G FRIEDMANN-UNIVERSES

G.1 *Friedmann-Lemaître-Robertson-Walker cosmologies*

Friedmann-Lemaître-Robertson-Walker spacetimes are highly symmetric solutions of the gravitational field equation for a particular matter distribution: Even though there is the cosmic large scale structure in the distribution of galaxies and strong inhomogeneities, fluctuations in the matter distribution are thought to subside approaching scales above a few hundred Mpc. This is summarised by the cosmological principle, which postulates that the matter and consequently the geometric properties of spacetime are homogeneous (they don't change as a function of position in the Universe) and isotropic (independent of the direction in which one observes the dynamics of spacetime). The high degree of symmetry in the matter distribution allows to find a non-vacuum solution to the gravitational field equation, and homogeneous and isotropic geometries sourced by ideal fluids constitute the class of FLRW-cosmologies. Observations of distant objects show that spacetime on these very large scales is dynamic.

Fundamental observers in a FLRW-spacetime are thought to be freely falling and are stationary with respect to their surrounding matter distribution. Their relative motion can be described by geodesic deviation, but every observer would naturally center a coordinate system on her or his position (allowed by symmetry) and perceive the properties of spacetime isotropically at every point. Is is perfectly possible that the world lines have intersected in the past (this was in fact the case!) and they might intersect in the future (which won't be the case according to our understanding). The first intersection point is called the Big Bang, and we'll come to the dynamic of congruences of geodesics at a later time.

A natural choice of the time coordinate is then the proper time τ of those observers, which need to be identical for every world line, again as a consequence of homogeneity, motivating the definition of synchronous time t. Spatial coordinates are defined to be comoving, meaning that every freely falling object stays at its respective coordinate. This defines a slicing of spacetime into spatial hypersurfaces of constant time, and a threading of spacetime in terms of world lines with a common passage of synchronous coordinate time.

The metric defines for an arbitrary set a measurable spacetime distance in form of the line element,

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{c}^2\,\mathrm{d}t^2 - g_{ij}\,\mathrm{d}x^i \mathrm{d}x^j \tag{G.424}$$

The orthogonality of spacetime slices and threads suggests the separation into the temporal and spatial part of the metric. The line element measures the length of a world line which is perceived by the observer as her or his elapsed proper time, $c^2\mathrm{d}\tau^2 = \mathrm{d}s^2 = c^2\mathrm{d}t^2$ if $\mathrm{d}x^i = 0$ for comoving observers, and therefore $\tau = t$: Synchronous, physical time is measured by clocks of the fundamental observers, and elapses identically for everyone.

A particle at rest follows a world line defined by

$$x^\alpha = \begin{pmatrix}\mathrm{c}t\\0\end{pmatrix} \quad\rightarrow\quad u^\alpha = \frac{\mathrm{d}}{\mathrm{d}\tau}x^\alpha = \frac{\mathrm{d}}{\mathrm{d}t}x^\alpha = \begin{pmatrix}\mathrm{c}\\0\end{pmatrix} \tag{G.425}$$

and the tangent u^μ needs to fulfil the geodesic equation - otherwise the particle could not be freely falling.

In fact,

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\ \mu\nu}\ u^\mu u^\nu = 0 \tag{G.426}$$

is fulfilled because the specific form of the tangents u^μ requires just a single Christoffel-symbol,

$$\Gamma^\alpha_{\ tt} = \frac{g^{\alpha\beta}}{2}\Big(\partial_t\, g_{t\beta} + \partial_t\, g_{\beta t} - \partial_\beta\, g_{tt}\Big) = 0 \tag{G.427}$$

which is necessarily zero: The changes of u^μ vanish and the particles stay at their comoving coordinates.

Symmetry requires that the metric g_{ij} can only be a function of t. In a frame where g_{ij} is diagonal isotropy must hold, too, so all three eigenvalues must be identical:

$$ds^2 = c^2\, dt^2 - a^2(t)\tilde{g}_{ij}\, dx^i dx^j \tag{G.428}$$

where $\tilde{g}_{ij}$ can be Euclidean, $\tilde{g}_{ij} = \delta_{ij}$, but it might as well be possible that the spatial submanifold has a constant (otherwise homogeneity would not hold) spatial curvature. Allowing for this case, the FLRW-line element assumes the shape

$$ds^2 = c^2\ dt^2 - a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\Big] \tag{G.429}$$

with the scale factor $a(t)$. The Euclidean case is recovered by $k = 0$. Spatially non-flat universes would have $k = +1$ if they are spherical with a positive curvature, and $k = -1$ if they are hyperbolical with a negative curvature.

