I GRAVITY FROM A VARIATIONAL PRINCIPLE

I.1 *Variational principles for particles and fields*

Variational principles play a huge role in theoretical physics, and only in the context of relativity becomes their true nature apparent: The Lagrange-function $\mathcal{L}$ is composed of invariants, and the Euler-Lagrange-equation carrying out the variation injects coordinates and generates a covariant equation of motion. There are fundamental properties of the Lagrange-function $\mathcal{L}$, for instance its convexity which makes sure that a global minimum for the variation exists and that the Legendre transform is well-defined, ultimately yielding the Hamilton-function $\mathcal{H}$ including possible conserved conjugate momenta.

While Hamilton's principle $\delta S = 0$ is straightforward to interpret for the motion of a particle as the arc length through spacetime, an analogous interpretation for fields is a bit more involved: After all, the field equation establishes a relation between the geometry of the field and the strength of the amplitudes and the source, so the variation is effectively searching among all field configurations for the single one that minimises the action. It is a curious property that vacuum solutions provide typically a lower bound on the action, for instance in electrodynamics: The Maxwell-action S is defined through the invariant Frobenius norm of $F_{\alpha\beta}$,

$$S = \frac{1}{4}\int d^4x\sqrt{-\det\eta}\,\eta^{\alpha\mu}\,\eta^{\beta\nu}\,F_{\alpha\beta}F_{\mu\nu} \tag{I.570}$$

integrated over spacetime, where we already introduced the covolume $\sqrt{-\det\eta}$ to make d^4x invariant under coordinate transforms. For vacuum solutions such as plane waves $\eta^{\alpha\mu}\,\eta^{\beta\nu}\,F_{\alpha\beta}F_{\mu\nu} \propto E_iE^i - B_iB^i = 0$ because for a wave the absolute values of E^i and B^i are equal. Incidentially, the (only) other quadratic invariant $\eta^{\alpha\mu}\,\eta^{\beta\nu}\,\tilde{F}_{\alpha\beta}F_{\mu\nu} \propto E_iB^i = 0$ as well, as the electric and magnetic fields are always perpendicular. Starting with squares of first derivatives of the potentials makes sure that one obtains a linear field equation which fulfils the superposition principle and excluding higher derivatives makes sure that the Ostrogradsky-theorem is respected and the Hamilton-function bounded from below.

As Lagrange-functions only ever appear as an integral in the action and as the Hamilton-principle makes a statement only about the action, any reformulation of the Lagrange-function by integration is permissible and should yield exactly the same equations of motion. For instance, a point particle would have an equivalent action if one writes

$$S = \int dt\,\mathcal{L} = \int dt\frac{1}{2}\delta_{ij}\,\dot{x}^i\dot{x}^j - \Phi(x) = -\int dt\frac{1}{2}\delta_{ij}\,\ddot{x}^i\dot{x}^j + \Phi(x) \tag{I.571}$$

if the boundary term arising in the integration by parts is neglected. But of course this form of the action calls for a generalised Euler-Lagrange equation that is capable of dealing with second derivatives $\ddot{x}^i$ of the trajectory $x^i(t)$. In fact, the variation for $\mathcal{L}(x^i, \dot{x}^i, \ddot{x}^i)$ is given by

$$\delta S = \int dt\,\frac{\partial\mathcal{L}}{\partial x^i}\delta x^i + \frac{\partial\mathcal{L}}{\partial\dot{x}^i}\delta\dot{x}^i + \frac{\partial\mathcal{L}}{\partial\ddot{x}^i}\delta\ddot{x}^i = \int dt\left(\frac{\partial\mathcal{L}}{\partial x^i} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial\mathcal{L}}{\partial\ddot{x}^j}\right)\delta x^j = 0 \tag{I.572}$$

with a single integration for the second and a double integration for the third term. In fact, this new Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}^i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial\mathcal{L}}{\partial\ddot{x}^j} = 0 \tag{I.573}$$

works perfectly: $\mathcal{L} = \frac{1}{2}x\ddot{x} + \Phi(x)$ has the derivatives

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{\ddot{x}}{2} + \partial\Phi, \quad \frac{\partial\mathcal{L}}{\partial\dot{x}} = 0, \quad \frac{\partial\mathcal{L}}{\partial\ddot{x}} = \frac{x}{2}, \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial\mathcal{L}}{\partial\ddot{x}} = \frac{\ddot{x}}{2} \tag{I.574}$$

which get assembled in the Euler-Lagrange equation to $\ddot{x}^i + \partial^i\Phi = 0$.

Almost exactly the same argument holds for a scalar field on a Euclidean background: The Lagrange-density $\mathcal{L} = 1/2\ \delta^{ij}\partial_i\Phi\partial_j\Phi - \mathrm{V}(\Phi)$ can be integrated by parts to yield the equivalent form,

$$\mathrm{S} = \int \mathrm{d}^3x\,\mathcal{L} = \int \mathrm{d}^3x\,\frac{1}{2}\delta^{ij}\,\partial_i\Phi\,\partial_j\Phi = -\int \mathrm{d}^3x\,\frac{1}{2}\Phi\delta^{ij}\,\partial_i\partial_j\Phi\cdot = -\int \mathrm{d}^3x : \frac{1}{2}\Phi\Delta\Phi \tag{I.575}$$

where again a generalised Euler-Lagrange equation is required for the variation $\delta\mathrm{S} = 0$,

$$\delta\mathrm{S} = \int \mathrm{d}^3x\,\frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial\partial_i\Phi}\delta\partial_i\Phi + \frac{\partial\mathcal{L}}{\partial\partial_i\partial_j\Phi}\delta\partial_i\partial_j\Phi \tag{I.576}$$

Single and double integration by parts while neglecting the boundary terms, where the variation is zero, yields

$$\delta\mathrm{S} = \int \mathrm{d}^3x\left(\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_i\frac{\partial\mathcal{L}}{\partial\partial_i\Phi} + \partial_i\partial_j\frac{\partial\mathcal{L}}{\partial\partial_i\partial_j\Phi}\right)\delta\Phi = 0 \tag{I.577}$$

from which one can read off the suitable second-order Euler-Lagrange equation. Going through the example again recovers conventional Poisson-equation $\Delta\Phi = \mathrm{dV}/\mathrm{d}\Phi = 4\pi\mathrm{G}\rho$ for $\mathrm{V}(\phi) = 4\pi\mathrm{G}\rho\Phi$.

