 \tag{B.66}$$

But the terms $\dot{x}^t$ are just the t-components of the velocities, which in the slow-motion limit $\dot{x}^\mu = (c, \upsilon^i)^t$, where proper time and coordinate time are identical, $t = \tau$ identical and consequently $\gamma = 1$. Then,

$$\ddot{x}^\alpha + \partial^\alpha \frac{\Phi}{c^2}\dot{x}^t \dot{x}^t = 0, \quad \text{suggesting that} \quad \Gamma^\alpha{}_{tt} \sim \partial^\alpha \frac{\Phi}{c^2} \tag{B.67}$$

by identifying the derivative of the potential with the Christoffel-symbol, consolidating the idea that Newton's equation of motion is the weak-field and slow-motion limit of the geodesic equation,

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu = 0 \tag{B.68}$$

If we try out an extremal principle for the trajectory as in classical mechanics and impose Hamilton's principle $\delta S = 0$ on an action integral

$$S = \int \mathrm{d}t\, \mathcal{L} \quad \text{with} \quad \mathcal{L} = \frac{1}{2}\dot{x}_i \dot{x}^i - \Phi \tag{B.69}$$

we end up with the Euler-Lagrange equation

$$\ddot{x}^i + \partial^i \Phi = 0 \tag{B.70}$$

from classical mechanics. In a similar calculation $\delta s = 0$ of the line-element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\mathrm{d}\lambda^2 \to \mathrm{d}s = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\mathrm{d}\lambda \quad \text{and} \quad s = \int \mathrm{d}s \tag{B.71}$$

provides the geodesic equation: Straight lines in the sense of autoparallelity are at the same time extremal in their arc length.

The affine parameter λ can be chosen arbitrarily as the geodesic equation is invariant under affine transforms of λ, $\lambda \to a\lambda + b$, but there are two practical choices: In the case of a massive particle which follows a time-like geodesic with $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu > 0$ one can choose proper time $\lambda = \tau$, such that the normalisation is given by $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = c^2$. Photons, on the other hand, follow null-geodesics with $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$, which is incompatible with proper time as an affine parameter. As parallel transport is with Levi-Civita connection is constructed to conserve norms, we can conclude that in both cases the normalisation of the tangent $u^\mu = \dot{x}^\mu$ for both τ or λ is conserved.

B.5 *Spacetime curvature*

The geodesic equation

$$\ddot{x}^\alpha + \Gamma^\alpha_{\ \mu\nu}\, \dot{x}^\mu \dot{x}^\nu = 0 \tag{B.72}$$

is unable to differentiate between inertial motion in the absence of gravity or freely-falling motion in a gravitational field: This is absolutely sensible because in both cases one would not *feel* or *measure* any acceleration, so the two situations are physically equivalent. This has profound implications which we should clarify: The Christoffel-symbol has 40 entries (For every choice of α, 4 in total, there are because of the symmetry 10 different choices in the index pair μ, ν), which are all measurable through the acceleration $\ddot{x}^\alpha$ for a given choice of $\dot{x}^\mu \dot{x}^\nu$, for both cases of massive and massless particles. Acceleration in this context is a non-uniform passage of the coordinates along the path of the particle, which should not be interpreted as a physical acceleration. In this sense, the geodesic equation only takes care of the non-uniformity of the coordinate choice and does *not* contain information about gravity or curvature! A good example might be inertial motion through Euclidean space in polar coordinates r, φ, and a situation where the particle moves off-centre relative to origin of the coordinate frame. There, the velocities $\dot{r}$ and $\dot{\varphi}$ are not constant and show accelerations $\ddot{r} \neq 0 \neq \ddot{\varphi}$, but clearly, there are no physical accelerations present.

To summarise this important point: Neither the metric, nor the geodesic equation, nor the covariant derivative and nor the Christoffel-symbols contain information about gravity, $\Gamma^{\alpha}{}_{\mu\nu} = 0$ does not imply the absence of gravity, and neither does $\nabla_\mu = \partial_\mu$. All these things are consequences of the coordinate choice. That is in fact sensible, as there is always a coordinate choice that sets *locally* $g_{\mu\nu}$ to $\eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu} = 0$, i.e. the metric becomes Minkowskian and the Christoffel-symbol vanishes.

Information about the gravitational field is contained in curvature, which is in Riemannian-geometry ultimately computed from the second derivatives of the metric and which can not be set to zero by a suitable coordinate transform in the general case. Curvature is present if covariant derivatives ∇_μ into different directions do not commute, or equivalently, if shifts $\delta x^\mu \nabla_\mu$ into different directions carried out after each other, affect the internal degrees of a vector or tensor. The non-commutativity of covariant derivatives directly defines the Riemann-curvature,

$$\left[\nabla_\mu \nabla_\nu\right] \upsilon^\alpha = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\upsilon^\alpha = R^{\alpha}{}_{\beta\mu\nu}\upsilon^\beta \tag{B.73}$$

It can be shown that the effect of parallel-transport around a loop would be a rotated vector $R^{\alpha}{}_{\beta\mu\nu}\upsilon^\beta$ relative to υ^α, where parallel-transport conserves the norm of the vector υ^α due to metric compatibility. This is in fact the best way to visualise the effect of $R^{\alpha}{}_{\beta\mu\nu}$ as an operator and to memorise the index structure. By definition, $R^{\alpha}{}_{\beta\mu\nu}$ is antisymmetric for every choice of μ, ν, and in the index pair α, β is is an antisymmetric rotation matrix. In 4 dimensions, $R^{\alpha}{}_{\beta\mu\nu}$ has 20 entries, as opposed to the 40 entries of $\Gamma^{\alpha}{}_{\mu\nu}$.

The Riemann-curvature vanishes in flat spaces

$$R^{\alpha}{}_{\beta\mu\nu} = 0 \tag{B.74}$$

in every coordinate choice, even though the Christoffel-symbols $\Gamma^{\alpha}{}_{\mu\nu}$ only vanish in Cartesian coordinates. Following the formal definition of curvature as the non-commutativity of shifts in different coordinate directions leads us to

$$\upsilon^\mu(x + \delta x) = \upsilon^\mu(x) - \Gamma^{\mu}{}_{\alpha\beta}(x)\upsilon^\alpha \delta x^\beta \tag{B.75}$$

and in a second step to

$$\begin{aligned} \upsilon^\mu((x + \delta\bar{x}) + \delta x) &= \upsilon^\mu(x + \delta\bar{x}) - \Gamma^{\alpha}{}_{\mu\nu}(x + \delta\bar{x})\upsilon^\alpha(x + \delta\bar{x})\delta x^\beta = \\ &\upsilon^\mu(x) - \Gamma^{\alpha}{}_{\mu\nu}\, v(x)\upsilon^\alpha(x)\delta\bar{x}^\beta - (\Gamma^{\alpha}{}_{\mu\nu}(x) + \underbrace{\partial_\gamma \Gamma^{\mu}{}_{\alpha\beta}\,\delta\bar{x}^\gamma}_{\text{Taylor}})(\upsilon^\alpha(x) - \Gamma^{\alpha}{}_{\gamma\delta}\,\upsilon^\gamma \delta x^\gamma)\delta x^\beta \end{aligned} \tag{B.76}$$

and therefore the order of the shifts matters:

$$\upsilon^\mu((x + \delta x) + \delta\bar{x}) \neq \upsilon^\mu((x + \delta\bar{x}) + \delta x) \tag{B.77}$$

Computing the difference shows that

$$\upsilon^\mu((x + \delta\bar{x}) + \delta x) - \upsilon^\mu((x + \delta x) + \delta\bar{x}) = R^{\mu}{}_{\alpha\beta\nu}\upsilon^\alpha \delta x^\beta \delta\bar{x}^\nu \tag{B.78}$$

with the Riemann-curvature-tensor $R^{\mu}{}_{\alpha\beta\nu}$

$$R^{\mu}{}_{\alpha\beta\nu} = \partial_{\beta}\Gamma^{\mu}{}_{\alpha\nu} - \partial_{\nu}\Gamma^{\mu}{}_{\alpha\beta} + \Gamma^{\mu}{}_{\delta\beta}\,\Gamma^{\delta}{}_{\alpha\nu} - \Gamma^{\mu}{}_{\delta\nu}\,\Gamma^{\delta}{}_{\alpha\beta} \tag{B.79}$$

which depends, as expected on the derivatives of the Christoffel-symbols as well as their "squares". But ultimately, due to the choice of a (pseudo-)Riemannian geometry, the curvature tensor can be computed from the metric and its first and second derivatives.

B.6 *Covariant divergence*

The idea of using the divergence for expressing conserved quantities like $g^{\alpha\mu}\nabla_{\alpha}j_{\mu} = 0$ for the electric charge or $g^{\alpha\mu}\nabla_{\alpha}T_{\mu\nu} = 0$ for the energy-momentum tensor is very central to physics. Formulated in a covariant way, it behaves properly as a tensor under coordinate transforms. The covariant divergence needs a pecular index-combination in the Christoffel-symbol, where two of the indices become equal.

$$\nabla_{\mu}\upsilon^{\mu} = \partial_{\mu}\upsilon^{\mu} + \Gamma^{\mu}{}_{\mu\alpha}\,\upsilon^{\alpha} \tag{B.80}$$

In particular, a Levi-Civita connection would have

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{g^{\mu\beta}}{2}\cdot\left[\partial_{\mu}\,g_{\beta\alpha} + \partial_{\alpha}\,g_{\mu\beta} - \partial_{\beta}\,g_{\mu\alpha}\right] = \frac{1}{2}\left[g^{\mu\beta}\,\partial_{\mu}\,g_{\beta\alpha} + g^{\mu\beta}\,\partial_{\alpha}\,g_{\mu\beta} - g^{\mu\beta}\,\partial_{\beta}\,g_{\mu\alpha}\right] \tag{B.81}$$

i.e. essentially

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{2}g^{\mu\beta}\,\partial_{\alpha}\,g_{\mu\beta} \tag{B.82}$$

Curiously, there is a relation between the covariant divergence and the covolume $g = \det(g_{\mu\nu})$. My third most favourite formula in theoretical physics says that

$$g = \det(g_{\mu\nu}) = \exp\ln\,\det(g_{\mu\nu}) = \exp\,\operatorname{tr}\,\ln(g_{\mu\nu}) \tag{B.83}$$

relating the logarithm of the determinant with the trace of the matrix-valued logarithm, which is easily checked in the principal axis frame. Then,

$$\partial_{\alpha}\,g = g\cdot\partial_{\alpha}\operatorname{tr}\,\ln(g_{\mu\nu}) = g\cdot\operatorname{tr}\,\partial_{\alpha}\,\ln(g_{\mu\nu}) = g\cdot\operatorname{tr}\left(g^{-1}\cdot\partial_{\alpha}\,g_{\mu\nu}\right) = g\cdot g^{\mu\nu}\cdot\partial_{\alpha}\,g_{\mu\nu} \tag{B.84}$$

using the linearity of the derivative as well as the inverse metric. With the derivative of the square root one then obtains

$$g^{\mu\nu}\,\partial_{\alpha}\,g_{\mu\nu} = \frac{1}{g}\partial_{\alpha}\,g, \quad \text{and therefore} \quad \frac{1}{2}g^{\mu\nu}\,\partial_{\alpha}\,g_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_{\alpha}\,\sqrt{-g}. \tag{B.85}$$

With this result one can write for the contracted Christoffel-symbol

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{\sqrt{-g}}\,\partial_{\alpha}\,\sqrt{-g} \tag{B.86}$$

and finally for the covariant divergence

$$\nabla_\mu\, \upsilon^\mu = \partial_\mu\, \upsilon^\mu + \Gamma^\mu{}_{\mu\alpha}\, \upsilon^\alpha = \partial_\mu\, \upsilon^\mu + \frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g}\cdot \upsilon^\alpha$$
$$\overset{\mu\leftrightarrow\alpha}{=} \partial_\mu \upsilon^\mu + \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\cdot \upsilon^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, \upsilon^\mu\right) \quad \text{(B.87)}$$

using the Leibnitz-rule. An interesting application of the covariant divergence is the wave equation

$$g^{\mu\nu}\, \nabla_\mu \nabla_\nu\, \phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, \partial^\mu\, \phi\right) = 0 \quad \text{(B.88)}$$

which is obviously not just $\partial_\mu \partial^\mu \phi = 0$; there is clearly an influence from the background onto wave propagation. For our particular case of FLRW-cosmologies, the covolume is quickly computed in comoving coordinates to be

$$\sqrt{-\det g} = ca^3 \quad \text{(B.89)}$$

with physical time t, and as

$$\sqrt{-\det g} = ca^4 \quad \text{(B.90)}$$

with conformal time η.

B.7 *Geodesic deviation: experiencing curvature*

A freely falling particle experiences perfect weightlessness and spacetime appears to be locally Minkowskian, $g_{\mu\nu} = \eta_{\mu\nu}$ with a vanishing first derivative $\partial_\alpha g_{\mu\nu} = 0$, which enables the *local* choice of Cartesian coordinates. But that does not imply that a second particle, likewise in a state of perfect free fall, moves at constant velocities relative to the first particle: This is exactly the idea of geodesic deviation. The relative distance δ^μ of two freely falling particles, each one following its geodesic, obeys

$$\frac{d^2\delta^\mu}{d\tau^2} = R^\mu{}_{\alpha\beta\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\delta^\nu \quad \text{(B.91)}$$

which follows from expanding $\Gamma(\bar{x})$ for the second particle in the geodesic equation in terms of $\Gamma(x)$ for the first geodesic. Only if the manifold is flat, the Riemann curvature $R^\alpha{}_{\beta\mu\nu} = 0$ vanishes, resulting in

$$\frac{d^2}{d\tau^2}\delta^\mu = 0 \quad \rightarrow \quad \frac{d}{d\tau}\delta^\mu = a^\mu \quad \rightarrow \quad \delta^\mu = a^\mu\tau + b^\mu \quad \text{(B.92)}$$

with two integration constants a^μ and b^μ: The particles would drift apart at a constant rate, and accelerations $\ddot{\delta}^\mu$ only appear if there is curvature. Please keep in mind, that this also applies for time component, as we use 4d coordinates. Classically we would get an analogous statement

$$\frac{d^2}{dt^2}\delta^i = -\underbrace{\partial^i\partial_j\Phi}_{\text{tidal tensor}}\ \delta^j = \partial^i g_j \delta^j \quad \text{(B.93)}$$

with no velocity dependence of the gravitational force and universal time instead of proper time. This underlines the idea that the tidal field tensor $\partial^i\partial_j\Phi$ is the Newtonian analogue of the Riemann curvature.

B.8 *Curvature invariants and curvature tensors*

The Riemann-curvature contains the complete information about curvature if the connection is chosen to be torsion-free and metric compatible, otherwise one would need the torsion tensor and the non-metricity scalar in addition. From the Riemann-curvature, one can compute further measures of curvature, which are physically relevant, such as the Ricci-tensor $R_{\beta\nu}$

$$R_{\beta\nu} = R^{\alpha}{}_{\beta\alpha\nu} = g^{\alpha\mu}R_{\alpha\beta\mu\nu} \tag{B.94}$$

and curvature scalars by complete contraction, for instance the Ricci-scalar R

$$R = R^{\alpha}{}_{\alpha} = g^{\beta\nu}R_{\beta\nu} \tag{B.95}$$

or the Kretschmann-scalar K

$$K = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}R_{\alpha\beta\gamma\delta}R_{\mu\nu\rho\sigma} \tag{B.96}$$

which are both coordinate-invariant measures of curvature.

Let's apply these ideas to a flat FLRW-cosmology, where the line element has the form

$$ds^2 = c^2dt^2 - a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)) \tag{B.97}$$

in terms of comoving coordinates and physical time. The trivial derivatives of the metric are

$$\partial_t g_{tt} = 0, \quad g_{t\alpha} = g_{\alpha t} = 0 \tag{B.98}$$

and the non-trivial derivatives can be summarised in the Christoffel-symbols

$$\Gamma^t{}_{\alpha\beta} = \frac{1}{2}\partial_t g_{\alpha\beta} \rightarrow \Gamma^t{}_{ij} = \frac{1}{2}\frac{d}{dt}a^2 = \dot{a}a = a^2H \tag{B.99}$$

with $H = \dot{a}/a$, and

$$\Gamma^i{}_{\alpha\beta} = \frac{1}{2a^2}\left(\partial_\beta g_{i\alpha} + \partial_\alpha g_{\beta i}\right) \rightarrow \Gamma^i{}_{it} = \Gamma^i{}_{ti} = \frac{1}{2a^2}2\partial_t g_{ii} = \frac{\dot{a}}{a} = H \tag{B.100}$$

For the Ricci-scalar of a flat FLRW-spacetime we get, by contracting $g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\mu\nu}$

$$R = 6a\left(\frac{H}{c}\right)^2(1-q) \quad \text{with} \quad q = -\frac{\ddot{a}a}{\dot{a}^2} \tag{B.101}$$

The metric usually has no units (since $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ and coordinates are usually chosen to have units of length, and consequently would ds have units of length, but please keep in mind that this is purely conventional), whereas curvature as composed of second derivatives (with respect to x^μ) have units of a inverse length squared, which results in the curvature scale

$$\frac{1}{\sqrt{R}} = \frac{c}{H} \tag{B.102}$$

ignoring pre-factors of order one: The curvature scale of a FLRW-cosmology $R^{-1/2}$ is the Hubble-scale c/H, implying that on scales larger than c/H one can see effects of strong gravity, whereas on scales smaller than c/H spacetime can be approximated to be Minkowksian. In fact, light propagation effects associated with horizons appear on this scale.

B.9 *Raychaudhuri-equation*

The Riemann-tensor as a complete characterisation of spacetime curvature decomposes into two distinct types of curvature: The Ricci-curvature contained in $R_{\mu\nu}$ and the Weyl-curvature $C_{\alpha\beta\mu\nu}$, both tensors having 10 entries in 4 dimensions. The Ricci-tensor $R_{\mu\nu}$ at one point in spacetime reflects the energy momentum tensor $T_{\mu\nu}$ at the same point, as a consequence of the field equation, and is necessarily only a function of time. As there are no spatial derivatives in a FLRW-geometry, we are not concerned with propagation effects of gravity, so the Weyl-tensor $C_{\alpha\beta\mu\nu}$ is zero, and we're dealing in FLRW-cosmologies with a system of pure Ricci-curvature.

The effects of Ricci- and Weyl-curvature on test particles can be understood in an extension to geodesic equation, which is known as the Raychaudhuri-equation. Here, one considers not a pair, but an entire cloud of freely falling test particles and monitors the change of volume or the change in shape of that cloud. Ricci-curvature, which FLRW-spacetimes carry exclusively, induce a pure change in volume while conserving shape, while Weyl-curvature does the opposite: It causes a cloud of test particles to change its shape while conserving the volume.

The idea of the Raychaudhuri-equation is to have a look at the time evolution of the area enclosed by a bundle of geodesics. The Riemann-curvature splits in two different parts:

- The Ricci-curvature changes the volume enclosed by a bundle of geodesics but keeps the shape (typical for FLRW-cosmologies)
- The Weyl-curvature changes shape but conserves the volume (typical for gravitational waves)

Let's try a classical approach: Two test particles at coordinates x'^i and x^i have a relative motion at velocity υ^i

$$x'^i = x^i + \upsilon^i \Delta t \tag{B.103}$$

The relative coordinate mapping is encapsulated in the Jacobian

$$\frac{\partial x'^i}{\partial x^j} = \delta^i_j + \frac{\partial \upsilon^i}{\partial x^j}\Delta t \tag{B.104}$$

which we could think of as the first-order Taylor expansion of of a matrix-valued exponential

$$\frac{\partial x'^i}{\partial x^j} = \exp\left(\frac{\partial \upsilon^i}{\partial x^j}\Delta t\right) \tag{B.105}$$

The coordinate mapping would introduce a change in volume elements given by the functional determinant

$$d^3x' = \det\left(\frac{\partial x'}{\partial x}\right) d^3x \tag{B.106}$$

where we can write for the logarithmic change

$$\ln d^3x' = \ln\det\left(\frac{\partial x'}{\partial x}\right) + \ln d^3x = \ln d^3x' = \mathrm{tr}\ln\exp\left(\frac{\partial v^i}{\partial x^j}\Delta t\right) + \ln d^3x = \Delta t\,\mathrm{tr}\frac{\partial v^i}{\partial x^j} + \ln d^3x \tag{B.107}$$

In this relation, one can identify $\mathrm{tr}(\partial v^i/\partial x^j)$ with the divergence of the velocity field: Intuitively, if this divergence is nonzero, the volume should change.

The Newton equation of motion for small time differences Δt is

$$v^i = -\partial^i\Phi\Delta t \tag{B.108}$$

which suggests for the velocity divergence

$$\partial_i v^i = -\partial_i\partial^i\Phi\Delta t = -\Delta\Phi\Delta t \tag{B.109}$$

with the Laplace-operator $\Delta = \partial_i\partial^i$, and therefore

$$\ln d^3x' - \ln d^3x = -\Delta\Phi(\Delta t)^2 \approx 8\pi G\rho\frac{(\Delta t)^2}{2} \tag{B.110}$$

Therefore, the logarithmic volume change is proportional to the density inside the volume and $\Delta\Phi \propto \rho$ from Poisson's equation. The volume change measures effectively the enclosed mass, and suggests that Riemann curvature and tidal field are analogous quantities, as well as the Ricci-curvature (as the trace of the Riemann curvature) and the Laplacian of the potential, and that both are coupled to the matter density as the source of the gravitational field.

The proper relativistic Raychaudhuri-equation makes a statement about the velocity divergence Θ,

$$\Theta = \nabla_\mu u^\mu = g^{\mu\nu}\nabla_\mu u_\nu \tag{B.111}$$

and states that for the time evolution that

$$\frac{d\Theta}{dt} = -\frac{\Theta^2}{3} - \underbrace{\sigma_{\mu\nu}\sigma^{\mu\nu}}_{=0\text{ in FLRW}} + \underbrace{\omega_{\mu\nu}\omega^{\mu\nu}}_{=0\text{ in FLRW}} - R_{\mu\nu}u^\mu u^\nu + \nabla_\mu\dot{u}^\mu. \tag{B.112}$$

The terms in the Raychaudhuri-equation are the shear,

$$\sigma_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{1}{3}\Theta p_{\mu\nu} - (\dot{u}_\mu u_\nu + u_\mu\dot{u}_\nu) \tag{B.113}$$

and the vorticity

$$\omega_{\mu\nu} = \nabla_\mu u_\nu - \nabla_\nu u_\mu - (\dot{u}_\mu u_\nu + u_\mu\dot{u}_\nu) \tag{B.114}$$

and finally the proper acceleration

$$\dot{u}^\mu = u^\nu \nabla_\nu u^\mu \tag{B.115}$$

which vanishes, if u^μ is tangent to the geodesic (from geodesic equation), i.e. with no non-gravitational accelerations.

FLRW-cosmologies have no preferred axis due to the isotropy postulate, resulting in vanishing vorticity ω and vanishing shear $\sigma = 0$ and therefore

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = -\frac{\Theta^2}{3} - \mathrm{R}_{\mu\nu} u^\mu u^\nu \tag{B.116}$$

for the evolution of the velocity divergence. By using comoving FLRW-coordinates we see no expansion as all, all galaxies stay in their spatial coordinate and only move in the time-direction. Consequently we choose $u^\mu = (c, 0)^t$ and can now compute the covariant divergence as

$$\Theta = \nabla_\mu u^\mu = \frac{1}{\sqrt{-\det g}} \partial_\mu \left(\sqrt{-\det g}\, u^\mu\right) \sim \partial_t \ln(\text{volume}) \tag{B.117}$$

Specifically, with the FLRW-metric in comoving coordinates

$$g_{\mu\nu} = \begin{pmatrix} c^2 & & & \\ & -a^2 & & \\ & & -a^2 & \\ & & & -a^2 \end{pmatrix} \tag{B.118}$$

suggesting for the velocity divergence, or equivalently, the volume evolution

$$\sqrt{-\det g} = ca^3 \tag{B.119}$$

and finally

$$\Theta = \frac{1}{a^3} \partial_t \left(a^3\right) = 3\frac{\dot{a}a^2}{a^3} = 3\frac{\dot{a}}{a} = 3\mathrm{H}(t) \tag{B.120}$$

such when replacing proper time by coordinate or cosmic time ($t = \tau$) as specifically allowed by FLRW-cosmologies one obtains:

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = -\frac{\Theta^2}{3} - \mathrm{R}_{\mu\nu} u^\mu u^\nu \tag{B.121}$$

with

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{a}}{a} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \tag{B.122}$$

In summary, the volume evolution of a FLRW universe is expressed in terms of the scale factor a and its derivatives, and depends on the term $\mathrm{R}_{\mu\nu} u^\mu u^\nu$, which we can provide through the field equation of gravity and which depends on the energy-momentum content of spacetime: It acts as the source of the gravitational field and introduces curvature, the Ricci-part of which affects the evolution of Θ. It is amazing to see how volume evolution of the FLRW-spacetime works in Newtonian gravity and relativity alike, and that the Raychaudhuri-equation makes a sensible statement about the evolution of the velocity divergence of comoving velocity (which one would naively visualise as a perfectly parallel vector field in comoving coordinates). Progress beyond this result is only possible if we assume a specific form of the field equation, in order to relate the Ricci-tensor to the energy momentum-tensor. Surprising as it may seem, the Raychaudhuri-equation is a purely geometric statement about the divergence of a vector field, and does not assume anything specific about the gravitational theory.

B.10 *Energy-momentum tensor*

The energy and momentum content of spacetime sources the gravitational field: Very similar to the case of Maxwell-electrodynamics which is the theory of electric and magnetic fields for charge-conserving systems, general relativity is the theory of gravity for energy and momentum conserving systems (although that can be only formulated locally in the form of $g^{\alpha\mu}\nabla_\alpha \mathrm{T}_{\mu\nu} = 0$ with the energy-momentum tensor $\mathrm{T}_{\mu\nu}$). In a fluid picture, energy and momentum conservation would characterise the dynamics of an ideal (i.e. inviscid) fluid, which can only have three properties: velocity u^μ, density ρ and pressure p, assembled into the energy momentum-tensor $\mathrm{T}^{\mu\nu}$.

$$\mathrm{T}^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu} \tag{B.123}$$

In a frame where the fluid is at rest, $u^\mu = (c, 0)^t$ and adopting locally flat, Cartesian coordinates one falls back on a diagonal form,

$$\mathrm{T}^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{B.124}$$

Ideal fluids are characterised by the conservation law

$$g^{\alpha\mu}\nabla_\alpha \mathrm{T}_{\mu\nu} = 0 \tag{B.125}$$

which is not straightforward to interpret, as it is a vectorial statement (in the index ν): This is in contrast to e.g. the law of charge conservation $g^{\alpha\mu}\nabla_\alpha \jmath_\mu = 0$ in electrodynamics, which is scalar.

It might be intuitive to project the vector $\nabla_\mu \mathrm{T}^{\mu\nu}$ onto the velocity u_ν and to a plane perpendicular to it, as u^μ is defining naturally a direction. Keeping in mind that the covariant derivative fulfils a Leibnitz-rule one gets:

$$\nabla_\mu \mathrm{T}^{\mu\nu} = \nabla_\mu \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + \left(\rho + \frac{p}{c^2}\right) \nabla_\mu (u^\mu u^\nu) - \nabla_\mu p g^{\mu\nu} \tag{B.126}$$

with

$$\nabla_\mu (u^\mu u^\nu) = u^\nu \nabla_\mu u^\mu + \underbrace{u^\mu \nabla_\mu u^\nu}_{=0 \text{ if } u^\mu \text{ tangent to a geodesic}} \tag{B.127}$$

while any covariant derivative of the metric (and its inverse) vanishes due to metric compatibility. Carrying out the projection

$$u_\nu \nabla_\mu T^{\mu\nu} = 0 \tag{B.128}$$

we end up at

$$u_\nu \nabla_\mu T^{\mu\nu} = \nabla_\mu \left(\rho + \frac{p}{c^2}\right) u^\mu \underbrace{u_\nu u^\nu}_{c^2} + \left(\rho + \frac{p}{c^2}\right) \underbrace{u_\nu u^\nu}_{c^2} \nabla_\mu u^\mu - \nabla_\mu p \underbrace{g^{\mu\nu} u_\nu}_{u^\mu} = 0 \tag{B.129}$$

which can be further simplified to

$$u_\nu \nabla_\mu T^{\mu\nu} = \nabla_\mu \rho u^\mu c^2 + \rho \nabla_\mu u^\mu c^2 + p \nabla_\mu u^\mu = \nabla_\mu (\rho u^\mu) c^2 + p \nabla_\mu u^\mu = 0. \tag{B.130}$$

The projection of $u_\nu T^{\mu\nu} = 0$ onto a plane perpendicular to u_ν yields the Euler-equation

$$\left(\rho + \frac{p}{c^2}\right) \nabla_\mu u^\nu u^\mu = \left(g^{\mu\nu} - \frac{u^\mu u^\nu}{c^2}\right) \nabla_\mu p \tag{B.131}$$

B.11 *Equation of state for ideal fluids*

Ideal fluids are characterised by just two quantities (apart from their velocity field u^μ): density ρ and pressure p, and often it is the case that the two are related by an equation of state which reflects internal properties of the substance. Like in the case of an ideal classical or relativistic gas there could be a proportionality

$$p = w \rho c^2 \tag{B.132}$$

with the equation of state parameter w, which in relativity is often assumed to be constant (Please remember that energy density and pressure have identical units!). In a frame where the fluid is at rest we would write $u^\mu = (c, 0)^t$ and covariant energy momentum conservation $\nabla_\mu T^{\mu\nu} = 0$ becomes in the choice of comoving coordinates,

$$\nabla_\mu (\rho u^\mu) + \frac{p}{c^2} \nabla_\mu u^\mu = 0 \tag{B.133}$$

The first term $\nabla_\mu (\rho u^\mu)$ can be taken apart with the Leibnitz-rule

$$\nabla_\mu (\rho u^\mu) = \nabla_\mu \rho \cdot u^\mu + \rho \nabla_\mu u^\mu = \partial_\mu \rho \cdot u^\mu + \rho \nabla_\mu u^\mu \tag{B.134}$$

resulting in

$$\partial_\mu \rho + \left(\rho + \frac{p}{c^2}\right) \nabla_\mu u^\mu = 0 \tag{B.135}$$

by substituting the equation of state $p = w \rho c^2$. The covariant divergence of the velocity u^μ is

$$\nabla_\mu u^\mu = \underbrace{\partial_\mu u^\mu}_{=0} + \Gamma^\mu{}_{\alpha\mu} u^\alpha = \underbrace{\Gamma^i{}_{it}}_{=H/c} u^t = 3\frac{\dot{a}}{a} = 3H(t) \tag{B.136}$$

and using the fact that ρ is homogeneous and only a function of time, there is just a single derivative left, $\partial_\mu\rho = \partial_t\rho = \dot{\rho}$. With the argument we arrive at the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1+w)\rho = 0 \tag{B.137}$$

which reflects covariant energy momentum-conservation $\nabla_\mu T^{\mu\nu} = 0$ in a FLRW-spacetime.

Separation of the variables and assuming a constant equation of state parameter w results in

$$\frac{d\rho}{\rho} = d\ln\rho = -3(1+w)\frac{da}{a} = -3(1+w)d\ln a \tag{B.138}$$

which can be solved by

$$\rho \propto a^{-3(1+w)} \tag{B.139}$$

In fact, non-relativistic matter with $w = 0$ would dilute $\rho \propto a^{-3}$ as it is simply dispersed over a larger volume a^3, but for relativistic matter with $w = 1/3$ there would be an additional redshifting effect leading to $\rho \propto a^{-4}$.

Sometimes you find a reformulation of continuity in this way: Multiplying eqn. B.137 with the volume a^3 yields

$$a^3\frac{d\rho}{da} + 3a^2\left(\rho + \frac{p}{c^2}\right) = 0 \quad \rightarrow \quad \frac{d}{da}(\rho c^2 a^3) = -p\frac{d}{da}(a^3) \tag{B.140}$$

with the interpretation that the energy density ρc^2 of the fluid changes if the volume changes, performing work against pressure p, reminiscent of the first law of thermodynamics, $dU = -p dV$: This is why the relation is sometimes called the adiabatic law.

General relativity is prepared to provide gravity for both fields and fluids, as both fields and fluids obey covariant conservation of energy and momentum, $\nabla_\mu T^{\mu\nu} = 0$ or $g^{\mu\alpha}\nabla_\mu T_{\alpha\nu} = 0$. But comparing the two cases fluids and fields, in the first case one would speak of the Poynting-law, and in the second case about the fluid mechanical equations: It is actually amazing that general relativity can provide gravity for such different concepts of matter.

B.12 *Relativistic fluid mechanics*

Let's start again at the expression for the energy-momentum tensor of an ideal fluid,

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu - g^{\mu\nu}p \tag{B.141}$$

of which one can directly compute the trace,

$$g_{\mu\nu}T^{\mu\nu} = \rho c^2 - 3p \tag{B.142}$$

keeping in mind that $g_{\mu\nu}u^\mu u^\nu = c^2$ for material particles, which approaches $p/(\rho c^2) = 1/3$ for relativistic, massless particles, for which $g_{\mu\nu}T^{\mu\nu} = 0$.

Computing the covariant divergence of $T^{\mu\nu}$ one gets

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho c^2 + p) \cdot u^\mu u^\nu + (\rho c^2 + p)\nabla_\mu(u^\mu u^\nu) - \nabla_\mu p \cdot g^{\mu\nu} \tag{B.143}$$

keeping in mind that the covariant derivative of the metric vanishes for a metric-compatible connection, in particular $\nabla_\mu g^{\mu\nu} = 0$ as well for the inverse metric.

Let's pursue this divergence and let's make a deliberate mistake by assuming that the fluid elements follow geodesics, i.e. that an autoparallelity condition applies to the velocities u^μ:

$$\nabla_\mu(u^\mu u^\nu) = \nabla_\mu u^\mu \cdot u^\nu + \underbrace{u^\mu \nabla_\mu u^\nu}_{=0} \tag{B.144}$$

such that the divergence of the velocity field is the only contributing term, which we've already encountered in the discussion of the Raychaudhuri-equation. Then,

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho c^2 + p) \cdot u^\mu u^\nu + (\rho c^2 + p)\nabla_\mu u^\mu \cdot u^\nu - \nabla_\mu p \cdot g_{\mu\nu} \tag{B.145}$$

Now we are projecting (as mentioned above) on the observer's world line with $u_\nu = \mathrm{d}x_\nu/\mathrm{d}\tau$, and by contraction

$$u_\nu \nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho c^2 + p) \cdot u^\mu \underbrace{u_\nu u^\nu}_{=1} + (\rho c^2 + p)\nabla_\mu \underbrace{u_\nu u^\nu}_{=1} - \nabla_\mu p \cdot \underbrace{g^{\mu\nu} u_\nu}_{=u^\mu} \tag{B.146}$$

and arrive the relativistic continuity equation

$$u_\nu \nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho c^2 u^\mu) + p\nabla_\mu u^\mu = 0 \tag{B.147}$$

in which we used the covariant conservation in the last step. In the non-relativistic limit with $p \ll \rho c^2$ this leaves us with only the first summand $\nabla_\mu(\rho c^2 u^\mu) = 0$ left. Furthermore, u^μ is given by $u^\mu = (1, \beta^i)^T$ with $\beta^i = \frac{\upsilon^i}{c}$ and of course $|\beta| \ll 1$ in the slow motion limit. Lastly, ∇_μ becomes ∂_μ by adopting locally Cartesian coordinates, so:

$$u_\nu \nabla_\mu T^{\mu\nu} = 0 = \nabla_\mu(\rho c^2 u^\mu) = \partial_{ct}(\rho c^2) + \partial_i(\rho c^2 \beta^i) \tag{B.148}$$

i.e. the classical continuity equation,

$$\partial_t \rho + \partial_i(\rho \upsilon^i) = 0. \tag{B.149}$$

In the real world it turns out that there are incompressible fluids which are characterised by $\partial_i \upsilon^i = 0$ (Please watch out: Incompressibility is a statement about the velocity field and has little to do with pressure!) and therefore the continuity equation becomes

$$\partial_t \rho + \partial_i \rho \cdot \upsilon^i = 0 \tag{B.150}$$

for incompressible fluids.

We can substitute the relativistic continuity equation back into the conservation law $\nabla_\mu T^{\mu\nu} = 0$ to arrive at

$$\nabla_\mu T^{\mu\nu} = \underbrace{\left(\nabla_\mu(\rho c^2 u^\mu) + p\nabla_\mu u^\mu\right)}_{=0 \text{ continuity}} u^\nu + \nabla_\mu p u^\mu u^\nu - \nabla_\mu p g^{\mu\nu} = \nabla_\mu p(u^\mu u^\nu - g^{\mu\nu}) = 0 \quad \text{(B.151)}$$

which states that there are no pressure gradients perpendicular to u^μ as u^μ is tangent to a geodesic (or otherwise the fluid doesn't follow a geodesic in equivalence to the previous expression). This is particularly relevant for FLRW-cosmologies, as it implies that there can not be any spatial gradients in pressure, $\nabla_i p = \partial_i p = 0$, in accordance to the Copernican-principle. If those gradient would exist, the motion of fluid elements can not be inertial.