G.2 *FLRW-cosmologies as maximally symmetric spacetimes*

FLRW-cosmologies are maximally symmetric spacetimes in what concerns the spatial part (also called a maximally symmetric 3-space), as one can write the Riemann-curvature as a function of the Ricci-scalar and the metric alone:

$$R_{\alpha\beta\mu\nu} = \frac{R}{12}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \tag{G.430}$$

which is traced back to the fact that there is no Weyl-curvature $C_{\alpha\beta\mu\nu} = 0$ and that the Ricci-tensor comes out proportional to the the metric, $R_{\beta\nu} = R/4\, g_{\beta\nu}$, self-consistent with $g^{\beta\nu}R_{\beta\nu} = R/4\, g^{\beta\nu}g_{\beta\nu} = R/4\, \delta^\beta_\beta = R$.

The physical reason for the absence of Weyl-curvature is not only that FLRW-solutions are non-vacuum solutions, but also that there are absolutely no propagation effects of gravity, as the densities on every spatial hypersurface are constant. Absence of Weyl-curvature implies conformal flatness and Minkowski-light cones in conformal coordinates, and it is the case that the scale factor $a(\eta)$ is exactly the conformal factor $\Omega(\eta)$.

G.3 *Conformal flatness of FLRW-cosmologies*

FLRW-cosmologies are systems with pure Ricci-curvature, and as their density on any spatial hypersurface is constant, the Weyl-tensor is necessarily zero: There are no propagation effects of gravity. As the Weyl-tensor vanishes, $C_{\alpha\beta\mu\nu} = 0$ the FLRW-spacetime is conformally flat and coordinates can be found where the metric can be written as

$$g_{\mu\nu} = \Omega^2(t)\eta_{\mu\nu}, \tag{G.431}$$

where the conformal factor is in this particular case only a function of time; and the suitable coordinate choice are conformal coordinates, the spatial part of which is usually called comoving. Specifically, the line element for a spatially flat FLRW-cosmology in physical time t and comoving coordinates r reads

$$\mathrm{d}s^2 = \mathrm{c}^2\,\mathrm{d}t^2 - a^2(t)\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{G.432}$$

If one defines conformal time $\mathrm{d}\eta = \frac{\mathrm{d}t}{a(t)} \to \eta = \int \frac{\mathrm{d}t}{a(t)} \neq t$ one obtains a new temporal coordinate different from physical time. While the length of the world line of a particle at rest is measured in terms of proper time τ,

$$c^2\mathrm{d}\tau^2 = \mathrm{d}s^2 = c^2\mathrm{d}t^2 \quad \to \quad \tau = t \tag{G.433}$$

such that proper time τ and coordinate time t come out equal, and must be equal everywhere due to the cosmological principle, conformal time intervals $\mathrm{d}\eta = \mathrm{d}t/a(t)$ have been short in the past and slow down as $a(t)$ expands, and catch up with $\mathrm{d}t$ today. In fact, the scale factor $a(t)$ plays the role of the conformal factor $\Omega(t)$, as in these coordinates the line element reads

$$\mathrm{d}s^2 = a^2(t)\left[\mathrm{c}^2\,\mathrm{d}\tau^2 - \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{G.434}$$

For radial (which can always be achieved using the cosmological principle, that allows to centre the coordinate frame on the observer such that $\mathrm{d}\theta = \mathrm{d}\phi = 0$ along the photon trajectory) light-like geodesics, one obtains

$$\mathrm{d}s^2 = a^2(t)\left[c^2\mathrm{d}t^2 - \mathrm{d}r^2\right] = 0 \tag{G.435}$$

and the scale factor as the conformal factor does not have any influence on light propagation, if measured in terms of comoving radial coordinate r and conformal time τ, in fact, in these coordinates one has perfectly conventional Minkowskian light cones, $c\mathrm{d}\eta = \pm\mathrm{d}r$ and from that, $c\eta = \pm r$. Whether the light cones expand to positive or negative infinity in terms of physical time instead of conformal time, depends on the relation $\mathrm{d}\eta = \mathrm{d}t/a(t)$ which might be divergent in which case a horizon appears.