Things get a bit more interesting with the Maxwell-field: The variation of the field can not be, in general, set to zero on the surface of a spacetime volume, because for instance a plane wave as a perfectly valid solution to the field equation exists for arbitrarily early and late times. But there is the freedom to pick a gauge, and in fact the surface terms can be set to zero by demanding the Lorenz-gauge $\partial^\mu\mathrm{A}_\mu = 0$ to be valid.

$$\mathrm{S} = \int \mathrm{d}^4x\,\sqrt{-\eta}\cdot\eta^{\alpha\mu}\eta^{\beta\nu}\,\mathrm{F}_{\alpha\beta}\mathrm{F}_{\mu\nu} = \ldots = -\int \mathrm{d}^4x\,\sqrt{-\eta}\cdot 2\cdot\eta^{\beta\nu}\mathrm{A}_\beta\Box\,\mathrm{A}_\nu \tag{I.578}$$

with the d'Alembert-operator $\Box = \eta^{\alpha\mu}\,\partial_\alpha\partial_\mu$. The generalised Euler-Lagrange equation needed to deal with the second-order action is

$$\delta\mathrm{S} = \int \mathrm{d}^4x\,\sqrt{-\eta}\cdot\left(\frac{\partial\mathcal{L}}{\partial\mathrm{A}_\alpha}\delta\mathrm{A}_\alpha + \frac{\mathcal{L}}{\partial\partial_\mu\mathrm{A}_\alpha}\delta\partial_\mu\mathrm{A}_\alpha + \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\mathrm{A}_\alpha}\delta\partial_\mu\partial_\nu\mathrm{A}_\alpha\right) \tag{I.579}$$

with $\delta\partial_\mu\mathrm{A}_\alpha = \partial_\mu\delta\mathrm{A}_\alpha$ and $\delta\partial_\mu\partial_\nu\mathrm{A}_\alpha = \partial_\mu\partial_\nu\delta\mathrm{A}_\alpha$. Then, integration by parts suggests

$$S = \int \mathrm{d}^4x\,\sqrt{-\eta}\left(\frac{\partial\mathcal{L}}{\partial A_\alpha} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\alpha} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu A_\alpha}\right)\delta A_\alpha = 0 \tag{I.580}$$

where again Hamilton's principle determines the Euler-Lagrange equation. Substitution yields the perfectly normal vacuum field equation for the potential A_α

$$\partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\Phi} = \Box\, A_\alpha = \eta^{\mu\nu}\,\partial_\mu\partial_\nu\, A_\alpha = 0 \tag{I.581}$$

in Lorenz-gauge. In summary, there are possible reformulations of the Lagrange-densities involving the product of the fields and its second derivative (please notice the locality here!), which give exactly the same field equation after variation. Technically, there are subtleties related to the boundary terms of the integration, which can be set to zero in certain gauges, for instance by assuming Lorenz-gauge $g^{\mu\nu}\nabla_\mu A_\nu = 0$ on the boundary for the Maxwell-field A_μ.

I.2 *Variational principles on manifolds*

Would it be possible to formulate a variational principle on a manifold? Clearly yes, but we would have to use the covariant derivative ∇_μ instead of the partial derivative ∂_μ as a general metric $g_{\mu\nu}$ as a globally Cartesian coordinate choice would not be possible. Let's try this with a scalar field first: Clearly, the action should be invariant under coordinate changes with a volume element $\mathrm{d}^4x\sqrt{-g}$, and the Lagrange-function should depend on ϕ, the covariant derivative $\nabla_\mu\phi$ and the metric $g_{\mu\nu}$ that mediates the geometry of the manifold:

$$S = \int_V \mathrm{d}^4x\,\sqrt{-g}\cdot\mathcal{L}\big(\phi, \nabla_\mu\,\phi, g_{\mu\nu}\big), \tag{I.582}$$

consisting of generally invariant scalars built from ϕ and $\nabla_\mu\phi$. Hamilton's principle $\delta S =$ would then imply for the variation that

$$\delta S = \int_V \mathrm{d}^4x\,\sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial\Psi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\delta\,\nabla_\mu\phi\right) = 0 \tag{I.583}$$

We would continue with the usual $\delta\,\nabla_\mu\phi = \nabla_\mu\delta\phi$ but reach an impasse when it comes to the integration by parts, as there is the covariant ∇_μ instead of the partial ∂_μ: We need a generalisation of the Gauß-theorem for manifolds:

$$\int_V \mathrm{d}^4x\,\sqrt{-g}\cdot\nabla_\mu\upsilon^\mu = \int_{\partial V}\mathrm{d}A_\mu\,\sqrt{|\gamma|}\,\upsilon^\mu \tag{I.584}$$

with the induced metric γ on the boundary ∂V,

$$\sqrt{-g}\,\Big|_{\partial V} \equiv \sqrt{|\gamma|} \tag{I.585}$$

The covariant divergence can be written as a conventional partial divergence with the covolume, such that

$$\int_V d^4x\,\sqrt{-g}\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{-g}\cdot v^\mu\right) = \int_V d^4x\,\partial_\mu\left(\sqrt{-g}\cdot v^\mu\right) = \int_{\partial V} dA_\mu\,\sqrt{|\gamma|}\cdot v^\mu \tag{I.586}$$

With these tools, one can write:

$$\int_V d^4x\,\sqrt{-g}\,\nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\delta\phi\right) = \int_{\partial V} dA_\mu\sqrt{|\gamma|}\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\delta\phi\right) \tag{I.587}$$

Considering

$$\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\delta\phi \equiv v^\mu \tag{I.588}$$

as the vector field v^μ, the product rule suggests that

$$= \int_V d^4x\,\sqrt{-g}\,\nabla_\mu\,\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\delta\phi + \int d^4x\,\sqrt{-g}\cdot\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\nabla_\mu\,\delta\phi \tag{I.589}$$

so that finally

$$\int_V d^4x\,\sqrt{-g}\,\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\nabla_\mu\,\delta\phi = -\int_V d^4x\,\sqrt{-g}\,\nabla_\mu\,\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\delta\phi \tag{I.590}$$

and the Euler-Lagrange equation on a manifold has exactly the same form as the conventional one, with a ∇_μ replacing the ∂_μ,

$$\nabla_\mu\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi} = \frac{\partial\mathcal{L}}{\partial\phi} \tag{I.591}$$

I.3 *Gauge transformations on manifolds and source terms*

Clearly, coordinate transformations and a nontrivial geometry can be dealt with as discussed in the previous chapter, but what about gauge transformations? Writing

$$\mathcal{L} \to \mathcal{L} + \nabla_\mu\,Q^\mu(\phi) \tag{I.592}$$

and having the transformation generated by Q^μ would imply that S becomes

$$S \to S + \int_V d^4x\,\sqrt{-g}\,\nabla_\mu\,Q^\mu \tag{I.593}$$

with the variation δS

$$\delta S \to \delta S + \int_V d^4x\,\sqrt{-g}\,\nabla_\mu\,\delta Q^\mu = \delta S + \int_V d^4x\,\sqrt{-g}\,\nabla_\mu\left(\frac{\partial Q^\mu}{\partial\phi}\delta\phi\right) \tag{I.594}$$

Clearly, invariance is only given if

$$\int_V d^4x\,\sqrt{-g}\,\nabla_\mu\left(\frac{\partial Q}{\partial\phi}\delta\phi\right) = \int_{\partial V} dA_\mu\cdot\sqrt{|\gamma|}\cdot\frac{\partial Q^\mu}{\partial\phi}\delta\phi = 0 \tag{I.595}$$

implying that the variation of the fields $\delta\phi = 0$ is valid on the boundary ∂V.