Let's restart by imposing no condition on geodesic motion of fluid elements. Then,

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho c^2 + p) \cdot u^\mu u^\nu + (\rho c^2 + p)\nabla_\mu(u^\mu u^\nu) - \nabla_\mu p \cdot g^{\mu\nu} = 0 \quad \text{(B.152)}$$

A useful auxiliary statement can be obtained from the normalisation of u_ν, where

$$0 = \nabla_\mu(\underbrace{u^\nu u_\nu}_{=1}) = u^\nu \nabla_\mu u_\nu + \nabla_\mu u^\nu \cdot u_\nu = 2u_\nu \nabla_\mu u^\nu \quad \text{(B.153)}$$

If one contracts the conservation law with u_ν, the second term $u^\mu \nabla_\mu u^\nu$ becomes $u^\mu u_\nu \nabla_\mu u^\nu = 0$. We therefore end up at

$$u_\nu T^{\mu\nu} = \nabla_\mu(\rho c^2 + p) \cdot u^\mu + (\rho c^2 + p)\nabla_\mu u^\mu - \nabla_\mu p u^\mu = \nabla_\mu(\rho c^2 u^\mu) + p\nabla_\mu u^\mu = 0 \quad \text{(B.154)}$$

even if the fluid follows non-geodesic motion. If we resubstitute back into the full, non-geodesic conservation equation we obtain

$$\nabla_\mu T^{\mu\nu} = (u^\mu u^\nu - g^{\mu\nu})\nabla_\mu p + (\rho c^2 + p)u^\mu \nabla_\mu u^\nu = 0 \quad \text{(B.155)}$$

with an additional term being present for non-geodesic motion caused by gradients in pressure.

C FLRW-COSMOLOGIES

C.1 *Dynamics of spacetime*

Over hundert years ago, E. Hubble did research on spectral lines of distant galaxies. He discovered that the spectral lines are shifted towards longer wave lengths, which he interpreted as a Doppler shift caused by the motion of galaxies away from us as observers. This recession motion increases proportionally to distance:

$$\upsilon = \mathrm{H}_0 r \tag{C.156}$$

with the Hubble-Lemaître constant $\mathrm{H}_0 = 10^5 h$ m/s/Mpc, and the Hubble-parameter $h = 0.68 \dots 0.72$, depending on the measurement method. For a galaxy 10 Mpc away from the Milky Way, the recession velocity would be $\beta = \upsilon/c \simeq 0.003$, which is easily measurable through spectroscopy. While the interpretation of a recession motion is absolutely valid in Newtonian cosmology, general relativity brings in a new concept, namely that the laws of Nature, in particular gravity, are fully covariant, i.e. that coordinate choice does not matter, and that different coordinate choices require different physical interpretations.

If one adopts physical coordinates, consisting of a static coordinate grid, through which the galaxies move isotropically as the Universe expands, one obtains for the relation between velocity and distance

$$\upsilon(r,t) = \mathrm{H}(t) r \tag{C.157}$$

which can only depend on time in fulfilment of the cosmological principle: Including any nonlinear dependence of r causes a violation of homogeneity: Starting from the continuity equation for the matter density ρ

$$\dot{\rho} + \partial_i \jmath^i = 0 \tag{C.158}$$

with the momentum density $\jmath^i = \rho\upsilon^i$. If the velocity fulfils the Hubble-law $\upsilon^i = \mathrm{H}_0 r^i$ it would imply for the divergence

$$\partial_i \jmath^i \overset{\text{isotropy}}{=} \frac{1}{r^2}\partial_r(r^2\rho\upsilon_r) = \frac{\rho\mathrm{H}}{r^2}\partial_r(r^3) = 3\mathrm{H}\rho \tag{C.159}$$

if one in addition assumes isotropy such that the velocity has only a radial dependence and using spherical coordinates to formulate the divergence. If ρ does not depend on r, as used in the last step, its time evolution will make sure that it will stay homogeneous. The situation would be fundamentally different if for instance $\upsilon = r^\alpha$, with $\alpha \neq 1$. Then,

$$\partial_i \jmath^i = \frac{1}{r^2}\partial_r(r^2\rho\upsilon_r) = \frac{\rho\mathrm{H}}{r^2}\partial_r(r^{2+\alpha}) = (2+\alpha)\mathrm{H}\rho r^{\alpha-1} \tag{C.160}$$

such that the time evolution of ρ depends on r, and the continuity equation can not uphold homogeneity, in violation of the cosmological principle.

In comoving coordinates the picture is different: The coordinate grid expands along with the flow of matter, and all particles stay at their comoving coordinate. We therefore differentiate between comoving coordinates x^i and physical coordinates

$r^i = a(t)x^i$, which are related through the scale factor $a(t)$, which itself is only a function of time t. The coordinate change of the physical coordinate with time is given by

$$\frac{\mathrm{d}r^i}{\mathrm{d}t} = \upsilon^i = \dot{a}x^i + a\dot{x}^i = \mathrm{H}(t)r^i + a\upsilon^i_{\mathrm{pec}} \tag{C.161}$$

with two possible contributions of the spectroscopically measured velocity υ^i: The cosmological part due to a nonzero $\mathrm{H}(t) = \dot{a}/a$ and a peculiar motion $\upsilon^i_{\mathrm{pec}}$ relative to the (comoving) coordinate grid: When considering truly fundamental observers and test particles, the peculiar velocity would be zero.

The FLRW-metric of a flat space is usually given in terms of the line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, which reads in comoving coordinates

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = c^2\mathrm{d}t^2 - a^2(t)\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{C.162}$$

so that the spatial part of the metric (here written down in Cartesian and in spherical coordinates) is scaled by the scale factor $a(t)^2$. The choice of comoving coordinates is uniquely suited to the symmetries of a FLRW-spacetime: Neither does the metric depend on position, nor does it single out any particular direction.

This form of the line element, however, is not the most general possible compatible with the cosmological principle: The spatial part of the spacetime can have a constant curvature such that the scaling of surfaces of spheres with their radii differs from the Euclidean prediction. Introducing a curvature parameter k we can write

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{C.163}$$

There is a peculiarity of the FLRW-spacetime that concerns the passage of time: While it is perfectly normal that time passes at a different rate at different locations in a gravitational potential, this is not the case in FLRW-cosmologies. In fact, the line element $\mathrm{d}s$ is perceived as the elapsed proper time $\mathrm{d}\tau$ by an observer,

$$c^2\mathrm{d}\tau^2 = \mathrm{d}s^2 \tag{C.164}$$

such that according to homogeneity, $\mathrm{d}\tau = \mathrm{d}t$ for the FLRW-spacetime: Every observer sees the same passage of time and the coordinate time t is equal to proper time τ. This has profound consequences, as it enables a universal definition of the age of the Universe, which necessarily needs to be equal for every observer.

C.2 *Light propagation on a FLRW-spacetime and redshift*

As the coordinate choice is arbitrary rates of change of coordinates should not be assigned any physical meaning, in particular if these velocities are compared to the speed of light c. Whether a cosmological object is visible or not depends on whether a geodesic line between that object and an observer exists or not, specifically for photons this must be a null-geodesic with a normalisation $\mathrm{d}s^2 = g_{\mu\nu}k^\mu k^\nu = 0$ of the wave vector $k^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$.

The null-property of the wave vector ensures that photons propagate dispersion-free in vacuum. In fact, writing k^μ in components

$$k^\mu = \begin{pmatrix} \omega/c \\ k^i \end{pmatrix} \tag{C.165}$$

with the angular frequency ω and the spatial wave vector k^i leads to the norm

$$g_{\mu\nu}k^\mu k^\nu = \left(\frac{\omega}{c}\right)^2 - k_i k^i = 0 \tag{C.166}$$

leads to a linear relation between angular frequency and wave number

$$\omega(k) = \pm ck \tag{C.167}$$

and in consequence to equal phase and group velocities,

$$\upsilon_{\mathrm{group}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = c \quad \text{and} \quad \upsilon_{\mathrm{phase}} = \frac{\omega}{k} = c. \tag{C.168}$$

Dispersion-free propagation of photons $\upsilon_{\mathrm{group}} = \upsilon_{\mathrm{phase}}$ is encoded by the fact that their wave vector k^μ is a null-vector.

The null-condition $\mathrm{d}s^2 = 0$ has a very intricate connection to FLRW-spacetimes, as they are conformally flat: The full Riemann-curvature decomposes into two contributions: Weyl-curvature and Ricci-curvature, and the FLRW-symmetries in fact make sure that cosmological solutions are of pure Ricci-curvature, as the Weyl-tensor vanishes identically. Spacetimes, in which this is the case, are conformally flat, as their metric can be written as a rescaled Minkowksi-metric with a conformal factor $\Omega(x^\mu)^2 > 0$, which is strictly positive,

$$g_{\mu\nu} = \Omega(x^\mu)^2\, \eta_{\mu\nu} \tag{C.169}$$

as conformal transformations leave they Weyl-tensor $C_{\alpha\beta\mu\nu}$ invariant and conserve in the FLRW-case its value of zero. Applied to cosmology, we would write for the line element

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = a^2(t)\left(c^2\frac{\mathrm{d}t^2}{a^2}\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) = a^2(t)\left(c^2\mathrm{d}\eta^2 - \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) \tag{C.170}$$

with a new time coordinate $\mathrm{d}\eta$, which is called conformal time:

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a} \tag{C.171}$$

and the scale-factor $a(t)$ is in fact the conformal factor $\Omega(x^\mu)$ which in our case only depends on time and not on the spatial coordinates.

In fact, $\mathrm{d}\eta$ is not uniformly passing unlike $\mathrm{d}t$. Only today with $a = 1$ time intervals in η and t are identical, and as $a < 1$ in the past, intervals in η have been larger than those in t.

This has two interesting consequences: Firstly, light-propagation in a conformally flat spacetime proceeds in a perfectly Minkowskian way as the conformal factor drops out in the null-condition:

$$ds^2 = a^2(t)\left(c^2 d\eta^2 - \gamma_{ij} dx^i dx^j\right) = 0 \tag{C.172}$$

Secondly, the conformal age of the Universe is in fact infinite even if the actual age of the Universe (defined as the physical time passing since the instant $a = 0$) is finite, as the coordinate axes of Minkowski-space stretch out to infinity. Because homogeneity of the FLRW-spacetime allow always to place the origin of the coordinate frame at the observer, all photons are radially moving, so one can write for the line element

$$ds^2 = \left(c^2 d\eta^2 - d\chi^2\right) = (c d\eta + d\chi)(c d\eta - d\chi) = d\upsilon dw = 0 \tag{C.173}$$

and define light cone coordinates $d\upsilon = c d\eta + d\chi$ and $dw = c d\eta - d\chi$, reminiscent of Kruskal-coordinates.

C.3 *Evolution of the Hubble-expansion with time*

Initially, the Hubble function $H(t)$ was introduced for parameterising the linear relationship between the recessional velocity υ and distance r, $\upsilon = H(t)r$, and with the definition $H(t) = \dot{a}/a$ we relate it to a Taylor-expansion of $a(t)$ at the current cosmic epoch t_0,

$$a(t) - a(t_0) = \frac{da}{dt}(t - t_0) + \frac{d^2 a}{dt^2}\frac{(t - t_0)^2}{2} \pm \ldots \tag{C.174}$$

which can be rewritten as

$$a(t) = a(t_0)\left(1 + H(t_0)(t - t_0) - q(t_0)H^2(t_0)\frac{(t - t_0)^2}{2}\right) \tag{C.175}$$

by renormalising everything with $a(t_0)$. Taking every function to be evaluated at t_0, $\dot{a}/a$ becomes the Hubble function and $\ddot{a}/a = a\ddot{a}/a^2 \times \dot{a}^2/\dot{a}^2 = -qH^2$ brings in the deceleration parameter, usually defined with a minus-sign:

$$q = -\frac{\ddot{a}a}{\dot{a}^2} \tag{C.176}$$

despite the fact that the Universe is currently accelerating, so $\ddot{a} > 0$ causes q to be negative, as both a and $\dot{a}^2$ are positive; this is in fact a historical remnant. In summary, H determines the current rate at which the scale factor changes as a function of time, and q states by how much that rate changes with time. It is interesting to realise that the Hubble-relation is valid at every instance in time simultaneously for every distance, but of course we do not observe the recession velocity of a distant galaxy at the time that the light was emitted - so as we look out into the distance along the past light cone, we see a record of the recession velocities.

C.4 *Field equation: coupling gravity to matter*

The curvature of spacetime is determined by the energy momentum tensor by means of the field equation

$$\underbrace{R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}}_{G_{\mu\nu}} = -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{C.177}$$

which equates the Einstein-tensor $G_{\mu\nu}$ to the energy momentum tensor $T_{\mu\nu}$ with Newton's gravitational constant G as a coupling constant, but there is an effect of gravity of empty space, too: Even if $T_{\mu\nu} \equiv 0$, the curvature is nonzero due to the presence of the cosmological constant Λ. Actually, this result is not totally surprising as the cosmological constant was already present in the most general linear field theory for a scalar field on a Minkowski-background in the first chapter. The gravitational field equation is unique, as shown by David Lovelock, as the most general (i) second-order partial differential equation in (ii) 4 dimensions, with (iii) covariant energy momentum conservation $\nabla_\mu T^{\mu\nu} = 0$, which establishes a (iv) local relationship between curvature and the source of gravitational field and lastly, if (v) the metric is the only dynamical degree of freedom, from which the curvature is derived. In particular, Lovelock's result makes sure that there are only two tensors, the Einstein-tensor $G_{\mu\nu}$ and the metric $g_{\mu\nu}$, that have vanishing divergences, the first as a consequence of the Bianichi-identity and metric due to metric compatibility. The field equation can be derived by a variation of the Einstein-Hilbert-Lagrange density

$$S = \int d^4x \sqrt{-\det g}(R - 2\Lambda) \tag{C.178}$$

with respect to the (inverse) metric: The choice of this Lagrange-density is unique, again according to Lovelock's theorem.

Within the highly symmetric solutions of general relativity discussed in every textbook cosmology plays a central role: FLRW-spacetimes are, due to the cosmological principle, systems of pure Ricci-curvature (with a vanishing Weyl-tensor); and as such they do not show any propagation effects of gravity. Because of the small value of the cosmological constant, its effect on the dynamics of spacetimes becomes only dominant on scales comparable to the observable Universe.

C.5 *FLRW-spacetimes and their dynamics*

FLRW-cosmologies are a solution to the gravitational field equation with homogeneity and isotropy as symmetries restricting the complexity of the solution, and for ideal fluids as sources. As the only degree of freedom left after imposing the FLRW-symmetries is the scale factor $a(t)$, one effectively ends up at ordinary (albeit nonlinear) differential equations: In fact, the Friedmann equations relate $a(t)$ and its first and second derivatives $\dot{a}$ and $\ddot{a}$ to the properties of the fluid, i.e. density ρ and pressure p, which have to be constant across spactime at a fixed time, in order to fulfil the cosmological principle, too.

Starting from the FLRW-metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ (determined through the defining property $g_{\mu\nu}g^{\nu\alpha} = \delta^\alpha_\mu$) one computes the Christoffel-symbols

$$\Gamma^\alpha{}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right) \tag{C.179}$$

under the choice of a metric compatible and torsion-free connection. Then, the Riemann-curvature $R^{\alpha}{}_{\beta\mu\nu}$ follows from derivatives and squares of the Christoffel-symbols, and the Ricci-curvature $R_{\beta\nu} = g^{\alpha\mu}R_{\alpha\beta\mu\nu}$

$$R_{tt} = 3\frac{\ddot{a}}{a} \tag{C.180}$$

$$R_{rr} = \frac{-c^2}{1-kr^2}\left(a\ddot{a} + 2\dot{a}^2 + 2c^2k\right) \tag{C.181}$$

$$R_{\theta\theta} = -\frac{c}{r^2}\left(a\ddot{a} + 2\dot{a}^2 + 2c^2k\right) \tag{C.182}$$

$$R_{\phi\phi} = R_{\theta\theta}\cdot\sin^2\theta \tag{C.183}$$

such that contraction $g^{\mu\nu}R_{\mu\nu} = R$ yields the Ricci-scalar,

$$R(t) = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{ck}{a^2}\right]. \tag{C.184}$$

Substituting the Ricci-tensor and Ricci-scalar into the field equation for an ideal fluid gives the two Friedmann-equations, first from the spatial part of the field equation,

$$H^2(a) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{c^2 a}{a^2} \tag{C.185}$$

as well as from the temporal part,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{p}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{C.186}$$

The combination of Newton's gravitational constant G and the Hubble-constant H_0 provides naturally a density scale

$$\rho_{\mathrm{crit}} = \frac{3H^2}{8\pi G} \tag{C.187}$$

which helps to re-express the densities of all fluids by dimensionless density parameters

$$\Omega_i = \frac{\rho_i}{\rho_{\mathrm{crit}}} \tag{C.188}$$

such that the first Friedmann-equation can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_m}{a^3} + \Omega_\Lambda\right) \tag{C.189}$$

by using $\rho \propto a^{-3}$ for matter. This can easily be extended to further fluids, characterised by their equation of state parameters $w = p/(\rho c^2)$.

We therefore can assign an Ω to k, for consistency

$$1 = \Omega_k + \sum_i \Omega_i \tag{C.190}$$

as otherwise $H(1) \neq H_0$. The curvature Ω_K vanishes if $\sum_i \Omega_i = 1$, in this limit the spatial part of spacetime would be a flat, Euclidean space.

C.6 *Gravitating fluids and their associated dynamics*

By coupling the dynamics of spacetime its energy-momentum content through the field equation, we can predict the time evolution of the scale factor $a(t)$ for a given density and equation of state parameter. While it is obvious that high matter or radiation densities should have a decelerating effect on spacetime, we should have a more detailed look into the effect of the equation of state. Setting up a spatially flat FLRW-cosmology with a single fluid ($\rho = \rho_{crit}$) and a constant equation of state parameter w leads to the realisation that equation of state w and deceleration q are connected by

$$3(1 + w) = 2(1 + q) \tag{C.191}$$

Clearly, a sign change in q takes place at $w = -1/3$: While decelerated universes $q < 0$ need to have equations of state of $w > -1/3$, accelerated universes $q > 0$ are characterised by very negative equations of state $w < -1/3$. Interestingly, a fully curved, empty universe with $q = 0$ has an effective equation of state of $w = -1/3$, in accordance with the a^2-scaling of Ω_K. It expands at a constant $\dot{a}$ as there are no gravitating substances to change the state of motion.

It might be surprising that the deceleration is stronger for photons than for non-relativistic matter, but it is the case that photons on the other hand are more strongly affected by gravitational fields, too: That's the famous factor 2 in gravitational lensing by which the accelerating effect of a gravitational field on a photon is larger compared to a non-relativistic test particle.

Whether the FLRW-spacetime has a finite age depends on whether substances with $w > -1/3$ have been dominating the expansion at early times. Curvature and all substances with more negative equations of state tend to lead to infinitely old universes. As the Universe expands, densities scale proprotional to $a^{-3((1+w)}$, so it is the case that the more negative an equation of state is, the slower the fluid dilutes in the course of the Hubble-expansion, the ultimate example being cosmological constant Λ with $w = -1$, leading to a constant energy density.

C.7 *Redshift and the Hubble-expansion*

We observe spectral lines of distant galaxies, which are shifted towards the red or rather to lower energies. One should not think of the effect as a loss of energy, rather than a transformation effect: Surely, there is a redshifting effect due to the motion of a source relative to the observer already in special relativity, and in addition a geometric effect due to changes in the metric in general relativity. The interpretation of redshifting as a transformation effect can not depend on the choice of coordinates, but of course the prediction has to be independent of a specifically adopted coordinate choice, and in the following derivation we should illustrate this. Due to conformal flatness of FLRW-universes it is best to work in conformal coordinates $(c\eta, \chi)$ which illustrate the Minkowskian causal structure:

$$ds^2 = a^2(t)\left(c^2 d\eta^2 - d\chi^2\right) = 0 \tag{C.192}$$

In these particular coordinates, the metric is the Minkowski-metric, preceeded by $a^2(t)$ as the overall conformal factor,

$$g_{\mu\nu} = a^2(t)\eta_{\mu\nu} = \begin{pmatrix} c^2a^2 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix} \tag{C.193}$$

Motion of photons along the geodesic conserves the normalisation of the wave vector k^μ, so that $g_{\mu\nu}k^\mu k^\nu = 0$ is maintained. A measurement of the frequency of a photon takes place when the photon is intercepted by a timelike observer with a tangent u^μ to her or his world line $x^\mu(\tau)$. The resulting frequency ω is given by the projection

$$\omega = g_{\mu\nu}u^\mu k^\nu \tag{C.194}$$

and is, as a scalar product, a general scalar and invariant under coordinate transforms, as requested for the result of measurement. Clearly, the observed frequency can be affected by the relative orientation of k^μ and u^μ, which is the special relativistic Doppler-effect, but also by a non-Minkowskian scalar product mediated by the metric.

While the wave vector of a photon in conformal coordinates is oblivious to changes in the geometry due to conformal flatness, and the normalisation of the wave vector is conserved in geodesic motion,

$$g_{\mu\nu}k^\mu k^\nu = 0 \tag{C.195}$$

the actual velocities of comoving observers are non-constant: The motion of a galaxy is timelike with the normalisation

$$g_{\mu\nu}u^\mu u^\nu = c^2 > 0 \tag{C.196}$$

and even though the galaxy stays at its comoving coordinate, it moves non-uniformly through spacetime with respect to conformal time! A galaxy at rest in the comoving frame has only a nonzero t-component in its velocity,

$$g_{tt}u^t u^t = c^2 \quad \text{implying} \quad u^t = \frac{c}{\sqrt{g_{tt}}} = \frac{c}{a} \tag{C.197}$$

which is, perhaps a bit surprisingly, changing as $a(t)$ evolves, until it reaches c today: But please keep in mind that in conformal coordinates we're dealing with a non-uniform passing time coordinate. Computing the projection between k^μ and u^μ for the frequency gives

$$\omega' = g_{\mu\nu}u^\mu k^\nu = g_{tt}u^t k^t = a^2\frac{c}{a}\frac{\omega}{c} = a\omega \tag{C.198}$$

which can be used to derive a relation for the shifted wave length, as $\omega = ck = c\frac{2\pi}{\lambda}$:

$$\lambda' = \frac{1}{a}\lambda \tag{C.199}$$

and therefore define the redshift z according to

$$z = \frac{\lambda' - \lambda}{\lambda} = \frac{1}{a} - 1 \tag{C.200}$$

and vice versa

$$a = \frac{1}{1+z} \tag{C.201}$$

with the convention that $a = 1$ and $z = 0$ for today. Lastly, I'd like to point out that the term redshift is perhaps not ideal: The entire spectrum of a source gets stretched by the scale factor a, and we should think of a shifting of the logarithmic wave length:

$$\ln\lambda = \ln(\lambda' a) = \ln\lambda' + \ln a = \ln\lambda' - \ln(1+z) \tag{C.202}$$

C.8 *Continuity equation and general relativity*

Einstein's field equation is prepared to conserve the energy-momentum-tensor

$$\nabla_\mu T^{\mu\nu} = 0 \tag{C.203}$$

with the energy in the time and the momenta in the spatial components. We arrived (using our covariant derivative) at

$$\partial_t\rho + 3H(t)(1+w)\rho = 0 \tag{C.204}$$

if the equation of state parameter w is constant in time. Pay attention to the fact, the $a(t)$ appears in the continuity equation even if the fluid is at rest in the comoving frame. In this continuity equation, the term $H = \dot{a}/a$ takes care of gravity, which is first of all surprising as there is no effect of Newtonian gravitational potentials on the continuity of classical fluid mechanics, only on the Euler-equation as an accelerating term. Clearly,

$$\partial_t\rho + \partial_i(\rho v^i) = 0 \tag{C.205}$$

does not depend on the gravitational potential Φ. In weak, static gravity on has the line-element

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right)dx_i dx^i \tag{C.206}$$

and see, that there are no gravitational effects whereas there are effects in the FLRW-metric. In the weakly perturbed metric (C.206) there is time-dilatation (which for us is not relevant, since t is the coordinate time). We have

$$\nabla_\mu(\rho u^\mu) + \frac{p}{c^2}\nabla_\mu u^\mu = 0 \tag{C.207}$$

which leaves us with the first term in the non-relativistic limit, as $p \ll \rho c^2$. Computing the covariant divergence then gives

$$\nabla_\mu(\rho u^\mu) = \partial_\mu(\rho u^\mu) + \Gamma^\mu{}_{\mu\alpha}(\rho u^\alpha). \tag{C.208}$$

Wherein the prefactors for the largest component, which is u^t, are 0 by construction, since $\Gamma^{\mu}{}_{\mu t} \sim \partial_t\Phi = 0$ for static fields, showing that there is no first-order influence of static, weak gravitational fields on the continuity equation: This is one instance where gravity really behaves differently than in a classical context.

C.9 *Construction of FLRW-universes for ideal fluids*

Gravity and the dynamical behaviour of the scale-factor $a(t)$ in a FLRW universe is sourced by an ideal fluid, at rest in the comoving frame: With the velocities of the fluid elements given by $u^{\mu} = (c, 0)^t$, the two only properties of the fluid to be specified are density ρ and pressure p, or equivalently, density ρ and equation of state parameter w. In many cases, w is constant in time and a genuine property of the fluid, such as $w = 0$ for nonrelativistic matter and $w = 1/3$ for photons. If there is just a single fluid with a constant equation of state, the density evolution is determined by the FLRW-background only and one obtains $\rho \propto a^{-3(1+w)}$.

The field equation reduces to the two Friedmann-equations under the assumption of the FLRW-symmetries, and as the field equation itself already respects covariant energy-momentum conservation $\nabla_{\mu}T^{\mu\nu} = 0$, is is automatically fulfilled. This implies that of the two Friedmann-equations and the continuity equation only two relations are truly independent. Commonly, such as in the ΛCDM-class of cosmological models one assumes that (i) all fluids are independent (i.e. there is no direct coupling or transition of energy from one fluid to another) and that (ii) the equation of state parameter is fixed through the properties of the fluid (we will encounter different examples later, such as quintessence) and governs the adiabatic, energy-momentum conserving behaviour of the fluid. Then, the Hubble function can be assembled by writing

$$H(a) = H_0\sqrt{\sum_i \frac{\Omega_i}{a^{3(1+w_i)}} + \frac{\Omega_K}{a^2}} \tag{C.209}$$

with the sum over the individual densities fixing the global curvature,

$$\sum_i \Omega_i = 1 - \Omega_K. \tag{C.210}$$

Statements on acceleration as done by the second Friedmann-equation can be computed by taking the derivative of $H(a)$, leading to the deceleration parameter q. The time evolution of the density parameters is determined from $\rho(a)$ of the respective fluids with their equation of state, and the time evolving critical density $\rho_{\text{crit}}(a)$, determined through the Hubble-function $H(a)$. Then,

$$\frac{\Omega_w(a)}{\Omega_w} = \frac{H_0^2}{a^{3(1+w)}H^2(a)} \tag{C.211}$$

which is illustrated for Ω_m $(w = 0)$, Ω_γ $(w = +1/3)$ and Ω_Λ $(w = -1)$ in Figure 1, clearly indicating phases, where the FLRW-dynamics is dominated by a single fluid, in order of descending w.

Auxiliary to the last argument, we can compute $\Omega_m(a)$ in its time evolution and compare it to the Hubble-function $H(a)$ for a range of dark energy models with differing w. It is very practical for this type of plot to scale out the behaviour of H in the matter-dominated phase, where it is $\propto a^{-3/2}$ and consider $a^{3/2}H(a)$. The result

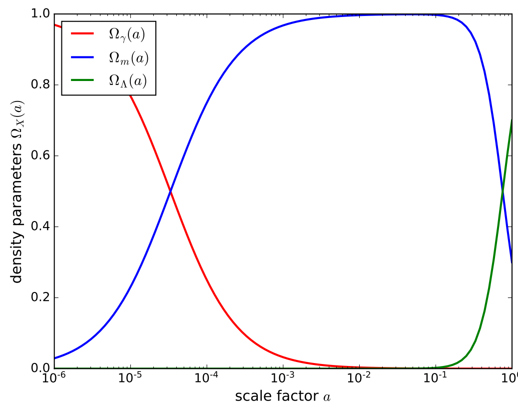


Figure 1: Density parameters $\Omega(a)$ *for radiation, matter and the cosmological constant*

is shown in Fig. 2, where the double logarithmic derivative $\mathrm{d}\ln \mathrm{H}/\mathrm{d}\ln a$ shows the effective power law behaviour of H.

C.10 *Cosmological distance measures*

Coordinate differences between objects are irrelevant, as the coordinate choice is completely arbitrary: For defining actual distances on needs to go through the metric which maps infinitesimal coordinate differences $\mathrm{d}x^\mu$ onto spacetime distances $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. The result of this operation is only differential, so any macroscopic distance measure involves an integration, and it would make a difference whether the coordinate differentials $\mathrm{d}x^\mu$ are part of a timelike or lightlike geodesic, so one would need to describe an actual experiment that defines the measurement of a distance on a metric manifold.

Perhaps most intuitive is the *proper distance* p, where one derives the distance from the light travel time, given infinitesimally by

$$\mathrm{d}p = c\mathrm{d}t \quad \text{with} \quad \mathrm{d}t = \frac{\mathrm{d}a}{a\mathrm{H}(a)} \tag{C.212}$$

so that p can be determined by integration as

$$p = c\int_a^1 \mathrm{d}a\,\frac{1}{a\mathrm{H}(a)} \tag{C.213}$$

and is naturally related to the amount of time passing between a and 1. Next, we define the *comoving distance* χ, which must never be confused with comoving coordinates! The null-condition for FLRW-universes reads

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2\mathrm{d}\chi^2 = 0 \tag{C.214}$$

χ would be the comoving coordinate differential, and integrating this up along a

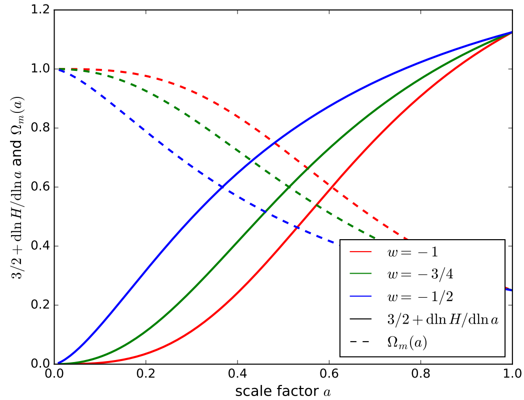


Figure 2: Matter density parameter $\Omega_m(a)$ and the logarithmic slope of the Hubble-function

null-geodesic yields

$$\chi = \int \mathrm{d}\chi = c \int \mathrm{d}t \, \frac{1}{a} = c \int_a^1 \mathrm{d}a \, \frac{1}{a^2 \mathrm{H}(a)} \tag{C.215}$$

An actually measurable distance indicator is the *angular diameter distance* d_A, as it incorporates an actual experimental setup: If one places an object of a known physical size dA at the distance d_A, it would subtend a (measurable) solid angle $\mathrm{d}\Omega$: In a spatially flat universe the two can be related by writing

$$\mathrm{d}\Omega = \frac{\mathrm{dA}}{d_\mathrm{A}^2} = \frac{\mathrm{dQ}}{\chi^2} \tag{C.216}$$

As physical size dA and comoving size dQ must be related by a factor of a^2, so must be d_A and χ: For consistency we get $d_a = a\chi$ in a flat, Euclidean universe. With a similar physical idea in mind, one can relate the apparent brightness of a source with its intrinsic luminosity: Spreading out the luminosity L of an object over a sphere with the *luminosity distance* d_L as a radius defines the flux f,

$$f = \frac{\mathrm{L}}{4\pi d_\mathrm{L}^2} = \frac{\mathrm{L}}{4\pi d_\mathrm{A}^2} a^4 \tag{C.217}$$

In metric spacetimes there is a general result between the angular sizes of objects and their surface brightnesses, called the Etherington-relation,

$$d_\mathrm{L} = \frac{d_\mathrm{A}}{a^2} \tag{C.218}$$

which helps to reformulate the apparent flux from a source in terms of comoving or angular diameter distance: The flux is distributed over a sphere with angular

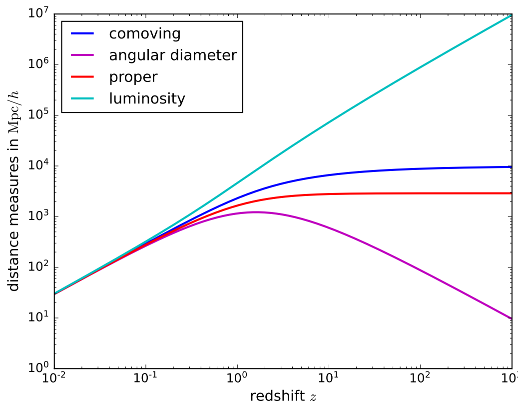


Figure 3: Luminosity distance, comoving distance, proper distance and angular diameter distance for a ΛCDM-cosmology

diameter distance d_A, but as we need to center this sphere on the source and not the observer, the quantity determining the area needs to incorporate a factor of a^2, as the Universe has become larger by a. Additionally, the arrival time of photons is stretched by a as well as their energies redshifted by the same factor. The distance measures are compared to each other for a vanilla ΛCDM-cosmology in Fig. 3.

C.11 *Age of FLRW-universes*

It is only sensible to speak about the age of the Universe, defined as the elapsed time between the instances $a = 0$ (possibly in the mathematical limit) and $a = 1$ (today), if this time interval is identical for all fundamental observers: This is in fact made sure by the FLRW-symmetries. Elapsed proper time τ of a fundamental observer who stays at her or his comoving coordinate with $d\chi = 0$ is given by $\tau = \int d\tau = \int ds/c = \int dt = t$, and therefore equal to the universally equal coordinate time. With the definition of the Hubble function $H = \dot{a}/a$, which implies that $dt = da/(aH)$ one can compute this time as

$$t = \int dt = \int_0^1 da \, \frac{1}{aH} \tag{C.219}$$

with $1/H_0$ setting the scale of the integral to be about $1/H_0 \simeq 10^{17}$ seconds. The exact number, and whether the integral itself is finite or not, depends on the cosmological model, i.e. the values of the density parameters Ω_i and of the gravitating fluid's equation of state parameters w_i. Let's go through a couple of specific examples with a single dominating fluid: A flat cosmology with only a cosmological constant $\Omega_\Lambda = 1$ and $w = -1$ has a constant Hubble-function, and consequently

$$t = \int_0^1 da \, \frac{1}{aH} = \frac{1}{H_0} \int_0^1 d\ln a = \frac{1}{H_0} \ln a \Big|_0^1 \to \infty \tag{C.220}$$

which is sensible as $a(t) \propto \exp(\mathrm{H}_0 t)$ is finite for all finite times and the instant $a = 0$ is never reached. A completely, fully hyperbolically curved universe with $\Omega_\mathrm{K} = 1$ and $w = -\frac{1}{3}$ has a Hubble function $\mathrm{H}(a) = \mathrm{H}_0/a$ and from that we obtain

$$t = \int_0^1 \mathrm{d}a \frac{1}{a\mathrm{H}} = \frac{1}{\mathrm{H}_0} \int_0^1 \mathrm{d}a = \frac{1}{\mathrm{H}_0} \tag{C.221}$$

and therefore a finite age! You can easily convince yourself that $w = -1/3$ is the boundary for the age of the Universe to be finite: Lower equation of state parameters make the integral diverge, and higher equation of state parameters cause the integral to converge. In fact, a flat, matter-filled universe with $\Omega_m = 1$ and $w = 0$ would have a Hubble-function with $\mathrm{H}(a) = \mathrm{H}_0 a^{-3/2}$ and therefore

$$t = \int_0^1 \mathrm{d}a \frac{1}{a\mathrm{H}} = \frac{1}{\mathrm{H}_0} \int_0^1 \mathrm{d}a \frac{1}{a^{-1/2}} = \frac{1}{\mathrm{H}_0} \frac{2}{3} a^{3/2} \Big|_1^0 = \frac{2}{3} \frac{1}{\mathrm{H}_0}, \tag{C.222}$$

again with a finite age.

C.12 *Causal structure of FLRW-spacetimes and cosmological horizons*

It is immediately obvious that a flat FLRW-spacetime stretches infinitely into the spatial directions but that, depending on the density parameters and the associated equation of state, could have existed only for a finite time, which implies that light from distant regions of the Universe could not yet have arrived at the location of an observer.

The particle horizon is the limit of the past light cone, caused by a finite time since $a = 0$. Working in conformal coordinates we compute the comoving distance as

$$\chi_\mathrm{PH} = c \int_{-\infty}^{\eta_0} \mathrm{d}\eta = c \int_0^{t_0} \mathrm{d}t \, \frac{1}{a}, \tag{C.223}$$

which is the maximum comoving distance from which a light signal could have reached us over the finite age of the Universe. Similarly, the future light cone has possibly a limit, corresponding to the maximum distance out to which we can send a light signal in the future: This is called the event horizon, whose comoving distance is given by

$$\chi_\mathrm{EH} = c \int_{\eta_0}^{+\infty} \mathrm{d}\eta = \int_{t_0}^{t_\mathrm{max}} \mathrm{d}t \, \frac{1}{a} \tag{C.224}$$

where the physical age of the Universe is finite in certain cosmological models.

Neither particle nor event horizon should be confused with the Hubble-sphere, which is defined by the physical distance r_Hubble at which the recession velocity υ reaches the speed of light,

$$c = \mathrm{H} r_\mathrm{Hubble} \quad \rightarrow \quad r_\mathrm{Hubble} = \frac{c}{\mathrm{H}} \tag{C.225}$$

which has today the value $c/H_0 \simeq 3$ Gpc/h. We can perfectly see objects from beyond the Hubble-radius; for instance the cosmic microwave background: All that matters for the visibility of a cosmological object is whether a null-geodesic between the object and observer can be drawn; uninterrupted by a horizon.