G.4 *Spatial curvature of FLRW-cosmologies*

Perhaps a bit surprisingly, spatial curvature $k \neq 0$ which affects the scaling of the surface of spheres with their comoving radii, does not imply deviations from conformal flatness as spacetime property: Homogeneity and isotropy as symmetries are still present, requiring the absence of Weyl-curvature, which in turn ensures conformal flatness. A general FLRW line element including spatial curvature is

$$\mathrm{d}s^2 = a^2\left[\mathrm{c}^2\mathrm{d}\eta^2 - \frac{1}{1-kr^2}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{G.436}$$

where $k = -1$ corresponds to negative, hyperbolic curvature, for which we define a new radial coordinate $r = \sinh\chi$ with the derivative $\mathrm{d}r/\mathrm{d}\chi = \cosh\chi$, implying

$$\mathrm{d}s^2 = a^2\left[c^2\,\mathrm{d}\eta^2 - \frac{\cosh^2\chi}{1+\sinh^2\chi}\mathrm{d}\chi^2 - \sinh^2\chi\cdot\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{G.437}$$

with $\cosh^2\chi \equiv 1 + \sinh^2\chi$, and one finds again Minkowski light cones for radial photon geodesics, $\mathrm{d}s^2 = a^2\cdot\left[c^2\,\mathrm{d}\eta^2 - \mathrm{d}\chi^2\right]$.

Similarly $k = +1$ corresponds to a spacetime with positive, spherical curvature. Definition of a new coordinate $r = \sin\chi$ with the derivative $\mathrm{d}r/\mathrm{d}\chi = \cos\chi$ then suggests for the line element,

$$\mathrm{d}s^2 = a^2\left[c^2\,\mathrm{d}\eta^2 - \frac{\cos^2\chi}{1-\sin^2\chi}\mathrm{d}\chi^2 - \sin^2\chi\cdot\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{G.438}$$

with $\cos^2\chi = 1 - \sin^2\chi$. Then again, radial photon geodesics will come out as Minkowskian. It is even possible to redefine conformal coordinates as light cone coordinates,

$$\mathrm{d}u = \frac{1}{2}(c\,\mathrm{d}\eta - \mathrm{d}\chi) \tag{G.439}$$

$$\mathrm{d}v = \frac{1}{2}(c\,\mathrm{d}\eta + \mathrm{d}\eta) \tag{G.440}$$

where the line element would read $\mathrm{d}s^2 = 4a^2\mathrm{d}u\,\mathrm{d}v$, implying $\mathrm{d}u\mathrm{d}v = 0$ for photons.

The geometric interpretation of spatial curvature $k \neq 0$ is a non-Euclidean scaling of areas and volumes of spheres with their comoving radius (at a fixed time). Embedding spatial part of a spherically curved FLRW-spacetime into a 4d Euclidean space can be done with the transformation

$$x = R\sin\chi\,\sin\theta\,\cos\phi \tag{G.441}$$

$$y = R\sin\chi\,\sin\theta\,\sin\phi \tag{G.442}$$

$$z = R\sin\chi\,\cos\theta \tag{G.443}$$

$$w = R\cos\chi \tag{G.444}$$

with the constraint $x^2+y^2+z^2+w^2 = R^2$, defining the manifold. With this embedding, one can compute the area A of a sphere with radius R,

$$A = \int \mathrm{d}\theta\,R\sin\chi\int \mathrm{d}\phi R\sin\chi\,\cos\theta = 4\pi R^2\sin^2\chi \tag{G.445}$$

as well as the volume V,

$$V = \int \mathrm{d}\chi\,R\cdot\int \mathrm{d}\theta\,R\sin\chi\int \mathrm{d}\phi\,R\sin\chi\cos\theta = 2\pi^2R^3 \tag{G.446}$$

Because $\sin^2\chi \leq 1$ always, one obtains for positively curved spherical FLRW-cosmologies surfaces that are smaller than that in a Euclidean space. Repeating the exercise for hyperbolic, negatively curved cosmologies yields $A = 4\pi R\sinh^2\chi$, and systematically larger areas, as well as a divergent volume V, both as $\chi \to \infty$.