Let's look at Maxwell electrodynamics as an intuitive example. Acting on the Lagrange-density

$$\mathcal{L} = \frac{1}{4}g^{\alpha\mu}g^{\beta\nu}\,F_{\alpha\beta}F_{\mu\nu} + +g^{\alpha\beta}\,A_\alpha\,\jmath_\beta \tag{I.596}$$

with a gauge transformation $A_\alpha \to A_\alpha + \nabla_\alpha\chi$ with a gauge function χ does not change the field tensor $F_{\alpha\beta}$: Formally it transitions to

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha \;\to\; F_{\alpha\beta} + \left(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right)\chi = F_{\alpha\beta} \tag{I.597}$$

but the additional term is zero as a consequence of the torsion-free condition $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$, making $\left(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right)\chi = \left(\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha\right)\chi = 0$. That implies that the gauge-transformed Lagrange-density becomes:

$$\mathcal{L} = \frac{1}{4}g^{\alpha\mu}g^{\beta\nu}\,F_{\alpha\beta}F_{\mu\nu} + g^{\alpha\beta}A_\alpha\jmath_\beta + g^{\alpha\beta}\,\nabla_\alpha\,\chi\cdot\jmath_\beta \tag{I.598}$$

with $g^{\alpha\beta}\,\nabla_\alpha\,\chi\cdot\jmath_\beta$ being an additional term. This term, however, is necessarily equivalent to

$$\int_V d^4x\sqrt{-g}\,g^{\alpha\beta}\nabla_\alpha\chi\cdot\jmath_\beta = \int_V d^4x\sqrt{-g}\,g^{\alpha\beta}\nabla_\alpha\left[\chi\cdot\jmath_\beta\right] - \int_V d^4x\sqrt{-g}\,\chi\cdot g^{\alpha\beta}\nabla_\alpha\jmath_\beta = 0 \tag{I.599}$$

where the first term vanishes as a boundary term and the second vanishes if charge is covariantly conserved, $g^{\alpha\beta}\,\nabla_\alpha\,\jmath_\beta = 0$.

The issue does not arise in the homogeneous Maxwell-equations. There, the covariant generalisation

$$g^{\alpha\mu}\,\nabla_\alpha\,\tilde{F}_{\mu\nu} = 0 \tag{I.600}$$

of the Bianchi identity

$$\nabla_\alpha\,F_{\mu\nu} + \nabla_\mu\,F_{\nu\alpha} + \nabla_\nu\,F_{\alpha\mu} = 0 \tag{I.601}$$

with the dual tensor $\tilde{F}_{\alpha\beta}$ is automatically gauge-independent, as $F_{\alpha\beta}$ and $\tilde{F}_{\alpha\beta}$ do not change under gauge transformations. The relation between the two are

$$\tilde{F}_{\alpha\beta} = -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\,F^{\mu\nu} \quad\text{and}\quad F_{\mu\nu} = +\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\tilde{F}^{\alpha\beta} \tag{I.602}$$

so that both become auto-dual, $\tilde{\tilde{F}}_{\alpha\beta} = F_{\alpha\beta}$,

$$\tilde{\tilde{F}}_{\alpha\beta} = -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\,\tilde{F}^{\mu\nu} = -\frac{1}{4}\epsilon_{\alpha\beta\mu\nu}\,\epsilon^{\mu\nu\gamma\delta}\,F_{\gamma\delta} = F_{\alpha\beta} \quad\text{with}\quad \epsilon_{\alpha\beta\mu\nu}\epsilon^{\mu\nu\gamma\delta} = -2!2!\cdot\delta^\gamma_\alpha\,\delta^\delta_\beta \tag{I.603}$$

I.4 *Invariant volume elements*

The integration measure for volumes needs to be independent of the coordinate choice. The transformation changes vectors according to

$$\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial x'^\mu}\,\mathrm{d}x'^\nu \tag{I.604}$$

but clearly that coordinate change implies for the volume element:

$$\mathrm{d}^n x = \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)\mathrm{d}^n x' \tag{I.605}$$

with the functional determinant as a prefactor. At the same time, the metric transforms like a rank-2 tensor,

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu}\,\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\mathrm{d}x'^\alpha\,\mathrm{d}x'^\beta = g'_{\alpha\beta}\,\mathrm{d}x'^\alpha \mathrm{d}x'^\beta \tag{I.606}$$

i.e. inverse to the vector and with two powers of the Jacobian for the determinant of the metric (as the line element is invariant):

$$\det(g'_{\alpha\beta}) = \det(g_{\mu\nu})\cdot\left(\det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)\right)^2 \tag{I.607}$$

implying the definition of an invariant volume element as

$$\mathrm{d}^n x\,\sqrt{-g} \rightarrow \mathrm{d}^n x'\cdot\det\mathrm{J}\cdot\frac{\sqrt{-g'}}{\sqrt{(\det\mathrm{J})^2}} = \mathrm{d}^n x'\sqrt{-g'} \tag{I.608}$$

with the functional determinant $\mathrm{J} = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)$.

It is important to realise that $\sqrt{-g}$ is a density, not a scalar, as $\mathrm{d}^n x\sqrt{-g}$ is scalar and therefore invariant under coordinate transformations. In particular,

$$\nabla_\mu\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\nabla_\mu g = \frac{1}{2\sqrt{-g}}\nabla_\mu\Big(\exp\operatorname{tr}\ln g_{\alpha\beta}\Big) = \\ \frac{-1}{2\sqrt{-g}}\,g\cdot\operatorname{tr}g^{-1}\cdot\nabla_\mu\,g = \frac{1}{2}\sqrt{-g}\cdot g^{\alpha\beta}\,\nabla_\mu\,g_{\alpha\beta} = 0 \tag{I.609}$$

as a consequence of metric compatibility of ∇_μ; but it is the case that $\nabla_\mu\sqrt{-g} \neq \partial_\mu\sqrt{-g}$ because of the missing scalar property of $\sqrt{-g}$: The covolume is a density rather than a scalar, and the reduction of the covariant derivative $\nabla_\mu\phi = \partial_\mu\phi$ for scalar fields is not applicable for $\sqrt{-g}$.

I.5 *Einstein-Hilbert: gravity from a variational principle*

Up to this point we postulated the gravitational field equation and convinced ourselves that it had properties desirable in a field equation. A variational principle would require the construction of an action for the metric

$$S = \int d^4x \, \sqrt{-g} \, \mathcal{L}\Big(g_{\alpha\beta}, \nabla_\mu \, g_{\alpha\beta}, \, \nabla_\mu \nabla_\nu g_{\alpha\beta}\Big) \tag{I.610}$$

composed of invariants such that after variation a covariant field equation is obtained. The Lagrange-density can in principle depend on the metric $g_{\alpha\beta}$ as the dynamical field itself and its first and second derivatives. There, $\nabla_\mu \, g_{\alpha\beta}$ is impossible to use as it always vanishes due to metric compatibility, so $\partial_\mu \, g_{\alpha\beta}$ or $\Gamma^\mu_{\alpha\beta}$ would be better alternatives, but we have already argued that the gravitational field should rather be contained in the second than the first derivatives of the metric: According to the equivalence principle, first derivatives would automatically be zero in a freely falling frame.

As invariants containing second derivatives, the Ricci-scalar $R = g^{\alpha\mu} g^{\beta\nu} \, R_{\alpha\beta\mu\nu}$ or the Kretschmann-scalar $K = g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma} \, R_{\alpha\beta\gamma\delta} R_{\mu\nu\rho\sigma}$ would be possible choices, although we would prefer R from the intuition on the contraction of freely falling clouds of point particles caused by Ricci-curvature. Perhaps a bit surprisingly, a straightforward constant Λ would be fine, too.