Of course, the integrals for particle and event horizon can be reformulated in terms of the scale-factor a, which might be more intuitive and which allows an easier judgement if the integrals converge or not. The expression for the Hubble-function $H(a)$

$$H(a) = H_0\sqrt{\frac{\Omega_\gamma}{a^4} + \frac{\Omega_m}{a^3} + \frac{\Omega_K}{a^2} + \Omega_\Lambda} = H_0\sqrt{\sum_i \frac{\Omega_i}{a^{3(1+w_i)}}} \tag{C.226}$$

suggest that, with the assumption of a monotonically increasing scale factor $\dot{a} > 0$ that the densities $\rho \sim a^{-3(1+w)}$ decrease if $w \geq -1$ and stay constant with $w = -1$. Therefore, the Universe goes typically through all fluids in decreasing order in the value of w in its evolution:

$$\Omega_\gamma \quad \rightarrow \quad \Omega_m \quad \rightarrow \quad \Omega_K \quad \rightarrow \quad \Omega_\varphi \quad \rightarrow \quad \Omega_\Lambda \tag{C.227}$$

Ω_γ and Ω_m are dominant at early times, resulting in decelerating expansion with $q > 0$, whereas in later times dark energy with Ω_φ and the cosmological constant Ω_Λ are dominant, which leads to accelerating expansion with $q < 0$. For any constant equation of state and a single dominating fluid at the critical density we would obtain

$$H = H_0 a^{-\frac{3(1+w)}{2}} = \frac{\dot{a}}{a} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \tag{C.228}$$

such that the integrand for the event- or particle horizon would become

$$\frac{\mathrm{d}t}{a} = \frac{\mathrm{d}a}{a^2}a^{\frac{3(1+w)}{2}} \tag{C.229}$$

and the integral would naturally depend on the equation of state as

$$\int \frac{\mathrm{d}t}{a} = \int \mathrm{d}a\, a^{\frac{3(1+w)}{2}-2} = \int \mathrm{d}a\, a^{\frac{3(w-1)}{2}} \sim a^{\frac{3(w+1)}{2}} \tag{C.230}$$

with a convergent solution at early times for $w > -1/3$ and at late times for $w < -1/3$. Particular problems would occur if the equation of state is more negative than -1: Then, a diverging scale factor $a \rightarrow +\infty$ is reached after a finite physical time. This event is called big rip.

D OBSERVATIONS OF FLRW-DYNAMICS

In this chapter we should have a look at possible observations in FLRW-Universes in which the expansion velocity is proportional to the distance ($\upsilon = \mathrm{H}(t)r$), specifically how the Hubble-Lemaître constant H_0 can be determined, and how the dynamic evolution of the Hubble-function due to the gravitational interaction can be observed.

D.1 *Hubble-Lemaître constant* H_0

The Hubble-Lemaître constant H_0 can be determined in observations of Cepheid variable stars: Those stars have (i) a known relation between their pulsation period and their intrinsic brightness, and (ii) are bright enough to be seen in distant galaxies. Combining the estimate of the intrinsic brightness with the observed apparent brightness one can estimate the distance, which scales with H_0, or equivalently, h. Similar methods based on luminosity estimates of galaxies with the Tully-Fisher or Faber-Jackson relation are superseeded in their accuracy by Cepheids.

D.2 *Spatial curvature* Ω_K

By using the angular diameter distance of an object with known physical size, we can determine whether there is curvature in our universe, as this would influence the observed angular diameter. From observations of CMB-fluctuations or baryon acoustic oscillation features in the distribution of galaxies, for which very precise models exist and whose comoving distance is known, one can predict their angular size and compare to the measured angular size. Measurements point towards very small values for curvature, $\Omega_k < 0.01$.

D.3 *Supernova measurements and acceleration q*

By comparing the apparent luminosity with a prediction of the intrinsic luminosity (supernovae of type Ia are very suitable for this purpose, as the released energy is almost constant) and a measurement of redshift one can determine the evolution of luminosity distance d_L with redshift z or scale factor $a = 1/(1+z)$:

$$d_\mathrm{L} = ca \int_a^1 \frac{\mathrm{d}a}{a^2 \mathrm{H}(a)} \tag{D.231}$$

for a spatially flat FLRW-cosmology. For a standard form of the Hubble-function

$$\mathrm{H} = \mathrm{H}_0 a^{-\frac{3(1+w)}{2}} \tag{D.232}$$

the above integral becomes divergent at the lower boundary if $w < -\frac{1}{3}$, corresponding to acceleration, and a supernova appears systematically darker. Typically, one would determine in for a model with two FLRW-fluids the matter density Ω_m and the equation of state w of the remaining dark energy fluid with density $1 - \Omega_m$, assuming a critical universe. The fact that supernovae appear systematically darker in accelerating universes is illustrated in Fig. 4, where models for the luminosity distance (and therefore, the distance modulus) for different Ω_m are compared to data.

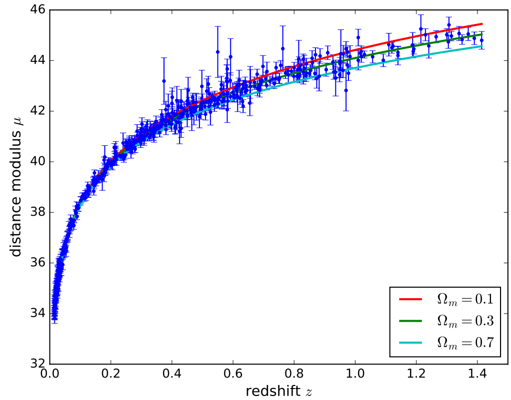


Figure 4: Supernova data and three different theoretical models for the distance modulus

The supernova data can be used to carry out a fit for Ω_m in a ΛCDM-cosmology, arriving at a value of $\Omega_m = 0.2785 \pm 0.013$, providing support for the existence of acceleration and the cosmological constant. The actual fit is shown in Fig. 5.

D.4 *(Finite) age of the Universe t_0*

The age of very old objects, for instance white dwarfs, one can put an lower bound on the age of the Universe,

$$t_0 = \int_0^1 \frac{\mathrm{d}a}{a\mathrm{H}} \tag{D.233}$$

which requires a period of decelerated expansion in the past to remain finite. Clearly, the magnitude of the integral is set by the inverse Hubble-Lemaître constant $1/\mathrm{H}_0$.

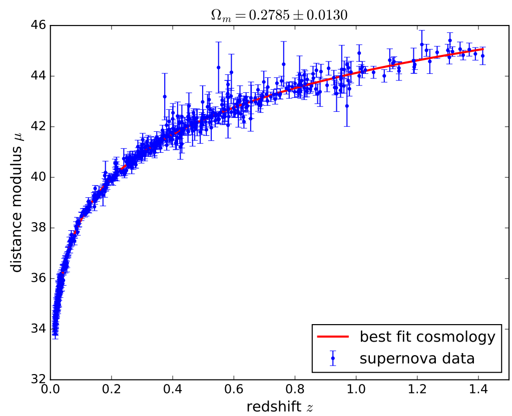


Figure 5: Supernova data and the best fitting ΛCDM-model

E THERMODYNAMICS AND COSMOLOGY

The thermal history of the Universe combines three aspects: Firstly, the decrease of temperature of the cosmic photon bath with increasing scale factor, which is mediated by the recession motion of particles at which scattering processes take place or as a straightforward transformation effect, secondly, the decrease in the corresponding energy scale at which particle processes take place such as the formation of light nuclei in the early Universe and the formation of atoms, and lastly the equilibration of particle ensembles.

E.1 *Temperature evolution and FLRW-dynamics*

The subject of this chapter is the relationship between the temperature of cosmological fluids, in particular photons, and the geometry of the Universe, i.e. the scale factor a. In physics it is a common approach that a new phenomenon is traced back to the most fundamental measurements we can take, time intervals and distances. In this spirit the effect of fields on charges in electrodynamics is explained by considering the acceleration of a test charge and general relativity itself is a theory of how the measurements of time intervals and distances is affected by the presence of gravitational fields. Likewise, temperature as a phenomenon can traced back to mechanical measurements by means of a Carnot-engine. A Carnot-engine is a cyclic engine which operates between two heat reservoirs at different temperature and converts heat into mechanical energy at a fixed efficiency which only depends on the ratio between the two temperatures. It can be used as a thermometer to determine the temperature of one reservoir relative to the other by determining the heat flux and the amount of mechanical work. Mechanical work can be measured purely by measurements of time intervals and distances, for instance, the mechanical work can be used to accelerate a test object of a given mass. From this point of view it is perhaps not surprising that temperatures are affected by changes in the metric, as they influence the basic measurements of distances and time-intervals.

The Universe is filled with a photon background in which the photons outnumber baryons by a factor of about 10^9, implying that the photon temperature governs many of the particle reactions until they can decouple from the photons under certain conditions. The photons are in thermal equilibrium and their temperature decreases while the Universe expands, in face the relationship between photon temperature T and the scale factor a is $T \propto a$. It is very important to realise, however, that photons can neither equilibrate nor make transitions to a new equilibrium temperature without interacting with matter: This is a direct consequence of electrodynamics, which is linear and does not include direct scattering processes between photons. The change in temperature of the photons is caused in emission and absorption processes or mere scattering processes with charged particles taking place in an expanding space: Due to the relative motion between e.g. atoms in which photon emission and absorption processes one realises a decrease in photon wavelength with the Hubble expansion, and therefore a decrease in energy due to a general relativistic Doppler-effect. This mechanical picture can be viewed in a very abstract way: Due to the relative motion between emitting and absorbing atoms the photon gas undergoes a thermodynamic change of state and is relaxed, accompanied by a decrease in temperature $T \propto a$.

Equilibration takes place in interactions of photons with matter in which the photon number is not conserved. Photons (and in fact all relativistic ensembles with massless particles) have the curious property that many properties including their number in an ensemble at equilibrium is determined by the temperature (and the chemical potential). The number of photons changes if the system is brought to a different temperature by a non-adiabatic process which is at contrast with an ensemble of atoms, which can at fixed particle number assume any temperature. In fact, the number of photons can fluctuate as interactions take place where single photons are absorbed and more than one photon is emitted. This implies that a grand-canonical description needs to be used for photons, where the particle number is not fixed and photons may be generated or destroyed.

For this chapter, keep in mind that the density ρ under the Hubble-expansion with the scale factor a behaves according

$$\rho \sim a^{-3(1+w)} \tag{E.234}$$

with an equation of state parameter w. In particular one gets for radiation with $w = +1/3$ the scaling $\rho \propto a^{-4}$, which is commonly interpreted as a scaling of volume which dilutes the number density by a^{-3} together with an additional redshifting by another factor of a, as the photons are scaled to longer waverlength by increasing a.

At first, let's have a look at the 'textbook derivation' of the temperature evolution $T(a)$. From statistical mechanics we know that ideal, relativistic gases (photons) have an adiabatic index of $\kappa = 4/3$. Further, the Hubble-expansion is adiabatic (because there are no heat fluxes that would transport thermal energy away, as heat fluxes would violate the cosmological principle and because there is no decay of particles into photons which would effectively constitute a source of thermal energy) and therefore the thermal energy content is unchanged $\delta Q = 0$. Using the adiabatic invariant

$$T \underbrace{V^{\kappa-1}}_{V^{\frac{1}{3}}=(a^3)^{\frac{1}{3}}=a} = \text{const.} \tag{E.235}$$

we obtain the important result

$$Ta = \text{const.} \qquad \text{or} \qquad T \sim \frac{1}{a}. \tag{E.236}$$

As second approach, let's try a (hopefully) more intuitive way: We start with the thermal energy $E = k_B T$ and use the dispersion relation $E = cp$ for relativistic particles like photons (effectively, this is the point where this derivation becomes compatible with the previous one: The dispersion relation is equivalent to the relativistic adiabatic index). Now using the de Broglie-relation $p = \frac{h}{\lambda}$, we end up using $\lambda \sim a$ from the Hubble-expansion at

$$E = k_B T = cp = \frac{ch}{\lambda} \sim \frac{1}{a} \tag{E.237}$$

From both of the above derivations it can be concluded, that as the Universe increases by a, the temperature drops by $1/a$. The temperature T as a function of comoving distance (or equivalently, conformal lookback time), is shown in Fig. 6, for different cosmological models.

E.2 *Cosmic microwave background*

The Universe is filled with a thermal ensemble of photons, whose temperature drops as the Universe expands. Depending on the physical picture one adopts, there are

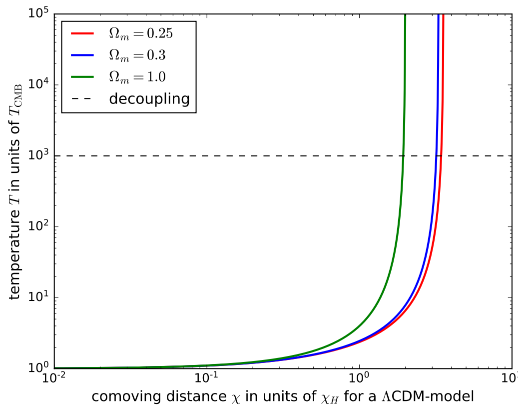


Figure 6: Temperature T *as a function of comoving distance* χ*, for 3 different* Λ*CDM cosmologies*

different views on the dependence of temperature with redshift: Clearly, the wavelength of each photon is redshifted with the Hubble-expansion and the photons are measured to have longer wavelengths at later times, and the ratio between observed wavelength to inital wavelength is proportional to $1/(1+z)$ or, equivalently, to the scale factor a. In order to link this change in wavelength to temperature, one can invoke three principes: Firstly, the momentum p of a photon in inversely proportional to wavelength λ, $p = h/\lambda$, with the Planck-constant h as the constant of proportionality, and secondly, the dispersion relation of photons is that of ultrarelativistic particles, $\mathrm{E} = cp$. If one then assign the thermal energy $\mathrm{E} = k_\mathrm{B}\mathrm{T}$ to an ensemble of photons in order to relate their typical energy to temperature, one obtains $k_\mathrm{B}\mathrm{T} = ch/\lambda$. As the wavelength λ is proportional to the scale factor, T must scale proportional to $1/a$. Using the photon dispersion relation $c = \lambda\nu$ which relates wavelength λ and frequency ν implies the inverse scaling of frequency, $\nu \propto a^{-1}$ and therefore $\nu \propto \mathrm{T}$.

The same result can be obtained in a very different physical picture: Considering the photon fluid as a thermodynamic substance and the Hubble-expansion as a (reversible) change in volume by a factor a^3, one would derive the change in temperature with the adiabatic relation. Adiabatic changes in state are characterised by the absence of a heat flux, and clearly such a heat flux would violate the FLRW-symmetry assumptions. For an adiabatic change in state the quantity $\mathrm{T}\mathrm{V}^{\kappa-1}$ is constant, with the volume V, the temperature T and the adiabatic index κ of the substance. Photons as ultrarelativisitic particles have $\kappa = 4/3$, implying the relation $\mathrm{T} \propto a^{-1}$ with $\mathrm{V} \propto a^3$.

There is a nice consistency between both pictures: If the Universe was filled with thermal non-relativistic particles, their adiabatic index of $\kappa = 5/3$ would imply a dependence $\mathrm{T} \propto a^{-2}$, which could likewise be derived by using a quadratic dispersion relation $\mathrm{E} = p^2/(2m)$: Together with the definition of thermal energy $\mathrm{E} = k_\mathrm{B}\mathrm{T}$ and the de Brogie-relation $p = h/\lambda$ this suggests $\mathrm{T} \propto a^{-2}$ as well. One should be careful in generalising this result to other substances: The adiabatic index of $\kappa = 5/3$ applies to non-relativistic particles with 3 translational degrees of freedom. If the Universe was filled with a diatomic gas it would be wrong to derive a scaling $\mathrm{T} \propto a^{-6/5}$ given its adiabatic index of $\kappa = 7/5$ on the basis of the three translational and two rotational degrees of freedom. Because only the translational degrees of freedom are affected (and the corresponding components of momentum redshifted), the gas would also cool down $\propto a^{-2}$.

It should be kept in mind that a photon gas always needs interactions with particles such as atoms to reach thermal equilibrium, because electrodynamics as a linear theory has perfect superposition and no scattering between the photons themselves (at least at the energies we are concerned with). Therefore, the increase in volume due to the Hubble expansion needs to be thought of as the increase in distance and the corresponding cosmological redshifting between emission and absorption of a photon at two locations: The photons are coupled to the Hubble expansion through scattering processes on advected particles.

The temperature of the photon background is sufficiently low at a scale factor of $a \simeq 10^{-3}$ to allow the formation of atoms from free nuclei and electrons. As the Universe becomes neutral scattering processes between photons and free electrons cease, the Universe becomes transparent to light and photons can propagate freely along straight lines: This corresponds to the release of the cosmic microwave background. Although, due to the FLRW-symmetries, the formation of atoms takes place at the same instant everywhere simultaneously, we perceive this process at a fixed distance isotropically around us: The spherical surface, from which the photons of the cosmic microwave background seem to emanate is called the surface of last scattering, or, the photosphere of the cosmic microwave background.

An estimate of the formation temperature of hydrogen atoms from the ionisation energy would correspond to about 10^4 Kelvin and not to the 3×10^3 Kelvin one finds in cosmology: In fact, the formation of atoms and therefore the release of the cosmic microwave background is delayed. The decoupling of the photons would be a very slow process, in which the rate of formation of atoms and their destrucion by photons with sufficient energy would slowly tilt towards the first process as the temperature decreases. Instead, there is a forbidden, two-photon transition from the $2s$-state to the ground state, which allows the generation of a photon population at a lower temperature along with a population of neutral atoms as they can not be reionised due to a deficit in photon energy.

The incredibly accurate data taken by the FIRAS-instrument onboard the COBE-satellite shows clearly that the CMB is described by a Planck-spectrum with proper Bose-Einstein statistics and not by an analogously constructed Wien-spectrum with Boltzmann-statistics, as illustrated by Fig. 7.

E.3 *Into and out of equilibrium*

Thermal equilibrium is maintained by collisions between particles, which implies a competition between two time-scales: The collision time scale t_c, at which particles exchange energy and momentum, and the Hubble-time scale, on which the temperature changes due to the expansion of the Universe: If $t_c \ll t_\mathrm{H}$, collisions between particles are frequent and thermal equilibrium can be maintained, but if $t_c \gg t_\mathrm{H}$, the system can drop out of thermal equilibrium. This happens necessarily at some point in the history of the Universe, because one can estimate t_H to be $t_\mathrm{H} = 1/\mathrm{H}(a) \propto a^2$ during radiation domination, whereas the collision rate $\Gamma = n\langle\sigma\upsilon\rangle$ with the number density n, the cross-section σ and the particle velocity υ implies a scaling of $t_c = \Gamma^{-1} \propto a^3$ due to the inverse proportionality to the particle number density. Therefore, $t_c/t_\mathrm{H} \propto a$ and thermal equilibrium can be maintained at early times, and can break down at late times.

Specifically, the time-evolution of the number density of particles follows a continuity equation,

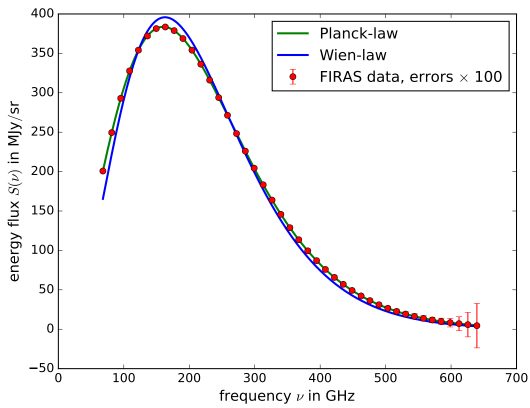


Figure 7: Spectrum of the cosmic microwave background as recorded by the FIRAS instrument onboard the COBE-satellite, with the best fitting Planck- and Wien-spectra in comparison

$$\dot{n} + \mathrm{div}(n\boldsymbol{\upsilon}) = 0 \tag{E.238}$$

which reduces to $\dot{n} + 3\mathrm{H}n = 0$ by substituting the Hubble-flow $\boldsymbol{\upsilon} = \mathrm{H}\boldsymbol{r}$, by assuming homogeneity of the particle density and by using that $\mathrm{div}\boldsymbol{r} = 3$. Then, the number density of particles has the solution $\mathrm{d}\ln n/\mathrm{d}t = -3\mathrm{H}$, which is solved by $n(t) \propto \exp(-3\mathrm{H}t)$ if H is constant, otherwise by $n(t) \propto \exp(-3\int \mathrm{d}t\mathrm{H})$. Relating this to the scale factor one can substitute the definition of the Hubble function, $\mathrm{H} = \dot{a}/a$, yielding $\mathrm{d}\ln n/\mathrm{d}t = -3\mathrm{d}\ln a/\mathrm{d}t$ with the solution $n \propto a^{-3}$, as expected: The substitution of the Hubble-law $\upsilon = \mathrm{H}r$ conserves homogeneity perfectly, and is in fact the only law that would allow this. As a proof, please remember that in an isotropic case one could substitute a generalised Hubble law $\upsilon \propto r^\alpha$ into the continuity equation, where the divergence is explicitly formulated in spherical coordinates,

$$\partial_i \upsilon^i = \frac{1}{r^2}\frac{\partial}{\partial r}(r^{2+\alpha}) = \frac{2+\alpha}{r^2} r^{1+\alpha} = \mathrm{H}(2+\alpha)r^{\alpha-1}, \tag{E.239}$$

which does not depend on r if $\alpha = 1$, implying that n can only change with time.

The time-evolution is modified if there are collisions present and if particles can be created in reactions,

$$\dot{n} + 3\mathrm{H}n = -\mathrm{Q} + \mathrm{S} = -\Gamma n\left(1 - \frac{n_\mathrm{T}^2}{n^2}\right) \tag{E.240}$$

with the collision rate $\mathrm{Q} = \langle\sigma\upsilon\rangle n^2$ and the particle creation rate S for which we make the ansatz $\mathrm{S} = \langle\sigma\upsilon\rangle n_\mathrm{T}^2$. Because both processes involve the collisions between particle pairs, the pair number density is relevant which is well approximated by the squared particle density. The particle density n should decrease if particles thermalise through collisions, which takes place at the rate Γ, and particles are created at the rate Γ from a thermal background, necessitating the proportionality to the density of thermal particles n_T. The number density of thermal particles n_T can be predicted from a dispersion relation and the suitable statistics.

Introducing the comoving number density $N = na^3$ with the derivative $\dot{N} = a^3(\dot{n} + 3Hn)$ yields

$$\dot{N} = -\Gamma N \left(1 - \frac{N_T^2}{N^2}\right) \tag{E.241}$$

which can be rewritten by replacing the time variable t with the scale factor a,

$$\frac{d\ln N}{d\ln a} = -\frac{\Gamma}{H}\left(1 - \frac{N_T^2}{N^2}\right) \tag{E.242}$$

by writing $\dot{N} = aH\mathrm{d}N/\mathrm{d}a$. In this relation, the competition of time scales is clearly expressed by the prefactor Γ/H, which is large if $t_c \ll t_H$ and collisions dominate, and conversely small if $t_c \ll t_H$, in which case the Hubble expansion dominates. This prefactor changes the rate at which n can change if $N \neq N_T$, and can effectively keep N constant even if $N \neq N_T$ in the limit $\Gamma \ll H$, for a dominating Hubble-expansion.

If a system is away from thermal equilibrium, the number N is larger than N_T, implying a positive bracket in the last equation, which causes N to decrease in time, meaning that the system is driven towards thermal equilibrium, which is reached at $N = N_T$ where the evolution of N stops. If conversely, N_T is larger than N, the sign switches and N can increase and the system can freeze out, if the prefactor Γ/H allows it.

E.4 *Photon background as a thermodynamical ensemble*

The properties of a cosmological radiation background can be understood from the properties of a quantum system at thermal equilibrium. Distributing the particles in phase space needs to respect the Friedmann-symmetries, so one assumes homogeneity in configuration space and isotropy in momentum space for any cosmological observer, while one is free to choose the distribution in momentum space as a function of energy and the dispersion relation $E(p)$ of the particles. Specifically, for photons one has as the ultrarelativitic dispersion relation $E(p) = cp$ with the momentum $p = h/\lambda$, and the phase space density $n(p,T) = 1/(\exp(cp/(k_B T)) - 1)$ for bosons.

An ideal gas of photons has the interesting property that its chemical potential μ vanishes and the corresponding fugacity $\exp(\mu/(k_B T))$ is equal to one: This is related to the fact that the photon number is not constrained, due to emission and absorption processes, which cause the number of photons in the system to fluctuate. In equilibrium the Helmholtz free energy $F = F(T,V,N)$ is at a minimum, as it describes the energy of a system in thermal equilibrium at a given temperature T, volume V and particle number N. Because F follows by a Legendre transform from the internal energy U, $F = U - TS$ replacing the entropy S by the temperature T one obtains for the differential $\mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V + \mu\mathrm{d}N$. The minimum condition implies that $\partial F/\partial N = \mu = 0$, meaning that the chemical potential for a system at constant temperature and volume vanishes, $\mu = 0$, in thermal equilibrium.

Radiation pressure and entropy of the thermal photon gas result from differentiation of the grand canonical potential $J(T,V,n)$, which describes a system at equilibrium at fixed temperature, constant chemical potential and not performing mechanical work. Specifically, the grand canonical potential $J(T,V,\mu)$ is defined as $J = U - TS - \mu N$ and by substituting the Euler-relation $U = TS - PV + \mu N$ it is readily shown to be $J = -PV$. The grand canonical potential has the differential $dJ = -SdT - PdV - Nd\mu$, which can be shown by substituting the Euler relation $dU = TdS - PdV + \mu dN$. It follows from the grand canonical partition sum Z by taking the logarithm,

$$J = -k_B T \ln Z. \tag{E.243}$$

The grand canonical partition sum Z is defined as

$$\ln Z = \frac{g}{(2\pi\hbar)^3} \int d^3x \int d^3p \, \ln\left(1 - \exp\left(-\frac{cp}{k_B T}\right)\right) \tag{E.244}$$

if the dispersion relation for ultrarelativistic particles $E(p) = cp$ is substituted for the energy and if their statistical weight is g, meaning that a single state can be occupied by g particles: Photons have spin 1, and being ultrarelativistic, there can only be two particles per state, $g = 2$. The expression for the grand canonical partition sum Z can be written in a closed form by integration by parts,

$$\ln Z = -\frac{gV}{(2\pi\hbar)^3} \int dp \, 4\pi p^2 \ln\left(1 - \exp\left(-\frac{cp}{k_B T}\right)\right) = \frac{gV}{(2\pi\hbar)^3} \int dp \, \frac{4\pi c}{3k_B T} p^3 \frac{1}{\exp\left(\frac{cp}{k_B T}\right) - 1} \tag{E.245}$$

while identifying the configuration space volume $V = \int d^3x$ and assuming isotropy in momentum space, and abbreviating $\beta = 1/(k_B T)$. The integration can be carried out by substituting $x = \beta cp$ and using the relation $\int dx \, x^n/(\exp(x) - 1) = \zeta(n+1)\Gamma(n+1)$,

$$\ln Z = \frac{gV}{(2\pi\hbar)^3} \frac{4\pi}{3} \left(\frac{k_B T}{c}\right)^3 \zeta(4)\Gamma(4). \tag{E.246}$$

The difference between the distribution functions for Bose-Einstein, Fermi-Dirac and Boltzmann statistics are shown in Fig. 8.

Already from the expression for Z it is apparent that the temperature must scale $\propto a^{-1}$. An adiabatic change of state implies that the system moves to a new temperature while the relative probabilites are unchanged: While the configuration space scales $\propto a^3$ and the momentum space $\propto a^{-3}$ due to the scaling of the photon momentum $p = h/\lambda \propto a^{-1}$, it is necessary for the temperature to scale $\propto a^{-1}$ in order for the partition sum to remain invariant. It is quite interesting to note that the rescaling of temperature is sufficient for particles obeying different dispersion relations, as long as this dispersion, i.e. the relation between energy and momentum is scale free. Any deviation from a power law would have the consequence that a rescaling affects high and low-energy particles differently, breaking the overall shape invariance under rescaling by a. In this way it is possible to derive simple scaling behaviours for ultrarelativistic particles with $E = cp$ or for classical particles with $E = p^2/(2m)$.

A very interesting illustration of the shape-invariance of the Planck-spectrum is Wien's displacement law: The shape of the spectrum itself defines a frequency scale, which needs to scale necessarily $\propto 1/a$ in order not to violate the dispersion relation. This is in fact realised in any definition of such a scale in the spectrum, for instance through the location of the maximum. $dS(\nu)/d\nu = 0$ yields a frequency

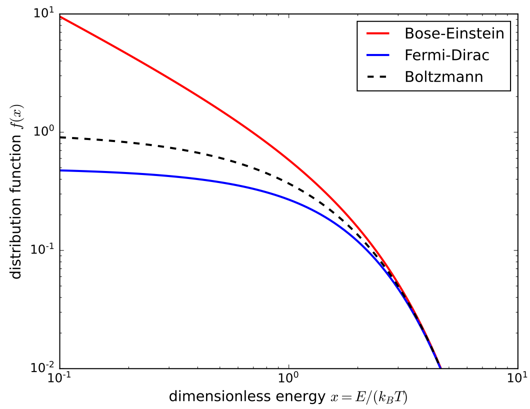


Figure 8: Bose-Einstein, Fermi-Dirac and Boltzmann-distribution functions

$\nu_{\max}$ = proportional to the temperature and hence proportional to $1/a$. Alternatively, one could consider the mean photon frequency $\bar{\nu} = \int \mathrm{d}\nu\, n(\nu)$, or the median frequency, which are all proportional to the temperature and hence inversely proportional to the scale factor a.

In the following we will derive the most important thermodynamical properties of a photon gas by an intuitive argument using a weighted integral over the occupation statistic and by a thorough derivation using the grand canonical partition sum: Starting with the internal energy one would use the phase space distribution $n(p, \mathrm{T})$ and the ultrarelativistic dispersion relation $\mathrm{E} = cp$ to collect the energy across the entire momentum space by carrying out the dp-integration, while the configuration space integration simply yields the volume of the system V:

$$\mathrm{U} = \frac{g\mathrm{V}}{(2\pi\hbar)^3} \int 4\pi p^2 \mathrm{d}p\, \mathrm{E}(p) \frac{1}{\exp\left(\frac{\mathrm{E}(p)}{k_B \mathrm{T}}\right) - 1} \tag{E.247}$$

where again isotropy in momentum space was assumed, $\mathrm{d}^3 p = 4\pi p^2 \mathrm{d}p$. The integral can be rewritten by integration by parts,

$$\int 4\pi p^2 \mathrm{d}p \ln\left(1 - \exp\left(\frac{cp}{k_B \mathrm{T}}\right)\right) = \int 4\pi \frac{p^3}{3} \mathrm{d}p \frac{1}{\exp\left(\frac{cp}{k_B \mathrm{T}}\right) - 1} \tag{E.248}$$

implying that $\ln \mathrm{Z} = \mathrm{U}/3$ and $p\mathrm{V} = \ln \mathrm{Z}$, i.e. the relation $p = \mathrm{U}/(3\mathrm{V})$ between pressure and internal energy as well as $\mathrm{J} = \mathrm{U}/3$. The total energy density of the radiation background is an expression of the Stefan-Boltzmann law. The total energy density can be evaluated to be equal to $\sigma_{\mathrm{SB}}\mathrm{T}^4$ with the Stefan-Boltzmann-constant σ_{SB}: This is in complete agreement with the fact that the number density $\int \mathrm{d}\nu\, n(\nu)$ is diluted $\propto a^{-3}$ and each photon's energy is redshifted by an additional factor of a^{-1}, resulting in a decrease of the energy density $\propto a^{-4}$, or equivalently, a proportionality of the energy density with T^4, as derived before.

The factor 1/3 in the relation between pressure and energy density follows from the same integral. The transfer of momentum onto a surface would be $2p\cos\theta$ under reflection and the flux of photons would be $c\cos\theta$. Therefore, assuming again isotropy of the photon momenta one would collect the total momentum transfer

$$P = \frac{g}{(2\pi\hbar)^3}\int d^3p\, 2cp\cos^2\theta\, n(p,T) = \frac{U}{3V} \tag{E.249}$$

by using spherical coordinates $d^3p = 2\pi p^2 dp \sin\theta d\theta d\phi$, where the azimuthal integration yields 2π and the polar one $1/3$, for the range of angles $0 \leq \theta \leq \pi/2$.

For evaluating the integrals which were needed for computing thermodynamical quantities one can use the following trick and rewrite the phase-space distribution function $n(p,t)$ as a geometric series starting at $m = 1$. In general, one has

$$\sum_{m=0}^{\infty} q^m = \frac{1}{1-q} \quad \rightarrow \quad q\sum_{m=0}^{\infty} q^m = \sum_{m=1}^{\infty} q^m = \frac{q}{1-q} = \frac{1}{\frac{1}{q}-1}, \tag{E.250}$$

and therefore

$$\frac{1}{\exp(x)-1} = \sum_{m=1}^{\infty}\exp(-mx). \tag{E.251}$$

Substituting into the expressions obtained above yields

$$\int dx\, \frac{x^{n-1}}{\exp(x)-1} = \int dx\, x^{n-1}\sum_{m=1}^{\infty}\exp(-mx). \tag{E.252}$$

The integral can be solved by substitution $y = mx$, $dy = m dx$,

$$\int dx\, x^{n-1}\sum_{m=1}^{\infty}\exp(-mx) = \sum_{m=1}^{\infty} m^n \int dy\, y^{n-1}\exp(-y) = \zeta(n)\Gamma(n), \tag{E.253}$$

where one can identify Riemann's ζ-function and the Γ-function in the last step.

The entropy can be determined by differentiation of the grand canonical potential with respect to temperature, $dJ = -SdT - pdV + \mu dN$, and consequently

$$S = -\frac{\partial J}{\partial T} = \frac{\partial}{\partial T}(k_B T\, \ln Z) = k\left(\ln Z + \frac{1}{z}\frac{\partial Z}{\partial(k_B T)}\right) \tag{E.254}$$

or, equivalently by using $U = TS - pV = TS - J$ (if $\mu = 0$), such that $S = (U+J)/T = 4U/(3T)$. Therefore, the total entropy S is conserved because VT^3 = const from these considerations, in accordance with the entropy being constant for an adiabatic reversible change of state. The entropy density S/V scales $\propto T^3$ and therefore $\propto a^{-3}$.

The total number N of particles can be derived through an integral over the phase space density,

$$\mathrm{N} = \frac{g\mathrm{V}}{(2\pi\hbar)^3}\int 4\pi p^2 \mathrm{d}p\ \frac{1}{\exp\left(\frac{cp}{k_\mathrm{B}\mathrm{T}}\right)-1} \tag{E.255}$$

or equivalently, by differentiation of the grand canonical potential with respect to μ,

$$\mathrm{N} = -\frac{\partial \mathrm{J}}{\partial \mu} = -k_\mathrm{B}\mathrm{T}\frac{\partial}{\partial \mu}\ln \mathrm{Z} \tag{E.256}$$

For this purpose one needs to include a chemical potential in the definition of ln Z,

$$\ln \mathrm{Z} = \frac{g\mathrm{V}}{(2\pi\hbar)^3}\int 4\pi p^2 \mathrm{d}p\ \ln\left(1-\exp\left(-\frac{cp+\mu}{k_\mathrm{B}\mathrm{T}}\right)\right) \tag{E.257}$$

which is set to zero after differentiating, yielding exactly the intuitive result. Evaluating the integrals shows the scaling of particle number density $\mathrm{N}/\mathrm{V} \propto a^{-3}$ due to the proportionality to T^3 and the conservation of the total number of particles N.

It suffices to replace the phase space density $n(p,\mathrm{T})$ by $n(p,\mathrm{T}) = 1/(\exp(cp/(k_\mathrm{B}\mathrm{T}))+1)$ for the description of (massless) neutrinos: In complete analogy one obtains expressions for the particle number density $n = \mathrm{N}/\mathrm{V}$, the entropy density $s = \mathrm{S}/\mathrm{V}$ and the energy density $u = \mathrm{U}/\mathrm{V}$ with identical scaling behaviours with temperature, but with different numerical prefactors due to the changed sign in the phase space density.

There is a very interesting catch in the physical properties of the Universe's photon and neutrino backgrounds: Their temperature is not equal. Due to the annihilation of electron-positron pairs into photons there has been a source of thermal energy in the photon background, lifting it's temperature to 2.736 Kelvin, in comparison to the neutrino background which is at equilibrium at a temperature of 1.95 Kelvins. As there is essentially no coupling between photons and neutrinos, the two would never really equilibrate.

E.5 *Quantum-statistics and classical statistics*

The Universe is filled with particles at thermal equilibrium, whose thermodynamic properties can be derived using quantum statistics, i.e. Bose-Einstein-statistics for particles with integer spin such as photons and Fermi-Dirac-statistics for particles with half-integer spin, for instance neutrinos. The quantum mechanical description is necessary in particular at low energies, and this energy is characterised by the thermal wavelength λ_th. If the particle separation is smaller than the thermal wavelength, quantum mechanical interference becomes important and the behaviour deviates from that of a classical system: In contrast to classical statistics, quantum mechanical particles show constructive interference in the case of bosons, if two particles are interchanged, and destructive interference in the case of fermions. This impacts on the occupation statistics, because there can be arbitrarily many bosons in a single state due to constructive interference whereas there can only be a single fermion due to destructive interference. There is no such restriction for classical particles as they are distinguishable: In their time evolution it is always possible to track them through phase space, and a state with interchanged particles is clearly different.

