H WEAK FIELD GRAVITY AND GRAVITATIONAL WAVES

H.1 *Weak field gravity and gravitational waves*

The gravitational field equation is a nonlinear, hyperbolic, partial differential equation. We have already encountered that the nonlinearity forbids the usage of a Green-function method for finding constructing solutions for a given $T_{\mu\nu}$, so that we can only hope to find solutions for very simple matter distributions such as the black hole solutions or the FLRW-cosmologies. While the gravitational field equation is certainly compatible with the Poisson-equation in the limit of small spacetime curvature for static matter distributions, it is a sensible question whether (i) there are gravitational effects that can be attributed to motion in the sourcing matter distribution, and (ii) the gravitational field can show dynamical behaviour on its own, in the form of wave-type propagating excitations: This would be natural for a hyperbolic PDE. If one replaces the Laplace operator $\Delta = \delta_{ij}\,\partial^i\partial^j$ in the Poisson-equation $\Delta\Phi = 4\pi G\rho$ (setting λ to zero for that instance) with the d'Alembert-operator $\Box = \eta_{\mu\nu}\,\partial^\mu\partial^\nu$ as the relativistic invariant constructed from ∂_μ, one obtains a typical wave equation $\Box\,\Phi = \partial_{ct}^2\Phi - \Delta\,\Phi = -4\pi G\rho$ with excitations travelling at the speed c away from the source ρ, irrespective of the frame: this is exactly the expression of hyperbolicity, i.e. the notion of a relativistically invariant light cone with wave-type excitations propagating along null-lines: Substitution of a plane wave $\Phi \propto \exp(\pm ik_\mu x^\mu)$ shows that $k_\mu k^\mu = 0$ and that $\omega = \pm ck$.

Incidentally (and I thank T. Baumgarte for this argument), requiring the matter distribution ρ to be homogeneous cancels the position-dependence of Φ, yielding $\partial_{ct}^2\Phi = -4\pi G\rho$, reminiscent of the second Friedmann-equation! This underlines the reasoning that depending on symmetry, black hole solutions, FLRW-solutions and wave-type solutions should naturally come out of the gravitational field equation at similar levels of symmetry (which deactivates certain derivatives), and that only in the limit of weak gravity one can expect to recover a pure wave equation.

H.2 *Nonlinearities in the field equation*

The gravitational field equation is naturally nonlinear due to the construction of the Ricci-curvature from the metric. This is pictorially summarised in the schematic

$$g_{\mu\nu} \quad\rightarrow\quad \Gamma^\alpha{}_{\mu\nu} \quad\rightarrow\quad R_{\alpha\beta\mu\nu} \quad\rightarrow\quad R_{\beta\nu} \quad\rightarrow\quad R \tag{H.508}$$

$$g\partial g \qquad \Gamma^2 \sim (g\partial g)^2 \qquad g^2\partial g \qquad g^3\partial g \tag{H.509}$$

$$\partial\Gamma \sim \partial(g\partial g) \qquad g\partial(g\partial g) \qquad g^2\partial(g\partial g) \tag{H.510}$$

where clearly contractions between the metric and its derivatives are needed for computing the curvature. If symmetries are present, the complexity is significantly reduced because in a suitably aligned coordinate system, the partial derivative of the metric with respect to the coordinate direction in which a symmetry is present, would be zero: We have encountered this in the case of the Schwarzschild solution and the FLRW-cosmologies. Additionally, both these solutions have defined natural scales, the Schwarzschild radius $r_s = 2GM/c^2$ and the Hubble distance c/H_0 (or, equivalently, the critical density $\rho_{crit} = 3H_0^2/(8\pi G)$). In contrast, classical gravity in more than three dimensions is scale free, as the potential follows a power law, as long as effects of the cosmological constant are neglected on small scales, $\ll 1/\sqrt{\Lambda}$, reiterating the argument that the cosmological constant is a perfectly admissible feature of classical gravity.

From a conceptual point of view, we will formally and not just by analogy join weak perturbations $h_{\mu\nu}$ of the otherwise Minkowskian metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad \left|h_{\mu\nu}\right| \ll 1 \tag{H.511}$$

with the gravitational potential Φ, the gravitomagnetic field A^i and the gravitational shear h_{ij}. It should be emphasised that in this process one loses general covariance as this decomposition with weak perturbations makes statements about individual entries of $h_{\mu\nu}$, and their smallness compared to one can only be made in a preferred coordinate system. There is, however, residual Lorentz-covariance pertaining to non-accelerated frames of reference, i.e. a transformation law of the form

$$h_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta\, h_{\alpha\beta} \tag{H.512}$$

with Lorentz-transforms $\Lambda_\mu{}^\alpha$.