Postulating the Einstein-Hilbert-Lagrange density as being the simplest, local invariant second-order action

$$S = \int d^4x \, \sqrt{-g} \, \big(R - 2\Lambda\big) \tag{I.611}$$

one can in fact derive the gravitational field equation through variation of the metric, $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. In the Ricci-scalar $R = g^{\mu\nu} R_{\mu\nu}$, however, there is the inverse metric $g^{\mu\nu}$ so actually one needs to vary with respect to that quantity, too. The two variations are related by

$$\delta\left(\delta^\mu_\nu\right) = \delta\left(g^{\mu\alpha} g_{\alpha\nu}\right) = \delta g^{\mu\alpha} \cdot g_{\alpha\nu} + g^{\mu\alpha} \cdot \delta g_{\alpha\nu} = 0 \tag{I.612}$$

where one can isolate $\delta^{\mu\alpha}$ by contraction with $g^{\nu\beta}$,

$$\delta g^{\mu\alpha} \, g_{\alpha\nu} \cdot g^{\nu\beta} = \delta g^{\mu\beta} = -g^{\nu\beta} g^{\mu\alpha} \, \delta g_{\alpha\nu} \tag{I.613}$$

with an additional minus-sign appearing.

Let's ignore the cosmological constant for a second, $\Lambda = 0$. Then, the variation δS of S becomes

$$\delta S = \int d^4x \left[\delta\sqrt{-g} \cdot R + \sqrt{-g} \cdot \delta g^{\mu\nu} \cdot R_{\mu\nu} + \sqrt{-g} \cdot g^{\mu\nu} \, \delta R_{\mu\nu}\right] \tag{I.614}$$

which requires a relation between $\delta\sqrt{-g}$ and $\delta g^{\mu\nu}$ as well as between $\delta R_{\mu\nu}$ and $\delta g^{\mu\nu}$, while the second term is already in good shape, being directly proportional to $\delta g^{\mu\nu}$. The variation of the covolume is done by

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}} \cdot \delta g = \frac{1}{2\sqrt{-g}} \cdot g \cdot g^{\mu\nu} \, \delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g} \cdot g_{\mu\nu} \, \delta g^{\mu\nu} \tag{I.615}$$

keeping in mind that $\ln g = \ln\det g_{\mu\nu} = \mathrm{tr}\ln g_{\mu\nu}$, such that $g = \exp\mathrm{tr}\ln g_{\mu\nu}$. δ acts like a derivative, so that g is reproduced as the derivative of the exponential, the trace is linear and the derivative of a matrix valued logarithm is given by the matrix inverse, multiplied with the internal derivative. Switching from $\delta g_{\mu\nu}$ to $\delta g^{\mu\nu}$ then introduces yet another minus sign. Collecting all results so far gives the intermediate formula

$$\delta S = \int d^4x\, \sqrt{-g} \cdot \left(R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}\right) \delta g^{\mu\nu} + \int d^4x\, \sqrt{-g}\, \delta R_{\mu\nu}\, g^{\mu\nu} \rightarrow 0 \qquad \text{(I.616)}$$

which is already very reminiscent of the field equation if Hamilton's principle $\delta S = 0$ is assumed, if only the last term was zero.

For continuing one needs the Palatini-identity, which relates the variation of the Ricci-tensor to covariant derivatives of the varied Christoffel-symbols, which I guess merits a few words of explanation. The Riemann-curvature $R_{\alpha\beta\mu\nu}$ is in general a function of $\Gamma^{\alpha}{}_{\mu\nu}$ and its derivatives $\partial_{\beta}\Gamma^{\alpha}{}_{\mu\nu}$, as suggested by parallel transport. In locally Cartesian coordinates $\Gamma^{\alpha}{}_{\mu\nu} = 0$ as in these coordinates partial derivatives of the metric vanish, but $\partial_{\beta}\Gamma^{\alpha}{}_{\mu\nu}$ are not necessarily zero. That implies that the Riemann-curvature only depends on the derivatives of the Christoffel-symbols but not on the squares. Secondly, the variation $\delta\Gamma^{\alpha}{}_{\mu\nu}$ of the Christoffel-symbols is a tensor, as the non-tensorial contributions drop out. And thirdly, $\nabla_{\mu} = \delta_{\mu}$ in locally Cartesian coordinates, as $\Gamma^{\alpha}{}_{\mu\nu} = 0$.

Putting everything together lets us write for the Riemann-tensor

$$\delta R^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\, \delta\Gamma^{\alpha}{}_{\beta\nu} - \partial_{\nu}\, \delta\Gamma^{\alpha}{}_{\beta\mu} = \nabla_{\mu}\, \delta\Gamma^{\alpha}{}_{\beta\nu} - \nabla\, \delta\Gamma^{\alpha}{}_{\beta\mu} \qquad \text{(I.617)}$$

and consequently for the Ricci-tensor

$$\delta R_{\beta\nu} = \nabla_{\mu}\, \delta\Gamma^{\mu}{}_{\beta\nu} - \nabla_{\nu}\, \delta\Gamma^{\mu}{}_{\beta\mu} \qquad \text{(I.618)}$$

which is the sought after Palatini-identity. Inspecting the surplus term of the Einstein-Hilbert action

$$\int d^4x\, \sqrt{-g} \cdot \delta R_{\mu\nu} \cdot g^{\mu\nu} = \int d^4x\, \sqrt{-g}\, g^{\beta\nu} \left[\nabla_{\mu}\, \delta\Gamma^{\mu}{}_{\beta\nu} - \nabla_{\nu}\, \delta\Gamma^{\mu}{}_{\beta\mu}\right] \qquad \text{(I.619)}$$

shows that both terms arising due to the Palatini-action are in fact covariant divergences, which would vanish when converted into surface integrals.

The cosmological constant Λ requires only the variation of the covolume in $S = \int d^4x\, \sqrt{-g} \cdot (2\Lambda)$, such that one gets:

$$\delta S = \int d^4x\, \delta\sqrt{-g}\, \Lambda = \int d^4x\, \sqrt{-g} \cdot \left(-\Lambda g_{\mu\nu}\right) \cdot \delta g^{\mu\nu} \qquad \text{(I.620)}$$

Finally, one finds that the variation of the Einstein-Hilbert-Lagrange density

$$S = \int d^4x\, \sqrt{-g} \left(R - 2\Lambda\right) \qquad \text{(I.621)}$$

in fact recovers the gravitational field equation (in vacuum)

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad \text{(I.622)}$$

Let's have a quick look at the non-relativistic limit of the Einstein-Hilbert-Lagrange density. Analogous pairs of quantities are Φ and $g_{\mu\nu}$, then ∂^i and $\Gamma^{\alpha}{}_{\mu\nu}$, as well as $\partial^i\partial^j\Phi$ and $R_{\alpha\beta\mu\nu}$ and finally $\Delta\Phi$ and $R_{\mu\nu}$. The weakly perturbed line element on an otherwise flat Minkowski background

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right)dx_i dx^i \tag{I.623}$$

can be used to extract the metric and to compute covolume through the determinant,

$$\det(g_{\mu\nu}) = -\left(1 + \frac{2\Phi}{c^2}\right)\left(1 - \frac{2\Phi}{c^2}\right)^3 = -\left[1 - \frac{6\Phi}{c^2} + \frac{2\Phi}{c^2} + \mathcal{O}\left(\left(\frac{\Phi}{c^2}\right)^2\right)\right] =\simeq -\left(1 - \frac{4\Phi}{c^2}\right) \tag{I.624}$$

such that the covolume becomes $\sqrt{-g} \simeq 1 - 2\Phi/c^2$ at lowest order. This means effectively, that in the classical, second-order Lagrange density for Newtonian gravity,

$$S = -\int d^3x\, \Phi\Delta\Phi = \int d^3x\, \delta_{ij}\partial^i\Phi\, \partial^j\Phi \tag{I.625}$$

the first factor of Φ could be thought of as a remainder of the covolume, while the second factor $\Delta\Phi$ appears as the Ricci-curvature. Integration by parts recovers the conventional form, which immediately poses the question if one could construct a gravitational action from squares of Christoffel symbols: This will be the Einstein-Palatini-action.