The thermal wavelength corresponds to the de-Broglie wavelength $\lambda = ch/\mathrm{E}$ of a photon with $k_\mathrm{B}\mathrm{T}$ of thermal energy, $\lambda_\mathrm{th} = ch/(k_\mathrm{B}\mathrm{T})$. It scales $\propto a$ with the scale factor,

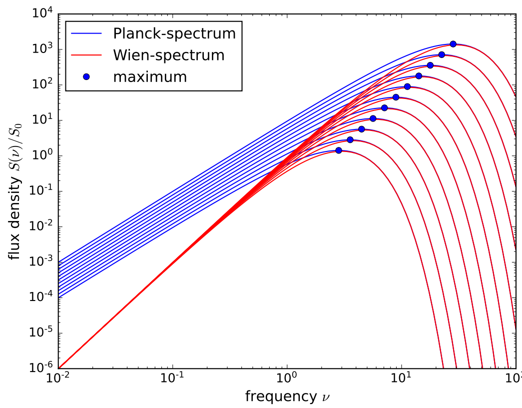


Figure 9: Planck- and Wien-spectra at different equilibrium temperatures

likewise the typical distance between particles of a given energy. Therefore, the photon gas is always characterised by the same Planck-distribution irrespective of scale factor, because for the same fraction of photons quantum mechanical interference is important, and the Hubble expansion will not affect the shape of the statistical distribution. The same argument holds for non-relativistic particles with a quadratic dispersion relation: $E = p^2/(2m)$, in which case the thermal wavelength would result in $\lambda_{th} = h/\sqrt{2mk_B T}$, which scales $\propto a$ in consistence with the scaling $T \propto a^{-2}$.

Planck- and Wien-spectra for different temperatures are compared to each other in Fig. 9, clearly showing an overabundance of photons at low energies in the correct quantum mechanical formulation relative to the classical prediction. In addition, the maxima show a clear linear trend to increase with increasing temperature as a manifestation of the Wien-displacement law.

E.6 *Radiation backgrounds*

Although the picture that the Universe is filled with photons, whose equilibrium temperature drops as the Universe expands is quite correct, it is worth pointing out two things: The change in wavelength or temperature is caused purely by the change in the metric, or if one adopts physical coordinates, by the a general relativistic Doppler-effect due to recession motion of the emitter. Because both the observer and the emitter in cosmology are following their world-lines in freely falling motion, one can be sure that locally for both the laws of special relativity are valid due to the equivalence principle. Because of the fact that in each frame all physical processes are determined by the laws of special relativity only, the redshifting effect on a photon can be unambiguously determined: In this respect, the interpretation would be that in the distant Universe atomic physics is exactly the same as it is here, and that we can measure a change in photon wavelength because we know the emission process under which a photon has been generated, for instance a certain atomic transition leading to a spectral line, and attribute the change in photon wavelength to the change in the metric between emission and absorption of a photon.

The Universe is filled with a homogeneous and isotropic radiation field in accordance with the symmetry assumption of the FLRW-metric. We perceive this photon background today as a blackbody radiation with an equilibrium temperature of $T_{CMB} = 2.725$ Kelvin. Looking along the backwards light cone towards earlier times, we perceive this temperature to be higher by a factor of $1/a$ (a is smaller than one in the past, implying a higher temperature) and there are physical processes, for instance emission and absorption processes with atoms, that take place at the corresponding temperature: The FLRW-symmetry assumptions make sure that at every time the photon background has the same temperature everywhere, but moving along the backward light cone of an observer one can see processes that are governed by temperature to set in at a certain redshift or, equivalently, distance relative to us.

For instance, atoms are formed in the Universe at a temperature of about 3000 Kelvin, and this formation of atoms takes place everywhere at the same age of the Universe, typically $10^{1}2$ seconds after the Big Bang. For an observer today, this temperature is reached going back by about 1000 units in redshift, or to a scale factor of $a = 10^{-3}$, in order for the Universe to reach this temperature relative to the temperature of the background today. Again due to the FLRW-symmetries, the temperature is reached on the surface of a sphere with a distance of about $3\chi_H$ centered on us, on which we can observe radiation from the formation of atoms. The notion that we are surrounded by a photosphere of the cosmic microwave background does not imply that our position as observers is special: In fact any other observer at a different position would see an identical photosphere in perfect spherical symmetry around them with the same radius today.

The effect of different cosmological models or choices of cosmological parameters on the evolution of the background temperature is only relevant if a physical distance or time is assigned to a scale factor, because for this assignment the Hubble function is needed, which includes all density parameters and equations of state. The comoving distance along the backward light cone to the CMB photosphere can be computed as an integral

$$\chi_{CMB} = c \int_{1}^{a_{CMB}} \frac{da}{a^2 H(a)} \tag{E.258}$$

with $a_{CMB} = 10^{-3}$.

In some calculations is is practical to use the temperature as a time-variable, which is possible due to the monotonic relationship between scale factor and temperature: $T/T_{CMB} = 1/a$ implies $dT/da = -T_{CMB}/a^2$. For instance, one might estimate the thickness of the photosphere intuitively for a certain value of ΔT, inside which the temperature drops enough for atoms to form:

$$\Delta\chi \simeq \left|\frac{d\chi}{dt}\right| \Delta T = \left|\frac{d\chi}{da}\frac{da}{T}\right| \Delta T = \frac{c}{H(a)}\frac{\Delta T}{T_{CMB}}. \tag{E.259}$$

If one very coarsely assumes in the next step that $\Delta T \simeq 0.1 T_{comb}$, one obtains $\Delta\chi \simeq 10^{-2.5}\chi_H$ with $T_{comb} = 3000$ Kelvin.

E.7 *Particle cosmology*

Extrapolating the dependence of temperature with the knowledge of the fact that the scale factor was much smaller in the past implies that the temperature in the early Universe was very high. There are two observations which support this idea, specifically, there is the cosmic microwave background on one side and the relative abundances of light chemical elements including their isotopes which are formed in the early Universe in a process called nucleosynthesis. Nucleosynthesis models constrain, in addition to nuclear reactions and the time passed between the initial and final temperatures, as well the relative abundances of neutrons and protons as its initial condition, with implications for baryongenesis at an even earlier stage.

E.7.1 Baryogenesis

In the course of the evolution of the early Universe, the temperature cools down sufficiently to allow the formation of baryons from quarks and gluons, i.e. there is a phase transition from the quark gluon-plasma to baryons such as protons and neutrons. At this point one can (and should) also ask the valid question, why there is more matter than antimatter in our Universe, for instance more protons than antiprotons, for which Sacharow has given three criteria:

1. The baryon number B has to be violated, e.g. by the asymmetric decay of a hypothetical X-particle precursing quarks and leptons,

$$X \to 2u \text{ } 51\% \text{ vs. } \to \bar{d} + e^{+} \text{ } 49\% \text{ } \Delta B = 0.177 \tag{E.260}$$

   in comparison to the decay of the anti-particle $\bar{X}$,

$$\bar{X} \to 2\bar{u} \text{ } 49\% \text{ vs. } \to d + e^{-} \text{ } 51\% \text{ } \Delta B = -0.157 \tag{E.261}$$

   ΔB is the baryon number weighted with the branching ratio.

2. CP- and P-symmetry have to be broken
3. The system has to be in thermal non-equilibrium

If all these criteria are fulfilled, baryons can outnumber anti-baryons. This is described e.g. as part of a grand unified theory of particle physics, it should be mentioned that all these theories are very uncertain.

E.7.2 Big bang nucleosynthesis

At a temperature scale of $\sim 10^{14}$ Kelvin the Universe experiences a phase transition at which protons and neutrons are formed from a plasma composed of quarks and gluons according to the rules of quantum chromodynamics, a quantum field theory that describes the interactions of these particles. Due to a slight mass difference between protons and neutrons (the neutron being more massive by about xxx Gev) one finds a slightly larger number of protons in the equilibrium of the β-process

$$n \leftrightarrow p + e^{-} + \bar{\nu}_e \tag{E.262}$$

After the formation of protons and neutrons the Universe continues to expand and to lower its temperature until temperatures are reached which allow the formation of light nuclei. Because neutrons are unstable with a lifetime of about 900 seconds, they partially decay until the formation of light nuclei starts at much lower temperatures. The neutron decay changes the abundance of protons significantly.

From the relation $T \sim \frac{1}{a}$ we can draw conclusions about the thermal history of our Universe as a was much smaller in history. For example at $a \sim 10^{-10}$ the corresponding temperature was $T \sim 10^{10}$K and therefore $\epsilon_{th} \sim MeV$, which allows nucleosynthesis in the early Universe shortly after the big bang. At $a \sim 10^{-3}$ the temperature was $T \sim 10^3$K or $\epsilon_{th} \sim e$V, which allows the formation of the first atoms. In the next chapters we will have a closer look at both mentioned periods.

For our initial conditions (at $\epsilon_{th} \sim$ GeV), the process

$$p + e^- \rightleftharpoons n + \nu_e \tag{E.263}$$

is allowed in both direction whereas after the freeze-out (T drops to $\epsilon_{th} \sim$ MeV) the process only happens from right to left (as known from 'normal' neutron decay with an $\bar{\nu}_e$). As the life-time of neutrons is $\sim$ 15 min, the rate $\frac{n}{p}$ drops from $\frac{n}{p} = 1$ to $\frac{n}{p} \sim \frac{1}{7}$ before the fusion to deuterium D (at $T \sim 2$MeV)

$$p + n \rightleftharpoons \mathrm{D} + \gamma \tag{E.264}$$

sets in. The backwards process from right to left results from high energetic photons, which cause the dissociation of the deterium again, therefore fusion only sets in at $\epsilon_{th} \sim 100ke$V energies.

A crucial point for creating heavier element is the 'deuterium-bottleneck', as there has to be a decent amount of deuterium while still having left over neutrons. At this point the next question to ask is: How much time was there for production of deuterium in the right temperature-window? The answer is: Not much, from abundance measurements (hyperfine structure) we know of $\frac{n_\mathrm{D}}{n_p} \sim 3.5 \cdot 10^{-5}$, this limitation only leads to a creation of very light elements in the big-bang nucleosynthesis up to $A \sim 5...7$.

Back at the big bang nucleosynthesis, one could compare the photon background to the neutrino background from the produced ν_e's. For the derivation of the neutrino background one has to consider that neutrinos are fermions and therefore has to exchange the Bose- to a Fermi-Dirac-distribution and ends up at a pretty similar result (Remember that the Fermi-Dirac-distribution can be written as a difference of two Bose-distributions at different temperatures) which we don't discuss here. Just prior to nucleosynthesis, photons are produced by annihilation

$$e^+ + e^- \rightarrow 2\gamma \tag{E.265}$$

with temperature ($T \sim 10^{10.5}$K) is set by the electron rest-mass. With the knowledge that the entropy of fermions $S_{\mathrm{fermion}} = \frac{7}{8} S_{\mathrm{boson}}$, $S \sim T^3$ and the assumption that the entropy is conserved, one receives for the above process (E.265)

$$S_\gamma = S'_\gamma + S'_{e^+} + S'_{e^-} \tag{E.266}$$

and with the entropy relations put in

$$\left(2\frac{7}{8} + 1\right) T'^3 = T^3 \tag{E.267}$$

one ends up at $T = 1.4T'$ which implies that $T_\gamma = 2$K for the today's neutrino background. We further can now have a look at the baryon to photon ration

$$n_b = \frac{\rho_b}{m_p} = \frac{1}{m_p} \Omega_b \underbrace{\frac{3H_0^2}{8\pi G}}_{=\rho_{\mathrm{crit}}} \tag{E.268}$$

Ω_b can be measured by X-ray observation of galaxy clusters and making use of the virial theorem. One obtains from these measurements $n_b \approx 1.1 \cdot 10^{-5} \Omega_b h^2 \mathrm{cm}^{-3}$ and $\Omega_b \approx 0.04$ or 10 atoms per cubic meter in the Universe today. Therefore the baryon to photon ratio is

$$\eta = \frac{n_b}{n_\gamma} \approx 2.7 \cdot 10^{-8} \Omega_b h^2 \approx 10^{-9} \tag{E.269}$$

n_γ in above's equation is received from the CMB-temperature and the usage of thermal equilibrium. With this result of approximate 10^9 more photons than atoms one can also imagine the first light elements being destroyed again by photodissociation.

F Early Universe and Cosmic Inflation

F.1 *Need for inflation and scales*

There are indications that the Universe underwent an episode of rapid, accelerated expansion at very early times, commonly referred to as cosmic inflation. Firstly, there is the horizon problem: If we consider thermal equilibrium in the early Universe, the horizon scale for this equilibrium is $c\Delta t$ with the time for equilibration being roughly equal to the travel time of photons. The observed homogeneity of the cosmic microwave background is therefore very surprising, it should be made of patches corresponding to the horizon size as the photons were set free. To make this more quantitative, one can have a look at the comoving horizon at the time when the CMB was generated, which was at a redshift of $z = 10^3$ or equivalently, a scale factor of $a = 10^{-3}$:

$$\chi_\mathrm{H} = c \int_0^{10^{-3}} \frac{\mathrm{d}a}{a^2 \mathrm{H}(a)} \approx 100 \text{ Mpc}/h \tag{F.270}$$

The comoving distance to the CMB is ~ 10 Gpc/h for ΛCDM. Taking the ratio of these two scales one arrives at an angular scale of

$$\Delta\Theta \sim \frac{1}{100} \sim 1^\circ. \tag{F.271}$$

This would be an estimate of the patch size for homogeneity on a small scales. This can be changed by including modification to the Hubble-function at early times, in particular by making it very small, such that the horizon scale becomes large as a consequence. Secondly, there is the flatness problem. As we know, the curvature Ω_K is smaller than $\Omega_\mathrm{K} \lesssim 0.01$, which is very small, but it grows in matter and radiation dominated phases. One can describe this in FLRW-cosmologies with fluids Ω_w with EOS-parameter w and curvature $\Omega_\mathrm{K} = 1 - \Omega_w$.

$$\mathrm{H}^2(a) = \mathrm{H}_0^2 \left(\frac{\Omega_w}{a^{3(1+w)}} + \frac{\Omega_\mathrm{K}}{a^2} \right) \tag{F.272}$$

wherein Ω_K's behaviour can be described like a fluid with $w = -\frac{1}{3}$. We can write

$$\frac{\Omega_w(a)}{\Omega_w} = \frac{\mathrm{H}_0^2}{a^{3(1+w)} \mathrm{H}^2(a)} \tag{F.273}$$

derived from

$$\Omega(a) = \frac{\rho(a)}{\rho_\mathrm{crit}(a)} \quad \text{with} \quad \rho_\mathrm{crit}(a) = \frac{3\mathrm{H}(a)^2}{8\pi \mathrm{G}} \tag{F.274}$$

Therefore we obtain for curvature in adiabatic evolution

$$\frac{\Omega_\mathrm{K}(a)}{\Omega_\mathrm{K}} = \frac{\mathrm{H}_0^2}{a^2 \mathrm{H}^2(a)} = \frac{1}{\frac{\Omega_w}{a^{3(1+w)-2}} + \Omega_\mathrm{K}} \tag{F.275}$$

which indicates directly the evolution of curvature in the presence of another fluid the model universe:

(1) if $3(1+w)-2=0$ then $w=-\frac{1}{3}$ and resulting no changes, as $q=0$, $\ddot{a}=0$ using $3(1+w)=2(1+q)$ for $\Omega=1$ and therefore $\Omega_K=$ const. Effectively, there is another fluid with $w=-1/3$ present and both fluids keep due to their analogous evolution the density parameters fixed at constant values.

(2) if $3(1+w)-2>0$ the resulting $\ddot{a}$ is smaller than 0, thus $q>0$ and in result Ω_K is increasing. An additional fluid with an equation of state more positive than $w=-1/3$ gives rise to a decelerating universe with an associated growth of curvature.

(3) if $3(1+w)-2<0$ the fluid EOS-parameter $w<-\frac{1}{3}$, further $q<0$ and $\ddot{a}>0$, in this configuration Ω_K is decreasing. This case is certainly interesting for us, as this drives Ω_K to small values, as a consequence of the dominating energy density of the additional fluid with an equation of state more negative than $w=-1/3$.

Thirdly, there is the scale problem, which arises if one tries to predict typical scales of the Universe from natural constants. In the Planck-system, constants are c, G and $\hbar$, whereas in the Hubble system we use c, G and Λ, and inflation catapults the Universe from a system that is described by the Planck length $l_P=\sqrt{\frac{G\hbar}{c}}=10^{-35}$ meters, the Planck time $t_P=\sqrt{\frac{G\hbar}{c^3}}=10^{-43}$ seconds and the Planck density $\rho_P=\frac{c^5}{G^2\hbar}=10^{96}\frac{kg}{m^3}$ to a state rather described by the Hubble length $l_H=\frac{1}{\sqrt{\Lambda}}=10^{25}$ meters, the Hubble-time $t_H=\frac{1}{c\sqrt{\Lambda}}=10^{17}$ seconds and the Hubble density $\rho_H=\frac{c^3}{\sqrt{\Lambda}G}=10^{-23}\frac{kg}{m^3}$, where we have made convenient use of the fact that the Universe today is flat and dominated by Λ (in fact, a yet unexplained coincidence). Very interestingly, there is a factor of 10^{60} appearing

$$\frac{l_H}{l_P}=10^{60} \quad \text{as well as} \quad \frac{t_H}{t_P}=10^{60} \tag{F.276}$$

suggesting a factor of 10^{120} between ρ_P and ρ_H. Perhaps a better way to phrase the scale problem is to ask why the Universe is so large an empty, and it is clear that accelerated expansion is able to achieve this, by making the Hubble-Lemaître parameter small and, by extension, giving the critical density a small value, too. All in all, these three problems are solved by having an early period of accelerated expansion, called cosmic inflation: it drives the curvature towards small values, shrinks the horizon and makes Universe large.

F.2 *Why is accelerated expansion (and stopping it) so difficult?*

General relativity provides gravity in the form of spacetime curvature for any energy momentum-tensor $T_{\mu\nu}$, which is covariantly conserved, $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu}=0$, and the trace $T=g^{\mu\nu}T_{\mu\nu}$ of the energy momentum tensor is proportional to the Ricci-curvature, as required by the trace of the entire field equation:

$$R_{\mu\nu}-\frac{R}{2}g_{\mu\nu}=-\frac{8\pi G}{c^4}T_{\mu\nu}-\Lambda g_{\mu\nu} \tag{F.277}$$

resulting from $g^{\mu\nu}R_{\mu\nu} = R$ as well as $g^{\mu\nu}g_{\mu\nu} = \delta^{\mu}_{\mu} = 4$. The trace of the energy momentum tensor is surely an invariant quantity but unlike electric charges which can have either of two possible signs, the energy momentum tensor is subjected to energy conditions, making sure that the energy momentum content of spacetime is bounded by zero from below and that gravity is attractive. Working with an ideal fluid

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - p g_{\mu\nu} \tag{F.278}$$

one can define the energy conditions through contractions with $T_{\mu\nu}$ and reexpressing them with density ρ and pressure p.

1. null energy condition ($\rho + p \geq 0$) resulting from $T_{\mu\nu}k^\mu k^\nu \geq 0$ for all fluids, if k^μ is a null-vector $g_{\mu\nu}k^\mu k^\nu = 0$.
2. weak energy condition ($\rho \geq 0$, matter density always positive) resulting from above's $T_{\mu\nu}u^\mu u^\nu \geq 0$, for time-like u^μ with $g_{\mu\nu}u^\mu u^\nu = c^2$ for the tangent $u^\mu = \mathrm{d}x^\mu \mathrm{d}\tau$ to a world line $x^\mu(\tau)$ of an observer.
3. strong energy condition ($\rho + 3p \geq 0$ for an ideal fluid) resulting from scalar $R_{\mu\nu}u^\mu u^\nu \geq 0$ for all fluids.

Therefore gravity is attractive and curves geodesics towards each other. The three conditions are subsets of each other and are related to each other by contraction of $k^\mu k^\nu$ or $u^\mu u^\nu$ with the field equation, similarly to the contraction with $g_{\mu\nu}$, and working best with an ideal fluid for $T_{\mu\nu}$. Thus it is very complicated to generate repulsive gravity, because all together $\rho \geq 0$ (weak), $\rho + p \geq 0$ (null) and $\rho + 3p \geq 0$ (strong) but for repulsion one needs $p < -\frac{1}{3}\rho$ (or $w < -\frac{1}{3}$) resulting in acceleration, $q > 0$.

Furthermore, it is clear that in the course of the Hubble expansion, the fluids will dominated in the order of descending value for their equation of state w: Once one has established accelerated expansion with a fluid $w < -1/3$, it is very difficult to return to e.g. matter domination with $w = 0$! Keeping in mind that $3(1 + w) = 2(1 + q)$ for a critical FLRW-universe with density parameter $\Omega = 1$ for a fluid with an arbitrary but constant equation of state w on would get a progression

$$\Omega_r \qquad \Omega_m \qquad \Omega_K \qquad \Omega_\Lambda \tag{F.279}$$

$$w = +\frac{1}{3} \qquad w = 0 \qquad w = -\frac{1}{3} \qquad w = -1 \tag{F.280}$$

$$q = 1 \qquad q = \frac{1}{2} \qquad q = 0 \qquad q = -1 \tag{F.281}$$

To make this explicity, we write down the evolution of the density parameter for a fluid with fixed equation of state w,

$$\frac{\Omega_w(a)}{\Omega_w} = \frac{H_0^2}{a^{3(1+w)}H^2(a)} \tag{F.282}$$

Comparing two such fluids with equations of state w and w' would result in

$$\frac{\Omega_{w'}(a)}{\Omega_w(a)} = \frac{\Omega_{w'}}{\Omega_w} \times a^{-3(w'-w)} \tag{F.283}$$

which increases if $w < w'$ and decreases if $w > w'$. Therefore, the fluid with the most negative equation of state will eventually dominate if the Hubble-function is monotonic: This result is actually very intuitive, as fluids with more negative equation of state parameters tend to have a slower evolution of ρ, such that they eventually dominate. The extreme case of this is Λ with a constant energy density, whose domination will be the natural target of the evolution of the Universe unless the densities of the other fluids are so high that they can halt the Hubble function or make the Universe recollapse.

Therefore, one needs a construction where the Universe is dominated by a fluid with sufficiently negative equation of state $w < -1/3$ such that curvature decreases, but which is able to return eventually back to being dominated by matter with $w = 0$ or radiation with $w = +1/3$, in agreement with observations at redshifts $z > 1$.

F.3 *Quintessence and dynamic dark energy*

Summarising the key results of the last two sections one sees that (i) accelerated expansion can be started with a fluid with a sufficiently negative equation of state but that (ii) it would be difficult to terminate the accelerated expansion and return to radiation- or matter-dominated, decelerated expansion. The solution to this problem is to construct a microscopic model behind the energy momentum tensor consisting of a self-interacting scalar field φ, called quintesence, which follows its own dynamics and which is gravitationally acting on the FLRW-background. Such a system has a dynamically evolving energy density and an equation of state and can terminate accelerated expansion naturally.

The Lagrange-density $\mathcal{L}$ of a scalar field φ on a possibly curved background with a metric $g_{\mu\nu}$ is given by

$$\mathcal{L}(\varphi, \nabla_\mu\varphi) = \frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi) \tag{F.284}$$

with a self-interaction potential $V(\varphi)$ including a mass term $V(\varphi) = m^2\varphi^2/2$. The Euler-Lagrange equation follows directly from variation of the action S

$$S = \int d^4x\sqrt{-\det g}\;\mathcal{L}(\varphi, \nabla_\mu\varphi) \tag{F.285}$$

where the covolume $\sqrt{-\det g}$ takes care of non-Cartesian coordinates. Hamilton's principle assumes that $\delta S = 0$ and therefore

$$\delta S = \int d^4x\sqrt{-\det g}\left(\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta(\nabla_\mu\varphi)\right) = \int d^4x\sqrt{-\det g}\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \nabla_\mu\partial\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\right)\delta\varphi \tag{F.286}$$

after an integration by parts, as done with the Gauss-theorem for integrations on manifolds,

$$\int_V d^4x\sqrt{-\det g}\;\nabla_\mu(av^\mu) = \int_{\partial V} dS_\mu\sqrt{\left|\det\gamma\right|}(av^\mu) = 0 \tag{F.287}$$

for a vector field υ and a scalar field a, which are assumed to reach values of zero on the integration boundary or at least asymptote towards zero fast enough. Formally, the co-volume $\sqrt{-\det g}$ gives rise to an induced measure $\sqrt{\det \gamma}$ (a co-area, in lack of a better expression) on the boundary $\partial\mathrm{V}$, as γ is the induced metric on $\partial\mathrm{V}$, $\gamma = g(\partial\mathrm{V})$. This leads to

$$\int_{\mathrm{V}} \mathrm{d}^4x\sqrt{-\det g}\,\nabla_\mu(a\upsilon^\mu) = \int_{\mathrm{V}} \mathrm{d}^4x\sqrt{-\det g}\,(\nabla_\mu a\cdot\upsilon^\mu + a\nabla_\mu\upsilon^\mu) \tag{F.288}$$

as the covariant derivative obeys the Leibnitz-rule, implying that if the surface integral vanishes due to fast enough decaying fields, that

$$\int_{\mathrm{V}} \mathrm{d}^4x\sqrt{-\det g}\,\nabla_\mu a\cdot\upsilon^\mu = -\int_{\mathrm{V}} \mathrm{d}^4x\sqrt{-\det g}\,a\nabla_\mu\upsilon^\mu \tag{F.289}$$

and everything looks like a straightforward integration by parts.

Deriving now all terms for the Euler-Lagrange equation gives first of all

$$\frac{\partial\mathcal{L}}{\partial\varphi} = -\frac{\mathrm{dV}}{\mathrm{d}\varphi} \tag{F.290}$$

because the potential V depends only on the field φ, as well as

$$\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)} = \frac{1}{2}\frac{\partial}{\partial(\nabla_\mu\varphi)}\left(g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi\right) = \frac{1}{2}g^{\alpha\beta}\left(\underbrace{\frac{\partial\nabla_\alpha\varphi}{\partial\nabla_\mu\varphi}}_{=\delta^\mu_\alpha}\nabla_\beta\varphi + \nabla_\alpha\varphi\underbrace{\frac{\partial\nabla_\beta\varphi}{\partial\nabla_\mu\varphi}}_{=\delta^\mu_\beta}\right) = g^{\alpha\mu}\nabla_\alpha\varphi \tag{F.291}$$

and further concluding that

$$\nabla_\mu\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)} = \nabla_\mu(g^{\alpha\mu}\nabla_\alpha\varphi) = g^{\alpha\mu}\nabla_\mu\nabla_\alpha\varphi \tag{F.292}$$

using metric compatibility of the covariant derivative. Therefore, the quintessence equation of motion for the field φ looks like a wave equation, or better, a covariant version of the Klein-Gordon equation,

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = -\frac{\mathrm{dV}}{\mathrm{d}\varphi} \tag{F.293}$$

driven by the self-interaction $\mathrm{V}(\varphi)$, which as stated before, may include a mass-term for the field φ. As this will facilitate the treatment later, we can rewrite the term $g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi$ as a covariant divergence for which there is a very practical formula:

$$g^{\mu\nu}\nabla_\mu\underbrace{\nabla_\nu\varphi}_{=\partial_\nu\varphi=\upsilon_\nu} = \nabla_\mu(g^{\mu\nu}\upsilon_\nu) = \nabla_\mu\upsilon^\mu = \frac{1}{\sqrt{-\det g}}\partial_\mu(\sqrt{-\det g}\,\upsilon^\mu) \tag{F.294}$$

making use of metric compatibility again and introducing the determinant g of the metric. For illustrative purposes we have defined the linear form $\upsilon_\mu = \partial_\mu\varphi = \partial_\mu\varphi$ as the field gradient in φ.

Restricting the background now to conform to the FLRW-symmetries on can determine the covolume to be $\sqrt{-\det g} = a^3$ and both spacetime and the field φ only possesses an evolution in the t-direction, such that $\partial_\mu \to \partial_t$. Then, the divergence becomes

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = \frac{1}{a^3}\partial_t(a^3\partial_t\varphi) = 3\frac{\dot{a}}{a}\partial_t\varphi + \partial_t^2\varphi \tag{F.295}$$

leading us finally to

$$\partial_t^2\varphi + 3\mathrm{H}(t)\partial_t\varphi = -\frac{\mathrm{dV}}{\mathrm{d}\varphi} \tag{F.296}$$

which is the Klein-Gordon equation for the field φ with self-interaction V. The FLRW-background manifests itself as the second term in eqn. (F.296), which is proportional to $\mathrm{H} = \dot{a}/a$: For large H it works like a damping term restricting the evolution of the field φ and is aptly named Hubble-drag. But please do keep in mind that there are no dissipative effects implied, the term purely arises because of the dynamic background.

F.4 *Gravity of the quintessence filed*

In the previous section we have derived the equation of motion of a scalar field on a FLRW-background and arrived at the Klein-Gordon-equation

$$\partial_t^2\varphi + 3\mathrm{H}(t)\partial_t\varphi = -\frac{\mathrm{dV}}{\mathrm{d}\varphi} \tag{F.297}$$

for the field evolution for a given background dynamics encapsulated in $\mathrm{H}(t)$. The background could be defined by a pre-determined Hubble-function $\mathrm{H}(t)$ with the field φ as a test object, but the more interesting case is certainly where the field φ itself exerts a gravitational effect onto the background, such that one deals with a coupled system of (i) the Klein-Gordon-equation for the evolution of φ and the (ii) Friedmann-equation sourced by the energy momentum-content of φ for the evolution of $\mathrm{H}(t)$.

If $\mathcal{L}$ depends on the field φ and its derivative $\nabla_\mu\varphi$, but not explicitly on the coordinates x^μ, then there is an associated covariant conservation law:

$$g^{\mu\nu}\nabla_\alpha\mathrm{T}_{\mu\nu} = 0 \tag{F.298}$$

Loosely speaking, because the definition of the field dynamics are invariant under shifts on the manifold, energy and momentum are conserved. A counter example would e.g. be a position- or time dependent change in the Lagrange-density of e.g. electrodynamics: Then, the energies of atomic lines would be different in different places of the Universe, and emission processes in the distant Universe would not be compatible with absorption processes in the Milky Way.

As in classical mechanics one can construct the Beltrami-identity

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta(\nabla_\mu\varphi) = \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta\varphi\right) - \nabla_\mu\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi \tag{F.299}$$

where on recognises the Euler-Lagrange equation for φ

$$\delta\mathcal{L} = \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta\varphi\right) + \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \nabla_\mu\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\right)}_{\text{Euler-Lagrange}=0}\delta\varphi \tag{F.300}$$

such that the final result for the variation of $\mathcal{L}$ caused by the field variation $\delta\varphi$ is given by

$$\delta\mathcal{L} = \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta\varphi\right) \tag{F.301}$$

For $\delta\varphi$ we construct an infinitesimal field variation $\delta\varphi$ through a coordinate shift

$$\varphi(x^\mu + \delta x^\mu) = \varphi(x^\mu) + \nabla_\nu\varphi(x^\mu)\delta x^\nu + \ldots \quad \rightarrow \quad \delta\varphi = \varphi(x^\mu + \delta x^\mu) - \varphi(x^\mu) = \nabla_\nu\varphi(x^\mu)\delta x^\nu \tag{F.302}$$

under which the Lagrange-density transforms according to

$$\mathcal{L}(\varphi, \nabla_\mu\varphi) \rightarrow \mathcal{L}(\varphi, \nabla_\mu\varphi) + \nabla_\nu\mathcal{L}\delta x^\nu \quad \rightarrow \quad \delta\mathcal{L} = \nabla_\nu\mathcal{L}\,\delta x^\nu \tag{F.303}$$

Now, we can write the variation $\delta\mathcal{L}$ as resulting from the field variation $\delta\varphi$, as there can not be a variation of the working principle of the field theory with coordinate itself, according to the assumption that the functional shape and therefore the working principle of the field φ is universal and would not depend on the coordinate x^μ:

$$\delta\mathcal{L} = g^{\mu\beta}\nabla_\mu\mathcal{L}\delta x_\beta = \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\delta\varphi\right) = \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\underbrace{\nabla_\nu\varphi\delta x^\nu}_{=g^{\alpha\beta}\nabla_\alpha\varphi\delta x_\beta}\right) \tag{F.304}$$

resulting in

$$\nabla_\mu\left(g^{\mu\beta}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}g^{\alpha\beta}\nabla_\alpha\varphi\right)\delta x_\beta = 0 \tag{F.305}$$

Identifying the term in the bracket in eqn. F.305 to be the energy-momentum tensor $T^{\mu\beta}$ shows the corresponding covariant conservation law

$$\nabla_\mu T^{\mu\beta} = 0 \tag{F.306}$$

for the energy-momentum tensor $T^{\alpha\beta}$, that results directly from the Lagrange-density $\mathcal{L}$ of the quintessence field φ

$$T^{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi)}g^{\beta\nu}\nabla_\nu\varphi - \mathcal{L}g^{\alpha\beta} \tag{F.307}$$

The explicit result $T^{\alpha\beta}$ for the scalar field φ by substituting its Lagrange-density

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi - V(\varphi) \tag{F.308}$$

into eqn. F.307, making use of

$$\frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi)} = \frac{1}{2}g^{\mu\nu}\left(\underbrace{\frac{\partial\nabla_\mu\varphi}{\partial\nabla_\alpha\varphi}}_{\delta^\alpha_\mu}\nabla_\nu\varphi + \nabla_\mu\varphi\underbrace{\frac{\partial\nabla_\nu\varphi}{\partial\nabla_\alpha\varphi}}_{\delta^\alpha_\nu}\right) = g^{\mu\alpha}\nabla_\mu\varphi \tag{F.309}$$

such that one arrives at an expression for the energy-momentum tensor as it is determined from the gradients $\nabla_\mu\varphi$ of the field and the strength $V(\varphi)$ of the field's self-interaction:

$$T^{\alpha\beta} = g^{\mu\alpha}g^{\beta\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\nabla_\mu\varphi\nabla_\nu\varphi + V(\varphi)g^{\alpha\beta} \tag{F.310}$$

It is instructive to interpret this result for the the energy-momentum tensor with that of an ideal fluid

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)u^\alpha u^\beta - pg^{\alpha\beta} \tag{F.311}$$

and possibly derive ρ and p from the terms ∇_φ and $V(\varphi)$: In particular for a FLRW-spactime with spatial homogeneity one should then be able to derive ρ and p, as they would result dynamically from solving the Klein-Gordon-equation and compute the evolution of the scale factor a from the Friedmann-equations, such that one has constructed a coupled dynamical system for φ and a, possibly with a dynamical relation between p and ρ, or, equivalently, a dynamically evolving equation of state $w = p/(\rho c^2)$.

Parameterising a FRLW-spacetime with comoving coordinates x^μ yields for the velocities $u^\mu = \mathrm{d}x^\mu/\mathrm{d}t = \mathrm{d}x^\mu/\mathrm{d}t = (c, 0)^t$ as tangents to the world lines of fluid elements simplifies the energy-momentum tensor tremendously: It will be diagonal (as the inverse metric $g^{\mu\nu}$ is diagonal is the FLRW-case) and have the tt-component

$$T^{tt} = \left(\rho + \frac{p}{c^2}\right)u^t u^t - pg^{tt} = \rho c^2 \tag{F.312}$$

with $g^{tt} = 1$, and the spatial ii-components

$$T^{ii} = \left(\rho c^2 + \frac{p}{c^2}\right)u^i u^i - pg^{ii} = 3\frac{p}{a^2} \tag{F.313}$$

as the spatial part of the inverse metric is $g^{ii} = -a^{-2}$ and $u^i = 0$ for comoving fluid elements.

Isolating these two components from the energy-momentum tensor for the field φ is straightforward in particular under the assumption of the FLRW-symmetries, where all spatial derivatives are zero and because the field φ is scalar, implying that $\nabla_\mu\varphi = \partial_\mu\varphi$ of which only $\partial_t\varphi$ is nonzero. Therefore, the density ρ must be

$$\rho c^2 = T^{tt} = g^{t\alpha}g^{t\beta}\nabla_\alpha\varphi\nabla_\beta\varphi - \frac{1}{2}g^{tt}g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi + V(\varphi)g^{tt} = \frac{1}{2}(\partial_t\varphi)^2 + V(\varphi) \tag{F.314}$$

and similarly for the spatial part yielding pressure p

$$3\frac{p}{a^2} = T^{ii} = g^{i\alpha}g^{i\beta}\nabla_\alpha\varphi\nabla_\beta\varphi - \frac{1}{2}g^{ii}g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi + V(\varphi)g^{ii} = 3a^{-2}\frac{1}{2}(\partial_t\varphi)^2 - 3a^{-2}V(\varphi) \tag{F.315}$$

which can be simplified to

$$p = \frac{1}{2}(\partial_t\varphi)^2 - V(\varphi) \tag{F.316}$$

Combining both results is a construction of the equation of state w

$$w = \frac{p}{\rho c^2} = \frac{\frac{1}{2}(\partial_t\varphi)^2 - V(\varphi)}{\frac{1}{2}(\partial_t\varphi)^2 + V(\varphi)} \tag{F.317}$$

which gives a direct indication of the gravitational effect of the field φ, as both ρ and p enter the gravitational field equation. In particular, if the evolution of the field is slow and therefore the kinetic term $\frac{1}{2}(\partial_t\varphi)^2$ is much less than the potential term $V(\varphi)$, one obtains for the equation of state is $w \sim -1$. Then, the gravitational effect of φ is identical to that of the cosmological constant Λ and the FLRW-spacetime is accelerating at $q = -1$, leading to exponential expansion.

In the course of time evolution with the Klein-Gordon equation

$$\partial_t^2\varphi + 3H(t)\partial_t\varphi = -\frac{dV}{d\varphi} \tag{F.318}$$

one would expect that $(\partial_t\varphi)^2$ increases at the expense of $V(\varphi)$, and that the equation of state evolves away from the value $w = -1$, as the slow-roll condition

$$\frac{1}{2}(\partial_t\varphi) \ll V(\varphi) \tag{F.319}$$

is violated. For instance, when $\frac{1}{2}(\partial_t\varphi)^2 \sim V(\varphi)$ is reached, the equation of state becomes $w = 0$, corresponding to a decelerated universe with $q = \frac{1}{2}$, as if it was filled with matter. Clearly, the quintessence field shows a variable gravitational effect on the FLRW-background, and in particular does it provide a mechanism of driving accelerated expansion to solve the flatness-, horizon- and scale-problems, and a natural way of stopping inflation and returning to normal expansion dominated by fluids with less negative equations of state.