H.3 *Gauging of the metric*

Transitions from one coordinate choice to another

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) \tag{H.513}$$

where the differential function $\xi^\mu(x)$ defines the transform. The corresponding Jacobian is given by

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \frac{\partial \xi^\mu}{\partial x^\nu} + \mathcal{O}\left(\partial^2 \xi\right) \tag{H.514}$$

with its inverse Jacobian

$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu - \frac{\partial \xi^\mu}{\partial x'^\nu} + \mathcal{O}\left(\partial^2 \xi\right) \tag{H.515}$$

so that

$$\frac{\partial x'^\mu}{\partial x^\nu} \cdot \frac{\partial x^\nu}{\partial x'^\beta} = \left(\delta^\mu_\nu + \frac{\partial \xi^\mu}{\partial x^\nu}\right) \cdot \left(\delta^\nu_\beta - \frac{\partial \xi^\nu}{\partial x'^\beta}\right) \simeq \underbrace{\delta^\mu_\nu\, \delta^\nu_\beta}_{\delta^\mu_\beta} - \underbrace{\delta^\mu_\nu \frac{\partial \xi^\nu}{\partial x'^\beta}}_{\frac{\partial \xi^\mu}{\partial x'^\beta}} + \underbrace{\delta^\nu_\beta \frac{\partial \xi^\mu}{\partial x^\nu}}_{\frac{\partial \xi^\mu}{\partial x^\beta}} = \delta^\mu_\beta \tag{H.516}$$

implying that we should not distinguish $\frac{\partial \xi^\mu}{\partial x^\beta}$ and $\frac{\partial \xi^\mu}{\partial x'^\beta}$ at this order. With this definition of a coordinate change, the metric transforms as

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \cdot \frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta} = \left(\delta^\alpha_\mu - \partial_\mu \xi^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu \xi^\beta\right)\left[\eta_{\alpha\beta} + h_{\alpha\beta}\right] \simeq$$

$$\eta_{\mu\nu} + h_{\mu\nu} - \delta^\alpha_\mu\, \partial_\nu\, \xi^\beta \cdot \eta_{\alpha\beta} - \partial_\mu\, \xi^\alpha\, \delta^\beta_\nu\, \eta_{\alpha\beta} = \eta_{\mu\nu} + h_{\mu\nu} - \partial_\nu\, \xi_\mu - \partial_\mu\, \xi_\nu. \tag{H.517}$$

This is the relation from which we can isolate the transformation rule of the perturbation

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \tag{H.518}$$

The inverse metric obeys $g^{\mu\beta} g_{\beta\nu} = \delta^\mu_\nu$ by definition, such that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{H.519}$$

is a good enough approximation at that order and correct to $\mathcal{O}(h^2)$, with the inverse Minkowski-metric being $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}\, h_{\alpha\beta}$.

$$g^{\mu\beta} g_{\beta\nu} = \left(\eta^{\mu\beta} - h^{\mu\beta}\right)\left(\eta_{\beta\nu} + h_{\beta\nu}\right) = \eta^{\mu\beta}\eta_{\beta\nu} + \eta^{\mu\beta} \cdot h_{\beta\nu} - h^{\mu\beta} \cdot \eta_{\beta\nu} + \mathcal{O}(h^2) \tag{H.520}$$

with $\eta^{\mu\beta}\eta_{\beta\nu} = \delta^\mu_\nu$, $\eta^{\mu\beta} \cdot h_{\beta\nu} = h^\mu{}_\nu = 0$ and $h^{\mu\beta}\eta_{\beta\nu} = h^\mu{}_\nu = 0$ at lowest order, $h^\mu{}_\nu = g^{\mu\alpha} h_{\alpha\nu} = \left(\eta^{\mu\alpha} - h^{\mu\alpha}\right) h_{\alpha\nu} \cong \eta^{\mu\alpha} h_{\alpha\nu}$. Effectively this implies that raising and lowering of indices is done with $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$, and that derivatives are replaced $\partial_\alpha g_{\mu\nu} = \partial_\alpha h_{\mu\nu}$ as $\eta_{\mu\nu}$ is constant in Cartesian coordinates.