I.6 *Palatini-variation: metric $g_{\mu\nu}$ and connection $\Gamma^{\alpha}{}_{\mu\nu}$*

There is an alternative approach to deriving the field equation from a variational principle where the metric and the connection are interpreted as independent fields: Then, the field equation and the Levi-Civita connection are simultaneous results of the variational principle.

$$S = \int d^4x\sqrt{-g}\, R = \int d^4x\, \sqrt{-g}g^{\beta\nu}\, R^{\alpha}{}_{\beta\alpha\nu} =$$

$$\int d^4x\, \sqrt{-g}g^{\beta\nu}\left[\partial_\alpha\Gamma^{\alpha}{}_{\beta\alpha} - \partial_\nu\Gamma^{\alpha}{}_{\beta\alpha} + \Gamma^{\alpha}{}_{\gamma\nu}\Gamma^{\gamma}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\gamma\nu}\Gamma^{\gamma}{}_{\beta\alpha}\right] \tag{I.626}$$

where there is no a-priori assumption about the relationship between the metric $g_{\mu\nu}$ and connection $\Gamma^{\alpha}{}_{\mu\nu}$. The variation with the metric borrows from the derivation in the previous section and gives directly the vacuum-field equation

$$\delta S = \int d^4x\, \delta\left(\sqrt{-g}\, g^{\beta\nu}\right) R_{\beta\nu} = 0 \rightarrow R_{\beta\nu} = 0 \tag{I.627}$$

as the Ricci-tensor $R_{\beta\nu}$ was taken to depend only on $\Gamma^{\alpha}{}_{\mu\nu}$ and $\partial_\beta\Gamma^{\alpha}{}_{\mu\nu}$, not on $g_{\mu\nu}$. Then, the variation with respect to the connection coefficients $\Gamma^{\alpha}{}_{\mu\nu}$ as the second independent field can be computed as follows. Firstly, on uses the Palatini-identity to get

$$\delta S = \int d^4x\, \sqrt{-g}\, g^{\beta\nu}\, \delta R_{\beta\nu} = \int d^4x\, \sqrt{-g}\, g^{\beta\nu} \cdot \left(\nabla_\mu\, \delta\Gamma^{\mu}{}_{\beta\nu} - \nabla_\nu\, \delta\Gamma^{\mu}{}_{\beta\mu}\right) \tag{I.628}$$

and rewrites it with the Leibnitz-rule: Please keep in mind that we did not yet make any assumption about e.g. metric compatibility, so terms of the type $\nabla_\mu g^{\mu\nu}$ are not automatically zero.

$$\delta S = \int d^4x\,\sqrt{-g}\cdot\nabla_\mu\left(g^{\beta\nu}\,\delta\Gamma^\mu{}_{\beta\nu}\right) - \nabla_\nu\left(g^{\beta\nu}\,\delta\Gamma^\mu{}_{\beta\mu}\right) = \\ -\int d^4x\,\sqrt{-g}\cdot\nabla_\mu\,g^{\beta\nu}\cdot\delta\Gamma^\mu{}_{\beta\nu} - \nabla_\nu\,g^{\beta\nu}\cdot\delta\Gamma^\mu{}_{\beta\mu} \quad \text{(I.629)}$$

The first two terms vanish as covariant divergences, as they can be rewritten as boundary integrals, leaving

$$\delta S = -\int d^4x\,\sqrt{-g}\cdot\left[\nabla_\mu\,g^{\beta\nu} - \delta^\beta_\mu\,\nabla_\alpha\,g^{\gamma\alpha}\right]\delta\Gamma^\mu{}_{\beta\nu} = 0 \quad \text{(I.630)}$$

Then, we realise that the Christoffel-symbol is symmetric in the lower two indices $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}$ if the connection was torsion free. The last equation has to be equal to zero as required by Hamilton's principle $\delta S = 0$, which then implies that the term in the square brackets, which does not have a perfect antisymmetry pertaining to the index pair μ, ν, has to vanish identically. From that one can conclude

$$\nabla_\mu\,g^{\beta\nu} = \frac{1}{2}\left[\delta^\beta_\mu\,\nabla_\alpha\,g^{\nu\alpha} + \delta^\beta_\nu\,\nabla_\alpha\,g^{\mu\alpha}\right] = 0 \quad \text{(I.631)}$$

after symmetrisation, and from that metric compatibility $\nabla_\mu g_{\beta\nu} = 0$, with the argument that

$$\nabla_\mu\left[g^{\alpha\beta}g_{\beta\gamma}\right] = \nabla_\mu\left(\delta^\alpha_\gamma\right) = 0 = \nabla_\mu g^{\alpha\beta}\cdot g_{\beta\gamma} + g^{\alpha\beta}\nabla_\mu g_{\beta\gamma} \quad \text{(I.632)}$$

implying that metric compatibility of the inverse metric is consistent with metric compatibility of the metric (please see Appendix X.1 for the detailed derivation).

These relations are sufficient to compute the Christoffel-symbol from the metric, as $\nabla_\mu\,g_{\beta\nu} = \partial_\mu\,g_{\beta\nu} - \Gamma^\alpha{}_{\mu\beta}\,g_{\alpha\nu} - \Gamma^\alpha{}_{\mu\nu}\,g_{\beta\alpha} = 0$ and the two cyclic permutations define already

$$\Gamma^\alpha{}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left[\partial_\mu\,g_{\beta\nu} + \partial_\nu\,g_{\mu\beta} - \partial_\beta\,g_{\mu\nu}\right]. \quad \text{(I.633)}$$

I.7 *Coupling to matter and generation of the energy momentum tensor*

The field equation needs to be coupled to energy and momentum in the form of energy momentum-tensor $T_{\mu\nu}$, such that curvature is induced into spacetime. A combined action including geometry and the material fields could be

$$S = \int d^4x\left(\sqrt{-g}\left[R - 2\Lambda\right] + \kappa\,L_m\right) \quad \text{(I.634)}$$

with an a-priori unknown coupling constant κ put as a prefactor to the Lagrange density L_m of the non-gravitational fields: Commonly, one calls this the matter-term, but actually it refers to any field that is defined on the spacetime.

