X MATHEMATICAL SUPPLEMENT

X.1 *Metric compatibility of the inverse metric*

Metric compatibility $\nabla_\alpha g_{\mu\nu} = 0$ of the metric $g_{\mu\nu}$ itself implies metric compatibility $\nabla_\alpha g^{\mu\nu} = 0$ of the inverse metric $g^{\mu\nu}$. This can be seen by starting from the definition of the inverse metric, $g^{\mu\tau} g_{\tau\nu} = \delta^\mu_\nu$ and have a covariant derivative act on this relation, keeping in mind that the covariant differentiation ∇_α obeys the Leibnitz-rule:

$$\nabla_\alpha g^{\mu\tau} g_{\tau\nu} = \nabla_\alpha g^{\mu\tau} \cdot g_{\tau\nu} + g^{\mu\tau} \nabla_\alpha g_{\tau\nu} = \nabla_\alpha \delta^\mu_\nu \qquad \text{(X.674)}$$

The covariant derivative of the Kronecker-symbol is determined from the fact that it is a tensor with a co- and a contravariant index, i.e.

$$\nabla_\alpha \delta^\mu_\nu = \partial_\alpha \delta^\mu_\nu + \Gamma^\mu{}_{\alpha\beta}\, \delta^\beta_\nu - \Gamma^\beta{}_{\alpha\nu}\, \delta^\mu_\beta = \partial_\alpha \delta^\mu_\nu + \Gamma^\mu{}_{\alpha\nu} - \Gamma^\mu{}_{\alpha\nu} = \partial_\alpha \delta^\mu_\nu \qquad \text{(X.675)}$$

renaming the indices in the second step. The Kronecker-δ is peculiar as a tensor, because it assumes the same values of 0 and 1 in every coordinate choice: One needs a Jacobian for μ and an inverse Jacobian for ν,

$$\delta^\mu_\nu \to \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \delta^\alpha_\beta = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\nu} = \frac{\partial x'^\mu}{\partial x'^\nu} = \delta^\mu_\nu \qquad \text{(X.676)}$$

so that the transformation does not have any influence on the tensor. Therefore, $\partial_\alpha \delta^\mu_\nu = 0$, and one gets

$$\nabla_\alpha g^{\mu\tau} \cdot g_{\tau\nu} + g^{\mu\tau} \nabla_\alpha g_{\tau\nu} = 0 \qquad \text{(X.677)}$$

Metric compatibility of the metric sets the second term to zero, so that one is left with

$$\nabla_\alpha g^{\mu\tau} \cdot g_{\tau\nu} = 0 \qquad \text{(X.678)}$$

from where one can isolate the metric compatibility condition for the inverse metric by contraction with $g^{\nu\beta}$

$$\nabla_\alpha g^{\mu\tau} \cdot g_{\tau\nu} g^{\nu\beta} = \nabla_\alpha g^{\mu\tau} \cdot \delta^\beta_\tau = \nabla_\alpha g^{\mu\beta} = 0 \qquad \text{(X.679)}$$

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There is a large number of excellent textbooks on relativity, and my script is not supposed to be a replacement for them. In no particular order I would like to mention:

- M.P. Hobson, G.P. Efstathiou, A.N. Lasenby: General Relativity: An Introduction for Physicists, Cambridge University Press, 2006
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