H.4 *Linearised gravitational field equation*

So far we have set up the metric as weak perturbation of the Minkowski-metric in Cartesian coordinates, determined the transformation properties and suitable approximations for the inverse metric. In this preferred frame with a particular coordinate choice we can continue to find a linearisation for curvature tensors, which are all ultimately computed from partial derivatives of the metric and by contractions with the metric.

The first step would be the Christoffel-symbols, where the inverse metric is replaced by the inverse Minkowski-metric,

$$\Gamma^\alpha{}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right) \simeq \frac{\eta^{\alpha\beta}}{2}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}\right) = \frac{1}{2}\left(\partial_\mu h^\alpha{}_\nu + \partial_\nu h_\mu{}^\alpha - \partial^\alpha h_{\mu\nu}\right), \tag{H.521}$$

renaming one of the indices, by writing $\partial_\alpha = \eta^{\alpha\beta}\,\partial_\beta$.

The Riemann-tensor is then derived in the limit that the dominating terms are the derivatives of the Christoffel symbols (in turn with the inverse Minkowski metric instead of the inverse actual metric), while the squared Christoffel-symbols are discarded,

$$R^\mu{}_{\alpha\beta\gamma} = \partial_\gamma \Gamma^\mu{}_{\alpha\beta} - \partial_\beta \Gamma^\mu{}_{\alpha\gamma} + \Gamma^\mu{}_{\delta\gamma}\Gamma^\delta{}_{\alpha\beta} - \Gamma^\mu{}_{\delta\beta}\Gamma^\delta{}_{\alpha\gamma} \tag{H.522}$$

Applying all simplifications then yields the final result for the Riemann-tensor,

$$R^{\mu}{}_{\alpha\beta\gamma} = \frac{1}{2}\partial_\gamma\left(\partial_\alpha\, h^{\mu}{}_{\beta} + \partial_\beta\, h_\alpha{}^{\mu} - \partial^\mu\, h_{\alpha\beta}\right) - \frac{1}{2}\partial_\beta\left(\partial_\alpha\, h^{\mu}{}_{\gamma} + \partial_\gamma\, h_\alpha{}^{\mu} - \partial^\mu\, h_{\alpha\gamma}\right) \quad \text{(H.523)}$$

so that finally one arrives at

$$R^{\mu}{}_{\alpha\beta\gamma} = \frac{1}{2}\left[\partial_\gamma\partial_\alpha\, h^{\mu}{}_{\beta} - \partial_\beta\partial_\alpha\, h^{\mu}{}_{\gamma} + \partial_\beta\partial^\mu\, h_{\alpha\gamma} - \partial_\gamma\partial^\mu\, h_{\alpha\beta}\right] \quad \text{(H.524)}$$

The contraction of the Riemann-tensor with the metric yields in a first step the Ricci-tensor, where we will use in this approximation the inverse Minkowski metric $\eta^{\mu\nu}$ as in the case of the Christoffel-symbols,

$$R_{\alpha\gamma} = \frac{1}{2}\left[\partial_\alpha\partial_\gamma\, h + \Box\, h_{\alpha\gamma} - \partial_\gamma\partial^\mu\, h_{\alpha\mu} - \partial_\mu\partial_\alpha\, h^{\mu}{}_{\gamma}\right] \quad \text{(H.525)}$$

where one can define the trace $h = h^{\mu}{}_{\mu}$ and recovers the d'Alembert-operator $\Box = \partial_\mu\partial^\mu$.

Further contraction of the Ricci-tensor with $\eta^{\mu\nu}$ gives the Ricci-scalar,

$$R = \frac{1}{2}\left[\partial^\alpha\partial_\alpha\, h + \Box\, h_\alpha{}^{\alpha} - \partial^\alpha\partial^\mu\, h_{\alpha\mu} - \partial^\alpha\partial^\mu\, h_{\alpha\mu}\right] \quad \text{(H.526)}$$

with a particular compact form using the trace h and the d'Alembert-operator $\Box$,

$$R = \Box\, h - \partial^\alpha\partial^\mu\, h_{\alpha\mu} \quad \text{(H.527)}$$

With these approximations, one can write down the field equation $R_{\mu\nu} - R/2\; g_{\mu\nu} = -8\pi G/c^4\; T_{\mu\nu}$ (setting $\Lambda = 0$ as it is not relevant on small scales) in the weak field limit.