Variation would recover the field equation

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda\, g_{\mu\nu} \tag{I.635}$$

which would work out if

$$\delta S_m = \delta \int d^4x\, L = \int d^4x \frac{\delta L}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = \frac{1}{2} \int d^4x \sqrt{-g}\, T_{\mu\nu} \delta g^{\mu\nu} = -\frac{1}{2} \int d^4x \sqrt{-g}\, T^{\mu\nu} \delta g_{\mu\nu} \tag{I.636}$$

by definition, with the energy momentum tensor $T_{\mu\nu}$ and the coupling constant $\kappa = 8\pi G/c^4$. Then, the symmetry of $T_{\mu\nu}$ is implied by $g^{\mu\nu}$, and the variation of S with respect to the (inverse) metric yields the correct field equation. Vice versa, this can only be consistent if

$$\frac{\delta L}{\delta g^{\mu\nu}} = \frac{\sqrt{-g}}{2} T_{\mu\nu} \tag{I.637}$$

I.8 *Dynamics of the energy-momentum tensor*

General relativity is the theory for the dynamics of spacetime for energy-momentum conserving fields, which is formulated in terms of the covariant divergence of the energy-momentum tensor $T_{\mu\nu}$,

$$g^{\alpha\mu}\, \nabla_\alpha\, T_{\mu\nu} = 0 \tag{I.638}$$

The variation in Hamilton's principle can be generated by an infinitesimal coordinate shift, which can have two important consequences: It should, applied to the matter-part of the action, reproduce covariant energy momentum conservation, as the working principle of the fields does not change across the manifold. Alternatively, it would as well generate a variation in the inverse metric, on which the Einstein-Hilbert-Lagrange density is built: Varying the gravitational part with respect to the inverse metric should yield the field equation, and varying the matter part the corresponding source of the gravitational field.

Infinitesimal coordinate shifts $x^\mu \to x'^\mu = x^\mu + \zeta^\mu(x)$ induce a change in the metric $g_{\mu\nu} \to g'_{\mu\nu}$ following

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta} \tag{I.639}$$

based on the Jacobians

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \frac{\partial}{\partial x'^\mu}\left(x'^\alpha - \zeta^\alpha\right) = \frac{\partial x'^\alpha}{\partial x'^\mu} - \frac{\partial \zeta^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu - \partial_\mu\, \zeta^\alpha \tag{I.640}$$

Therefore, the metric changes according to

$$g'_{\mu\nu}(x') = \left(\delta^\alpha_\mu - \partial_\mu \zeta^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu \zeta^\beta\right) \cdot g_{\alpha\beta} = \delta^\beta_\nu g_{\alpha\beta} - \delta^\alpha_\mu \partial_\nu \zeta^\beta g_{\alpha\beta} - \delta^\beta_\nu \partial_\mu \zeta^\alpha g_{\alpha\beta} + \mathcal{O}(\zeta^2) \tag{I.641}$$

such that at order ζ^2 the changed metric is given by

$$g'_{\mu\nu}(x') = g_{\mu\nu} - \partial_\nu\, \zeta^\beta\, g_{\mu\beta} - \partial_\mu\, \zeta^\alpha \cdot g_{\alpha\nu} \tag{I.642}$$

The induced variation in $g_{\mu\nu}$ due to the coordinate change is given by

$$\delta g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) + g'_{\mu\nu}(x') - g'_{\mu\nu}(x') = \left[g'_{\mu\nu}(x') - g_{\mu\nu}(x)\right] - \left[g'_{\mu\nu}(x') - g'_{\mu\nu}(x)\right] \quad (I.643)$$

Subsituting eqn. I.642 and the Taylor expansion $g'_{\mu\nu}(x') - g'_{\mu\nu}(x) \simeq \partial_\alpha\, g'_{\mu\nu}\zeta^\alpha$

$$\delta g_{\mu\nu} = -g_{\mu\beta}\cdot\partial_\nu\,\zeta^\beta - g_{\alpha\nu}\,\partial_\mu\,\zeta^\alpha - \partial_\alpha\,g_{\mu\nu}\cdot\zeta^\alpha \quad (I.644)$$

if the approximation $\partial_\alpha\,g_{\mu\nu} = \partial_\alpha\,g'_{\mu\nu}$ is done

Replacing the partial derivatives ∂_μ with covariant ones ∇_μ according to

$$\nabla_\mu\,\zeta^\alpha = \partial_\mu\zeta^\alpha + \Gamma^\alpha{}_{\mu\tau}\,\zeta^\tau \quad (I.645)$$

as ζ^α is a vector yields

$$g_{\mu\beta}\partial_\nu\zeta^\beta + g_{\alpha\nu}\partial_\mu\zeta^\beta + \partial_\alpha g_{\mu\nu}\zeta^\alpha =$$
$$g_{\mu\beta}\nabla_\nu\zeta^\beta - g_{\mu\beta}\Gamma^\beta{}_{\nu\tau}\,\zeta^\tau + g_{\alpha\nu}\nabla_\mu\zeta^\alpha - g_{\alpha\nu}\Gamma^\beta{}_{\mu\tau}\,\zeta^\tau + \nabla_\alpha g_{\mu\nu}\zeta^\alpha + \left(\Gamma^\tau{}_{\alpha\mu}\;g_{\tau\nu} + \Gamma^\tau{}_{\alpha\nu}\;g_{\mu\tau}\right)\zeta^\alpha \quad (I.646)$$

where the metric compatibility condition $\nabla_\alpha g_{\mu\nu} = 0$ has been substituted. Two pairs of Christoffel-symbols drop out, leaving

$$\delta g_{\mu\nu} = -\left(g_{\alpha\nu}\nabla_\mu\zeta^\alpha + g_{\mu\beta}\nabla_\nu\zeta^\beta\right) = -\left(\nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu\right) \quad (I.647)$$

This result can be substituted into the variation δS_m of the part of the action S_m describing the material fields,

$$\delta S_m = -\frac{1}{2}\int\,d^4x\,\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} = +\frac{1}{2}\int\,d^4x\,\sqrt{-g}T^{\mu\nu}\left(\nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu\right) \quad (I.648)$$

and using the Leibnitz-rule to orient the ∇_μ-differentiations to $T_{\mu\nu}$ rather than ζ_μ,

$$\delta S_m = -\frac{1}{2}\int\,d^4x\,\sqrt{-g}\left(\nabla_\mu\,T^{\mu\nu}\cdot\zeta_\nu + \nabla_\nu\,T^{\mu\nu}\cdot\zeta_\mu\right) = \int\,d^4x\,\sqrt{-g}\,\nabla_\mu\,T^{\mu\nu}\cdot\zeta_\nu \quad (I.649)$$

by exploiting the symmetry of the expression, and if the variation on the boundary vanishes, to be assumed when the Gauß-theorem is applied,

$$\int_V\,d^4x\,\sqrt{-g}\,\nabla_\mu\left[T^{\mu\nu}\,\zeta_\nu\right] = \int_{\partial V}\,dA_\mu\sqrt{|\gamma|}\,T^{\mu\nu}\,\zeta_\nu = 0 \quad (I.650)$$

Let's try out this relation for a straightforward scalar field ϕ with a self-interaction or a coupling $V(\phi)$, as the easiest example of a non-gravitational field serving as a model for the matter content of the theory. Variation of the action

$$S_\phi = \int\,d^4x\,\sqrt{-g}\left(\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\right) \quad (I.651)$$

with respect to $g^{\mu\nu}$ should recover the energy momentum tensor $T_{\mu\nu}$. In fact, there are two dependences on the metric, the covolume $\sqrt{-g}$ and the contraction $g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi$ in the kinetic term, such that the variation becomes

$$\delta S_\phi = \int d^4x\,\sqrt{-g}\Big[\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\Big] + \delta\sqrt{-g}\cdot\Big[\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\Big]g^{\mu\nu} \tag{I.652}$$

using the relation

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}}\,g\,g^{\mu\nu}\,\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} \tag{I.653}$$

for the variation of the covolume. Rewriting the variation yields

$$\delta S_\phi = \int d^4x\,\sqrt{-g}\Big[\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\,\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\right)\Big]\delta g^{\mu\nu} \tag{I.654}$$

Naturally, we obtain the energy momentum tensor

$$T_{\mu\nu} = \nabla_\mu\phi\,\nabla_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\,\nabla_\alpha\phi\,\nabla_\beta\phi - V(\phi)\right) \tag{I.655}$$

for a scalar field, by comparing eqn. I.654 with

$$\delta S_\phi = \int d^4x\,\sqrt{-g}\frac{1}{2}T_{\mu\nu}\,\delta g^{\mu\nu} \tag{I.656}$$

which is the correct form, that could otherwise be obtained by Legendre-transform or by taking the Lie-derivative of the Lagrange-function.