F.5 *Slow-roll approximation*

Cosmic inflation as driven by the scalar field φ, if it should solve the horizon and flatness problems, has to provide accelerated expansion through a negative enough equation of state and take care that this period of accelerated expansion lasts long enough. These two conditions are ultimately requirements on the potential $V(\varphi)$, usually formulated in terms of the two slow-roll parameters ϵ and η:

$$\epsilon = \frac{1}{8\pi G}\left(\frac{d\ln V}{d\varphi}\right)^2 \quad\text{and}\quad \eta = \frac{1}{24\pi G}\left(\frac{1}{V}\frac{d^2V}{d\varphi^2}\right) \tag{F.320}$$

which are essentially logarithmic derivatives of the quintessence potential $V(\varphi)$. If ϵ and η are small, the potential has a small slope and is weakly curved, implying that the time evolution of φ is weak, slow-roll is maintained for a long time, and exponential, accelerated expansion is maintained, such that a low spatial curvature can be realised and the horizon becomes large enough.

A sufficiently negative equation of state parameter w for accelerated expansion is generated by the slow-roll condition itself, $\frac{1}{2}(\partial_t\varphi)^2 \ll V(\varphi)$. This condition implies directly for the first Friedmann-equation that

$$H^2(t) = \frac{8\pi G}{3}\rho c^2 = \frac{8\pi G}{3}\left(\frac{1}{2}(\partial_t\varphi)^2 + V(\varphi)\right) \quad\rightarrow\quad H^2(t) = \frac{8\pi G}{3}V(\varphi) \tag{F.321}$$

where we used the slow-roll in the last step. The acceleration $\ddot{a}$ can be derived from the latter equation by differentiating it with respect to t, yielding

$$2H\partial_t H = \frac{8\pi G}{3}\partial_t\varphi\frac{dV}{d\varphi} \tag{F.322}$$

by application of the chain rule to $\partial_t V(\varphi(t))$. The slow-roll approximated Klein-Gordon equation F.321 for the FRLW-background

$$3H\partial_t\varphi = -\frac{dV}{d\varphi} \tag{F.323}$$

implies the condition

$$\partial_t H = -4\pi G(\partial_t\varphi)^2 \ll 4\pi G V(\varphi) \tag{F.324}$$

as an expression for slow roll, on the basis of the potential and constrains the evolution of the Hubble-function H. This allows now to formulate the slow-roll parameters ϵ and η defined in eqn. F.320 in their dependence on the potential $V(\varphi)$.

The square of the approximate Klein-Gordon equation,

$$(3H\partial_t\varphi)^2 = \left(\frac{dV}{d\varphi}\right)^2 \tag{F.325}$$

together with the Friedmann-equation for H^2

$$3^2\frac{8\pi G}{3}V(\varphi)^2(\partial_t\varphi)^2 = \left(\frac{dV}{d\varphi}\right)^2 \tag{F.326}$$

shows that

$$(\partial_t\varphi)^2 = \frac{1}{24\pi G}\left(\frac{1}{V}\frac{dV}{d\varphi}\right)^2 = \frac{1}{24\pi G}\left(\frac{d\ln V}{d\varphi}\right)^2 \equiv \epsilon \ll 1 \tag{F.327}$$

where the slow-roll parameter $\epsilon \ll$ ensures that the kinetic term $(\partial_t\varphi)^2/2$ stays small.

Differentiating the approximate Klein-Gordon equation with respect to t yields

$$3(\partial_t H\partial_t\varphi + H\partial_t^2\varphi) \simeq 3\partial_H\partial_t\varphi = \frac{d^2V}{d\varphi^2}\partial_t\varphi \tag{F.328}$$

where we neglect $H\partial_t^2\varphi$ over $\partial_t H\partial_t\varphi$ and which we divide with $\partial_t\varphi$ for

$$3\partial_t H = -\frac{d^2V}{d\varphi^2} \tag{F.329}$$

But because of the fact that $\partial_t H = -4\pi G(\partial_t\varphi)^2 \ll 4\pi GV(\varphi)$ as derived above, one can conclude that the second slow-roll parameter η,

$$(\partial_t\varphi)^2 = \frac{1}{12\pi GV}\frac{d^2V}{d\varphi^2} \equiv \eta \ll 1 \tag{F.330}$$

must be small compared to one as well.

F.6 *Accelerated expansion in the late Universe*

To what limit the accelerated expansion at the current time is related to quintessence at early times is unclear, but the mechanism works in both cases: at early times, as cosmic inflation and at late times as dark energy. Whether inflation in the early Universe is initiated by randomly setting the right initial conditions for the field φ (the exact mechanism of this is still unclear), achieving domination of φ in the late Universe at redshifts below unity in a natural way is equally difficult. Many dark energy models link accelerated expansion to other physical processes, for instance, the acquisition of mass in neutrinos.

F.7 *Seeding of cosmic structures in inflation*

Apart from solving the flatness and horizon problems, cosmic inflation provides a mechanism for seed fluctuations from which the cosmic large-scale can grow: The exact mechanism is quite technical, but the fundamental idea is that the comoving horizon $c/(aH)$ shrinks during the accelerated expanding phase. Fluctuations in the metric with a fixed comoving wave length are initialised at the instant when they leave the (shrinking) horizon, at an amplitude that is given by the so-called Bunch-Davies vacuum, which corresponds to the ground state amplitude of the field φ.

The amplitude of these perturbation in φ and the associated fluctuations in the metric $\delta\Phi$ are roughly given by $\sqrt{\langle\delta\Phi^2\rangle} \simeq H^2/V(\varphi)$, which is roughly constant while the expansion is exponential. One can now relate fluctuations in the potential Φ to fluctuations in the density field by invoking the Poisson-equation which reads in Four-space $k^2\Phi(k) = -\delta(k)$.

Then, the relation

$$|\delta(k)|^2 \propto k^4\,|\delta\Phi|^2 \propto k^3 P(k) \tag{F.331}$$

for the variance of the density field fluctuations in Fourier-space applies, which is related to the variance in the potential fluctuations. If $|\delta\Phi|^2$ is constant as predicted by the constant Hubble-function, the spectrum $P(k)$ must be $\propto k$ to give a consistent scaling.

In reality, there are tiny deviations from perfect exponential expansion, of the order of the slow-roll parameters ϵ and η. As a consequence, there is a minute evolution of the Hubble-function and the amplitude $\sqrt{\langle\delta\Phi^2\rangle}$ becomes a function of time. As the comoving horizon evolves, that time-dependence can be converted into a scale dependence, which effectively makes $P(k) \propto k^{n_s}$ with $n_s \simeq 0.96$, deviating slightly from unity, by a quantity of the order of the slow-roll parameters.

G FLUID MECHANICS

G.1 *fluid mechanics as a continuum theory*

The motion of matter on large scale and for small perturbations can be described by fluid mechanics, such that the evolution of the cosmic density field and the cosmic velocity field is determined through the equations of fluid mechancis, namely the continuity and the Navier-Stokes equation, both with gravity as the driving force of structure formation. For the purpose of this book we restrict ourselves to nonrelativistic fluid mechanics with a Newtonian description of gravity and Galilean relativity. The motion of a fluid is primarily determined by the continuity and the Navier-Stokes equation, which determine the time evolution of the density and the velocity fields, respectively. Fluid mechanics is a continuum theory, because it considers the fluids as continuous media without any microscopic structure, and as such it can only describe fluid elements which are large enough that they contain a large number of particles. The description of collisionless systems under the influence of gravity is conceptually not clear, because (i) individual particles can gain very large velocities in many-body-interactions, such that the particle density might not be sufficient to define a smooth fluid through averaging of particle properties and because (ii) self-gravitating systems produce structures on small scales, which are not wiped out by collisions such that in the averaging process in deriving smooth fields information on the phase-space structure is lost.

It is very important to notice that both the continuity and the Navier-Stokes equations are nonlinear, as both involve products between the density and the velocity, and between the velocity and gradients of the velocity, respectively. In addition, the equation of state $p(\rho)$, if present in the Navier-Stokes equation, can be nonlinear as well and can, in addition depend on other quantities, for instance the entropy density s or temperature T, leading to additional terms in particular in the vorticity equation. Alternative, one can choose to work with the momentum density $\rho\boldsymbol{\upsilon}$ instead of the velocity $\boldsymbol{\upsilon}$, which would render the continuity equation linear but would make the gravitational force in the Navier-Stokes equation nonlinear.

G.2 *From relativistic to non-relativistic fluid mechanics*

Energy-momentum conservation in the covariant form $\nabla_\mu T^{\mu\nu} = 0$ is equivalent to relativistic fluid mechanics of ideal fluids. In the non-relativistic limit with slow velocities $|\upsilon| \ll c$ on a Minkowski-background with $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\alpha{}_{\mu\nu} = 0$. In the non-relativistic limit, $p \ll \rho c^2$ and the motion of the fluid elements proceeds essentially only in $\mathrm{d}t$-direction:

$$\rho c^2\underbrace{(\partial_t\beta^j + (\beta^i\partial_i)\beta^j)}_{=u^\mu\nabla_\mu u^\nu = u^\mu\partial_\mu u^\nu} = \rho(\partial_t\upsilon^j + \upsilon^i\partial_j\upsilon^j) = -\partial^j p \quad \text{or} \quad \partial_t\boldsymbol{\upsilon} + (\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon} = -\frac{\nabla p}{\rho} \tag{G.332}$$

which is exactly the non-relativistic Euler-equation. Including gravity requires to use $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$ with a corresponding nonzero Christoffel-symbol. In the weak-field limit $|\Phi| \ll c^2$ on has the line element

$$\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right)\mathrm{d}x_i\mathrm{d}x^i \tag{G.333}$$

where only the first term contributes as the displacements in the spatial $\mathrm{d}x^i$-directions are small:

$$g_{tt} = 1 + \frac{2\Phi}{c^2} \tag{G.334}$$

The gravitational acceleration is computed from the Christoffel symbols

$$\Gamma^{\alpha}_{\ \mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu g_{\beta\nu} - \partial_\nu g_{\mu\beta} + \partial_\beta g_{\mu\nu}\right) \tag{G.335}$$

where in the weak-field limit the inverse metric is replaced by the (inverse) Minkowski metric $g^{\alpha\beta} = \eta^{\alpha\beta}$ but of course the gradients $\partial_\beta g_{\mu\nu}$ are kept. In static gravitational fields $\partial_t g_{\alpha\beta} = 0$ and only nonzero spatial derivatives $\partial_i g_{\alpha\beta} = \frac{2}{c^2}\partial_i \Phi \delta_{\alpha\beta}$, from which one would expect gradients $\partial^j\Phi$ to appear:

$$u^\mu(\nabla_\mu u^\nu) = u^\mu(\partial_\mu u^\nu + \Gamma^{\nu}_{\ \mu\alpha}\, u^\alpha) = u^\mu \partial_\mu u^\nu + \Gamma^{\nu}_{\ \mu\alpha}\, u^\mu u^\alpha \tag{G.336}$$

The three terms naturally correspond to gravitational acceleration in an inhomogeneous field, to the Coriolis- and centrifugal accelerations:

$$\Gamma^{\nu}_{\ \mu\alpha} = \Gamma^{\nu}_{\ tt}\underbrace{u^t}_{c}\underbrace{u^t}_{c} + \underbrace{\Gamma^{\nu}_{\ ti}\, u^t u^i + \Gamma^{\nu}_{\ it}\, u^i u^t}_{2\Gamma^{\nu}_{\ it}\, c v^i} + \Gamma^{\nu}_{\ ij}\underbrace{u^i}_{v^i}\underbrace{u^j}_{v^j} \tag{G.337}$$

The first term is clearly dominating for small velocities

$$u^\mu \nabla_\mu u^\nu = u^\mu\partial_\mu u^\nu + \Gamma^{\nu}_{\ \mu\alpha}\, u^\mu u^\alpha = \partial_t v^j + (v^i\partial_i)v^j - \Gamma^{j}_{\ tt}\, c^2 = -\frac{1}{\rho}\partial^j p \tag{G.338}$$

with the Christoffel-symbol $\Gamma^{j}_{\ tt}$

$$\Gamma^{j}_{\ tt} \simeq \frac{\eta^{jk}}{2}\left(\partial_t g_{tk} + \partial_t g_{kt} - \partial_k g_{tt}\right) = -\partial^j\frac{\Phi}{c^2} \tag{G.339}$$

as only the last term $g_{tt} = 2\Phi/c^2$ contributes and the first two terms vanish, because of the assumption of static gravitational fields. At the same time, the terms $\Gamma^{i}_{\ tj}$ and $\Gamma^{i}_{\ jk}$ offer a natural and consistent way to incorporate other inertial accelerations. So the final result is the non-relativistic Euler-equation with gravity

$$\partial_t v^j + v^i\partial_i v^j = -\frac{1}{\rho}\partial^j p - \partial^j\Phi \quad \text{or} \quad \partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi \tag{G.340}$$

It is quite interesting that the nonlinearities in the fluid-mechanical equations have a relativistic origin, and that one needs empirical reasoning to make sense of them in classical mechanics. The advective term $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ is interpreted as the rate of change of the velocity at a fixed point in the laboratory frame as the flow sweeps new fluid elements to this point which may carry a different velocity (the velocity the fluid element has had upstream an infinitesimal time in the past), while only $\partial_t v$ is the proper rate of change of the flow velocity, measured in terms of coordinate time instead of proper time.

G.3 *Continuity*

The continuity equation is an expression of the conservation of matter. If the density field changes in a volume element at a fixed point it must necessarily be because fluxes have converged and have transported matter into that element:

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{\upsilon}) = 0. \tag{G.341}$$

The interpretation of the continuity equation is particularly clear if one applies the Gauss-theorem:

$$\int_V \mathrm{d}V\, \partial_t \rho = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathrm{d}V\, \rho = \frac{\mathrm{d}}{\mathrm{d}t} M = -\int_V \mathrm{d}V\, \mathrm{div}(\rho \boldsymbol{\upsilon}) = -\int_{\partial V} \mathrm{d}\mathbf{A} \cdot (\rho \boldsymbol{\upsilon}), \tag{G.342}$$

such that the mass M changes if there are fluxes through the surface of the volume element. The continuity equation is nonlinear because the definition of the flux $\rho\boldsymbol{\upsilon}$ involves the product of two fields.

G.4 *Navier-Stokes equation*

The Navier-Stokes equation is the equation of motion for fluid elements as a generalisation of Newton's third axiom,

$$\partial_t \boldsymbol{\upsilon} + (\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon} = -\frac{\nabla p}{\rho} - \nabla\Phi + \mu\Delta\boldsymbol{\upsilon}, \tag{G.343}$$

as it relates the acceleration of a fluid element with the specific force density. Relevant forces include pressure gradients, gradients in the gravitational potential or viscous forces. The Navier-Stokes-equation seems to have the shape of an evolution equation, but in fact it originates together with the continuity equation from a relativistic conservation equation $\partial_\mu T^{\mu\nu} = 0$ with the energy-momentum-tensor $T^{\mu\nu}$ of the fluid. In a chosen reference frame it is possible to separate the conservation equation in the time-part containing the conservation of mass and a spatial part with the conservation of momentum.

The time derivative of the velocity, as required by Newton's equation of motion, is computed for a field which depends on time and on position. In components one would write

$$\frac{\mathrm{d}}{\mathrm{d}t}\upsilon_i(r_j, t) = \partial_t \upsilon_i + \frac{\partial r_j}{\partial t}\frac{\partial \upsilon_i}{\partial r_j} \tag{G.344}$$

With the subsitution of the derivative $\partial_t r_j = \upsilon_j$ one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\upsilon_i(r_j, t) = \partial_t \upsilon_i + \upsilon_j \frac{\partial \upsilon_i}{\partial r_j}. \tag{G.345}$$

Rewriting this expression yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\upsilon} = \partial_t \boldsymbol{\upsilon} + (\boldsymbol{\upsilon}\cdot\nabla)\boldsymbol{\upsilon}. \tag{G.346}$$

Therefore, the nonlinearity $(\boldsymbol{\upsilon}\cdot\nabla)\boldsymbol{\upsilon}$ originates purely from the choice of a fixed coordinate frame, relative to which the fluid moves: The derivative $\partial_t \boldsymbol{v}$ would indicate the acceleration or the rate of change of velocity with time of fluid elements which pass in succession through a fixed position $\boldsymbol{x}$ in space, while the so-called convective derivative $\mathrm{D}_t = \partial_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ describes the acceleration of a single fluid element as it moves around, combining the time-derivative ∂_t with the rate of change of velocity with position $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ projected onto the velocity-components themselves. By choosing instead of a fixed Euler-frame a coordinate frame which moves along with the fluid, referred to as the Lagrange-frame, the fluid equation of motion becomes linear, by introducing comoving, Lagrangian coordinates $\boldsymbol{r} = \boldsymbol{r}_0 + \int \mathrm{d}t\ \boldsymbol{v}$ and reexpressing all derivatives.

Both viscosity and pressure originate from collisions between the particles from which the fluid is composed. The viscosity is usually modelled on the Lamé'-viscosity coefficients and is able to dissipate kinetic energy from the fluid by friction if velocity gradients or shear flows $\partial_i v_j$ are present. If there is such a phenomenon, one needs an analogous energy equation to keep track of the evolution of the energy content of the fluid, in particular because the equation of state might show a dependence on e.g. temperature or entropy density. We will only consider ideal fluids without viscosity, because they approximate dark matter well due to its collisionlessness, and cover the phenomenology of baryonic fluids at low densities.

G.5 *Ideal versus viscous fluid mechanics*

In contrast to the kinematical terms in fluid mechanics and in contrast to gravity, effects associated with the microscopic properties of the fluid itself need to have a phenomenological description. In fact, how bulk properties like fluid-mechanical pressure and viscosity would be determined from the microscopic interactions between the particles that the fluid consists of, is yet not fully understood.

The differential change $\mathrm{d}\boldsymbol{v}$ of the velocity in a fluid is to first order proportional to the displacement

$$\mathrm{d}\boldsymbol{v} = (\mathrm{d}\boldsymbol{r}\nabla)\boldsymbol{v} \quad \rightarrow \quad \mathrm{d}v^i = \partial_j v^i \mathrm{d}x^j \tag{G.347}$$

defining the velocity tensor, which is conveniently decomposed into a symmetric part (shear) and the antisymmetric part (vorticity)

$$\partial_j v^i = \underbrace{\frac{1}{2}(\partial_j v^i + \partial_i v^j)}_{\epsilon_j{}^i} + \underbrace{\frac{1}{2}(\partial_j v^i - \partial_i v^j)}_{\omega_j{}^i} \tag{G.348}$$

Again, this idea is very similar to the Raychauduri-equation: The volume change is given by

$$\mathrm{d}V \sim \mathrm{div}\boldsymbol{v} \sim \partial_i v^i = \epsilon^i{}_i = \mathrm{tr}(\epsilon) \tag{G.349}$$

such that the trace of the velocity tensor induces a change in volume of a fluid element. Incompressible flows have the unique property that the divergence of their velocity field is always zero, and hence there can not be any change in the volume of fluid elements.

In a phenomenological model one can now relate shears in a fluid to stresses and pressure: In general, the stress tensor σ_{ij} is the ith component of the force acting on a surface element with normal vector into the jth direction: As such, stresses and pressure have the same unit of force normalised by area. One can decompose the stress tensor into the isotropic part $p\delta_{ij}$ and the anisotropic contribution σ'_{ij}

$$\sigma_{ij} = \sigma'_{ij} - p\delta_{ij} \tag{G.350}$$

where

$$\mathrm{tr}(\sigma) = p\ \mathrm{tr}(\delta_{ij}) = -3p. \tag{G.351}$$

so that pressure gets the interpretation of isotropic stress.

Furthermore, the stress tensor is also symmetric $\sigma_{ij} = \sigma_{ji}$. This can be shown by setting up a counter example which turns out to be aphysical: If stresses act on two faces of a cube with volume $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$, one introduces a torque torque M_x if the stresses are unequal, in contradiction to $\sigma_{ij} = \sigma_{ji}$,

$$M_x = \sigma_{zy}(\mathrm{d}x\mathrm{d}z)\mathrm{d}y - \sigma_{yz}(\mathrm{d}x\mathrm{d}y)\mathrm{d}z = (\sigma_{zy} - \sigma_{yz})\mathrm{d}V \tag{G.352}$$

With a Newtonian equation of motion $M = I\ddot{\phi}$ with the inertia $I = (\mathrm{d}y^2 + \mathrm{d}z^2)\mathrm{d}V$ for rotation around the x-axis one would obtain the angular acceleration

$$\ddot{\phi} = \frac{M}{I} \sim V^{-\frac{2}{3}} \tag{G.353}$$

Therefore, for $V \to 0$ the volume term $V^{-\frac{2}{3}}$ diverges, which leads to the conclusion that the angular acceleration $\ddot{\phi}$ diverges, too: Accelerations for the smallest torques would assume arbitrarily high values, which would be aphysical. A way out is the condition $\sigma_{yz} = \sigma_{zy}$ and a symmetric stress tensor σ_{ij}.

G.5.1 Bulk and shear viscosity

With the shear as the differential velocity field into which a fluid is embedded and the stress as the reaction of a fluid element to this external shear it is reasonable to assume a linear relationship between these two symmetric tensors: This is the foundational idea of a Newtonian fluid, if in addition the response of the fluid element is instantaneous to the external shear. The shear tensor ϵ_{ij} and the stress tensor σ'_{ij} are related in Lamé parameterisation by introducing two coefficients η and ξ,

$$\sigma'_{ij} = 2\eta\left(\epsilon_{ij} - \frac{\mathrm{tr}(\epsilon)}{3}\delta_{ij}\right) + \xi\ \mathrm{tr}(\epsilon)\delta_{ij} \tag{G.354}$$

with $\mathrm{tr}(\epsilon) = \partial_i v^i - \mathrm{div}\boldsymbol{v}$ is the divergence of the velocity field. The first term parameterises a reaction of the fluid in form of anisotropic stresses to the traceless shear, which would be realised for instance if there is a shearing motion of fluid layers against each other, motivating the term shear viscosity for η. But there is likewise a reaction of the fluid to changes in volume beyond the effects of pressure mediated by the equation of state: The bulk viscosity ξ parameterises for this case the magnitude of anisotropic stresses.

Again, in flows consisting of purely collisionless dark matter, microscopic stresses and effects of viscosity are not present, but there are, like in the case of pressure, collective effects with emulate these.

G.5.2 Viscous fluid mechanics

The effects of pressure and viscosity can obviously change the state of motion of a fluid element, as expressed by the momentum density $\rho\boldsymbol{\upsilon}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}V\,\mathrm{div}(\rho\boldsymbol{\upsilon}) = -\int_V \mathrm{d}V\,\rho\nabla\Phi + \int_{\partial V}\mathrm{d}\mathbf{A}\,\sigma \tag{G.355}$$

such that apart from bulk forces $\rho\nabla\Phi$ acting on the fluid element as a whole there are stresses as surface forces σ. The first term in the momentum equation can be reformulated as a surface integral, too, yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}V\,\mathrm{div}(\rho\boldsymbol{\upsilon}) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial V}\mathrm{d}\mathbf{A}\,\rho\boldsymbol{\upsilon} = \int_{\partial V}\mathrm{d}\mathbf{A}\,\rho\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} \tag{G.356}$$

in a Lagrangian frame that moves along with the flow: Following the fluid element in this way tracks the momentum evolution as forces are acting on its surface, and because there is no exchange of matter with the environment of a fluid element, the time derivative only acts on the velocity. The stresses acting on the surface of the volume element are given by

$$\left(\int_{\partial V}\mathrm{d}\mathbf{A}\,\sigma\right)_i \overset{(\mathrm{d}\mathbf{A})_i=\mathrm{d}A n_i}{=} \int_{\partial V}\mathrm{d}A\,\sigma_{ij}n_j = \int_V \mathrm{d}V\,\frac{\partial}{\partial x^j}\sigma_{ij} = \left(\int_V \mathrm{d}V\,\nabla\sigma\right)_i \tag{G.357}$$

Substituting back gives

$$\rho\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} = \rho\left(\frac{\partial\boldsymbol{\upsilon}}{\partial t} + (\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon}\right) = -\rho\nabla\Phi + \nabla\sigma \tag{G.358}$$

Introducing viscosity and pressure

$$(\nabla\sigma)_i = (\nabla\sigma')_i - \frac{\partial}{\partial x^j}(p\delta_{ij}) = (\nabla\sigma' - \nabla p)_i \tag{G.359}$$

leads to the expression

$$\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} = \frac{\partial}{\partial t}\boldsymbol{\upsilon} + (\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon} = -\nabla\Phi - \frac{\nabla p}{\rho} + \frac{1}{\rho}\nabla\sigma' \tag{G.360}$$

If now viscosity is parameterised by the Lamé-coefficients η and ξ

$$(\nabla\sigma')_i = \frac{\partial}{\partial x_j}\sigma'_{ij} = \eta\frac{\partial^2 \upsilon_i}{\partial x_j^2} + \left(\xi - \frac{\eta}{3}\right)\frac{\partial}{\partial x_i}\underbrace{\frac{\partial \upsilon_k}{\partial x_k}}_{=\mathrm{div}\,\boldsymbol{\upsilon}} \tag{G.361}$$

and if the fluid is incompressible with the condition $\mathrm{div}\boldsymbol{\upsilon} = 0$, the bulk viscosity is irrelevant and one arrives at the Navier-Stokes equation

$$\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\nabla\Phi - \frac{1}{\rho}\nabla p + \mu\Delta\boldsymbol{v} \tag{G.362}$$

with the kinematic viscosity $\mu = \eta/\rho$.

G.6 *Fluid mechanical similarity and scaling relations*

Since nobody has found a general solution to the Navier-Stokes-equation, one wants to use some properties of mechnical similarity to bring the Navier-Stokes-equation to an already solved case. One might argue at this point, that classical fluid mechanics is scale-free from fundamental theory, but scales can enter through macroscopic properties of the fluid. Therefore we have some 'typical' behaviour of flows and can use corresponding scale symmetries. For this we first need to look for a dimensionless form of the Navier-Stokes-equation. To do so, we rescale

$$x \to x^* = \frac{x}{L} \qquad t \to t^* = \frac{t}{T} \tag{G.363}$$

as well as

$$\upsilon \to \upsilon^* = \frac{\upsilon}{V} \quad p \to p^* = \frac{p}{P} \quad g \to g^* = \frac{g}{G} = \frac{\nabla\Phi}{G} \tag{G.364}$$

It'd be important to realise that the scaling with L and T is relevant for derivatives in the fluid mechanical equations, but that V as a scale for the velocity is not automatically L/T: There can be high-velocity flows that vary only slowly with time or position, and vice versa.

Defining dimensionless derivatives is possible by writing

$$\frac{\partial}{\partial t} = \frac{\partial t^*}{\partial t}\frac{\partial}{\partial t^*} = \frac{1}{T}\frac{\partial}{\partial t^*} \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{\partial x^*}{\partial x}\frac{\partial}{\partial x^*} = \frac{1}{L}\frac{\partial}{\partial x^*} \tag{G.365}$$

Rewriting the entire Navier-Stokes equation for incompressible flows in terms of dimensionless variables and dimensionless derivatives gives

$$\frac{\rho V}{T}\frac{\partial}{\partial t^*}\boldsymbol{v}^* + \rho\frac{V^2}{L}(\boldsymbol{v}^*\nabla)\boldsymbol{v}^* = -\frac{P}{L}\nabla^* p^* - \rho G\underbrace{\nabla^*\Phi^*}_{=g^*} + \frac{\eta V}{L^2}\Delta^*\boldsymbol{v}^* \tag{G.366}$$

As all prefactors are equal in their units to $\frac{\rho V^2}{L}$ one can divide this factor out and arrive at

$$\underbrace{\frac{L}{TV}}_{St}\frac{\partial}{\partial t^*}\boldsymbol{v}^* + (\boldsymbol{v}^*\nabla)\boldsymbol{v}^* = -\underbrace{\frac{p}{\rho V^2}}_{Eu}\nabla^* p^* - \underbrace{\frac{GL}{V^2}}_{Fr^{-2}}\nabla^*\Phi^* + \underbrace{\frac{\eta}{\rho V L}}_{Re^{-1}}\Delta^*\boldsymbol{v}^* \tag{G.367}$$

which defines the scaling numbers:

- Strouhal-number $St = \frac{L}{TV}$ - proper acceleration
- Euler-number $Eu = \frac{p}{\rho V^2}$ - pressure vs. kinetic energy density
- Froude-number $Fr = \sqrt{\frac{V}{GT}}$ - potential vs. kinetic energy density

- Reynolds-number $\mathrm{Re} = \frac{\mu}{\mathrm{VL}}$ - magnitude of viscous forces

Working with the dimensionless form of the Navier-Stokes equation implies that the information about the actual physical properties of the system is replaced with the four scaling numbers. If two flows on physically different scales have the same scaling numbers, one must be able to map them onto each other by a similarity or scaling transform. This implies that there should be a classification of fluid mechanical problems into categories according to the dominating scaling numbers. Again, dark matter poses the conceptual problem how the Euler- and Reynolds-numbers should be defined, with the absence of microscopic interactions between the particles there is no pressure and no viscosity.

G.7 *Gravity and the Poisson-equation*

The gravitational force in the fluid-mechanical equations

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\nabla\Phi \tag{G.368}$$

could be determined through the Poisson equation,

$$\Delta\Phi = 4\pi G\rho, \tag{G.369}$$

and describes gravity in the weak field limit and at distances smaller than the Hubble-distance such that retardation effects do not play a role. In addition, all additional gravitational effects on and by moving objects are neglected: In summary, the equation is valid for $|\Phi| \ll c^2$, $|\upsilon| \ll c$ and on scales $\ll c/\mathrm{H}_0$.

Due to the fact that it is the same density field ρ which is driven in its evolution by gradients $\nabla\Phi$ in the gravitational potential and which is at the same time sourcing the gravitational potential through the Poisson equation speaks of cosmic structure formation as a self gravitating phenomenon: Heuristically, a perturbation in the matter distribution generates a potential, which attracts matter from the surrounding of the perturbation, making it stronger. Then, the potential becomes deeper and the fields amplify, such that more matter is falling towards the perturbation, making it grow rapidly and at an exponential rate with time, if the influence of the background cosmology is neglected.

G.8 *Wave-type solutions and the Jeans-scale*

Pressure gradients have an influence on the evolution of the velocity field, and they typically lead to wave-type solutions: Compressing the medium builds up pressure, causing the medium to re-expand:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} \tag{G.370}$$

In order to construct a determined system of differential equations one would need to specify a relation between pressure p and density ρ, i.e. an equation of state, which accompanies the equation of continuity. Then, there are three relations (Euler, continuity and equation of state) for three fields ρ, $\boldsymbol{v}$ and p. For collisionless dark matter, though, pressure would not exist.

Wave-like phenomena are, because they fulfil the superposition principle, obtained as solutions to the linearised Navier-Stokes equation. Linearisations involve perturbing the dynamical fields away from their averages $\rho = \rho_0 + \delta\rho$, $p = p_0 + \delta p$ and $\boldsymbol{v} = \delta\boldsymbol{v}$. Therefore,

$$\partial_t \delta\rho + \rho_0 \mathrm{div}\,\delta\boldsymbol{v} = 0 \quad \text{as well as} \quad \partial_t \delta\boldsymbol{v} + \frac{1}{\rho_0}\nabla\delta p = 0 \tag{G.371}$$

Taking the time-derivative of the continuity equation and the divergence of the Navier-Stokes euqation defines the wave equation

$$\partial_t^2 \delta\rho - \underbrace{\left.\frac{\partial p}{\partial\rho}\right|_{\rho_0}}_{=c_s^2} \Delta\delta\rho = 0 \tag{G.372}$$

if one introduces an equation of state

$$\delta p = \left.\frac{\partial p}{\partial\rho}\right|_{\rho_0} \delta\rho \tag{G.373}$$

The derivative $c_s^2 = \partial p/\partial\rho$ defines sound speed inside the medium and depends typically on the thermodynamic change of state, e.g. isothermal and adiabatic.

Combining both gravity and pressure leads to an interesting concept: the Jeans-scale. If a system of size R and density ρ collapses under its own gravity, we can associate a free-fall time scale with the collapse, estimated to be

$$\tau_{ff} = \frac{1}{\sqrt{G\rho}} \tag{G.374}$$

and it can provide pressure support on the time scale of the sound-crossing time

$$\tau_s = \frac{R}{c_s} \tag{G.375}$$

Now, comparison between the two time scales suggests that if $\tau_{ff} \ll \tau_s$, the system collapses as pressure support can not be established fast enough, and if $\tau_{ff} \ll \tau_s$, the system is stabilised by pressure against gravity. Re-expressing the time scale as a length scale lets us define the Jeans-length $R_J = c_s\tau_{ff}$, and the associated Jeans-mass

$$M_J = \frac{4\pi}{3}\rho R_J^3 = \frac{4\pi}{3}\frac{c_s^3}{\sqrt{G^3\rho}} \tag{G.376}$$

In systems with masses exceeding M_J defined for a given c_s and ρ gravity is dominant over pressure and the system collapses, vice versa, in low-mass systems below M_J, pressure is able to provide support against gravity. Again, these concepts are irrelevant for systems consisting of dark matter only, due to its collisionlessness and the absence of pressure terms from the fluid mechanical equations.

G.9 *Vorticity equation*

The vorticity tensor is the antisymmetric part of velocity tensor $\partial \upsilon_j / \partial x_i$

$$\omega_{jk} = \frac{1}{2}(\partial_j \upsilon_k - \partial_k \upsilon_j) \tag{G.377}$$

and the vorticity-vector ω_j can be written as

$$\omega^i = \epsilon^{ijk}\partial_j \upsilon_k = \epsilon^{ijk}\omega_{jk} \tag{G.378}$$

or as $\boldsymbol{\omega} = \mathrm{rot}\boldsymbol{\upsilon}$.

The vorticity evolution can be deduced from the Navier-Stokes equation

$$\partial_t \boldsymbol{\upsilon} + (\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon} = -\frac{\nabla p}{\rho} - \nabla\Phi + \mu\Delta\boldsymbol{\upsilon} \tag{G.379}$$

by application of the operation rot to the equation and by using

$$(\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon} = \nabla\frac{\boldsymbol{\upsilon}^2}{2} - \boldsymbol{\upsilon} \times \underbrace{\nabla \times \boldsymbol{\upsilon}}_{=\omega} \tag{G.380}$$

arriving at

$$\partial_t \boldsymbol{\omega} - \mathrm{rot}(\boldsymbol{\upsilon} \times \boldsymbol{\omega}) = \mu\mathrm{rot}(\Delta\boldsymbol{\upsilon}) \tag{G.381}$$

For an equation of state where pressure only depends on density, $p = p(\rho)$, the pressure term assumes the shape

$$\mathrm{rot}\left(\frac{\nabla p}{\rho}\right) = \frac{\mathrm{rot}\nabla p}{\rho} - \frac{1}{\rho^2}\nabla p \times \nabla\rho = 0 \tag{G.382}$$

making use of the chain rule in $\nabla p(\rho) = \frac{\partial p}{\partial \rho}\nabla\rho$. The Leibnitz-rule applied to $\boldsymbol{\upsilon} \times \boldsymbol{\omega}$ suggests

$$\mathrm{rot}(\boldsymbol{\upsilon} \times \boldsymbol{\omega}) = (\boldsymbol{\omega}\nabla)\boldsymbol{\upsilon} - (\boldsymbol{\upsilon}\nabla)\boldsymbol{\omega} + \boldsymbol{\omega}\underbrace{\mathrm{div}\ \boldsymbol{\upsilon}}_{=0} + \boldsymbol{\upsilon}\underbrace{\mathrm{div}\ \boldsymbol{\omega}}_{=0} \tag{G.383}$$

for incompressible fluids where $\mathrm{div}\boldsymbol{\upsilon} = \partial_i \upsilon^i = 0$, and because $\mathrm{div}\boldsymbol{\omega} = \epsilon_{ijk}\partial^i\partial^j \upsilon^k = 0$ always. Then, making use of

$$\mathrm{rot}(\Delta\boldsymbol{\upsilon}) = \mathrm{rot}(\nabla\underbrace{\mathrm{div}\boldsymbol{\upsilon}}_{=0} - \mathrm{rot}\ \mathrm{rot}\boldsymbol{\upsilon}) = -\mathrm{rot}\ \mathrm{rot}\ \mathrm{rot}\boldsymbol{\upsilon} = \mathrm{rot}\ \mathrm{rot}\boldsymbol{\omega} = \Delta\boldsymbol{\omega} - \nabla\underbrace{\mathrm{div}\ \boldsymbol{\omega}}_{=0} = \Delta\boldsymbol{\omega} \tag{G.384}$$

one arrives at a relation featuring again an advective derivative

$$\partial_t \boldsymbol{\omega} + (\boldsymbol{\upsilon} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{\upsilon} + \mu\Delta\boldsymbol{\omega} \tag{G.385}$$

Combining all results gives the vorticity equation, as a dynamical equation for the vorticity field $\boldsymbol{\omega}$:

$$\partial_t\omega + (\boldsymbol{v}\nabla)\omega = -\omega\mathrm{div}\boldsymbol{v} + \frac{\nabla\rho\times\nabla p}{\rho^2} + \mu\Delta\omega \tag{G.386}$$

which has the form of a convection-diffusion equation. The vorticity equation has a convective derivative of the form $\partial_t\boldsymbol{\omega} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega}$, implying that the vorticity is advected in its own velocity field which is given by inverting the definition $\boldsymbol{\omega} = \mathrm{rot}\boldsymbol{v}$ by means of the law of Biot-Savart,

$$\boldsymbol{v}(\boldsymbol{r}) = \int \mathrm{d}^3r'\ \omega(\boldsymbol{r}')\times\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{G.387}$$

illustrating that the vorticity field needs to be known in the entire volume for converting back to the velocity field, as an expression of the nonlocal properties of this term. Secondly, the sourcing of the vorticity field can take place through the baroclinic term $\nabla p\times\nabla\rho$, if the density gradient and the pressure gradient are not parallel. Gravity alone is not able to source vorticity because as a scalar field, it can not decide about the orientation of the vorticity vector: $\mathrm{rot}\nabla\Phi = 0$, which immediately suggests the question why spiral galaxies should be rotating, if their dynamics is dominated by gravity. Lastly, the term $\mu\Delta\boldsymbol{v}$ causes in conjunction with the term $\partial_t\omega$ a diffusion of vorticity with the viscosity μ as the diffusion coefficient.