By redefining the amplitude $h_{\mu\nu}$ one can reach a significant simpification: The trace referse

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} \quad \text{(H.528)}$$

has the properties

$$\bar{h} = \eta^{\mu\nu}\, h_{\mu\nu} - \frac{h}{2}\eta^{\mu\nu}\eta_{\mu\nu} = h - \frac{h}{2}\cdot 4 = -h \quad \text{(H.529)}$$

as well as

$$\bar{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} + \frac{h}{2}\eta_{\mu\nu} = h_{\mu\nu} \quad \text{(H.530)}$$

such that $h_{\mu\nu}$ is given by

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{h}{2}\eta_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu} \quad \text{(H.531)}$$

because $\bar{h} = -h$.

Then, the linearise gravitational field equation becomes

$$\square\, \bar{h}_{\alpha\gamma} + \eta_{\alpha\gamma}\, \partial^{\mu}\partial^{\nu}\, \bar{h}_{\mu\nu} - \partial_{\alpha}\partial^{\mu}\, \bar{h}_{\mu\nu} - \partial_{\gamma}\partial^{\mu}\, \bar{h}_{\alpha\mu} = -\frac{16\pi G}{c^4} T_{\alpha\gamma} \tag{H.532}$$

After linearising the field equation, introducing a Minkowskian background and redefining the amplitudes there is still the freedom for picking a particular gauge, where the choice of the Lorenz-gauge would naturally come to mind. The gauge choice should be able to simplify the field equation further, discarding all terms apart from $\square\, \bar{h}_{\alpha\gamma}$.

Now introducing the Lorenz-gauge and replacing $\bar{h}$ by $\bar{h}'$

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{h'}{2}\eta_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} - \frac{\eta_{\mu\nu}}{2}\left(h - 2\eta^{\alpha\beta}\, \partial_{\alpha}\, \xi_{\beta}\right) \tag{H.533}$$

such that one arrives at:

$$\bar{h}'_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\xi - \partial_{\nu}\xi_{\mu} - \frac{h}{2}\eta_{\mu\nu} + h_{\mu\nu}\, \eta^{\alpha\beta}\, \partial_{\alpha}\xi_{\beta} \tag{H.534}$$

Applying ∂^{ν} to the equation then gives:

$$\partial^{\nu}\, \bar{h}'_{\mu\nu} = \partial^{\nu}\, \bar{h}_{\mu\nu} - \partial^{\nu}\partial_{\mu}\xi_{\nu} - \partial^{\nu}\partial_{\mu}\xi_{\nu} - \partial^{\nu}\partial_{\nu}\xi_{\mu} + \eta_{\mu\nu}\, \partial^{\nu}\eta^{\alpha\beta}\, \partial_{\alpha}\xi^{\beta} \tag{H.535}$$

Using the definitions $\partial^{\nu}\partial_{\nu} = \square$, as well as $\eta_{\mu\nu}\partial^{\nu}\eta^{\alpha\beta}\, \partial_{\alpha}\, \xi_{\beta} = \partial_{\mu}\partial^{\beta}\, \xi_{\beta}$ one arrives finally at

$$\partial^{\nu}\bar{h}'_{\mu\nu} = \partial^{\nu}\bar{h}_{\mu\nu} - \square\, \xi_{\mu} \tag{H.536}$$

such that, with the gauge choice $\square\, \xi_{\mu} = \partial^{\nu}\, \bar{h}_{\mu\nu}$ implying $\partial^{\nu}\bar{h}'_{\mu\nu} = 0$, the linearised field equation in Lorenz-gauge reads

$$\square\, \bar{h}'_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \tag{H.537}$$

which is is perfect agreement with the expectations: $T_{\mu\nu}$ sources perturbations in $\bar{h}'_{\mu\nu}$ in a linear, Lorentz-covariant wave equation with propagation along the light cones: gravitational waves!

H.5 *Vacuum solutions of the linearised field equation*

Vacuum solutions $T_{\mu\nu} = 0$ of the linearised field equation $\square\, \bar{h}_{\mu\nu} = 0$ with the Lorenz gauge condition $\partial^{\nu}\, \bar{h}_{\mu\nu} = 0$ very naturally call for plane wave solutions, in complete analogy to the vacuum Maxwell-equation $\partial^{\mu}F_{\mu\nu} = 0$. By substituting $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ and assuming Lorenz gauge $\partial^{\mu}A_{\mu} = 0$ one obtains $\partial^{\mu}\partial_{\mu}A_{\nu} = \square A_{\nu} = 0$, which is likewise solved by planed waves, where the gauge condition makes sure that the vector potential A_{μ} is oriented perpendicular to the wave vector k^{μ}, justifying the expression transverse gauge.