2do: redo with induced metric

G.5 *Cosmological redshift*

The dynamics of the FLRW-spacetime has the effect that photons arrive at an observer redshifted lower frequency (or higher wavelength), caused by the changing geometry between emission and observation. To make the point that the lower frequency caused by the increase in scale factor is a transformation effect, we can try the following: Photon propagation is most conveniently described in conformal coordinates, where absolutely no property of the photon changes with time. What changes, however, is the definition of the scalar product that is needed to project the wave vector of the photon k^μ onto the world lines of the emitter and observer represented by the tangent u^μ, thereby defining the frequency $\omega = g_{\mu\nu}u^\mu k^\nu$. ω is a physical observable and comes out, as a scalar, independent of any coordinate choice for $g_{\mu\nu}$, u^μ and k^μ.

In conformal coordinates metric reads

$$g_{\mu\nu} = \begin{pmatrix} +a^2 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix} \tag{G.447}$$

such that $g_{\mu\nu} = \eta_{\mu\nu}$ at $a = 1$, i.e. today, and provides the scalar product for projecting k^μ onto u^μ.

The motion of galaxies is purely timelike along the ct- or $c\eta$-direction, and in conformal coordinates every galaxy and every observer stays at their comoving coordinate: $\mathrm{d}r = 0$. The tangent $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau = \mathrm{d}x^\mu/\mathrm{d}t$ is normalised to c^2, so we get:

☠ *Conformal time η is not an affine parameter, so we can't directly parameterise the world line with η. But it's perfectly permissible to take $u^\mu = \mathrm{d}x^\mu/\mathrm{d}t$ as a vector and do a coordinate transform switching from physical to conformal time.*

$$g_{\mu\nu}\, u^\mu u^\nu = c^2 = g_{\eta\eta}\, u^\eta u^\eta \quad \to \quad u^\eta = \frac{c}{\sqrt{g_{\eta\eta}}} = \frac{c}{a} \tag{G.448}$$

because of the motion along the $c\eta$-direction only the η-entry of u^μ is nonzero. Photons follow null geodesics, so

$$c^2\,\mathrm{d}\eta^2 - \mathrm{d}r^2 = 0 \quad \to \quad c\eta = \pm r \tag{G.449}$$

with a wave vector $k^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$, using an affine parameter λ. k^μ is normalised to zero, $g_{\mu\nu}k^\mu k^\nu = 0$ and has the entries $k^\mu = (\omega/c, k)^t$. Then, the projection of the wave vector k^μ onto the tangent of the world lines of comoving systems u^μ is given by

$$\omega' = g_{\mu\nu}k^\mu u^\nu = a^2 k^\eta u^\eta = a^2 \cdot \frac{\omega}{c}\frac{c}{a} = a\omega \tag{G.450}$$

ω' is the frequency today, where $a = 1$ by convention. Reformulating the result in terms of wave length with the dispersion $\omega = ck$ (coming from $g_{\mu\nu}k^\mu k^\nu = (\omega/c)^2 - k^2 = 0$) and $k = 2\pi/\lambda$ then implies

$$\lambda' = \frac{\lambda}{a} \tag{G.451}$$

such that the redshift z is defined as

$$z = \frac{\lambda' - \lambda}{\lambda} = \frac{1}{a} - 1 \quad \rightarrow \quad a = \frac{1}{1+z}. \tag{G.452}$$

G.6 *Cosmological horizons and causal structure*

The introduction of conformal coordinates brushes over the fact that depending on the cosmology photons are only given a finite time to propagate and can only reach finite physical distances, both coming from the finite past or traveling into a possibly finite future. Effectively, we ask the question whether there are limits to the light cones, which are not apparent in terms of conformal coordinates. The particle horizon is the limitation of the past light cone caused by a finite age of the Universe. The maximum distance a photon could have traveled since $a = 1$ is given by

☠ *Please keep in mind that for a vanilla model with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$ this is in fact case! But arbitrary FLRW-models could realise anything.*

$$r_{\mathrm{PH}} = c\int_{t_i}^{t_0} \frac{\mathrm{d}t}{a} = c\int_{-\infty}^{0} \mathrm{d}\eta \tag{G.453}$$

where for an actual computation one needs $H = \dot{a}/a$. The origin of the conformal coordinate system in time is conveniently chosen to be $\eta = 0$ today. The event horizon is the maximum distance that light emitted today could possibly cover in the future:

$$r_{\mathrm{EH}} = c\int_{t_0}^{t_f} \frac{\mathrm{d}t}{a} = c\int_{0}^{+\infty} \mathrm{d}\eta \tag{G.454}$$

where it is clear that the behaviour of $1/a(t)$ is the decisive quantity that causes the integrals to converge or to diverge, while the $a(t)$ relation itself as a solution to the Friedmann-equation depends on all gravitating fluids and their properties ρ and w.