G.10 *Effective processes in collisionless systems*

Even though dark matter does not show elastic collisions between the particles and even though there is no microscopic origin of pressure and viscosity, there can be collective processes of groups of dark matter particles, emulating pressure and viscosity. After all, we observe that dark matter dominated objects are stable against their own gravity, due to the random motion of the particles, which acts as an effective pressure term in a hdyrostatic equilibrium. Similarly, we observe how systems like galaxies slow down if they enter a high density environment, by a process called dynamical friction.

H COSMIC STRUCTURE FORMATION

H.1 *Structure formation equations*

Structure formation with cold dark matter is driven by self-gravity of cosmic structures that have been seeded by cosmic inflation as inhomogeneities in the density field. At the highest degree of simplification, the dark matter density is subjected to fluid mechanics but without effects of pressure and viscosity (as they would derive from the microscopic interactions between the particles). While the background on which structure formation takes place, is a dynamics spacetime conforming to the FLRW-symmetries, structure formation is well captured in the Newtonian limit, with both Newtonian gravity in the form of a potential Φ, $|\Phi| \ll c^2$ and with non-relativistic velocities $|\upsilon| \ll c$ in the comoving frame.

The formation of cosmic structure is a phenomenon that only involves weak, Newtonian gravitational fields, slowly moving matter and scales much smaller than the Hubble scale. Therefore, we are going to use a Newtonian description of gravity on the relativistic FLRW-background, a nonrelativistic equation of motion and neglect retardation effects due to the finite propagation speed of the gravitational field as well as gravitative effects on moving objects such as gravitomagnetic forces.

As coordinates, we use the conformal time η and comoving coordinates x^i as those coordinates are particularly suited for FLRW-spacetimes, implying that the rate of change of physical coordinate $r = ax$ with physical time gives rise to two contributions in velocity:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \dot{a}x + a\dot{x} = a\mathrm{H}x + a\dot{x} = a\mathrm{H}x + \upsilon \tag{H.388}$$

with the peculiar velocity υ relative to the Hubble flow. Clearly, both terms would contribute to a measurement of redshift. The peculiar velocity υ would likewise be the rate of change of comoving coordinate with conformal time,

$$\upsilon = a\dot{x} = a \underbrace{\frac{\mathrm{d}\eta}{\mathrm{d}t}}_{=1/a} \frac{\mathrm{d}x}{\mathrm{d}\eta} = \frac{\mathrm{d}x}{\mathrm{d}\eta} \tag{H.389}$$

Comoving coordinates have the advantage that the advection of matter due to the Hubble-expansion is absorbed by the coordinates, and we only need to consider relative motion of particles with respect to the comoving coordinate frame. Being a hydrodynamical self-gravitating phenomenon, structure formation is described in the this comoving frame by the system of differential equations composed of (i) the continuity equation

$$\frac{\partial}{\partial\eta}\delta + \mathrm{div}\left[(1+\delta)\boldsymbol{\upsilon}\right] = 0, \tag{H.390}$$

which relates the time-evolution of the density field to the divergence of the matter fluxes $\boldsymbol{\jmath} = (1+\delta)\boldsymbol{\upsilon}$, (ii) the Euler-equation

$$\frac{\partial}{\partial\eta}\boldsymbol{\upsilon} + a\mathrm{H}\boldsymbol{\upsilon} + (\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon} = -\nabla\Phi, \tag{H.391}$$

which describes the evolution of the peculiar velocity field $\boldsymbol{\upsilon}$ from the gradient $\nabla\Phi$

of the peculiar gravitational potential Φ, acting on a fluid element, and finally (*iii*) the comoving Poisson-equation

$$\Delta\Phi = \frac{3}{2}\Omega_m(\eta)\,(a\mathrm{H})^2\delta = \frac{3\mathrm{H}_0^2\Omega_m}{2a}\delta, \tag{H.392}$$

which gives the gravitational potential Φ induced by the matter distribution δ (Newton's constant has been replaced with the definition of the critical density, $\rho_{\mathrm{crit}} = 3\mathrm{H}_0^2/(8\pi\mathrm{G})$ and the density parameter $\Omega_m = \bar{\rho}/\rho_{\mathrm{crit}}$. In the last step, we used the adiabatic relation

$$\frac{\Omega_m(a)}{\Omega_m} = \frac{\mathrm{H}_0^2}{a^{3(1+w)}\mathrm{H}(a)^2} \tag{H.393}$$

while setting $w = 0$ for nonrelativistic matter.

The three equations are sufficient to describe the dynamics of the three relevant fields δ, $\boldsymbol{v}$ and Φ, because there are no dissipative and pressure forces due to the collisionlessness of dark matter, and it is not necessary to track the energy balance or to introduce and an equation of state parametrising the pressure-density relation.

H.2 *Linearised equations on an expanding background*

Linearisation of the structure formation equations by substituting a perturbative expansion and neglecting all terms involving products of two or more fields. This methods yields the linearised continuity equation,

$$\frac{\partial}{\partial\eta}\delta + \mathrm{div}\boldsymbol{v} = 0, \tag{H.394}$$

and the linearised Euler-equation,

$$\frac{\partial}{\partial\eta}\boldsymbol{v} + a\mathrm{H}\boldsymbol{v} = -\nabla\Phi, \tag{H.395}$$

which are valid as long as the deviation from the mean density is small, $|\delta| \ll 1$. The Newtonian Poisson-equation is always linear, or the superposition principle of classical gravity would not apply.

The three linearised relationships between δ, $\boldsymbol{v}$ and Φ can be combined into the growth-equation: By taking the divergence of the Euler-equation and the time-derivative of the continuity-equation one can eliminate $\partial\mathrm{div}\boldsymbol{v}/\partial\eta$ and re-substitute the continuity equation to obtain an expression

$$\frac{\partial^2}{\partial\eta^2}\delta + a\mathrm{H}\frac{\partial}{\partial\eta}\delta = \Delta\Phi \tag{H.396}$$

where, after substitution of the Poisson-equation for $\Delta\Phi$ all spatial derivatives have vanished. This implies that structure growth in the linear regime is homogeneous and can not depend on position. It merely scales all amplitudes in the density field with a factor that only depends on time, $\delta(\mathbf{x},\eta) = \mathrm{D}_+(a)\delta(\mathbf{x},\eta=0)$, and this factor is commonly referred to as the growth function $\mathrm{D}_+(a)$.

One can continue to replace the time derivatives with respect to conformal time η by derivatives with respect to the scale factor a to obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2}D_+(a) + \frac{1}{a}\left(3 + \frac{\mathrm{d}\ln H}{\mathrm{d}\ln a}\right)\frac{\mathrm{d}}{\mathrm{d}a}D_+(a) = \frac{3}{2a^2}\Omega_m(a)D_+(a). \tag{H.397}$$

where the dependence on the background cosmology is clearer, and reflects the change of the Hubble function, i.e. acceleration or deceleration in the Hubble rate $H(a)$ as well as the change of the background matter density with time. The homogeneity of the growth is the reason why e.g. inflationary models of structure formation can be investigated by observations of the statistical properties of the large-scale structure today: Even though inflation takes place at incredibly high redshifts of $z \simeq 10^{30}$, the cosmic structure is conserving the density field perfectly as long as it is linearly evolving.

Homogeneous structure formation corresponds to independently growing Fourier modes,

$$\delta(\mathbf{x}, a) = D_+(a)\delta(\mathbf{x}, a = 1) \longrightarrow \delta(\boldsymbol{k}, a) = D_+(a)\delta(\boldsymbol{k}, a = 1), \tag{H.398}$$

which conserves every statistical property of the initial conditions, in particular Gaussianity. The Gaussianity of the initial density perturbations is a consequence of inflation, where a large number of uncorrelated quantum fluctuations are superimposed, yielding a Gaussian amplitude distribution due to the central limit theorem. In fact, homogeneous growth in the linear regime is the reason why investigation of inflationary processes in structure is possible by observing the large-scale structure today, even after the cosmic time $1/H_0$ has passed.

A convenient way for approximating the growth function is the γ-parameter, introduced by in the study of peculiar velocities:

$$\frac{\mathrm{d}\ln D_+}{\mathrm{d}\ln a} \simeq \Omega_m(a)^\gamma, \tag{H.399}$$

with $\gamma \simeq 0.6$ in ΛCDM. Solving this equation for the growth function yields

$$D_+(a) = \exp\left(\int_0^a \mathrm{d}\ln a\, \Omega_m(a)^\gamma\right). \tag{H.400}$$

In dynamic dark energy models, γ can be approximated by $\gamma \simeq 0.55 + 0.05(1 + w(z = 1))$ with the dark energy equation of state parameter taken at unit redshift. The effect of adding a fluid with a negative equation of state is a slower growth in the recent cosmic past and a faster growth in the remote past (if the growth function is normalised to unity today). Solutions for $D_+(a)$ for different dark energy cosmologies are compared in Fig. 10.

H.3 *Peculiar velocity field*

Matter streams in the large-scale structure drive structure formation: If they converge, they transport matter into a volume and increase the local density, according to the continuity equation. In order to investigate the properties of the velocity field one can carry out a Helmholtz-decomposition into its curl and gradient components $\theta = \mathrm{div}\boldsymbol{\upsilon}$ and $\omega = \mathrm{rot}\boldsymbol{\upsilon}$. From the Euler-equation one obtains and evolution equation for the divergence of the matter fluxes ,

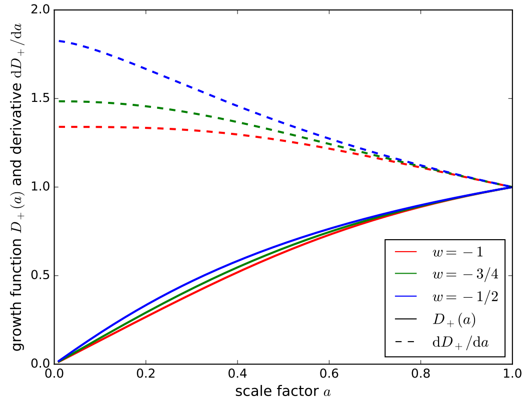


Figure 10: Growth functions $D_+(a)$ *for different dark energy cosmologies, as well as the derivative* dD_+/da

$$\frac{\partial}{\partial\eta}\theta + aH\theta + \frac{3H_0^2\Omega_m}{2a}\delta = 0 \tag{H.401}$$

and the corresponding equation for the vorticity ω,

$$\frac{\partial}{\partial\eta}\omega + aH\omega = 0. \tag{H.402}$$

With the definition of the differential of the conformal time, $da = a^2 H d\eta$, one immediately notices that $d\ln\omega = -d\ln\ a$, and hence $\omega \propto 1/a$ in the matter dominated phase: Vorticity can not be generated in linear structure formation in collisionless fluids, and the flows are necessarily laminar. The divergence θ can be linked to the evolution of the density field using the continuity equation,

$$\theta = -aH\frac{d\ln D_+}{d\ln a}\delta, \tag{H.403}$$

which underlines the fact that in the linear regime of structure formation, the velocity field is the gradient of a potential. At the same time, eqn. H.403 suggests that a natural scale for the velocity divergence is the comoving Hubble-rate aH.

H.4 *Linear structure formation*

The linear growth equation is given by

$$\frac{d^2D}{da^2} + \frac{1}{a}\left(3 + \frac{d\ln H}{d\ln a}\right)\frac{dD}{da} = \frac{3}{2a^2}\Omega_m(a)D(a) = 0. \tag{H.404}$$

Therefore, linear cosmic structure formation is governed by magnitude and time evolution of two terms: the density of matter as given by $\Omega_m(a)$ and the term $3 + d\ln H/d\ln a$ describing a change in the expansion rate. This latter term is sometimes referred to as Hubble-drag, but although the interpretation as a drag term is formally correct it does not represent the physical picture correctly. In particular it would be wrong to formulate a time-scale for Hubble expansion $1/H(t)$ and compare it to a time scale $t = 1/\sqrt{G\rho}$ because the structure in the overdensity field δ are invariant in shape and amplitude under Hubble-expansion as both densities $\rho(\boldsymbol{x}, a)$ and $\bar{\rho}$ scale identically $\propto a^{-3}$. The relevant physical mechanism is an acceleration of matter relative to the Hubble expansion and a change in the expansion velocity, i.e. an acceleration or deceleration in the cosmological model. This is apparent when writing the growth equation with e.g. the scale factor a as an evolution parameter. In this case, the Hubble-drag term reflects a derivative of the Hubble-expansion with a, and the term $3 + \mathrm{d}\ln H/\mathrm{d}\ln a$ is in fact equal to $2 - q$, with the deceleration parameter $q = -\ddot{a}a/\dot{a}^2$.

Linear structure formation is scale invariant, at a rate determined purely by the FLRW-cosmology through q and H, which determines the evolution of Ω_m and hence of the strength of gravitational fields through the relation

$$\frac{\Omega_m(a)}{\Omega_m} = \frac{H_0^2}{a^3 H(a)^2} \tag{H.405}$$

as a consequence of the continuity equation for normal matter with $w = 0$, which itself is a consequence of conserved 4-momentum $\nabla_\mu T^{\mu\nu} = 0$. As such, it allows the investigation of the the cosmological model through the Hubble function and its derivative if measurements of the amplitude of structures as a function of scale factor or redshift are available. Redshift information is crucial because the same amplitude of cosmic structures is reached in different cosmologies at different times, and this information would be impossible to disentangle without redshift information.

The influence of the two terms $3 + \mathrm{d}\ln H/\mathrm{d}\ln a$ and $2 - q$ on the growth equation are straightforward to understand in the context of standard cosmologies with two relevant fluids, with dark matter dominating at early and dark energy dominating at late times. In these cosmologies the universe makes a transition from deceleration to acceleration, which is reflected by the growth rate $D(a)$. During matter domination, the Hubble function scales $H \propto a^{-3/2}$ which transitions in the course of cosmic evolution to dark energy domination, where in the extreme case of a cosmological constant $H = \text{const}$. The derivative $3 + \mathrm{d}\ln H/\mathrm{d}\ln a$ would change from 3/2 at early times to 3 at late times, therefore slowing down structure formation. A similar behaviour is found in the matter density, which starts at the value $\Omega_m = 1$ in matter domination and drops to 0 when the dark energy component dominates. In summary, there are now two reasons why structure formation stops at late times under the influence of a cosmological constant: The driving term involving Ω_m, which originates from the Poisson-equation, becomes very small and the damping term $3+\mathrm{d}\ln H/\mathrm{d}\ln a$ assumes the largest possible value.

There are certain cosmologies, where the growth equation has particularly simple solutions. For instance, in a critical FLRW-universe with a constant $\Omega_m = 1$ requires $D(a) = a$. By substitution into the comoving Poisson-equation one immediately sees that the Newtonian potentials Φ scale with D_+/a and are in this particular cosmological model constant in linear structure formation.

Therefore, structures grow proportional to the scale factor. For a general cosmology one can at least infer the asymptotic behaviour by making a power law ansatz for D as a function of scale factor at early times, $D \propto a^\alpha$ and consider solution to the resulting quadratic equation in α, while the exact solution for an arbitrary cosmology defined in terms of $H(a)$ or $q(a)$, or, in terms of the density parameters and their equations of state, is only possible numerically. It is sufficient to formulate the ansatz as a proportionality $D \propto a^\alpha$ because the growth equation is a linear differential equation. Physically, this means that structure growth continues irrespective of the amplitudes of the density field.

To begin, we consider the entire linear growth equation again in the $\Omega_m = 1$-cosmology, which yields as a characteristic polynomial $\alpha^2 + \alpha/2 - 3/2 = 0$, which is solved by $\alpha_+ = 1$ and $\alpha_- = -3/2$: The growth is proportional to the scale factor, as already found by direct substitution, $D_+(a) \propto a$, with a secondary solution $D_-(a) \propto a^{-3/2}$: Due to the fact that the growth equation is of second order in a one expects two solution branches, which need to be combined by linear combination with suitable coefficients such that the boundary condition $D(a) = 1$ at $a = 1$ is met. Usually one neglects the branch $D_-(a)$ because it decreases rapidly.

In addition, it is possible to illustrate the behaviour of the growth equation of individual terms are set to zero and are therefore disfunctional. For instance, the growth in a cosmology with an arbitrary but constant deceleration parameter q, but where gravity in structure formation has been switched off leads with the same ansatz $D(a) \propto a^\alpha$ to a characteristic polynomial $\alpha(\alpha + 1 - q) = 0$ with the two solutions $D_+ = \mathrm{const}$ for $\alpha = 0$ and $D_- \propto a^{q-1}$. Taking this to extremes, the dark energy dominated universe with $q = -1$ and $\Omega_m = 0$ has $\alpha(\alpha + 2) = 0$, implying a constant growing mode $D_+ = \mathrm{const}$ and a fast decaying mode $D_+ \propto a^{-2}$: structure growth is frozen and the amplitudes reached at the point of dark energy domination are conserved from that point on.

Conversely, in an artificial inconsistent universe with a constant expansion rate (vanishing deceleration $q = 0$) and gravitational fields generated by the large-scale structure with $\Omega_m = 1$ one would obtain $\alpha + \alpha - 3/2 = 0$, with the solutions $\alpha = (-1 \pm \sqrt{7})/2$ with a growing $\alpha > 1$ and a decaying solution $\alpha < 1$. Clearly, this is the prototype solution to the differential equation, where the two solutions are modified in any consistent cosmology relative to their actual deceleration and matter density, including their evolution.

H.5 *Nonlinear structure formation*

As long the structure formation is linear, the growth is homogeneous and conserves the Gaussianity of the initial conditions. Nonlinear structure formation implies inhomogeneous growth and the emergence of non-Gaussian features, which is illustrated by a number of arguments: Non-linearity implies inhomogeneity, because if e.g. a void reaches underdensities close to $\delta \simeq -1$ (corresponding to $\rho \simeq 0$), the linearisation fails and the growth has to slow down locally. Inhomogeneity implies non-Gaussianity because the initially Gaussian distribution $p(\delta)\mathrm{d}\delta$ becomes wider with increasing amplitudes δ, but the density δ can not be more negative than -1, requiring the amplitude distribution $p(\delta)\mathrm{d}\delta$ to become asymmetric and to acquire a nonzero skewness. For completing the argument one immediately notices that in inhomogeneous growth, i.e. a position dependence of the growth rate $D_+(\boldsymbol{x}, a)$, the Fourier-modes $\delta(\boldsymbol{k}, a)$ become coupled, violating the central limit theorem such that the superposition of Fourier-modes yields a non-Gaussian amplitude distribution.

- linearity $\leftrightarrow$ homogeneity
    - There are no spatial derivatives in the growth equation, and therefore, the growth must be homogeneous $\delta(\boldsymbol{x}, a) = D_+(a)\delta(\boldsymbol{x})$. Only nonlinear terms would bring in spatial derivatives and make the growth position dependent.

  - If the density field is close to $\delta = -1$ somewhere, the growth needs to slow down locally, which leads to different structure formation rates at different positions which eventually breaks homogeneity.

- linearity $\leftrightarrow$ Gaussianity

  - Linear growth introduces a scaling with a function D_+ which itself is a linear transform and therefore preserves statistical properties.
  - Again, if δ approaches -1, the initially Gaussian distribution starts to become asymmetric, as it generates potentially very large positive values for δ but has to be zero for $\delta < -1$.

- homogeneity $\leftrightarrow$ Gaussianity

  - Homogeneous growth $\delta(\boldsymbol{x}, a) = D_+(a)\delta(\boldsymbol{x})$ implies independent growth of all Fourier-modes $\delta(\boldsymbol{k}, a) = D_+(a)\delta(\boldsymbol{k})$, as the Fourier-transform is linear. If a large amount of statistically independent Fourier-modes is superimposed (by inverse Fourier-transform), the resulting δ is a Gaussian distribution.
  - Inhomogeneous growth $\delta(\boldsymbol{x}, a) = D_+(\boldsymbol{x}, a)\delta(\boldsymbol{x})$ results in a convolution in Fourier-space

$$\delta(\boldsymbol{k}, a) = \int \frac{d^3k'}{(2\pi)^3} D_+(\boldsymbol{k} - \boldsymbol{k}', a)\delta(\boldsymbol{k}') \tag{H.406}$$

    with a position-dependent growth rate, which breaks the statistical independence by coupling Fourier-modes. Then, the resulting distribution can not be Gaussian anymore.

H.6 *Eulerian perturbation theory*

The non-linearities in the continuity and Euler-equation make a closed analytical solution impossible. It is possible, however, to obtain a perturbative solution to the structure formation equations, which contains the mode coupling mechanism and describes the generation of non-Gaussianities in nonlinear structure formation. The non-linearities in the continuity- and the Euler-equation translate to convolutions of the density and the velocity fields in Fourier space which couple the individual Fourier modes, violating the central limit theorem and therefore violating Gaussianity. It is worth noting that in the perturbative expansion each field $\delta^{(n)}$ grows homogeneously at the rate $D_+^n(a)$, but the sum does not.

Applying a perturbative solution means to write out perturbation series for δ and Θ in terms of powers of the linear solutions

$$\delta(\boldsymbol{x}, t) = \sum_n \delta^{(n)}(\boldsymbol{x}, t) \quad \text{and} \quad \Theta(\boldsymbol{x}, t) = \sum_n \Theta^{(n)}(\boldsymbol{x}, t) \quad \text{where} \quad \Theta = \frac{\mathrm{div}\boldsymbol{\upsilon}}{aH} \tag{H.407}$$

and substituting them into the fully nonlinear equations:

$$\partial_\tau \delta + \mathrm{div}((1+\delta)\boldsymbol{\upsilon}) = 0 \quad \text{and} \quad \partial_\tau \boldsymbol{\upsilon} + aH\boldsymbol{\upsilon} + (\boldsymbol{\upsilon}\nabla)\boldsymbol{\upsilon} = -\nabla\Phi \tag{H.408}$$

where the comoving divergence is computed for the second equation. Differential equations become algebraic in Fourier space, therefore continuity reads

$$\partial_\tau\delta(\boldsymbol{k}) + \Theta(\boldsymbol{k}) = -\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\Theta(\boldsymbol{k}_1)\delta(k_2)\delta_D(\boldsymbol{k}-\boldsymbol{k}_{12})\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2) \tag{H.409}$$

and similarly, the Euler-equation becomes

$$\partial_\tau\Theta(\boldsymbol{k})+aH\Theta(\boldsymbol{k})+\frac{3}{2}\Omega_m(aH)^2\delta(\boldsymbol{k}) = -\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\Theta(\boldsymbol{k}_1)\Theta(\boldsymbol{k}_2)\delta_D(\boldsymbol{k}-\boldsymbol{k}_{12})\beta(\boldsymbol{k}_1,\boldsymbol{k}_2) \tag{H.410}$$

keeping in mind that products in real space become convolutions in Fourier-space, here expressed by introducing the Dirac-δ_D function. The derivatives are expressed with

$$\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{1}{k_1^2}\boldsymbol{k}_{12}\boldsymbol{k}_1 \tag{H.411}$$

as well as

$$\beta(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{1}{2k_1^2k_2^2}\boldsymbol{k}_{12}\boldsymbol{k}_1\boldsymbol{k}_2 \tag{H.412}$$

with the abbreviation $\boldsymbol{k}_{12} = \boldsymbol{k}_1 + \boldsymbol{k}_2$. Substitution of the perturbation series yields a recursive relation

$$\delta_n(\boldsymbol{k}) = \int d^3q_1\ldots\int d^3q_n\delta_D(\boldsymbol{k}-\boldsymbol{q}_{1\ldots n})F_n(\boldsymbol{q}_1\ldots\boldsymbol{q}_n)\delta_1(\boldsymbol{q}_1)\ldots\delta_n(\boldsymbol{q}_n) \tag{H.413}$$

for the density field, as well as

$$\Theta_n(\boldsymbol{k}) = \int d^3q_1\ldots\int d^3q_n\delta_D(\boldsymbol{k}-\boldsymbol{q}_{1\ldots n})G_n(\boldsymbol{q}_1\ldots\boldsymbol{q}_n)\delta_1(\boldsymbol{q}_1)\ldots\delta_n(\boldsymbol{q}_n) \tag{H.414}$$

for the velocity divergence. Here, F_n is a function of $F_n(F_{n-1}, G_{n-1})$ and the same for G_n, all defined inductively starting at $F_1 = G_1 = 1$.

The lowest order symmetrised solutions for F_n are $F_1 = 1$ and

$$F_2(\boldsymbol{q}_1,\boldsymbol{q}_2) = \frac{5}{7}+\frac{\mu}{2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right)+\frac{2}{7}\mu^2 \quad \text{with} \quad \mu = \frac{\boldsymbol{q}_1\cdot\boldsymbol{q}_2}{q_1q_2} \tag{H.415}$$

being the cosine of the angle between $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Assuming $q_1 = q_2$ for simplicity, the mode coupling function F_2 attains the largest value of $F_2 = 2$ if the wave vectors are parallel ($\mu = +1$), an intermediate value of $F_2 = 5/7$ if $\boldsymbol{q}_1 \perp \boldsymbol{q}_2$ ($\mu = 0$) and the smallest value of $F_2 = 0$ if the the wave vectors are antiparallel ($\mu = -1$). Varying the wave numbers at fixed separation angle μ shows that F_2 is smallest if $q_1 = q_2$, and that the mode coupling increases if the wave numbers are chosen differently. From this point of view, mode-coupling bears resemblance to a resonance phenomenon, where modes with identical direction of propagation experience the strongest coupling. The perturbative solution to the system of equations eqns. (H.390) and (H.391) in terms of a perturbation series in δ and υ is possible due to their renormalisation properties, which hold exactly in the case of SCDM ($\Omega_m = 1$, $\Omega_\varphi = 0$) and approximately for dark energy cosmologies. In these cosmologies, the mode coupling kernels themselves acquire a slow time dependence.

In application to statistics, any correlation function of nonlinear fields can reduced to a higher-order correlation function of the linearly evolving fields, which obey Gaussian statistics, integrated over momentum space with the mode coupling function as a weighting function. While odd n-point correlation functions of Gaussian random fields are equal to zero, even n-point functions can be decomposed into products of two-point functions by virtue of the Wick-theorem,

$$\langle \delta(\boldsymbol{k}_1)\ldots\delta(\boldsymbol{k}_n)\rangle = \sum_{\mathrm{pairs}} \prod_{i,j\in\mathrm{pairs}} \langle \delta(\boldsymbol{k}_i)\delta(\boldsymbol{k}_j)\rangle \tag{H.416}$$

for which a proof can be found in e.g. and which constitutes an extension of the well-known relation $\langle\delta^{2n}\rangle = (2n-1)!!\langle\delta^2\rangle^n$ for the higher moments of a Gaussian random variable δ with $\langle\delta\rangle = 0$.

H.7 *Dark matter in astrophysical systems*

With the idea, that all forms of matter, including dark matter, are effected in the same way by gravity as commanded by the equivalence principle of general relativity one would conclude in a range of astrophysical system that the strength of gravitational field can not be explained by luminous matter alone.

H.7.1 Rotation curves of galaxies

Setting up circular orbits for stars in a galactic disk in the gravitational potential of a galaxies would use the condition

$$\frac{\upsilon^2}{r} = \frac{\mathrm{d}\Phi}{\mathrm{d}r} \quad\rightarrow\quad \upsilon^2 = r\frac{\mathrm{d}\Phi}{\mathrm{d}r} \tag{H.417}$$

The gravitational potential Φ would result from solving the Poisson-equation for the total matter density ρ

$$\Delta\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho \tag{H.418}$$

With a matter profile $\rho \propto 1/r^2$ one would obtain, after multiplying with r^2, integrating and multiplying with $\frac{1}{r^2}$ the result

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = 4\pi G\frac{1}{r^2}\int \mathrm{d}r\,\frac{1}{r^2}r^2 = \frac{4\pi G}{r} = \frac{\upsilon^2}{r} \tag{H.419}$$

implying that the rotational velocity υ does not depend on radius r anymore, suggesting the idea that the galactic disc is embedded into a much larger dark matter halo with density $\rho \propto 1/r^2$, which sources the gravitational potential, and which naturally reproduces the observed flat rotation curves.

There are theories that modify dynamical laws in the regimes of really small accelerations which can reproduce galaxy rotation curves even if the gravitational potential is sourced by the visible matter only. The scale at which these MO(dified) N(ewtonian) D(ynamics) theories change the equations of motion is for accelerations close to $a_0 \simeq 10^{-10}\mathrm{m/s}^2$, which corresponds to the acceleration experienced by the Solar System on its orbit around the Milky Way centre. An example of a rotation curve in a low surface-brightness galaxy is provided by Fig. 11.

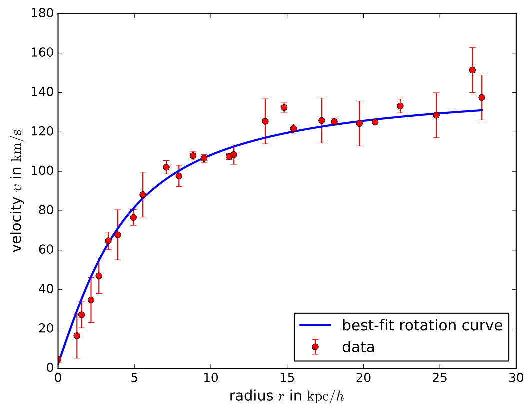


Figure 11: Rotation curve of the galaxy U11616 with an fit to the rotational velocity as a function of radius

H.7.2 Virial equilibria of clusters of galaxies

On the scale of galaxy clusters one again notices similar mismatch: The velocities of the galaxies are too large to be compatible with the gravitational potential if only visible matter should contribute to it. From the positions $\boldsymbol{q}_i$ and the momenta $\boldsymbol{p}_i$ of all galaxies in a cluster one defines the virial G,

$$G = \sum_i \boldsymbol{p}_i \boldsymbol{q}_i \tag{H.420}$$

with the time derivative

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t}\boldsymbol{q}_i + \boldsymbol{p}_i \frac{\mathrm{d}\boldsymbol{q}_i}{\mathrm{d}t} = \sum_i \mathbf{F}_i \boldsymbol{q}_i + m \sum_i \dot{\boldsymbol{q}}_i \dot{\boldsymbol{q}}_i \tag{H.421}$$

where Newton's equation of motion $\mathrm{d}\boldsymbol{p}_i/\mathrm{d}t = \mathbf{F}_i$ and the definition of momentum $\boldsymbol{p}_i = m\dot{\boldsymbol{q}}_i$ was substituted. Particularly in systems with potentials of power-law shape allow a very compact statement: If Φ is the mutual interaction potential of the particle j onto particle i

$$\Phi(\boldsymbol{q}_i, \boldsymbol{q}_j) \propto |\boldsymbol{q}_i - \boldsymbol{q}_j|^n \tag{H.422}$$

one can find

$$\sum_i \mathbf{F}_i \boldsymbol{q}_i = -\frac{1}{2}\sum_i \sum_j \frac{\mathrm{d}\Phi(\boldsymbol{q}_i, \boldsymbol{q}_j)}{\mathrm{d}q_{ij}} \frac{\left|\boldsymbol{q}_i - \boldsymbol{q}_j\right|^2}{q_{ij}} = -\frac{1}{2}\sum_i \sum_j \frac{\mathrm{d}\Phi(\boldsymbol{q}_i, \boldsymbol{q}_j)}{\mathrm{d}q_{ij}} q_{ij} \tag{H.423}$$

and in particular for homogeneous potentials of order n that

$$\frac{1}{2}\sum_i \sum_j \frac{\mathrm{d}\Phi(\boldsymbol{q}_i, \boldsymbol{q}_j)}{\mathrm{d}q_{ij}} q_{ij} = \frac{n}{2}\sum_i \sum_j \Phi(\boldsymbol{q}_i, \boldsymbol{q}_j) = \frac{n}{2} V \tag{H.424}$$

with $\mathrm{V} = \sum_i \sum_j \Phi(\boldsymbol{q}_i, \boldsymbol{q}_j)$ and $\mathrm{T} = m/2 \sum_i \boldsymbol{q}_i^2$. We can therefore conclude that

$$\frac{\mathrm{dG}}{\mathrm{d}t} = 2\mathrm{T} - n\mathrm{V} \tag{H.425}$$

The time averaging yields

$$\left\langle \frac{\mathrm{dG}}{\mathrm{d}t} \right\rangle = \frac{1}{\Delta t} \int_0^{\Delta t} \mathrm{d}t\, \frac{\mathrm{dG}}{\mathrm{d}t} \leq \frac{1}{\Delta t} |\mathrm{G}_{\max} - \mathrm{G}_{\min}| \tag{H.426}$$

if G has a finite range of values, typically realised for systems bounded in the phase space coordinates, the average vanishes in the limit $\Delta t \to \infty$, and therefore the virial theorem applies,

$$2\langle \mathrm{T} \rangle = n \langle \mathrm{V} \rangle \tag{H.427}$$

For Newtonian gravity with a Coulomb-potential we insert $n = -1$ and get

$$\langle \mathrm{T} \rangle = -\frac{1}{2} \langle \mathrm{V} \rangle \tag{H.428}$$

as well as a negative total energy

$$\langle \mathrm{T} \rangle + \langle \mathrm{V} \rangle = \langle \mathrm{T} \rangle - 2\langle \mathrm{T} \rangle = -\langle \mathrm{T} \rangle < 0 \tag{H.429}$$

indicating a bound system. $\langle \mathrm{T} \rangle$ can be measured from the velocity of the galaxies inside the cluster and the potential $\langle \mathrm{V} \rangle$ can be determined from the total mass and size, typically $\langle \mathrm{V} \rangle \sim \mathrm{M}/\mathrm{R}$. Observations, either of the peculiar velocity of galaxies and a mass estimate based on luminosity, or of X-ray temperature and luminosity, show a striking mismatch between data and theory and one would need a n of a few hundred to reconcile $\langle \mathrm{T} \rangle$ with $\langle \mathrm{V} \rangle$, which is clearly at odds with Newtonian gravitational potentials, or alternatively, that there is much more gravitating matter present in these systems compared to luminous matter.

H.7.3 Gravitational lensing

Substituting non-relativistic particles with relativistic photons for probing gravitational potentials leads to the topic of gravitational lensing. Photons travel along null-geodesics of spacetime, which would be lines with vanishing $\mathrm{d}s^2$ for instance on a Minkowski-background

$$\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right) \mathrm{d}x_i \mathrm{d}x^i = 0 \tag{H.430}$$

slightly curved by a (static) gravitational potential $|\Phi| \ll c^2$. It is sufficient to work with a perturbed Minkowski-metric instead of a FLRW-metric because of conformal flatness of the background: With a suitable choice of conformal time as a coordinate light propagation is impervious to the background dynamics and identical to that in special relativity.

The effective speed of propagation of light is the rate at which the coordinates pass as a function of time,

$$c' = \frac{\mathrm{d}|x^i|}{\mathrm{d}t} = c\sqrt{\frac{1+\frac{2\Phi}{c^2}}{1-\frac{2\Phi}{c^2}}} \simeq c\left(1-\frac{2\Phi}{c^2}\right) \tag{H.431}$$

where we used the approximation $1/(1+\epsilon) \approx 1-\epsilon$ for $|\epsilon| \ll 1$. With $c' \neq c$ it is suggestive to define a refractive index

$$n = \frac{c'}{c} \approx 1 - \frac{2\Phi}{c^2} \tag{H.432}$$

The factor of 2 in the effective propagation speed is typical for relativistic particles like photons, on which the effects of gravitational fields is stronger compared to non-relativistic particles. In fact, gravitational time dilation for non-relativistic particles is determined through the interpretation of the line element $\mathrm{d}s$ as the elapsed proper time $\mathrm{d}\tau$

$$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 \tag{H.433}$$

such that

$$\mathrm{d}\tau^2 = \left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1-\frac{2\Phi}{c^2}\right)\mathrm{d}x_i\mathrm{d}x^i \tag{H.434}$$

If the velocities are small, the displacement in the $\mathrm{d}x^i$-directions are small compared to those into the $\mathrm{d}t$-direction:

$$\mathrm{d}\tau^2 = \left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 \quad \rightarrow \quad \mathrm{d}\tau \simeq \left(1+\frac{\Phi}{c^2}\right)\mathrm{d}t \tag{H.435}$$

with the approximation $\sqrt{1+2\epsilon} \simeq 1+\epsilon$, again for $|\epsilon| \ll 1$. Comparing these two results with Fermat's principle for photons and Hamilton's principle for the motion of massive particles now shows that the effect of gravitational fields on photons is twice as large as that on non-relativistic particles.

Gravitational lensing would be described by the geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}k^\alpha = -\Gamma^\alpha_{\ \mu\nu}k^\mu k^\nu \quad \text{with} \quad k^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{H.436}$$

where the wave vector k^μ is tangent to the trajectory $x^\mu(\lambda)$ and normalised to zero. Using the invariance of geodesics under affine reparameterisation we can choose λ to yield $k^t = 1$, $k_i k^i = -1$, such that $k_\mu k^\mu = (k^t)^2 - k_i k^i = 0$.