Plane waves of the form $\bar{h}_{\mu\nu} \sim \exp(\pm i\eta^{\alpha\beta}\, k_{\alpha}x_{\beta})$ have to have a light-like wave vector $\rightarrow \eta_{\mu\nu}\, k^{\mu}k^{\nu} = 0$ such that the propagation in the gravitational field takes place along the light cone without any dispersion at all. It should be emphasised that in the limit of linearised gravity that we are dealing with there the light cone is defined by the background alone, $g_{\mu\nu}k^\mu k^\nu = 0$ which becomes in the preferred coordinate system $\eta_{\mu\nu}k^\mu k^\nu = 0$ and that there is no effect of the gravitational field of the wave back onto the propagation of the wave.

H.6 *Stationary sources and gravitomagnetism*

Stationary sources are peculiar as there is no time dependence is the source and hence none in the gravitational field. As a consequence, there is no corresponding retardation in the Green-function and the perturbation to the metric $\bar{h}_{\mu\nu}$ can be computed from the source $T_{\mu\nu}$:

$$\bar{h}_{\mu\nu}(x) = -\frac{4G}{c^4} \cdot \int d^3x' \, \frac{T_{\mu\nu}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{H.538}$$

There is a tremendous simplification in the energy momentum tensor if taken in the non-relativistic limit, where $p \ll \rho c2$:

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - p \cdot g_{\mu\nu} \simeq \rho \, u_\mu u_\nu = \begin{pmatrix} \rho \, c^2 & \rho \, c u_i \\ \rho \, c u_j & \rho u_i u_j \end{pmatrix} \tag{H.539}$$

Solving for the metric perturbations then suggests a sourcing of Φ through ρc^2

$$\Phi(\boldsymbol{x}) = -G \cdot \int d^3x' \, \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{H.540}$$

suggesting that $\bar{h}_{tt} = \frac{4\Phi}{c^2}$, as well as of a vectorial contribution A_i

$$A_i(\boldsymbol{x}) = -\frac{4G}{c^2} \cdot \int d^3x' \, \frac{\rho(\boldsymbol{x}') \cdot u_i(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{H.541}$$

from ρu_i, appearing as $\bar{h}_{it} = \bar{h}_{ti} = \frac{A_i}{c}$. In order to construct the metric we need to revert back to $h_{\mu\nu}$,

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2} \, \eta_{\mu\nu} \tag{H.542}$$

and discard contributions to $\bar{h}_{ij}$, which is valid for small velocities $\beta \ll 1$. Then, the trace is simply given by the Newtonian potential, $\bar{h} = \bar{h}_{tt} \rightarrow h_{\mu\nu} = \pm\frac{2\Phi}{c^2}$ and the full line element reads

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \, dt^2 + 2A_i \, dtx^i - \left(1 - \frac{2\Phi}{c^2}\right) dx_i dx^i. \tag{H.543}$$

When computing the Christoffel-symbols from this metric, which would be needed for e.g. the geodesic equation

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\ \mu\nu} \, u^\mu u^\nu = 0 \tag{H.544}$$

describing the motion of a test particle, one realises that the scalar, Newtonian potential Φ appears in $\Gamma^i_{\ tt}$ while the vectorial potential A_i influences $\Gamma^i_{\ tj}$ and $\Gamma^i_{\ jk}$ terms, i.e.

that one needs a nonzero velocity u^i to notice them, and that those terms will be proportional to the velocity at first and second power (and as inertial accelerations those would be exactly the Coriolis acceleration and the centrifugal acceleration). Velocity-dependent accelerations in relativistic motion are very typical, and in analogy to the Lorentz-force in electrodynamics these accelerations are called gravitomagnetic accelerations.

H.7 *Wave equation and Lorenz-gauge condition*

Gravitational waves are a typical consequence of the hyperbolic gravitational field equation. After a suitable linearisation procedure and after writing the amplitudes with the trace reverse, one obtains the wave equation

$$\Box\, \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{H.545}$$

in the Lorenz gauge, $\partial^\mu\, \bar{h}_{\mu\nu} = 0$. The waves necessarily follow null-geodesics which illustrates why in our consideration about the most general classical theory of gravity the parameter m was set to zero: Otherwise, the wave equation would have read $\Box\Phi = m^2\Phi$ such that for the wave vector $\eta_{\mu\nu}k^\mu k^\nu = m^2 > 0$ and would therefore lie inside the light cone. In addition, propagation of wave would not be dispersion-free.