G.7 *Friedmann-equations*

Substituting the energy-momentum tensor $T_{\mu\nu}$ into the gravitational field equation and solving for $g_{\mu\nu}$ which in turn is needed for the motion of the fluid according to $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu}$ is a nice example of how gravity, geometry and motion work together. Our starting point is the FLRW-metric

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\cdot\left[\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{G.455}$$

with the choice of using physical time t (identical to proper time τ of comoving observers) and comoving distance r as coordinates: this is referred to as the synchronous gauge, as the metric is constant on a spatial hypersurface defined through a constant value of t. We have already made the point that the FLRW-spacetime is conformally flat and has only Ricci-curvature. There is a single parameter, k, which determines the spatial geometry on a spatial hypersurface, and the only dynamic degree of freedom is the scale factor $a(t)$, which changes the distance definition on each hypersurface, moving from t to another time t'. It is a convention to set $a = 1$ today - there is a priori no particular instant in time defined singled out by the FLRW-metric, so we may bring in this human element.

Substitution of this metric into the field equation

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{G.456}$$

with a homogeneous and isotropic ideal fluid (the symmetries of the fluid need to be consistent with the symmetries of the metric, and the fluid can only be ideal as it otherwise would not obey local energy momentum conservation) yields the Friedmann-equations as dynamical equations for $a(t)$.

Turning to the energy-momentum tensor $T_{\mu\nu}$ as the source of the gravitational field and its covariant energy momentum conservation $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ (which we have already shown to be equivalent to the equations of relativistic fluid mechanics on a possibly curved background), one realises that the cosmological principle requiring a homogeneous and isotropic fluid makes sure that the Euler-equation is trivially fulfilled: There are no spatial gradients in p that would accelerate the fluid by non-gravitational forces. In fact,

$$\left(\rho + \frac{p}{c^2}\right) g^{\alpha\mu}\, u_\mu \nabla_\alpha\, u_\nu = -g^{\alpha\mu}\left(\frac{u_\mu u_\nu}{c^2} - g_{\mu\nu}\right)\nabla_\alpha p \tag{G.457}$$

suggests that the relevant driving gradient in p gets projected onto a plane perpendicular to u^μ. The FLRW-symmetries disallow $\partial_i p$ in this hyperplane, but do not restrict $\partial_t p$. That component however, is in our coordinate choice perpendicular to the hyperplane, so it can not affect the motion of the fluid. From that we conclude that $g^{\alpha\mu}\, u_\mu \nabla_\alpha\, u_\nu = 0$, which is just the autoparallelity condition: The fluid elements follow geodesics.

The continuity equation, however, is not trivial and reads

$$g^{\alpha\mu}\left[\nabla_\alpha\left(\rho c^2 u_\mu\right) + p\nabla_\alpha u_\mu\right] \tag{G.458}$$

Rewriting it in terms of a divergence

$$\nabla_\mu\left(\rho c^2 u^\mu\right) + p\nabla_\mu u^\mu = 0 \tag{G.459}$$

using metric compatibility and using the divergence formula bringing in the covolume $\sqrt{-g}$

$$\partial_\mu\left(\sqrt{-g}\left(\rho c^2 u^\mu\right)\right) + p\partial_\mu\left(\sqrt{-g}u^\mu\right) = 0 \tag{G.460}$$

reduces to a considerably more simple shape using comoving coordinates: There are only derivatives ∂_t and only $u^t = c$, while $\sqrt{-g} = ca^3$:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{G.461}$$

There is an intuitive but potentially misleading reinterpretation of the continuity equation in the form of the adiabatic equation: While the mathematics is certainly correct, the physical interpretation is a bit problematic. Establishing the relation between the $\mathrm{d}p$ and $\mathrm{d}a$ differentials,

$$\mathrm{d}\rho + 3\left(\rho + \frac{p}{c^2}\right)\frac{\mathrm{d}a}{a} = 0 \tag{G.462}$$

and multiplying with a^3 one can use the Leibnitz-rule to write

$$d\left(\rho c^2 a^3\right) = -p\, d(a^3) \tag{G.463}$$

which seems to suggest that the change in energy, given by the energy density multiplied with the volume a^3 is equal to the work done by changing the volume against the pressure p, reminiscent of the first law of thermodynamics. Please keep in mind, however, that pressure enters the field equations as a source of gravity and that there are no gradients in p that could perform work.