The geodesic equation is an implicit relation: one needs to know the trajectory x^μ as the integral curve over k^μ to be able to evaluate the Christoffel-symbol $\Gamma^\alpha_{\ \mu\nu}$ at the right location: In actual numerical application it needs to be solved as a differential equation. To circumvent this, one uses the Born-approximation and assumes that the deflections are small, such that the change of the wave vector δk^α are computed relative to fixed tangents k^μ resulting from a solution of the geodesic equation for the background. Then, only the perturbations $\delta\Gamma^\alpha_{\ \mu\nu}$ determine the deflection:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta k^\alpha = -\delta\Gamma^\alpha_{\ \mu\nu}k^\mu k^\nu \tag{H.437}$$

The changes to the propagation direction can be integrated up directly

$$\delta k^\alpha = -k^\mu k^\nu \int \mathrm{d}\lambda\, \delta\Gamma^\alpha_{\mu\nu}(\lambda) \tag{H.438}$$

The perturbed Christoffel-symbols $\delta\Gamma^\alpha_{\mu\nu}$ are given by the usual relation

$$\delta\Gamma^\alpha_{\mu\nu} = \frac{\eta^{\alpha\beta}}{2}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}\right) \tag{H.439}$$

as derivatives of the weakly perturbed metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{and} \quad g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu} \simeq \eta^{\mu\nu} \tag{H.440}$$

If we assume that the perturbations correspond to Newtonian gravitational potentials, as the perturbed Christoffel-symbols contain gradients of Φ. To evaluate the geodesic equation further we can assume that the unperturbed propagation proceeds into the z-direction and that the gradients in Φ deflect the photons into the perpendicular directions:

$$\delta k^i = -k^\mu k^\nu \int \mathrm{d}\lambda\, \delta\Gamma^i_{\mu\nu} = -\int \mathrm{d}\lambda\left(\delta\Gamma^i_{tt} + 2\delta\Gamma^i_{tz} + \delta\Gamma^i_{zz}\right) \tag{H.441}$$

Inspecting the explicit expressions for the Christoffel-symbol yields

$$\delta\Gamma^i_{tt} = \frac{\eta^{i\beta}}{2}\left(\partial_t h_{\beta t} + \partial_t h_{t\beta} - \partial_\beta h_{tt}\right) = \frac{1}{2}\partial^i h_{tt} = \frac{1}{c^2}\partial^i\Phi \tag{H.442}$$

as well as

$$\delta\Gamma^i_{tz} = \frac{\eta^{i\beta}}{2}\left(\partial_t h_{\beta z} + \partial_z h_{z\beta} - \partial_\beta h_{tz}\right) = 0 \tag{H.443}$$

and

$$\delta\Gamma^i_{zz} = \frac{\eta^{i\beta}}{2}\left(\partial_z h_{\beta z} + \partial_z h_{z\beta} - \partial_\beta h_{zz}\right) = \frac{1}{2}\partial^i h_{zz} = \frac{1}{c^2}\partial^i\Phi \tag{H.444}$$

making heavy use of the diagonal form of the metric and its inverse, and ignoring derivatives along the direction of propagation z. Collecting all results gives for the gravitational light deflection angle $\hat{\alpha} = \delta k^i/k^z \simeq \delta k^i$

$$\delta k^i = -\frac{2}{c^2}\int \mathrm{d}\lambda\, \partial^i\Phi \tag{H.445}$$

In the lensing deflection, the scale of the potential Φ is set by c^2, and the factor 2 originates from the fact that photons as relativistic test particles are more sensitive to gravitational potentials than massive particles. The relevant gradients of Φ are those perpendicular to the line of sight.

H.8 *Properties of dark matter*

A number of experiments (rotation curves of galaxies, virial equilibria in galaxy clusters, gravitational lensing, amplitude of CMB temperature fluctuations) suggests the existence of non-baryonic dark matter. Dark matter is significantly more abundant than normal matter, as $\Omega_m/\Omega_b \simeq 7$, and has extreme properties. Apart from exotic models of macroscopic dark matter such as primordial black holes, many cosmologists suspect dark matter to be made up from yet undetected elementary particles, for instance by WIMPs in the TeV-range, or by ultra-light axions. The dark matter particles are required to interaction by the weak force and by gravity, and they are required to have these properties:

- Dark matter is **cold**, meaning that there is little or none thermal motion of the dark matter particles. Therefore, this kind of dark matter is non-relativistic, and as there is no thermal motion of the particles, any structures on small scales seeded by cosmic inflation is preserved: Neither diffusive motion of the dark matter particles themselves nor radiation pressure can break up small-scale structures.
- Dark matter is, well, **dark** and shows no signs of interactions through electromagnetism: There are no annihilation or decay processes of dark matter producing photons, nor are there effects of radiation pressure on dark matter particles.
- In fact, the only appreciable interaction of dark **matter** is gravitational, and its presence manifests itself in rotation curves, virial equilibria, gravitational lensing or in the amplitude of CMB-fluctuations.
- Dark matter is **collisionless**, meaning that there is only a very small cross-section for elastic collisions, as demonstrated e.g. by the bullet cluster: In this system, ob observes a merging of two clusters at high velocity, where the dark matter component as mapped out by lensing is unperturbed in the passage of the two clusters, whereas the baryonic component is not, which clearly indicates differences in the fluid mechanics of the two components: It is not possible to predict fluid properties like pressure and viscosity from the microscopic interaction of particles for dark matter.

H.9 *Spherical collapse of dark matter haloes*

The gravitational dynamics of a homogeneous sphere of matter under its own gravity is, due to its high degree of symmetry, one of the few exactly solvable systems, in Newtonian gravity as well as in general relativity. A spherically symmetric perturbation would initially follow the Hubble-expansion, but its own gravity would slow down the local expansion rate, ultimately stalling the perturbation and decoupling it from the Hubble-flow, before it collapses on itself. During the collapse one can expect that virialisation processes take place such that a stabilised bound state is reached, in which the baryonic component can cool and form stars.

In classical gravity the radius R of a spherical perturbation of mass M follows the Newtonian equation of motion

$$\ddot{R} = -\frac{GM}{R^2} \tag{H.446}$$

The instant t at which the radius stalls, $\dot{R} = 0$, defines the moment of turn-around. With a_a and R_a as scale factor and radius at turn-around, respectively, on defines the dimensionless variables $x = \frac{a}{a_a}$ and $y = \frac{R}{R_a}$.

If we assume for simplicity a flat, matter-dominated FLRW background with $\Omega_m = 1$ the Hubble function is given by

$$H = \frac{\dot{a}}{a} = H_0 a^{-3/2} \tag{H.447}$$

And, due to flatness and matter-domination, $\rho = \rho_{\mathrm{crit}}$, and we can define a dimensionless parameter

$$\tau = H_a t = H(a_a)t = H_0 a_a^{-3/2} t \tag{H.448}$$

which allows to re-express the dynamical equations for x as

$$x' = \frac{dx}{d\tau} = \frac{dt}{d\tau}\frac{dx}{dt} = \frac{1}{H_a}\dot{x} = \frac{1}{H_a}\frac{\dot{a}}{a_a} = \frac{1}{H_a}\frac{a}{a_a}\frac{\dot{a}}{a} = \frac{H}{H_a}x \tag{H.449}$$

substituting the Friedmann-equation in the last step. Similarly,

$$\ddot{R} = -\frac{GM}{R^2} = -\frac{4\pi G}{3}\rho_a R_a^3 \frac{1}{R^2} \tag{H.450}$$

with the background density $\rho_a = \frac{3H_a^2}{8\pi G}\xi$ the density contrast at turn around $\xi > 1$. In a similar way, the dynamical equation for y can be rewritten

$$y'' = -\frac{\xi}{2y^2} \tag{H.451}$$

with the natural initial conditions $y'\big|_{x=1} = 0$ and $y\big|_{x=0} = 0$. The collapse equations are solved analytically through

$$\frac{dx}{d\tau} = x^{-1/2} \quad \rightarrow \quad d\tau = x^{1/2}dx \quad \rightarrow \quad \tau = \frac{2}{3}x^{3/2} + c \tag{H.452}$$

as well as

$$y' = \pm\sqrt{\xi}\sqrt{\frac{1}{y} - 1} \tag{H.453}$$

which can be combined to

$$\tau = \frac{1}{\sqrt{\xi}}\left(\frac{1}{2}\arcsin(2y-1) - \sqrt{y - y^2} + \frac{\pi}{4}\right) \tag{H.454}$$

At turn around $x = y = 1$ one obtains $\tau = \frac{2}{3}$ leading to $\xi = \left(\frac{3\pi}{4}\right)^2$. The density inside the halo results from the ratio

$$\Delta = \left(\frac{x}{y}\right)^3 \approx 1 + \underbrace{\frac{3}{5}y}_{=\delta} \tag{H.455}$$

If we now extrapolate the density to $x = 1$ by $\delta_a = \frac{\delta}{x} = \frac{3y}{5x}$ and use

$$\frac{1}{x} = \left(\frac{3\tau}{2}\right)^{-2/3} \approx \left(\frac{3\pi}{4}\right)^{2/3}\frac{1}{y} \quad \rightarrow \quad \delta_a = \frac{3}{5}\left(\frac{3\tau}{4}\right)^{2/3} \tag{H.456}$$

we receive the time $\tau = \frac{4}{3}$ of the collapse. From this one can deduce a linear growth up to the critical density δ_c

$$\delta_c = 2^{2/3}\delta_a \approx 1.69 \tag{H.457}$$

at which the collapse sets in.

H.10 *Mass function of dark matter haloes*

The central result on spherical collapse was the overdensity of $\delta_c \simeq 1.69$ for a perturbation to collapse in its own gravitational field against the Hubble-expansion of the background. This number can be used to determine the number of objects such as clusters or galaxies per comoving volume that can form from initial conditions with suitably high initial densities. The formalism for achieving this was discovered in three different contexts: Assuming that the noise in an electric circuit is described by a one-dimensional Gaussian random field, the probability for a peak in the voltage exceeding a certain threshold would result from the spectrum of the fluctuations. Similarly, the occurrences of waves on the surface of the ocean above a certain threshold would likewise result from the fluctuation statistics of a Gaussian random field, now in two dimensions. And lastly, objects like galaxies form if the density exceeds the threshold for spherical collapse, and how often this happens in a comoving volume in a Gaussian random field is an application of the same idea in three dimensions.

A spherical perturbation of radius R encloses the mass M

$$M = \frac{4\pi}{3}R^3\Omega_m\rho_{\text{crit}} \quad \rightarrow \quad R = \sqrt[3]{\frac{3M}{4\pi\Omega_m\rho_{\text{crit}}}} \tag{H.458}$$

with the ambient density $\Omega_m\rho_{\text{crit}}$, $\rho_{\text{crit}} = 3H_0^2/(8\pi G)$, such that each mass M corresponds to a length scale R(M). If we now filter the density field δ by convolution with a filter W_R of spatial size R(M)

$$\bar{\delta}(\boldsymbol{x}) = \int d^3x' W_{R(M)}(|\boldsymbol{x} - \boldsymbol{x}'|)\delta(\boldsymbol{x}'), \tag{H.459}$$

then the convolved density field $\bar{\delta}$ consists of fluctuations that are massive enough that they can form objects of mass M by spherical collapse. In Fourier-space, the convolution relation reads

$$\bar{\delta}(\boldsymbol{k}) = W_R(\boldsymbol{k})\delta(\boldsymbol{k}) \tag{H.460}$$

with the Fourier-transform $W_R(\boldsymbol{k})$ of the filter function. The convolution as a linear operation does not change fundamentally the distribution of the density field amplitude, but changes the variance. Working with a Gaussian distribution

$$p(\bar{\delta}, a) = \frac{1}{\sqrt{2\pi\sigma_R^2(a)}} \exp\left(-\frac{1}{2}\left(\frac{\bar{\delta}}{\sigma_R(a)}\right)^2\right) \tag{H.461}$$

where the variance is growing in linear structure formation according to with the relation

$$\sigma_R^2(a) = \sigma_R^2(\text{today})D_+^2(a) \tag{H.462}$$

With this distribution we can ask how often in a fixed comoving volume the smoothed density field reaches amplitudes sufficient for spherical collapse, i.e. where the condition $\bar{\delta} > \delta_c$ is fulfilled. The probability of finding those is equal to the volume fraction filled with halos of mass M,

$$F(M, a) = \int_{\delta_c}^{\infty} d\bar{\delta}\, p(\bar{\delta}, a) = \frac{1}{2}\mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_R(a)}\right) \tag{H.463}$$

with the complementary error function erfc(). One determines the halo-distribution by differentiation

$$\frac{\partial F}{\partial M} = \frac{dR}{dM}\frac{dF}{dR}\frac{\delta_c}{\sigma_R D_+}\frac{d\ln\sigma_R}{dM}\exp\left(-\frac{1}{2}\left(\frac{\delta_c}{\sigma_R D_+}\right)^2\right) \tag{H.464}$$

because $\frac{d}{dx}\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\exp(-x^2)$. To obtain the comoving number density we divide the by halos occupied volume fraction by the halo-volume $\frac{4\pi}{3}R^3$ and get

$$n(M, a) = \frac{\rho_0}{\sqrt{2\pi}}\frac{\delta_c}{\sigma_R D_+}\frac{d\ln\sigma_R}{d\ln M}\exp\left(-\frac{1}{2}\left(\frac{\delta_c}{\sigma_R D_+}\right)^2\right)\frac{1}{M} \tag{H.465}$$

The mass function or Press-Schechter function $n(M, a)$ is a valuable source of cosmological information as it is sensitive to the shape of the CDM-spectrum $P(k)$ through the variance σ_R^2 and its derivative $d\sigma^2/dM$. Practical numbers to remember are about 100 clusters of galaxies above $5 \times 10^{13} M_\odot/h$ in a volume of $(100\mathrm{Mpc}/h)^3$, and about 10^4 galaxies with masses between $10^{11} M_\odot/h$ and $10^{12} M_\odot/h$ in the same volume. An important caveat is that the number of haloes per comoving volume is not observable, and neither would be comoving distance, but redshift is straightforwardly observable. Then, a cosmological probe could be the number of haloes observed within a fixed solid angle $\Delta\Omega$ between two redshifts $z_{\min}$ and $z_{\max}$

$$N = \frac{\Delta\Omega}{4\pi}\int_{z_{\min}}^{z_{\max}} dz\, \frac{dV}{dz}\int_{M_{\min}(z)}^{\infty} dM\, n(M, a(z)) \tag{H.466}$$

where the minimal mass $M_{\min}(z)$ for an object to be detectable is commonly determined by the observational technique. But in almost all cases, the magnitude of

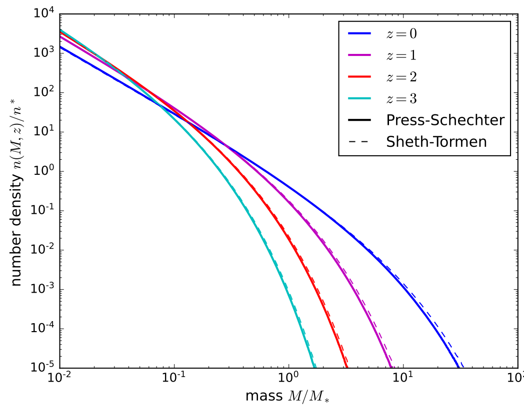


Figure 12: Halo mass function $n(\mathrm{M}, z)$ at different redshifts

observable properties of haloes, like luminosity or temperature, scale with halo mass. The comoving volume evolves with redshift z according to

$$V = \frac{4\pi}{3}\chi^3(a(z)) \quad \rightarrow \quad \frac{\mathrm{d}V}{\mathrm{d}z} = \frac{\mathrm{d}a}{\mathrm{d}z}\frac{\mathrm{d}\chi}{\mathrm{d}a}\frac{\mathrm{d}V}{\mathrm{d}\chi} \tag{H.467}$$

Due to $\chi = c\int \mathrm{d}a/(a^2\mathrm{H}(a))$ and $a = 1/(1+z)$ this expression becomes

$$\frac{\mathrm{d}\chi}{\mathrm{d}a} = \frac{c}{a^2\mathrm{H}} \quad \text{as well as} \quad \frac{\mathrm{d}a}{\mathrm{d}z} = \frac{1}{(1+z)^2} = a^2 \quad \text{and therefore,} \quad \frac{\mathrm{d}V}{\mathrm{d}z} = \frac{c}{\mathrm{H}}4\pi\chi^2 \tag{H.468}$$

The halo mass function $n(\mathrm{M}, z)$ is shown in Fig. 12 for a ΛCDM-cosmology, in two different parameterisations. Clearly, the most massive halos only appear at late times in the Universe.

I STATISTICS IN COSMOLOGY

I.1 *Statistical description of structure and random fields*

Structures in the Universe require a statistical description: On large scales, they look statistically identical and similar in every direction, and predictions from cosmological theories such as structure formation concern statistical properties rather than for instance individual formation scenarios for single galaxies, the principal exception being our own Milky Way. Suitable tools for a statistical description of e.g. the density field are random fields: There, one specifies a distribution of field amplitudes and possible correlations between them taken at different points. Conceptually, a Gaussian distribution such as

$$p(\delta(\boldsymbol{x})) = \frac{1}{\sqrt{2\pi\langle\delta(\boldsymbol{x})^2\rangle}}\exp\left(-\frac{1}{2}\frac{\delta(\boldsymbol{x})^2}{\langle\delta(\boldsymbol{x})^2\rangle}\right) \tag{I.469}$$

predicts values of the field δ taken at a specified point $\boldsymbol{x}$ for an ensemble of statistically equivalent universes. Over this ensemble, the variance $\langle\delta(\boldsymbol{x})^2\rangle$ is defined. Now, descriptive statistics always concerns moments or cumulants of δ over this ensemble of universe, and the same is true of symmetries like statistical isotropy or homogeneity, that is invariance of statistical quantities if $\boldsymbol{x}$ is rotated or shifted. Naturally, there is only one Hubble-volume in which we can carry out observations, such that accessing the ensemble for computing statistical quantities is impossible. But there is the concept of ergodicity, implying that one can construct estimates for ensemble averages from volume averages, provided that the random field is Gaussian and has a continuous spectrum, both concepts will be explained below. Gaussian random fields, i.e. a Gaussian distribution of field amplitudes are of particular relevance in cosmology, as at least at early times there are very good indications that all fields have Gaussian statistical properties.

The fluctuations of the cosmic density field $\delta(\boldsymbol{x})$, which are defined as the relative deviation of the density field $\rho(\boldsymbol{x})$ from the mean background density $\langle\rho\rangle = \Omega_m\rho_{\mathrm{crit}}$,

$$\delta(\boldsymbol{x}) = \frac{\rho(\boldsymbol{x}) - \langle\rho\rangle}{\langle\rho\rangle}, \tag{I.470}$$

are assumed to be Gaussian with a certain correlation length, meaning that the probability of finding the amplitudes $\delta_1 \equiv \delta(\boldsymbol{x}_1)$ and $\delta_2 \equiv \delta(\boldsymbol{x}_2)$ and positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in a hypothetical ensemble of universes is given by a multivariate Gaussian probability density,

$$p(\delta_1), \delta_2) = \frac{1}{\sqrt{(2\pi)^2\det(\mathrm{Q})}}\exp\left[-\frac{1}{2}\left(\begin{array}{c}\delta_1\\ \delta_2\end{array}\right)^t \mathrm{Q}^{-1}\left(\begin{array}{c}\delta_1\\ \delta_2\end{array}\right)\right] \tag{I.471}$$

with the covariance matrix Q:

$$\mathrm{Q} = \left(\begin{array}{cc}\langle\delta_1\delta_1\rangle & \langle\delta_1\delta_2\rangle\\ \langle\delta_2\delta_1\rangle & \langle\delta_2\delta_2\rangle\end{array}\right) \tag{I.472}$$

The off-diagonal variance in Q is the correlation function $\xi(\boldsymbol{x}_1,\boldsymbol{x}_2) \equiv \langle\delta_1\delta_2\rangle$ of the random field, which describes how fast with increasing distance $|\boldsymbol{x}_2 - \boldsymbol{x}_1|$ the field loses its memory on the amplitude at $\boldsymbol{x}_1$. A length scale in $\xi(\boldsymbol{x}_1,\boldsymbol{x}_2)$ can be interpreted as a correlation length. Due to the Cauchy-Schwarz inequality,

$$\langle\delta_1\delta_2\rangle^2 \leq \langle\delta_1^2\rangle\langle\delta_2^2\rangle \quad\rightarrow\quad r = \frac{\langle\delta_1\delta_2\rangle}{\sqrt{\langle\delta_1^2\rangle\langle\delta_2^2\rangle}} \tag{I.473}$$

the correlation function is always smaller than the geometrical mean of the variances at a single point, i.e. the covariance Q is positive definite, and the Pearson correlation coefficient $|r|$ is smaller than unity. Therefore, the Cauchy-Schwarz inequality makes sure that the distribution I.471 is normalisable, as the determinant det(Q) is ensured to be positive.

Clearly, if $\xi(\boldsymbol{x}_1, \boldsymbol{x}_2)$ vanishes the Gaussian probability density separates,

$$p(\delta_1, \delta_2) = p(\delta_1)p(\delta_2) \tag{I.474}$$

and the amplitudes are uncorrelated: The covariance Q becomes diagonal if c is zero, which has two important consequences: (i) The determinant factorises,

$$\det(\mathrm{Q}) = \langle\delta_1^2\rangle\,\langle\delta_2^2\rangle \tag{I.475}$$

as well as (ii) the quadratic form in the exponent of the distribution I.471,

$$\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}^t \mathrm{Q}^{-1}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} = \frac{\delta_1^2}{\langle\delta_1^2\rangle} + \frac{\delta_2^2}{\langle\delta_2^2\rangle}. \tag{I.476}$$

The Pearson correlation coefficient r vanishes simultaneously with the correlation function $\xi(\boldsymbol{x}_1, \boldsymbol{x}_2)$. It is sensible that the correlations in a random field decrease with increasing distance between the points where the amplitudes are measured and correlated, therefore, in the limit $r \to \infty$ we get $\xi \to 0$ as well as $r \to 0$, such that $p(\delta_1, \delta_2) = p(\delta_1)p(\delta_2)$ for sufficiently separated points.

The knowledge of the variance is sufficient because all moments of a Gaussian distributed random variable with zero mean are proportional to the variance, $\langle\delta^{2n}\rangle \propto \langle\delta^2\rangle^n$. Hence the characteristic function $\varphi(t) = \int \mathrm{d}\delta p(\delta)\exp(\mathrm{i}t\delta) = \sum_n\langle\delta^n\rangle(\mathrm{i}t)/n!$ only requires the estimation of the variance $\langle\delta^2\rangle$ for reconstructing $p(\delta)\mathrm{d}\delta$ from the moments $\langle\delta^{2n}\rangle$ by inverse Fourier transform.

If the correlation function $\xi(\boldsymbol{r}) = \langle\delta_1\delta_2\rangle$ only depends on the separation vector $\boldsymbol{r} = \boldsymbol{x}_2 - \boldsymbol{x}_1$, the density field has homogeneous fluctuation properties: Pictorially, this is a case where the fluctuations are similar (and in fact, statistically equivalent) at every point in space. In this case it is convenient to transform to Fourier space,

$$\delta(\boldsymbol{k}) = \int \mathrm{d}^3x\,\delta(\boldsymbol{x})\exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{x}) \quad\leftrightarrow\quad \delta(\boldsymbol{x}) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\delta(\boldsymbol{k})\exp(+\mathrm{i}\boldsymbol{k}\boldsymbol{x}), \tag{I.477}$$

and to consider the variance between two Fourier modes $\delta(\boldsymbol{k}_1)$ and $\delta(\boldsymbol{k}_2)$

$$\langle\delta(\boldsymbol{k}_1)\delta^*(\boldsymbol{k}_2)\rangle = (2\pi)^3\delta_\mathrm{D}(\boldsymbol{k}_1 - \boldsymbol{k}_2)\mathrm{P}(\boldsymbol{k}_1) \quad\text{with}\quad \mathrm{P}(\boldsymbol{k}) = \int\mathrm{d}^3r\,\xi(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{r}). \tag{I.478}$$

Therefore, Fourier modes of **homogeneous** random fields are mutually independent and their variance in Fourier-space defines the power spectrum $\mathrm{P}(\boldsymbol{k})$ as the Fourier transform of the correlation function $\langle\delta_1\delta_2\rangle$. If, in addition, the random field is **isotropic**, $P(k)$ only depends only the wave number k instead of the wave vector $\boldsymbol{k}$. Pictorially, this would be a random field whose fluctuation properties are identical in every direction. An intuitive counter-example wound be waves of the surface of the ocean close to a beach, where the wave fronts are roughly parallel to the seafront and isotropy, which one would expect from the open ocean, is broken.

In this case, the angular integrations in eqn. I.478 can be carried out by introducing spherical coordinates in Fourier-space, yielding:

$$P(k) = 2\pi \int_0^\infty r^2 \mathrm{d}r\, \xi(r) j_0(kr), \tag{I.479}$$

with the spherical Bessel function of the first kind $j_0(kr)$ of order $\ell = 0$, being equal to

$$j_0(kr) = \mathrm{sinc}(kr) = \frac{\sin(kr)}{kr} \tag{I.480}$$

Cosmological inflation provides a mechanism for generating Gaussian fluctuation fields with the spectrum $P(k)$,

$$P(k) \propto k^{n_s} T^2(k) \tag{I.481}$$

with the CDM transfer function $T(k)$. $T(k)$ describes the scale-dependent suppression of the growth of small-scale modes between horizon-entry and matter-radiation equality by the Meszaros-mechanism. It is well approximated with the polynomial fit of the type

$$T(q) = \frac{\ln(1 + 2.34q)}{2.34q} \left(1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4\right)^{-\frac{1}{4}}, \tag{I.482}$$

The asymptotic behaviour of the transfer function is such that $T(k) \propto$ const for $k \ll 1$ and $T(k) \propto k^{-2}$ at $k \gg 1$, such that $P(k) \propto k^{n_s}$ on large scales and $P(k) \propto k^{n_s-4}$ on small scales. The wave vector is rescaled with the shape parameter $\Gamma \simeq \Omega_m h$, which corresponds to the horizon size at the time of matter-radiation equality $a_{\gamma m}$, and describes the peak shape of the CDM power spectrum $P(k)$. There are weak corrections due to a nonzero baryon density Ω_b

$$\Gamma = \Omega_m h \, \exp\left[-\Omega_b \left(1 + \frac{\sqrt{2h}}{\Omega_m}\right)\right], \tag{I.483}$$

where Γ is measured in units of $(\mathrm{Mpc}/h)^{-1}$, such that $q = k/\Gamma$ is a dimensionless wave vector. The spectrum is usually normalised to the variance of the linearly evolved density field at zero redshift on a scale of $R = 8$ Mpc/h,

$$\sigma_R^2 = \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}k\, k^2 P(k) W^2(kR), \tag{I.484}$$

with a Fourier transformed spherical top hat filter function,

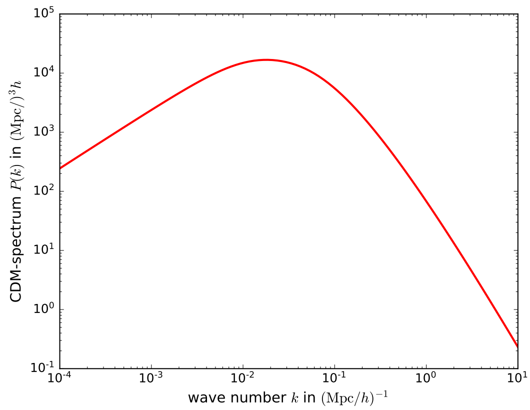


Figure 13: CDM-spectrum $\mathrm{P}(k)$ *today for linear growth*

$$W(x) = \frac{3j_1(x)}{x} \tag{I.485}$$

$j_1(x)$ is the spherical Bessel function of the first kind of order $\ell = 1$.

This particular definition of σ_R^2, along with the fact that the power spectrum has the dimension of a volume, motivates the definition of the dimensionless power spectrum $\Delta^2(k) \propto k^3 \mathrm{P}(k)$,

$$\Delta^2(k) = \frac{k^3}{2\pi^2}\mathrm{P}(k) \quad \rightarrow \quad \sigma_R^2 = \int_0^\infty \mathrm{d}\ln k\, \Delta^2(k) W^2(kR), \tag{I.486}$$

such that $\Delta^2(k)$ reflects the fluctuation variance per logarithmic band in k, $\mathrm{d}\sigma_R^2/\mathrm{d}\ln k \propto \Delta^2$. It is common to normalise $\mathrm{P}(k)$ by the variance σ_8^2 on scales of comoving $R = 8$ Mpc/h, and typical values are $\sigma_8 = 0.8 \ldots 0.9$. The spectrum $\mathrm{P}(k)$ is shown in Fig. 13 for linear evolution at the current cosmic epoch.

I.2 *Fluctuations on the sky and Limber-projections*

A Gaussian random field $\gamma(\theta)$ on the celestial sphere can be characterised by the correlation function

$$C_{\gamma\gamma}(\alpha) = \langle \gamma(\theta)\gamma^*(\theta') \rangle \tag{I.487}$$

with the separation $\alpha = \sphericalangle(\theta, \theta')$, because a Gaussian distribution is determined by the variance, following the argument using the characteristic function of a distribution outlined in Sect. **??**. The averaging brackets $\langle \ldots \rangle$ denote a hypothetical ensemble average over realisations of the random field, but can be replaced by spherical averages for estimating the correlation function because of the ergodicity of the ensemble provided that the random process has a continuous correlation function up to cosmic variance. The correlation function and the angular power spectrum can be converted into each other under transformations with Legendre polynomials $P_\ell(\cos\alpha)$,

$$C_{\gamma\gamma}(\ell) = 2\pi \int \mathrm{d}\cos\alpha\, C_{\gamma\gamma}(\alpha)P_\ell(\cos\alpha) \quad \leftrightarrow \quad C_{\gamma\gamma}(\alpha) = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)C_{\gamma\gamma}(\ell)P_\ell(\cos\alpha) \tag{I.488}$$

using the orthonormality of the Legendre-polynomials $P_\ell(\cos\alpha)$:

$$\int_{-1}^{+1} \mathrm{d}x\, P_\ell(x)P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}. \tag{I.489}$$

Fluctuations of a quantity like the sky temperature $\tau(\theta)$ or the galaxy density $\gamma(\theta)$ on the celestial sphere with homogeneous fluctuation properties can be decomposed in using the spherical harmonics $Y_{\ell m}(\theta)$, because they are a complete orthonormal set of basis functions:

$$\gamma(\theta) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\gamma_{\ell m}Y_{\ell m}(\theta) \quad \leftrightarrow \quad \gamma_{\ell m} = \int_{4\pi}\mathrm{d}\Omega\, \gamma(\theta)Y^*_{\ell m}(\theta). \tag{I.490}$$

The orthonormality relation of the spherical harmonics $Y_{\ell m}(\theta)$ reads

$$\int_{4\pi}\mathrm{d}\Omega\, Y_{\ell m}(\theta)Y^*_{\ell' m'}(\theta) = \delta_{\ell\ell'}\delta_{mm'}, \tag{I.491}$$

and is not identical to the completeness relation:

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} Y_{\ell m}(\theta)Y^*_{\ell m}(\theta') = \delta(\theta-\theta'), \tag{I.492}$$

because the spherical harmonics $Y_{\ell m}(\theta)$ are a discrete basis system. Orthonormality and completeness are identical in the case of a continuous basis system like the plane waves $\exp(\pm ikx)$ of the Fourier transform. The variance of the spherical harmonics expansion coefficients $\gamma_{\ell m}$ can be related to the angular power spectrum,

$$\langle\gamma_{\ell m}\gamma^*_{\ell' m'}\rangle = \int_{4\pi}\mathrm{d}\Omega\int_{4\pi}\mathrm{d}\Omega'\, C_{\gamma\gamma}(\alpha)Y_{\ell m}(\theta)Y^*_{\ell' m'}(\theta'), \tag{I.493}$$

by substituting the decomposition eqn. (I.490) and using the definition of the correlation function eqn. (I.487). The correlation function $C_{\gamma\gamma}(\alpha)$ can be replaced with the angular spectrum $C_{\gamma\gamma}(\ell)$, and the Legendre polynomial can be substituted with the spherical harmonic's addition theorem, $\alpha = \sphericalangle(\hat\theta, \hat\theta')$:

$$\sum_{m=-\ell}^{+\ell} Y_{\ell m}(\theta)Y^*_{\ell m}(\theta') = \frac{2\ell+1}{4\pi}P_\ell(\cos\alpha). \tag{I.494}$$

Using the orthonormality relation twice and contracting the Kronecker δ-symbols yields the final result

$$\langle\gamma_{\ell m}\gamma^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}\, C_{\gamma\gamma}(\ell), \tag{I.495}$$

i.e. that the variance of the expansion coefficients $\gamma_{\ell m}$ is equal to the angular spectrum $C_{\gamma\gamma}(\ell)$ and that there is no cross-correlation between coefficients on different angular scale ℓ or different propagation direction m in the case of homogeneous and isotropic fields.

The Limber equation is used for relating the fluctuation statistics of the 3d source field to the fluctuation statistics of the projected observable. Both observables, the iSW-temperature perturbation $\tau(\theta)$ and the tracer density $\gamma(\theta)$ are derived as line of sight projections from the source fields $\varphi(\chi\theta, \chi)$ and $\delta(\chi\theta, \chi)$ with weighting functions $W_\tau(\chi)$ and $W_\gamma(\chi)$:

$$\gamma(\theta) = \int_0^{\chi_H} d\chi\, W_\gamma(\chi)\delta(\chi\theta, \chi) \quad \text{and} \quad \tau(\theta) = \int_0^{\chi_H} d\chi\, W_\tau(\chi)\varphi(\chi\theta, \chi) \tag{I.496}$$

The angular correlation function $C_{\gamma\gamma}(\alpha)$ can be then related to the correlation of the source field $\delta(\theta\chi, \chi)$:

$$C_{\gamma\gamma}(\alpha) = \int_0^{\chi_H} d\chi\, W_\gamma(\chi) \int_0^{\chi_H} d\chi'\, W_\gamma(\chi') \int dk\, k^2 P(k, \chi, \chi') \int_{4\pi} d\Omega_k \exp(i\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}')), \tag{I.497}$$

with the spatial comoving coordinates $\boldsymbol{x} = (\theta\chi, \chi)$ and the solid angle element $d\Omega_k$ in Fourier space. The power spectrum $P(k, \chi, \chi')$ follows from the Fourier transform of the correlation function of the source field,

$$\langle \gamma(\theta\chi, \chi)\gamma^*(\theta'\chi', \chi')\rangle = \int \frac{d^3k}{(2\pi)^3} P(k) \exp(i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')) = \int dk k^2 P(k) \int_{4\pi} d\Omega_k \exp(i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')) \tag{I.498}$$

In order to solve the angular integration, one can take advantage of the Rayleigh expansion of a plane wave in terms of spherical waves:

$$\exp(i\boldsymbol{k}\boldsymbol{x}) = 4\pi \sum_{\ell=0}^{\infty} i^\ell j_\ell(kx) \sum_{m=-\ell}^{+\ell} Y_{\ell m}(\hat{k}) Y^*_{\ell m}(\theta). \tag{I.499}$$

The angular integration can be carried out while substituting the orthonormality relation of the spherical harmonics,

$$\int_{4\pi} d\Omega_k \exp(i\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}')) = (4\pi)^2 \sum_{\ell=0}^{\infty} j_\ell(k\chi) j_\ell(k\chi') \sum_{m=-\ell}^{+\ell} Y_{\ell m}(\theta) Y^*_{\ell m}(\theta') = 4\pi \sum_{\ell=0}^{\infty} j_\ell(k\chi) j_\ell(k\chi')\, (2\ell + 1)\, P_\ell(\cos\alpha) \tag{I.500}$$

where in the last step the addition theorem has been used, yielding

$$C_{\gamma\gamma}(\alpha) = 4\pi \int_0^{\chi_H} d\chi W_\gamma(\chi) \int_0^{\chi_H} d\chi' W_\gamma(\chi') \int dk k^2 P(k, \chi, \chi') \sum_{\ell=0}^{\infty} j_\ell(k\chi) j_\ell(k\chi')(2\ell+1) P_\ell(\cos\alpha) \tag{I.501}$$

Inverting the expression for the angular correlation function $C_{\gamma\gamma}(\ell)$ by multiplying both sides with $P_{\ell'}(\cos\alpha)$ and integrating over $d(\cos\alpha)$ results in

$$C_{\gamma\gamma}(\ell) = (4\pi)^2 \int_0^{\chi_H} d\chi\, W_\gamma(\chi) \int_0^{\chi_H} d\chi'\, W_\gamma(\chi') \int dk\, k^2 P(k, \chi, \chi') j_\ell(k\chi) j_\ell(k\chi') \tag{I.502}$$

by using the orthonormality relation of the Legendre-polynomials. The expression can be further simplified if $P(k, \chi, \chi')$ is slowly varying in comparison to the spherical Bessel functions, i.e. if the angles involved are small, which corresponds to approximating the sky as being flat. In this case $P(k, \chi, \chi')$ can be moved in front of the dk-integration, which can then be carried out using the orthogonality relation of the spherical Bessel functions,

$$\int_0^{\infty} k^2 dk\, j_\ell(k\chi) j_\ell(k\chi') = \frac{\pi}{2\chi^2} \delta_D(\chi - \chi'). \tag{I.503}$$

We can approximately set $P(k) \simeq P(\ell/\chi)$, giving the final result

$$C_{\gamma\gamma}(\ell) \simeq \int_0^{\chi_H} \frac{d\chi}{\chi^2} W_\gamma^2(\chi) P(k = \ell/\chi, \chi). \tag{I.504}$$

A slightly better approximation to the correct result in spherical coordinates can be obtained by replacing $k = \ell/\chi$ with $k = (\ell + 1/2)/\chi$. The small-angle approximation of the Limber eqn. (I.504) generally overestimates the angular power spectrum in comparison to the correct solution in eqn. (I.502).

I.3 *Cosmic microwave background anisotropies*

The spectrum $C_{TT}(\ell)$ of the CMB-fluctuations is given in Fig. 14.