The superposition principle applies to such a linear field equation and one can introduce plane waves as fundamental Fourier-modes:

$$h_{\mu\nu}(x) = \int \frac{d^3k}{(2\pi)^3}\, A_{\mu\nu}(\boldsymbol{k})\, \exp\left(\pm i\eta_{\alpha\beta}\, k^\alpha x^\beta\right) \tag{H.546}$$

with amplitudes $A_{\mu\nu}(\boldsymbol{k})$, and perhaps it's worth pointing out that in the context of a flat Minkowski-background with Cartesian coordinates the tuple x^μ is indeed a vector. While the $\Box$-operator generates a perfectly normal retardation,

$$\bar{h}_{\mu\nu}(x) = -\frac{4G}{c^4} \cdot \int d^3x'\, \frac{T_{\mu\nu}(\boldsymbol{x}', ct - |\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{H.547}$$

captured by the Green-function, it would be unnecessary to distinguish distances from different points of the source to the observer, $|\boldsymbol{x} - \boldsymbol{x}'| \sim r$ for all $\boldsymbol{x}'$, defining the compact source approximation:

$$\bar{h}_{\mu\nu}(x) = -\frac{4G}{c^4 \cdot r} \cdot \int d^3x'\, T_{\mu\nu}(\boldsymbol{x}', ct - r) \tag{H.548}$$

with a common retardation. It should be kept in mind that gravitational waves as vacuum solutions to the field equation only exhibit Weyl-curvature and that the Birkhoff-theorem forbids spherically symmetric gravitational waves, as spherically symmetric vacuum solutions need to be static.

H.8 *Plane gravitational waves in traceless transverse gauge*

The wave equation fixes the wave vector k^μ to be lightlike, $\eta_{\mu\nu}k^\mu k^\nu = \omega^2/c^2 - k^2 = 0$, so for a propagation along the z-axis of a Cartesian coordinate frame one would write $k_\mu = (k, 0, 0, -k)^t$, so that the Lorenz-gauge condition $\partial^\mu\, \bar{h}_{\mu\nu} = 0$ makes sure that the amplitudes obey $A_{\mu\nu}k^\mu = 0$ so that they are confined to the (x, y)-plane of the coordinate system. Any further gauge transformation

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} - \partial_\mu\,\xi_\nu - \partial_\nu\,\xi_\mu \tag{H.549}$$

defined through the gauge function $\xi_\mu(x)$ does not interfere with the Lorenz-gauge if it obeys $\Box\,\xi_\mu = 0$, because due to the condition $\Box\,\xi_\mu = \partial^\alpha\,\bar{h}_{\alpha\mu}$ the gauging condition is maintained.

A specific choice for the gauge function would be $\xi_\mu = \epsilon_\mu\,\exp(\mathrm{i}\,k_\alpha x^\alpha)$ with constant ϵ_μ, which would obviously fulfil $\Box\,\xi_\mu = 0$ as a wave, and it would have the effect to change the gravitational wave amplitude to

$$A'_{\mu\nu} = A_{\mu\nu} - \mathrm{i}\epsilon_\mu k_\nu - \mathrm{i}\epsilon_\nu k_\mu + \mathrm{i}\epsilon_\alpha k^\alpha \cdot \eta_{\mu\nu} \tag{H.550}$$

Effectively, the new gauge introduces coordinates that oscillate along with the gravitational wave, and the best way to visualise this would be to draw the analogy to comoving coordinates. Specifically, the amplitudes in this coordinate frame with the null-vector k^μ become

$$A'_{tt} = A_{tt} - \mathrm{i}k\,(\epsilon_t + \epsilon_z) \quad A'_{tx} = A_{tx} - \mathrm{i}k\epsilon_x \tag{H.551}$$

$$A'_{xx} = A_{xx} - \mathrm{i}k\,(\epsilon_t - \epsilon_z) \quad A'_{ty} = A_{ty} - \mathrm{i}k\epsilon_y \tag{H.552}$$

$$A'_{yy} = A_{yy} - \mathrm{i}k\,(\epsilon_t - \epsilon_z) \quad A'_{xy} = A_{xy} \tag{H.553}$$