I.9 *Symmetries on manifolds: Lie-derivatives and the Killing equation*

Spacetime as a manifold can have symmetries; whether they a particular choice of coordinates is compatible with them or not. Up to this point we have always relied on our intuition about choosing coordinates in which the symmetries became apparent in a very clear way, for instance the Schwarzschild coordinates for a spherically symmetric, static spacetime. But general covariance of relativity does not require that we find the best coordinate choice, instead, it should be possible to make a statement about symmetry without recursing to particular, properly adjusted coordinates; there should be a perfectly valid Schwarzschild solution for oscillating cylindrical coordinates, too. As all observables are associated with scalars, the coordinate choice does not matter for the prediction of measurable physical quantities.

Additionally, there should be conserved quantities along with any symmetry of a system as predicted by Noether's theorem. It is worth pointing out that certain statements are impossible or do not contribute substantially to statements on symmetry: concerning motion through manifolds, $g_{\mu\nu}u^\mu u^\nu = c^2$ or $g_{\mu\nu}k^\mu k^\nu = 0$ are expressing causality or define the choice of a sensible affine parameter rather than conservation, and symmetries of the metric are certainly not expressed by $\partial_\alpha g_{\mu\nu} = 0$ because of its unclear transformation properties, nor by $\nabla_\alpha g_{\mu\nu} = 0$, which is always true for a Levi-Civita connection.

Instead, we would require a new derivative, the Lie derivative $(\mathcal{L}_a g)_{\mu\nu} = 0$, which states that there is an isometry present: The metric does not change under shifts in the direction of a vector a^μ, as an expression of a spacetime symmetry. Ideally, we can link this new derivative to the already defined covariant derivative and possibly derive a relationship which allows us to find coordinates adopted to a spacetime with a given symmetry.

Imagine two distinct points $P(x^\mu)$ and $P'(x'^\mu)$ with coordinates x^μ and x'^μ, respectively. Then, the coordinates of the two points are related in general by

$$x'^\mu = x^\mu - \epsilon a^\mu \quad \text{and differentially, by} \quad \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu - \epsilon\,\partial_\nu\, a^\mu \tag{I.657}$$

where ϵ controls the infinitesimal shift into the direction a^μ. Any vector field v^μ then transforms according to

$$v'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}\, v^\nu(x) = \left(\delta^\mu_\nu - \epsilon\partial_\nu\, a^\mu\right)v^\nu(x) = v^\mu(x) - \epsilon\partial_\nu\, a^\mu v^\mu(x) \tag{I.658}$$

Clearly, $v'^\mu(x') - v^\mu(x)$ is not a vector because the two v refer to physically different points on the manifold, so we could apply a Taylor-expansion

$$v'^\mu(x') = v'^\mu(x) + (x'-x)^\nu\,\partial_\nu\, v'^\mu(x) + \ldots = v'^\mu(x) - \epsilon a^\nu\,\partial_\nu\, v'^\mu(x) \tag{I.659}$$

such that

$$v^\mu(x) = v'^\mu(x') + \epsilon\partial_\nu\, a^\mu v^\nu(x) = v'^\mu(x) - \epsilon a^\nu\,\partial_\nu\, v'^\mu + \epsilon\,\partial_\nu\, a^\mu \cdot v^\nu(x) \tag{I.660}$$

and we can define the Lie-derivative $\left(\mathcal{L}_a v\right)^\mu$

$$\lim_{\epsilon\to 0} \frac{v^\mu(x) - v'^\mu(x)}{\epsilon} = -a^\nu\,\partial_\nu\, v'^\mu + \partial_\nu\, a^\mu \cdot v^\nu \equiv \left(\mathcal{L}_a v\right)^\mu \tag{I.661}$$

of the vector field v^μ in the direction a^μ with all terms at order ϵ. If defined for linear forms, the Lie-derivative picks up a different sign in the Jacobian,

$$\left(\mathcal{L}_a v\right)_\mu = +a^\nu\,\partial_\nu\, v_\mu + \partial_\mu\, a^\nu \cdot v_\nu \tag{I.662}$$

and applied to a rank-2 tensor such as the metric one obtains

$$\left(\mathcal{L}_a g\right)_{\mu\nu} = g_{\mu\lambda} \cdot \partial_\nu\, a^\lambda + g_{\lambda\nu}\,\partial_\mu\, a^\lambda + a^\lambda\,\partial_\lambda\, g_{\mu\nu} \tag{I.663}$$

It is very important to realise that up to this point we did not use the concept of parallel transport nor the covariant derivative, but only partial derivatives. In fact, symmetries of vector or tensor fields on a manifold exist and and quantifiable with the Lie-derivative even when there is no differential structure and no parallel transport. But of course, one would like to define the Lie-derivative in a way that it becomes compatible with the covariant derivative, and that is in fact one motivation for Levi-Civita connections:

$$\nabla_\nu \upsilon^\mu \cdot a^\nu - \nabla_\nu a^\mu \cdot \upsilon^\nu = \partial_\nu \upsilon^\mu \cdot a^\nu - \partial_\nu a^\mu \cdot \upsilon^\nu - \left(\Gamma^\mu{}_{\kappa\lambda} - \Gamma^\mu{}_{\lambda k}\right) \cdot \upsilon^\kappa \cdot a^\lambda = \partial_\nu \upsilon^\mu \cdot a^\nu - \partial_\nu a^\mu \cdot \upsilon^\nu \tag{I.664}$$

if the connection is torsion free, $\Gamma^\mu{}_{k\lambda} = \Gamma^\mu{}_{\lambda k}$, and the covariant expression falls back onto the partial one. Applied to the metric this would mean that

$$\left(\mathcal{L}_a g\right)_{\mu\nu} = g_{\mu\lambda} \nabla_\lambda a^\lambda + g_{\lambda\nu} \nabla_\mu a^\lambda + a^\lambda \cdot \nabla_\lambda g_{\mu\nu} = \nabla_\nu a_\mu + \nabla_\mu a_\nu \tag{I.665}$$

with the last of the terms being canceled by metric compatibility $\nabla_\lambda g_{\mu\nu} = 0$, and using the index-lowering property of the metric. Again, we should be able to compute the Lie-derivative of the metric purely with partial derivatives instead of covariant ones. Indeed, replacing ∇ with ∂ and the Christoffel-symbols yields

$$\begin{aligned}\left(\mathcal{L}_a g\right)_{\mu\nu} &= \\ & g_{\mu\lambda}\left[\partial_\nu a^\lambda + \Gamma^\lambda{}_{\nu\kappa} a^\kappa\right] + g_{\lambda\nu}\left[\partial_\mu a^\lambda + \Gamma^\lambda{}_{\mu\kappa} a^\kappa\right] + a^\lambda\left[\partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\lambda\mu} g_{\kappa\nu} - \Gamma^\kappa{}_{\lambda\nu} g_{\mu\kappa}\right] = \\ & g_{\mu\lambda}\, \partial_\nu a^\lambda + g_{\lambda\nu}\, \partial_\mu a^\lambda + a^\lambda \cdot \partial_\lambda g_{\mu\nu}\end{aligned} \tag{I.666}$$

because of the pairwise cancellation in the expression

$$g_{\mu\lambda}\Gamma^\lambda{}_{\nu\kappa} a^\kappa + g_{\lambda\nu}\Gamma^\lambda{}_{\mu\kappa} a^\kappa - \Gamma^\kappa{}_{\lambda\mu} g_{\kappa\nu} a^\lambda - \Gamma^\kappa{}_{\lambda\nu} g_{\mu\kappa} a^\lambda = 0 \tag{I.667}$$

.