But the argument suggests a new question: Where is the limitation in the relation between pressure and energy density? Taking the trace of the field equation yields for the Ricci-scalar R

$$R(t) = \frac{8\pi G}{c^4}\, T + 4\Lambda \tag{G.464}$$

with the trace of the energy-momentum tensor T

$$T = g^{\mu\nu} \cdot \left[\left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - p\, g_{\mu\nu}\right] = \rho c^2 - 3p \tag{G.465}$$

such that one arrives at

$$R(t) = \frac{8\pi G}{c^4}\left(\rho c^2 - 3p\right) + 4\Lambda \tag{G.466}$$

which suggests that the Ricci-curvature is positive for all fluids with equation of state $w < +1/3$, and in the absence of a cosmological a fully radiation dominated Universe would have a vanishing Ricci-scalar, $R = 0$! Of course that is a direct consequence of the masslessness of the photon that already makes sure that $T = 0$, and would not imply that there is no Ricci curvature at all: The Ricci tensor would still be non-vanishing.

Then, one needs the Ricci-tensor $R_{\mu\nu}$ as well as the Ricci-scalar $R(t)$ for the field equation, following the chain $g_{\mu\nu} \rightarrow \Gamma^{\alpha}{}_{\mu\nu} \rightarrow R_{\alpha\beta\mu\nu} \rightarrow R_{\beta\nu} \rightarrow R$, for which there is really no shortcut (apart from the Cartan-formalism). It's important to realise that the Ricci-tensor comes out proportional to the metric, as required for maximally symmetric spacetimes, and therefore diagonal in our choice of coordinates,

$$R_{tt} = 3\frac{\ddot{a}}{a} \tag{G.467}$$

$$R_{rr} = \frac{-c^2}{1-kr^2}\left(a\ddot{a} + 2\dot{a}^2 + 2c^2 k\right) \tag{G.468}$$

$$R_{\theta\theta} = -\frac{c}{r^2}\left(a\ddot{a} + 2\dot{a}^2 + 2c^2 k\right) \tag{G.469}$$

$$R_{\phi\phi} = R_{\theta\theta} \cdot \sin^2\theta \tag{G.470}$$

such that contraction $g^{\mu\nu} R_{\mu\nu} = R$ yields the Ricci-scalar,

$$R(t) = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{ck}{a^2}\right]. \tag{G.471}$$

Substitution into the gravitational field equation $G_{\mu\nu} = -8\pi G/c^4\, T_{\mu\nu} - \Lambda g_{\mu\nu}$ and separating $\dot{a}$ from $\ddot{a}$ then yields the standard form of the Friedmann-equations

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1 + 3w)\,\rho + \frac{\Lambda c^2}{3} \tag{G.472}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = +\frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{c^2 k}{a^2} \tag{G.473}$$

which relate the evolution of the scale factor $a(t)$ to the presence of gravitating fluids, curvature and the cosmological constant. In parallel, covariant energy momentum conservation $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ yields in these coordinates the adiabatic equation

☠ Covariant energy-momentum conservation is already built into the gravitational field equation, so the adiabatic equation is not independent of the Friedmann equations.

$$\frac{\dot{\rho}}{\rho} + 3(1 + w)\frac{\dot{a}}{a} = 0, \tag{G.474}$$

from which the evolution of standard fluids with constant equation of state w can directly be read off: The equation is equivalent to $\partial_t \ln\rho = -3(1 + w)\partial_t \ln a$, which is solved to be $\rho \propto a^{-3(1+w)}$, so one obtains naturally $\rho \propto a^{-3}$ for matter, $\rho \propto a^{-4}$ for radiation and a constant ρ for the cosmological constant.