While transversality $A_{\mu\nu}\,k^\nu = 0$ fixes the relation

$$A_{\mu\nu}\,k^\nu = k \cdot A_{\mu t} - k \cdot A_{\mu z} = k\,(A_{\mu t} - A_{\mu z}) = 0 \tag{H.554}$$

to $A_{\mu t} = A_{\mu z}$. Then, the particular choice of the constants ϵ_μ : $A'_{tt} = A'_{tx} = A'_{ty} = 0$, $A'_{xx} = -A'_{yy}$ implies

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{H.555}$$

which is referred to as the traceless transverse gauge, because of $\eta^{\mu\nu}A_{\mu\nu} = 0$ and $k^\mu A_{\mu\nu} = 0$. The shape of the central section of the matrix pertaining to the (x, y)-plane suggests the ansatz $a(t,z)\,\sigma^{(1)} + b(t,z)\,\sigma^{(3)} \sim \bar{h}_{\mu\nu}$ illustrating that there should be two polarisation modes, with oscillatory functions $a(t)$ and $b(t)$, such that the line element assumes the form

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 - a(t,z)\left[\mathrm{d}x^2 - \mathrm{d}y^2\right] - 2\,b(t,z)\,\mathrm{d}x\mathrm{d}y \tag{H.556}$$

i.e. effectively a Minkowski line element with periodic deformations in the plane transverse to the propagation direction. In comparison, the FLRW-line element for a flat Universe is given by

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t) \cdot \left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right] \tag{H.557}$$

which suggests that the two functions $a(t)$ and $b(t)$ should be thought of as scale factors, relating the comoving coordinates in the (x,y)-plane (which is actually the role of the traceless transverse gauge) to physical distances. Of course, the analogy does not go further than that as the two solutions could not be more different: FLRW-universes are systems of pure Ricci curvature and the effects of Λ are important, while gravitational waves are vacuum solutions with pure Weyl-curvature.

The motion of test particles is given by the geodesic equation $\mathrm{d}u^\alpha/\mathrm{d}\tau + \Gamma^\alpha_{\mu\nu}\, u^\mu u^\nu = 0$, and if the particle is initially at rest, $u^\mu = (c,0)^t$ one would obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}u^\alpha = \frac{\mathrm{d}}{\mathrm{d}t}u^\alpha = -\Gamma^\alpha_{\mu\nu}\, u^\mu u^\nu = -c^2\,\Gamma^\alpha_{tt} = -\frac{c}{2}\eta^{\alpha\beta}\left[\partial_t h_{\beta t} + \partial_t h_{t\beta} - \partial_\beta h_{tt}\right] = 0 \quad \text{(H.558)}$$

confirming that the test particles are indeed at rest in the traceless transverse (comoving) coordinate frame. That of course does not mean that the physical distance between the particles does not change! Physical distances, as measured for instance at $\mathrm{d}t = 0$ or along the light cone $\mathrm{d}s = 0$ oscillate as given by $a(t)$ and $b(t)$.

H.9 *Huygens' principle and elementary waves*

There is a fundamental difference in the propagation of (spherical) waves in spacetimes with different dimensionalities. A plane wave obviously obeys the wave equation, for instance for a scalar field ϕ one gets

$$\eta^{\mu\nu}\,\partial_\mu\partial_\nu\,\phi = \Box\,\phi = \left[\partial_{ct}^2 - \sum_{i=1}^{n}\partial_i^2\right]\phi = 0 \quad \text{(H.559)}$$

with a light cone condition $\eta^{\mu\nu}k_\mu k_\nu = 0$, obtained by substitution of $\phi \propto \exp(\pm \mathrm{i}k_\mu x^\mu)$. If one now asks whether a spherical wave obeys a light cone condition, too, i.e. whether the radius r of a spherical wave front is given by $r = ct$, the answer would depend on the number of dimensions that spacetime has. This is in contrast to plane waves, because in fact eqn. H.559 always reduces to a wave equation in one temporal and one spatial dimension by orienting the coordinate system in the direction of k^i.

Spherical symmetry reduces the Laplace-operator to contain only derivatives along the r-direction, such that $\delta^{ij}\,\partial_i\partial_j\psi = \Delta_n\,\phi = \sum_{i=1}^{n}\partial_i^2\phi = 0$, with the definition $r^2 = \delta_{ij}\,x^i x^j = x_i x^i = \sum_{i=1}^{n} x_i^2$ typical for Euclidean space. A spherically symmetry blast wave would then increase its radius r as a function of time t, such that the system is effectively 2-dimensional. But even though it would be reasonable to assume that r and t fulfil a light cone condition $ct - r = 0$ we shall see that this is only the case in $1+1$ and $3+1$ dimensions!