If $\left(\mathcal{L}_a g\right)_{\mu\nu} = 0$ for a given shift field a^μ then the spacetime possesses a certain symmetry and $\nabla_\nu a_\mu + \nabla_\mu a_\nu = 0$. Then, a^μ is called a Killing-vector. There is a weird relationship between Killing vectors and the Riemann-curvature. For any vector we have the definition of curvature through the non-commutability of second covariant derivatives,

$$\left(\nabla_\kappa \nabla_\lambda - \nabla_\lambda \nabla_\kappa\right) a_\mu = R^\tau{}_{\mu\kappa\lambda} a_\tau = R_{\tau\mu\kappa\lambda} a^\tau \tag{I.668}$$

from which we can construct

$$\begin{aligned}&\nabla_\kappa\left[\nabla_\nu a_\mu - \nabla_\mu a_\nu\right] + \nabla_\nu\left[\nabla_\mu a_\kappa - \nabla_\kappa a_\mu\right] + \nabla_\mu\left[\nabla_\kappa a_\nu - \nabla_\nu a_\kappa\right] = \\ &\qquad \left(R_{\tau\mu\nu\kappa} + R_{\tau\nu\kappa\mu} + R_{\tau\kappa\mu\nu}\right] a^\tau = 0\end{aligned} \tag{I.669}$$

which necessarily vanishes due to the algebraic Bianchi-identity. From the Killing-condition $\nabla_\nu a_\mu + \nabla_\mu a_\nu = 0$ we get $\nabla_\mu a_\nu = -\nabla_\nu a_\mu$, so we can change the sign in every second term,

$$\begin{aligned}&\nabla_\kappa\left(\nabla_\nu a_\mu + \nabla_\nu a_\mu\right) + \nabla_\nu\left(\nabla_\mu a_\kappa + \nabla_\mu a_\kappa\right) + \nabla_\mu\left(\nabla_\kappa a_\nu + \nabla_\kappa a_\nu\right) = \\ &\qquad 2\left[\nabla_\kappa \nabla_\nu a_\mu + \nabla_\nu \nabla_\mu a_\kappa + \nabla_\mu \nabla_\kappa a_\nu\right] = 0\end{aligned} \tag{I.670}$$

Inspecting the result $\nabla_\kappa \nabla_\nu a_\mu + \nabla_\nu \nabla_\mu a_\kappa + \nabla_\mu \nabla_\kappa a_\nu = 0$ in more detail we can carry out this treatment: Let's keep the first term unchanged, but switch the indices $\mu \leftrightarrow \kappa$ in the second term. Because of the Killing-condition, this can be done as $\nabla_\mu a_\kappa + \nabla_\kappa a_\mu = 0$, so one picks up a minus-sign. The analogous index switch can be performed on the last term. $\kappa \leftrightarrow \mu$ is possible because $\nabla_\kappa a_\nu + \nabla_\nu a_\kappa = 0$, again introducing a minus sign:

$$\nabla_\kappa \nabla_\nu a_\mu + \nabla_\nu \nabla_\mu a_\kappa + \nabla_\mu \nabla_\kappa a_\nu = \nabla_\kappa \nabla_\nu a_\mu - \nabla_\nu \nabla_\kappa a_\mu - \nabla_\mu \nabla_\nu a_\kappa \tag{I.671}$$

The first two terms are just double covariant derivatives with interchanged order applied to the vector a_μ which yields the Riemann-curvature: Making this identification yields the Killing-equation

$$\nabla_\mu \nabla_\nu a_\kappa = \Big(\nabla_\nu \nabla_\kappa - \nabla_\kappa \nabla_\nu\Big) a_\mu = \mathrm{R}^\tau{}_{\mu\kappa\nu} a_\tau \tag{I.672}$$

The Killing-equation is a tool of determining the Killing-vectors a^μ for a spacetime with a given metric $g_{\mu\nu}$: Think of it as an eigenvalue equation, which yields the shift-vectors for any spacetime where the covariant derivatives and the Riemann-curvature are given in an arbitrary coordinate choice, and effectively isolate the spacetime symmetries in the form of the set of a_μ. If the connection is of the Levi-Civita type, both the covariant derivative ∇ as well as the Riemann-curvature are completely computable from $g_{\mu\nu}$, so that all ingredients of the Killing equation for a given metric are present.

Euclidean space, for instance, has two types of symmetries: shifts and rotations. By using intuition and introducing global Cartesian coordinates one simplifies everything tremendously as $g_{\mu\nu} = \delta_{\mu\nu}$, $\Gamma^\alpha{}_{\mu\nu} = 0$ such that $\nabla_\mu = \partial_\mu$ and of course $\mathrm{R}^\tau{}_{\kappa\mu\nu} = 0$. Then, the Killing-equation reduces to $\nabla_\mu \nabla_\nu a_\kappa = 0 = \partial_\mu \partial_\nu a_\kappa$ and one can search for solutions to $\partial_\mu \partial_\nu a_\kappa = 0$, which are obviously given by $a_\kappa = q_{\nu\kappa} x^\kappa + p_\kappa$ with 6 constants $q_{\nu\kappa}$ (due to the antisymmetry $q_{\nu\kappa} = -q_{\kappa\nu}$, from the Lie-derivative) and 3 constants p_κ, corresponding to the rotations and shifts, respectively.

There is a tight connection between Killing-vectors $\nabla_\nu a_\mu + \nabla_\mu a_\nu = 0$ expressing an isometry of spacetime and geodesics, which are defined through their autoparallelity condition $u^\nu \nabla_\nu u^\mu = 0$. If the scalar product $a_\mu u^\mu$ is shifted by $u^\lambda \nabla_\lambda$ into the direction of u^λ, we obtain

$$u^\lambda \nabla_\lambda \Big[a_\mu \cdot u^\mu\Big] = u^\lambda \Big[\nabla_\lambda a_\mu \cdot u^\mu + a_\mu \nabla_\lambda u^\mu\Big] = \nabla_\lambda a_\mu \cdot u^\mu u^\lambda + a_\mu \cdot u^\lambda \nabla_\lambda u^\mu = 0 \tag{I.673}$$

as $\nabla_\lambda a_\mu \cdot u^\mu u^\lambda = 0$ because of the antisymmetry $\nabla_\nu a_\mu = -\nabla_\mu a_\nu$ and $u^\lambda \nabla_\lambda u^\mu = 0$ because of geodesic motion. Hence, the projection of the tangent u^μ onto the Killing vector field a^μ is conserved along the geodesic.