Fig. 15 shows the size of the CMB photosphere from the moment of decoupling until today, for three different ΛCDM cosmologies.

I.4 *Secondary anisotropies in the cosmic microwave background*

I.4.1 Gravitational lensing of the CMB

There is a very interesting gravitational lensing effect in the cosmic microwave background: A typical lensing deflection that photons from the CMB experience amounts to a few arcminutes, which is small compared to the typical scales on which the temperatures in the cosmic microwave backgrounds vary, which is roughly on the degree-scale. Therefore, one expects a small distortion of the CMB-fluctuation pattern, as hot and cold patches are deformed by roughy a percent. There is no energy input into the CMB by the gravitational lensing effect, if one assumes the gravitational potentials to be static (which is a good approximation but which is ultimately flawed

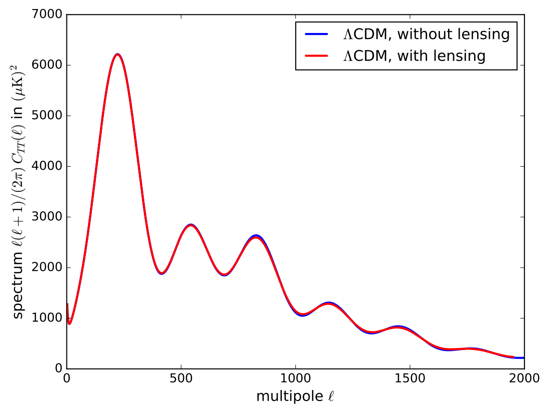


Figure 14: Angular spectrum $C_{TT}(\ell)$ of the temperature anisotropies of the CMB

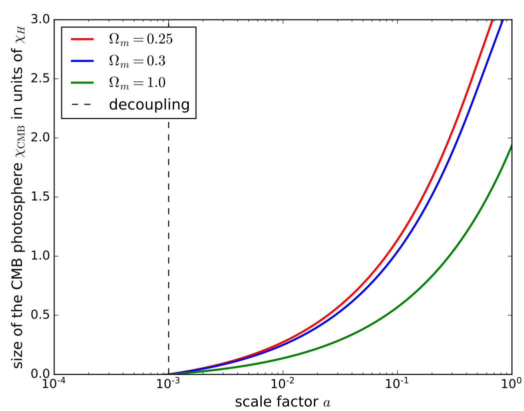


Figure 15: Comoving radius of the CMB photosphere as a function of scale factor, for three ΛCDM cosmologies

because of the integrated Sachs-Wolfe effect). Lensing, being a gravitational effect, can not differentiate between photons of different energy and is completely achromatic, as well as perfectly conserving photon density, energy flux and spectral distribution. Therefore, we expect that lensing conserves the Planck-distribution of the photons of the CMB as a thermal source. Because the lensing effect only redistributes photons, it could not generate structures in a completely isotropic CMB, contrarily, it needs structures to work on.

In gravitational lensing in astronomy it is rarely the case that one can access the unlensed situation, where the Solar eclipse of 1919 is a very notable exception. In almost all other cases of gravitational lensing, one has to make an assumption about the unlensed case in order to detect the gravitational lensing effect. Obviously, one would like to make assumptions that are as weak as possible and generic from a physical point of view. Gravitational lensing changes the statistical properties of the cosmic microwave background and breaks statistical homogeneity as a symmetry. With the assumption of a statistically homogeneous unlensed CMB one can quantify the magnitude of the broken statistical symmetry and therefore measure the weak lensing effect. It is practical to define a dimensionless amplitude $T(\boldsymbol{\theta})$ of the temperature fluctuations $T(\boldsymbol{\theta})$ in the cosmic microwave background relative to the mean temperature $T_{\mathrm{CMB}} = 2.725$ Kelvin,

$$T(\boldsymbol{\theta}) = \frac{T(\boldsymbol{\theta}) - T_{\mathrm{CMB}}}{T_{\mathrm{CMB}}}, \tag{I.505}$$

Statistical homogeneity has a very interesting implication for the Fourier-modes $T(\boldsymbol{\ell})$ of the temperature field $T(\boldsymbol{\theta})$. Defining

$$T(\boldsymbol{\ell}) = \int d^2\theta\, T(\boldsymbol{\theta}) \exp(-i\boldsymbol{\theta}\boldsymbol{\ell}) \quad \leftrightarrow \quad T(\boldsymbol{\theta}) = \int \frac{d^2\ell}{(2\pi)^2}\, T(\boldsymbol{\ell}) \exp(+i\boldsymbol{\theta}\boldsymbol{\ell}), \tag{I.506}$$

one can ask the question how the Fourier-modes are correlated, if there is a nonzero correlation in configuration space. In fact,

$$\langle T(\boldsymbol{\ell}) T(\boldsymbol{\ell}')^* \rangle = (2\pi)^2 T_D(\boldsymbol{\ell} - \boldsymbol{\ell}') C_{TT}(\ell), \tag{I.507}$$

with the Dirac-function T_D,

$$T_D(\ell) = \int d^2\theta\, \exp(+i\boldsymbol{\theta}\boldsymbol{\ell}). \tag{I.508}$$

$C_{TT}(\ell)$ is the spectrum of the temperature fluctuations and is given by

$$C_{TT}(\ell) = \int d^2\theta\, \xi(\theta) \exp(-i\boldsymbol{\theta}\boldsymbol{\ell}), \tag{I.509}$$

where in the case of statistically isotropic fields the integration can be simplified according to $\boldsymbol{\theta}\boldsymbol{\ell} = \theta\ell\cos\varphi_\ell$ and $d^2\theta = \theta d\theta d\varphi_\ell$ by introducing polar coordinates. Therefore, one observes uncorrelated Fourier-modes in the case of statistically homogeneous fields. As we will see, the signature of CMB-lensing is to change the fluctuation statistics of the cosmic microwave background. In particular, it will make a statistically homogeneous CMB statistically inhomogeneous and will generate correlations $\langle T(\boldsymbol{\ell}) T(\boldsymbol{\ell}')^* \rangle \neq 0$ even if $\boldsymbol{\ell} \neq \boldsymbol{\ell}'$. Under the assumption that the unlensed CMB has been statistically homogeneous, which is a weak assumption that is supported by models of cosmic inflation, any measurement of these correlations would be an indication for the gravitational lensing effect.

As the photons of the CMB propagate to us, they have to transverse the cosmic large-scale structure and experience gravitational lensing. As they are deflected by gravitational potentials, they seem to change their propagation direction: Instead of measuring the temperature field $T(\theta)$ in the direction θ, the arrival direction is changed to $\theta + \boldsymbol{\alpha}$, where the deflection angle $\boldsymbol{\alpha}$ is the gradient of the lensing potential $\psi(\theta)$:

$$T(\theta) \rightarrow \hat{T}(\theta) = T(\theta + \boldsymbol{\alpha}) \tag{I.510}$$

These deflections distort the pattern of hot and cold patches in the CMB, which can be quantified in a statistical way. To this purpose, assuming that the deflections are small compared to the angular size of structures in the microwave background, one can expand the temperature field in a Taylor-series,

$$\hat{T}(\theta) = T(\theta + \boldsymbol{\alpha}) = T(\theta) + \sum_i \alpha_i \partial^i T + \frac{1}{2}\sum_{ij} \alpha_i \alpha_j \partial^i \partial^j T + \dots. \tag{I.511}$$

Computing a correlation function of the lensed temperature field yields

$$\langle \hat{T}(\theta)\hat{T}(\theta')\rangle = \langle T(\theta + \boldsymbol{\alpha})T(\theta' + \boldsymbol{\alpha}')\rangle \tag{I.512}$$

and consequently

$$= \langle T(\theta)T(\theta')\rangle + \sum_i \sum_k \langle \alpha_i(\theta)\alpha_k(\theta')\rangle \times \langle \partial_i T(\theta)\partial'_k T(\theta')\rangle + 2\sum_{ij} \langle \alpha_i(\theta)\alpha_j(\theta)\rangle \times \langle \partial^2_{ij} T(\theta)T(\theta')\rangle + \dots \tag{I.513}$$

if one assumes that the deflection field is uncorrelated with the temperature field, and that the distribution of the lensing deflection angle components are symmetric with zero mean. Both assumptions are physically sensible, because the deflecting large-scale structure responsible for the gravitational lensing effect is separated by a very large distance from the CMB, and because the structures responsible for lensing do not define a preferred direction. The two terms appearing as a correction to the unlensed temperature fluctuations can be interpreted as a correlated deflection $\langle \alpha_i(\theta)\alpha_k(\theta')\rangle$ where the lensing deflection $\alpha_i(\theta)$ at θ is not independent from the deflection $\alpha_k(\theta')$ at θ', and as an effect caused by a second-order deflection $\langle \alpha_i(\theta)\alpha_j(\theta)\rangle$ at a single point. Especially the last effect can be visualised by imagining that CMB photons reaching us are deflected by some amount into a random direction, leading to a blurring of the CMB. In fact, the lensed CMB has less structure compared to the unlensed one, as the blurring causes the contrast of structures to decrease.

Transforming the Taylor-series to Fourier-space then yields

$$\hat{T}(\boldsymbol{\ell}) = T(\boldsymbol{\ell}) + \\ i\int \frac{d^2\ell_1}{(2\pi)^2} \sum_i \alpha_i(\boldsymbol{\ell}_1)(\boldsymbol{\ell}-\boldsymbol{\ell}_1)_i T(\boldsymbol{\ell}-\boldsymbol{\ell}_1) - \\ \int \frac{d^2\ell_1}{(2\pi)^2}\int \frac{d^2\ell_2}{(2\pi)^2} \sum_{ij} \alpha_i(\boldsymbol{\ell}_1)\alpha_j(\boldsymbol{\ell}_2)(\boldsymbol{\ell}-\boldsymbol{\ell}_1-\boldsymbol{\ell}_2)_i(\boldsymbol{\ell}-\boldsymbol{\ell}_1-\boldsymbol{\ell}_2)_j T(\boldsymbol{\ell}-\boldsymbol{\ell}_1-\boldsymbol{\ell}_2) + \ldots, \tag{I.514}$$

applying the two properties of Fourier transforms, i.e. that products become convolutions and that every derivative ∂_i generates a prefactor $i\boldsymbol{\ell}_i$. Furthermore, one can replace the deflection angle $\boldsymbol{\alpha}(\boldsymbol{\ell})$ by the derivative $-i\boldsymbol{\ell}\psi(\boldsymbol{\ell})$ of the lensing potential ψ. Inspection of the last relationship shows that the lensed CMB temperature field is given by the unlensed field, with a series of corrections that involve n-fold derivatives of T, contracted with n factors of the lensing deflection field $\boldsymbol{\alpha}$.

Assembling a correlation function $\langle\hat{T}(\boldsymbol{\ell})\hat{T}(\boldsymbol{\ell}')^*\rangle$ of the lensed CMB then yields a series of correction terms to the correlation function $\langle T(\boldsymbol{\ell})T(\boldsymbol{\ell}')^*\rangle$ of the unlensed CMB. If one assumes that the structures that are responsible for gravitational lensing are separated by a large distance from the structures that cause the temperature fluctuations of the CMB, one can again factorise the mixed correlation functions

$$\langle\alpha_i(\boldsymbol{\ell}_1)\alpha_j(\boldsymbol{\ell}_2)T(\boldsymbol{\ell}-\boldsymbol{\ell}_1-\boldsymbol{\ell}_2)T(\boldsymbol{\ell}')\rangle = \langle\alpha_i(\boldsymbol{\ell}_1)\alpha_j(\boldsymbol{\ell}_2)\rangle \times \langle T(\boldsymbol{\ell}-\boldsymbol{\ell}_1-\boldsymbol{\ell}_2)T(\boldsymbol{\ell}')\rangle, \tag{I.515}$$

and using the assumption, that the lensing deflection field is isotropic, implying that the distributions of each of the components of α is symmetric with mean zero, sets $\langle\alpha_i\rangle = 0$. Then, one obtains for the correlations of $\hat{T}$ in Fourier space:

$$\langle\hat{T}(\boldsymbol{\ell})\hat{T}(\boldsymbol{\ell}')^*\rangle = \langle T(\boldsymbol{\ell})T(\boldsymbol{\ell}')^*\rangle \\ + \int \frac{d^2\ell_1}{(2\pi)^2}\int \frac{d^2\ell_1'}{(2\pi)^2} \sum_i\sum_k (\boldsymbol{\ell}-\boldsymbol{\ell}_1)_i(\boldsymbol{\ell}'-\boldsymbol{\ell}_1')_k\langle\alpha_i(\boldsymbol{\ell}_1)\alpha_k(\boldsymbol{\ell}_1')\rangle \times \langle T(\boldsymbol{\ell}-\boldsymbol{\ell}_1)T(\boldsymbol{\ell}'-\boldsymbol{\ell}_1')\rangle \\ +2\int \frac{d^2\ell_1}{(2\pi)^2}\int \frac{d^2\ell_2}{(2\pi)^2} \sum_{ij} (\boldsymbol{\ell}-\boldsymbol{\ell}_1-\boldsymbol{\ell}_2)_i(\boldsymbol{\ell}-\boldsymbol{\ell}_1-\boldsymbol{\ell}_2)_j\langle\alpha_i(\boldsymbol{\ell}_1)\alpha_j(\boldsymbol{\ell}_2)\rangle \times \langle T(\boldsymbol{\ell}-\boldsymbol{\ell}_1-\boldsymbol{\ell}_2)T(\boldsymbol{\ell}')\rangle. \tag{I.516}$$

The two correction terms that appear at second order have a clear physical interpretation. They involve the correlations $\langle\alpha_i(\boldsymbol{\ell}_1)\alpha_k(\boldsymbol{\ell}_1')\rangle$ and $\langle\alpha_i(\boldsymbol{\ell}_1)\alpha_j(\boldsymbol{\ell}_2)\rangle$, showing that the correlations of the temperature field in fact get changed due to correlations in the deflection field, which can both be traced back to the spectrum $C_{\alpha\alpha}(\ell)$. Both correction terms introduce correlations between $\boldsymbol{\ell}$ and $\boldsymbol{\ell}'$ as an expression of breaking of statistical homogeneity. The effect is proportional to $C_{\alpha\alpha}(\ell)$, such that the lensing effect can be measured in a quantitative way.

I.4.2 Thermal and kinetic Sunyaev-Zel'dovich effect

Of all secondary CMB-anisotropies, the thermal Sunyaev-Zel'dovich effect is the most subtle: There is a redistribution of the CMB-photons in energy in scattering processes

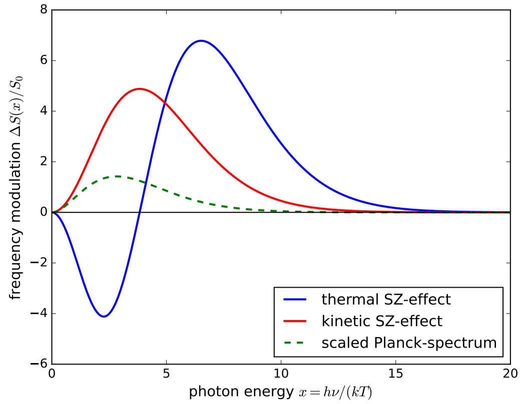


Figure 16: Spectral modulation of the CMB due to the thermal and kinetic Sunyaev-Zel'dovich effects

with electrons in galaxy clusters, as illustrated by Fig. 16. Essentially, Compton-collisions between the CMB-photons and electrons of the intra-cluster medium put the CMB as a very cold reservoir into thermal contact with the intra-cluster medium of a galaxy cluster as a very hot reservoir. Consequently, there will be a flow of thermal energy from the hot electron gas to the cold photon gas, causing a spectral distortion of the CMB: This is illustrated in Fig. 16, where the peculiar modulation of the CMB-spectrum if it is observed through a galaxy cluster is depicted. There is a secondary Sunyaev-Zel'dovich effect caused by the bulk motion of the cluster itself: In the cluster's rest frame, the CMB appears anisotropic, and likewise the radiation pressure exerted on it through Compton collisions, causing effectively the cluster to be slowed down until it comes to rest in a frame where the CMB appears isotropic. The kinetic energy of the cluster is transfered to the CMB, and therefore one perceives photons of higher energy from the direction of a cluster that is approaching the observer.

I.4.3 Integrated Sachs-Wolfe effect

The integrated Sachs-Wolfe effect is essentially a gravitational lensing effect: In the same way as spatial gradients $\partial^i\Phi$ of the gravitational potential Φ have an influence on the direction of propagation k^i of the photons, the time derivative $\partial^t\Phi$ changes the frequency (or colour) of photon. Again, working with a Newtonian perturbation on a flat, Minkowksian background

$$\mathrm{d}s^2 = (\eta_{\mu\nu} + h_{\mu\nu})\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 + \frac{2\Phi}{c^2}\right)\mathrm{d}\eta^2 + \left(1 - \frac{2\Phi}{c^2}\right)\mathrm{d}\boldsymbol{x}^2 \tag{I.517}$$

one can write for the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\left|h_{\mu\nu}\right| \ll 1$ in this preferred frame. Photons follow null-geodesics defined by $k_\mu k^\mu = 0$ where $k^\mu = (k^0, \boldsymbol{k})^t$ as the wave vector is tangent to $x^\mu(\lambda)$; it is parameterised by an affine parameter λ and for convenience normalised to $k^0 = 1$ and $\boldsymbol{k}^2 = 1$. Again, it is sufficient to consider a static background because of the conformal invariance of null-geodesics, which do not change under conformal transformations of the type $g_{\mu\nu} \to a^2 g_\mu$ of the metric $g_{\mu\nu}$. Effectively, this amounts to ignoring cosmological redshifts while focusing on the gravitational interaction.

The geodesic equation, which describes the change δk^α in k^α due to gravitational interaction now reads:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta k^\alpha = -\delta\Gamma^\alpha_{\ \mu\nu}\ k^\mu k^\nu \tag{I.518}$$

where the Christoffel symbol of the weakly perturbed metric transforming the time-component of k^α is given by

$$\delta\Gamma^t_{\ \mu\nu} = -\frac{1}{2}\left[\partial_\nu h_{\mu t} + \partial_\mu h_{\nu t} - \partial_t h_{\mu\nu}\right]. \tag{I.519}$$

In this approximation of $\delta\Gamma^t_{\ \mu\nu}$, the multiplication with the metric $g_{\mu\nu}$ was dropped because it would give rise to terms quadratic in the perturbation $h_{\mu\nu}$.

The first two terms give rise to the conventional Sachs-Wolfe effect, and the last term with the time derivative $\partial_t h_{\mu\nu}$ of the metric perturbation $h_{\mu\nu}$ causes the iSW-effect. Substitution into the geodesic equation yields:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta k^t = -\frac{1}{2}\,\partial_t h_{\mu\nu}\,k^\mu k^\nu \tag{I.520}$$

The energy shift δk^0 of a photon is given by subsequent integration,

$$\delta k^0 = \frac{1}{c^2}\int \mathrm{d}\lambda\,\left[(k^t)^2 + \boldsymbol{k}^2\right]\frac{\partial\Phi}{\partial\eta} = \frac{2}{c^2}\int \mathrm{d}\lambda\,\frac{\partial\Phi}{\partial\eta} \tag{I.521}$$

such that the energy perturbation is a measure of the integrated growth rate along the line of sight. Curiously, the iSW-effect is a direct probe of dark energy, as $\partial\Phi/\partial\eta$ vanishes in flat cosmologies with only matter, $\Omega_m = 1$.

The integral should be evaluated along the photon geodesic, but one assumes Born's approximation, such that the energy shift is obtained perturbatively while the geodesic remains characterised by the conditions $(k^0)^2 = 1$ and $\boldsymbol{k}^2 = 1$ mentioned above. In a cosmological context, the photon geodesic $\mathrm{d}s^2 = 0$ is given by $\mathrm{d}\chi = c\mathrm{d}t/a = c\mathrm{d}\eta$ with the conformal time η such that η is would be the natural choice for the affine parameter λ in the comoving frame. η and λ are linearly related such that their ratio can be absorbed in the normalisation of k. As a purely gravitational interaction, the iSW-effect conserves the spectral distribution of photons: Due to the equivalence principle, gravity treats all photons in the same way, which is true for the iSW-effect and lensing alike.

I.5 *Weak gravitational lensing by the large-scale structure*

In Sect. H.7.3 we have already derived the deflection angle $\hat{\alpha}$ in gravitational light deflection

$$\hat{\alpha}^i = -\frac{2}{c^2}\int \mathrm{d}\lambda\,\partial^i\Phi \tag{I.522}$$

into the direction of the gradients of Φ perpendicular to the line of sight. To be exact, $\hat{\alpha}$ is the change in direction between the spatial wave vector entering and leaving the gravitational potential, but not yet the change in position as observed on the sky. If a source is situated at a comoving distance χ_s and the gravitational potential acting as the light deflector is at a comoving distance χ, the change in position α^i of the source being at position θ^i without lensing is given by

$$\alpha = \theta' - \theta \tag{I.523}$$

so that the source appears at θ'. Writing $\theta = x/\chi_s$, $\theta' = (x+d)/\chi_2$ and $\hat{\alpha} = d/(\chi_s - \chi)$ suggests that the angular displacement on the sky generated by lensing is

$$\alpha = \left(1 - \frac{\chi}{\chi_s}\right)\hat{\alpha} \tag{I.524}$$

where $\hat{\alpha}$ is computable with eqn. I.522. With comoving distance as the integration variable and by rewriting the spatial as an angular derivative $\chi\partial^i = \partial_\theta$ we get

$$\alpha^i = \partial^i_\theta\psi \quad \text{with} \quad \psi = 2\int \mathrm{d}\chi\,\frac{\chi_s - \chi}{\chi_s\chi}\frac{\Phi}{c^2} \tag{I.525}$$

defining the lensing potential ψ.

In this entire discussion it would be important to realise that the change in propagation direction $\hat{\alpha}^i$ is only defined because the lens is embedded in a flat spacetime (or at least a conformally flat spacetime). Then, there is a parallel transport around the lens through essentially flat space as a reference wave vector, to which one can compare the actual wave vector that has been properly parallel transported through the gravitational potential along the physical trajectory, defining a deflection angle.

Clearly, one needs to make an assumption about the unlensed situation: Generally one does not know the positions of objects without lensing (Eddington's Solar eclipse from 1919 being a very notable exception). Instead, one could try to observe a differential deflection across the image of a distant object like galaxy: If the deflection field depends on position, there is such a differential deflection and one observes a change in shape of the image. Similarly, one could invoke Raychaudhuri's equation, as the light bundle of the galaxy forms a geodesic congruence. Then, changes in shape and size of the light bundle are related to the tidal gravitational fields, or relativistically speaking, to the curvature experienced by the light bundle. With this idea, if the observable are galaxy shapes, a weak assumption about the unlensed situation would be uncorrelated shapes, which would get coherently distorted, as light bundles from neighbouring galaxies would experience similar tidal distortions.

For changes in shape and size to emerge one needs variations of the deflection field α^i across the image of a galaxy, and as α^i is defined as the gradient of the lensing potential ψ, the changes are induces by second derivatives $\psi_{ij} = \partial_i\partial_j\psi$ of ψ. The index pair ij runs over x and y and as partial derivatives interchange, ψ_{ij} is a real symmetric 2×2 matrix. A suitable basis system are the real-valued Pauli matrices,

$$\psi_{ij} = \sum_n a_n\sigma^{(n)}_{ij} \quad \text{with} \quad a_n = \frac{1}{2}\sum_{ij}\sigma^{(n)}_{ji}\Psi_{ij} \tag{I.526}$$

The role of the three different components of $\psi_{ij} = \partial_i \partial_j \Psi$ are:

- convergence $\kappa = a_0 = \frac{1}{2}(\partial_i \partial_j \psi \delta_{ij}) = \Delta\psi/2$, which changes the angular size of a galaxy isotropically, i.e. by the same amount in the x and y-direction
- shear $\gamma_+ = a_1 = \frac{1}{2}\sigma^{(1)}_{ij}\psi_{ij} = \frac{1}{2}(\partial_x^2\psi - \partial_y^2\psi)$, which elongates the image in x-direction while compressing it in the y-direction
- shear $\gamma_\times = a_3 = \frac{1}{2}\sigma^{(3)}_{ij}\psi_{ij} = \partial_x\partial_y\psi$, which stretches an image into the $(x+y)$-direction while compressing in the $(x-y)$-direction

The two components of shear are often combined into a single complex number $\gamma = \gamma_+ + \mathrm{i}\gamma_\times$.

The convergence κ provides a mapping of the matter density:

$$\kappa = \frac{1}{2}\Delta_\theta \Psi = \Delta_\theta \int_0^{\chi_s} \mathrm{d}\chi \, \frac{\chi_s - \chi}{\chi_s \chi} \frac{\Phi}{c^2} = \int_0^{\chi_s} \mathrm{d}\chi (\chi_s - \chi) \frac{\chi}{\chi_s} \Delta_x \frac{\Phi}{c^2} \tag{I.527}$$

using $\Delta_\theta = \chi^2 \Delta_x$ and the small angle approximation $x = \theta\chi$. Substituting the Poisson-equation

$$\Delta \frac{\Phi}{c^2} = \frac{3\Omega_m}{2\chi_H^2} \frac{\delta}{a} \tag{I.528}$$

yields

$$\kappa = \frac{3\Omega_m}{2\chi_H^2} \int_0^{\chi_s} \mathrm{d}\chi \, \frac{\chi_s - \chi}{\chi_s} \chi \frac{D_+}{a} \delta_0 \tag{I.529}$$

Statistically, line of sight expressions like $\kappa = \int_0^{\chi_s} \mathrm{d}\chi \, W(\chi)\delta$ can be used in Limber's equation to give the angular spectrum of the shear or convergence fields,

$$C^{\kappa\kappa}(\ell) = \int_0^{\chi_s} \frac{\mathrm{d}\chi}{\chi^2} W(\chi)^2 \, P(k = \ell/\chi) \tag{I.530}$$

as a function of the spectrum $P(k)$ of the source field, in our case δ. The spectrum $C^{\gamma\gamma}(\ell)$ is identical to that of κ.

A quantification of shape could be the ellipticity ϵ measured in terms of the second moments of the brightness distribution $I(\theta)$

$$Q_{ij} = \int \mathrm{d}^2\theta \, I(\theta) \, \theta_i \theta_j \tag{I.531}$$

from which one defines the ellipticity

$$\epsilon = \frac{Q_{xx} - Q_{yy}}{Q_{xx} + Q_{yy}} + 2\mathrm{i} \frac{Q_{xy}}{Q_{xx} + Q_{yy}} \tag{I.532}$$

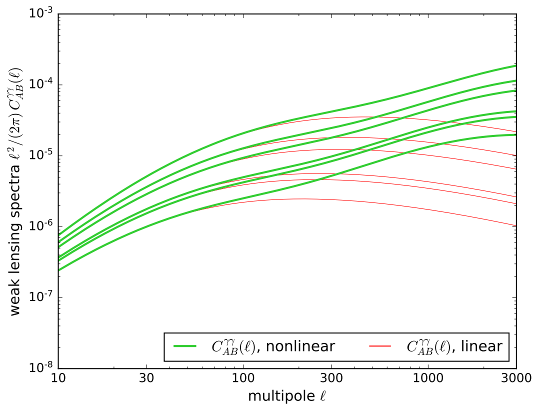


Figure 17: Tomographic spectra $C^{\gamma\gamma}(\ell)$ of the weak lensing shear

Similarly to lensing shear γ, ellipticity is a tensor with two components, and has the property to be invariant under rotations of π, as one can easily imagine by rotating an actual ellipse, and there is a practical notational advantage to combine both components into a complex ellipticity. In the weak lensing limit, the shear γ operates on ellipticity according to

$$\epsilon \to \epsilon + \gamma \tag{I.533}$$

such that an estimate of correlation functions with the observable ϵ provides an estimate of γ, if there is no intrinsic correlation between the ellipticities without lensing. The angular spectrum $C^{\gamma\gamma}(\ell)$ is shown in Fig. 17, for a so-called tomographic measurement, where the galaxies are divided up in redshift intervals.

I.6 *Bayes-inference in cosmology*

Science knows two types of truths: empirical truths correspond to reproducible, objective observations, and logical truths to statements that are derived from axioms in a mathematically consistent way. The way in which science operates is by making predictions for theories, and comparing them to observations, possibly discarding the theories in the process: As such, science is a self-correcting process guided by deduction and inference. Here, inference refers to deriving statistical statements about model parameters from data, or about the validity of entire model classes. The issue in this is that Nature provides data only with an added experimental error or by providing only finite amounts of data with a restricted statistical power: After all, the Hubble volume is finite. Therefore, an observation can not tell in an absolute sense if a theory is true, rather, it provides confidence regions and statistical error estimates, and only allows to differentiate theories that differ by significantly more than the inherent error of the measurement.

One might wonder how randomness in a measurement comes about: but after all, it is simply the result of all variables in the measurement process that can not be controlled in the experiment, because the experimental setup itself is perfectly predictable by the laws of Nature, as it is clearly part of Nature and does not exist in a transcendent way, and a better experiment will essentially allow a better control, resulting in reduced errors. Recording a set of values y_i in a measurement that are Gaussian distributed with error σ_i allows to compute the likelihood $\mathcal{L}(\{y_i\}|\theta_\mu)$, here with the simplifying assumption that all data points are statistically independent,

$$\mathcal{L}(\{y_i\}|\theta_\mu) \propto \prod_i \exp\left(-\frac{1}{2}\left(\frac{y_i - y(x_i)}{\sigma_i}\right)^2\right) = \exp\left(-\frac{1}{2}\sum_i\left(\frac{y_i - y(x_i)}{\sigma_i}\right)^2\right) \tag{I.534}$$

that one would make the measurement if the values result from a theoretial model $y(x)$ with model parameters θ_μ. A likelihood is, for all intents and purposes, a probability as it is a number obeying Kolmogorov's axioms. But there is an important difference in perspective: Usually, one imagines in a probability that there is a fixed random experiment that is able to produce outcomes at different probability, but in a likelihood there is a fixed outcome (the data values y_i) for which one considers now variable models $y(x)$ that differ by the value of their model parameters θ_μ. For the Gaussian error process as in eqn. I.534 it is possible to work with the χ^2-functional instead, which is linked to the likelihood by

$$\mathcal{L}(\{y_i\}|\theta_\mu) \propto \exp\left(-\frac{\chi^2(\theta_\mu)}{2}\right) \quad \text{with} \quad \chi^2 = -\sum_i\left(\frac{y_i - y(x_i)}{\sigma_i}\right)^2 \tag{I.535}$$

Therefore, the likelihood is a function of the model parameters and depends of course on the actual data set. Now, one suspects the true model parameters in the value that maximises $\mathcal{L}$ (or minimises χ^2), as the data that one has is most easily generated by the true model: This is exactly the principle of maximum likelihood. At the same time, eqn. I.535 shows that the origin of least squares-rule originates from the Gaussian error in the data.

But there is a very important catch: The likelihood $\mathcal{L}(\{y_i\}|\theta_\mu)$ is able to quantify how probable it would have been to observe the data for a given choice of θ_μ, but what one actually would like to know is the distribution $p(\theta_\mu|\{y_i\})$ of the model parameters given the observation of the data points. For interchanging the random variable and the condition one needs to use the Bayes-theorem:

$$p(\theta_\mu|\{y_i\}) = \frac{\mathcal{L}(\{y_i\}|\theta_\mu)}{p(\{y_i\})}\, p(\theta_\mu) \tag{I.536}$$

In Bayes' reasoning, the posterior distribution $p(\theta_\mu|\{y_i\})$, i.e. the distribution of the model parameters taking the data into account is given by the likelihood $\mathcal{L}(\{y_i\}|\theta_\mu)$, for which one needs the model to predict the data and the knowledge on the error process, and the prior distribution $p(\theta_\mu)$, which reflects the uncertainty in the model parameters before one has carried out the experiment, normalised by the evidence $p(\{y_i\})$,

$$p(\{y_i\}) = \int \mathrm{d}^n\theta\, \mathcal{L}(\{y_i\}|\theta_\mu)\, p(\theta_\mu) \tag{I.537}$$

which is the probability to obtain the data in the first place given the prior information. If a new experiment is carried out, the posterior distribution from the last experiment would serve as a prior for the next experiment.

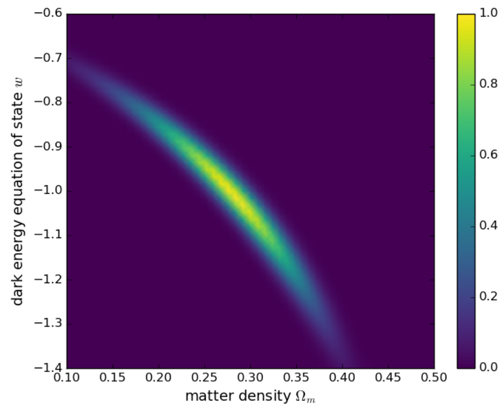


Figure 18: Direct gridded evaluation of the supernova likelihood in the parameters Ω_m and the dark energy equation of state w.

As one essentially multiplies peaked distributions in this process, the resulting distribution will be more peaked as the original ones, indicating that the uncertainty on a model parameter has been reduced by including more data.

Even though the posterior or the likelihood are computable for a given model $y(x)$ parameterised by θ_μ and for a given data set $\{y_i\}$, this is in practise numerically very challenging for highly-dimensional parameter spaces. Instead, one uses the Metropolis-Hastings algorithm (or more efficient variants of it) to generate samples θ_μ that are distributed according to the posterior distribution $p(\theta_\mu|\{y_i\})$.

In the Metropolis-Hastings algorithm one performs a random walk in parameter space on a potential given by the logarithmic likelihood (or the sum of logarithmic likelihood and logarithmic prior, those quantities exist for distributions from the exponential family). For evaluating the random walk, one takes a step from θ_μ to $\theta_\mu + \delta_\mu$, where δ_μ is a random vector from the so-called proposal distribution. Comparing $\mathcal{L}(\theta_\mu)$ with $\mathcal{L}(\theta_\mu + \delta_\mu)$ with the logarithmic likelihood ratio

$$r = \ln \frac{\mathcal{L}(\theta_\mu + \delta_\mu)}{\mathcal{L}(\theta_\mu)} \tag{I.538}$$

gives two possible options: Either $r > 0$, in which case one allows the process to jump $\theta_\mu \rightarrow \theta_\mu + \delta_\mu$, as the new point is a better explanation for the data. Or, $r < 0$, in which one accepts the step to a point with lower likelihood only in $\exp(r)$ of all cases. In this way, the samples θ_μ will follow the distribution $\mathcal{L}(\theta_\mu)$. The comparison between the gridded evaluation of a supernova likelihood in Fig. 18 with the Metropolis-Hastings evaluated result in Fig. 19 at a fraction of the computational cost is striking.

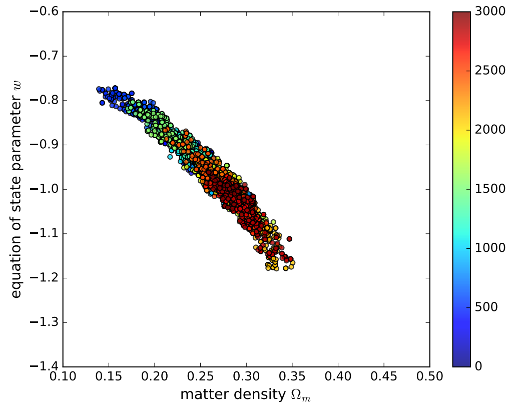


Figure 19: Samples ($n = 3 \times 10^3$) from the supernova likelihood in the parameters Ω_m and w. The colour indicates the number of the sample resulting from the Metropolis-Hastings random walk.

ACKNOWLEDGEMENTS

Ana Marta Machado de Pinho and Marco Hübner deserve my gratitude for helping with the typesetting, my colleagues Burkhard Fuchs, Matteo Maturi, Tilman Plehn and Markus Pössel for lecturing with me, and Robert Reischke, Elena Sellentin, Sarah Konrad, Hans-Martin Rieser, Maximilian Düll and Carsten Littek for accompanying the lecture as tutors. And it is a pleasure to thank Jens Jasche, Philipp M. Merkel, Francesco Pace, ln(a) Sellentin and Britta Zieser, who always understood things really, really well and always found a new perspective. If you're interested, please visit cosmology question of the week for weekly appearing questions on cosmology!

There is a large number of excellent textbooks on cosmology, and my script is not supposed to be a replacement for them. In no particular order I would like to mention:

- M.P. Hobson, G.P. Efstathiou, A.N. Lasenby: General Relativity: An Introduction for Physicists, Cambridge University Press, 2006
- A.R. Liddle, D.H. Lyth: Cosmological Inflation and the Large-Scale Structure, Cambridge University Press, 2000
- V. Mukhanov: Physical Foundations of Cosmology, Cambridge University Press, 2005
- L. Amendola, S. Tsujikawa: Dark Energy - Theory and Observations, Cambridge University Press, 2010
- E. Guyon, J.-P. Hulin, L. Petit, C.D. Mitescu: Physical Hydrodynamics, Oxford University Press, 2001
- R. Durrer: The Cosmic Microwave Background, Cambridge University Press 2008

I would like to acknowledge the lecture LaTeX-class vhbelvadi.com/latex-lecture-notes-class/ by V.H. Belvadi.

INDEX

The lecture notes give an overview of modern cosmology: After introducing the necessary concepts from general relativity, the FLRW-class of cosmological models is discussed, with emphasis on dark energy. Cosmic structure formation, the necessity of dark matter and the interplay between statistics and nonlinear fluid mechanics are treated in detail. The physics behind cosmological observations that have led to the standard model of cosmology is explained, in particular supernovae, the cosmic microwave background and gravitational lensing.

About the Author

Björn Malte Schäfer works at Heidelberg University on problems in modern cosmology, relativity, statistics, and on theoretical physics in general.

UNIVERSITÄT HEIDELBERG
ZUKUNFT
SEIT 1386