From the derivatives

$$\partial_i r = \frac{x_i}{r} \quad \text{and} \quad \partial_i^2 r = \frac{1}{r} - \frac{x_i^2}{r^3} = \frac{r^2 - x_i^2}{r^3} \quad \text{(H.560)}$$

we can derive that

$$\sum_i(\partial_i r)^2 = \frac{1}{r^2}\cdot\sum_i x_i^2 = 1 \quad \text{and} \quad \sum_i \partial_i^2 r = \frac{1}{r}\sum_i 1 - \frac{1}{r^3}\sum_i x_i^2 = \frac{n-1}{r} \quad \text{(H.561)}$$

such that the double derivatives ∂_i^2 in the wave equation can be written as

$$\partial_i^2\, \phi = \partial_i\,(\partial_i\, r\partial_r\, \phi) = \partial_i^2 r \cdot \partial_r \phi + (\partial_i\, r)^2 \cdot \partial_r^2 \phi \tag{H.562}$$

Summing over i gives the Laplace-operator needed for the wave equation, then reformulated in radial derivatives,

$$\Delta\phi = \sum_i \partial_i^2 \phi = \sum_i \partial_i^2 \phi = \frac{n-1}{r}\partial_r\, \phi + \partial_r^2 \phi \tag{H.563}$$

so that the wave equation for a spherical wave reads

$$\Box\, \phi = \partial_{ct}^2 \phi - \partial_r^2\, \phi - \frac{n-1}{r}\partial_r\, \phi = 0 \tag{H.564}$$

with an additional term $\partial_r \phi / r$ containing a first derivative. If it was not for that term, spherical waves in any number of dimensions would behave like plane waves, which is the case $n = 1$.

The asymptotic behaviour of the wave can be isolated by setting $\psi(r) \simeq r^k \cdot \phi(r)$ with a negative exponent k, as the amplitude is expected to decrease with increasing distance. Reformulating the wave equation in terms of ψ instead of ϕ gives

$$\partial_r \psi = r^k\, \partial_r\, \phi + k \cdot r^{k-1} \phi \quad \text{and} \quad \partial_r^2 \psi = r^k\, \partial_r^2 \phi + 2 \cdot k\, r^{k-1} \partial_r\, \phi + k(k-1) \cdot r^{k-2} \phi \tag{H.565}$$

arriving by division with r^k at

$$\frac{1}{r^k}\partial_r^2\, \psi = \partial_r^2\, \phi + \frac{2k}{r} \cdot \partial_r\, \phi + \frac{k(k-1)}{r^2}\phi \tag{H.566}$$

If the energy flux is proportional to the squared amplitudes ϕ^2 and if it is conserved when integrated over shells of radius r which in turn have an area $\propto r^{n-1}$ in n spatial dimensions, the amplitudes need to scale as

$$\psi(r) = r^{\frac{n-1}{2}} \cdot \phi(r) \tag{H.567}$$

suggesting that $k = (n-1)/2$. Substitution of that particular scaling then

$$\frac{1}{r^{\frac{n-1}{2}}} \cdot \partial_r^2\, \psi = \partial_r^2 \phi + \frac{n-1}{r}\partial_r\, \phi + \frac{(n-1)(n-3)}{4r}\phi \tag{H.568}$$

and finally

$$\partial_{ct}^2 \psi = \partial_r^2 \psi - \frac{(n-1)(n-3)}{4r^2}\psi \tag{H.569}$$

which is a truly surprising result: One recovers the archetypical wave equation in 1 and 3 spatial dimensions as the last term vanishes, but there will be in general additional effects from that term in propagation problems. Spherical waves in 3 + 1 dimensions behave in every aspect as plane waves as their radius obeys a light cone condition $ct - r = 0$ as as their propagation is therefore dispersionless. In spacetimes with other dimensionality one would see through numerical computation that there is no infinitesimally thin wave front, instead the entire bubble with radius $r = ct$ is filled with nonzero amplitudes, as not all partial waves propagate at the same speed. Formally, solutions to the spherical wave can be constructed with a power series ansatz, as eqn. H.568 is a differential equation of the Bessel-type.