For a single dominating fluid at critical density it is possible to relate the equation of state directly to the deceleration

$$q = -\frac{\ddot{a}a}{\dot{a}^2} \tag{G.475}$$

with this funky relationship:

$$3(1 + w) = 2(1 + q) \tag{G.476}$$

such that the following picture emerges:

$$w \qquad q \tag{G.477}$$

$$+\frac{1}{3} \quad +1 \quad \text{relativistic particles, e.g. photons} \tag{G.478}$$

$$\pm 0 \quad +\frac{1}{2} \quad \text{non-relativistic matter} \tag{G.479}$$

$$-\frac{1}{3} \quad 0 \quad \text{pure curvature, empty universe, like a fluid } w = -\frac{1}{3} \tag{G.480}$$

$$-1 \quad -1 \quad \Lambda \sim \text{like a fluid with eos } w = -1 \tag{G.481}$$

Fluids with positive equation of state have an attractive effect and slow down the expansion of the Universe, but as a increases, they get diluted: $\rho \propto a^{-3(1+w)}$ is a decreasing function for all w strictly larger than -1. Therefore any expanding Universe will work its way towards smaller values equation of state as time passes. But as soon as $w < -1/3$ something interesting happens as the deceleration changes its sign: $q > 0$ for all fluids with $w < -1/3$, such that the expansion of the Universe gets accelerated if the Universe has gotten large enough, that the densities are sufficiently small. Weirdly, an empty and therefore maximally hyperbolically curved universe, expands at a constant velocity: $q = 0$ for $w = -1/3$, and therefore $\ddot{a} = 0$, from which one integrates $\dot{a}$ to be constant and a to be a linear function in time: There is no gravity that changes the state of motion. While this may seem as an odd result, please keep in mind that in a completely empty (and therefore hyperbolic universe) there is no matter content that could by its gravitational action change the state of motion of spacetime! Or, if you prefer a fancy argument, one can invoke the Birkhoff-theorem: There is no gravitational dynamic outside a spherically symmetric matter distribution: Surely, FLRW-universes are isotropic, and because there is nothing inside, one deals with a vacuum solution, and therefore, the universe is in a state of inertial motion.

The logarithmic derivative of the scale factor as a function of time defines the Hubble-Lemaître-function

$$H(a) = \frac{\dot{a}}{a} \tag{G.482}$$

which defines the critical density ρ_{crit} as a scale. Multiplying the first Friedmann-equation G.473 with $1 = H_0^2/H_0^2$ yields

$$\left(\frac{\dot{a}}{a}\right)^2 = +\frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{c^2 k}{a^2} \tag{G.483}$$

so that we can identify

$$\rho_{crit} = \frac{8\pi G}{3H_0^2} \tag{G.484}$$

with a numerical value of about 10^{-26}kg/m^3, roughly a few ten atoms per cubic metre. Equivalent to the density scale is the Hubble-length

$$\chi_H = \frac{c}{H_0} \tag{G.485}$$

roughly 10^{25}m in size. Redefining the terms in the Friedmann-equation by introducing the density parameters Ω_X

$$\Omega_\rho = \frac{\rho}{\rho_{crit}}, \quad \Omega_\Lambda = \frac{\Lambda}{3}\left(\frac{c}{H_0}\right)^2, \quad \Omega_k = -k\left(\frac{c}{H_0}\right)^2 \tag{G.486}$$

(please watch out for the minus-sign in the definition of Ω_k: negative curvature $k < 0$ has a positive Ω_k!) brings the first Friedmann-equation in the standard shape

$$H^2(t) = H_0^2 \cdot \left[\frac{\Omega_\rho}{a^3} + \Omega_\Lambda + \frac{\Omega_k}{a^2}\right] \tag{G.487}$$

which helps us to understand the meaning of critical density: As $H(t) = H_0$ at $a = 1$ necessarily,

$$\Omega_m + \Omega_k + \Omega_\Lambda = 1 \tag{G.488}$$

so that spatial curvature can only arise if the densities do not add up to the critical density. It seems natural that the gravitational field equation links the dynamics of the metric and therefore geometric properties of spacetime to the gravitating effect of all substances, but interestingly, we can use the field equation as well to assign properties of material substances such as ρ and p (or equivalently w) to a geometric property (curvature) or a phenomenon of gravity Λ.

G.8 *Cosmological constant* Λ

The numerical value of the cosmological constant $\Lambda = 10^{-50}\mathrm{m}^{-2}$ implies that it can only play a substantial role on scales of 10^{25}m an above, corresponding to the size c/H_0 of the observable Universe. The first Friedmann-equation shows that ultimately a continued expansion will necessarily lead to a Λ-dominated Universe,

$$\frac{\dot{a}}{a} = \mathrm{H}_0\sqrt{\frac{\Omega_m}{a^3} + \frac{\Omega_k}{a^2} + \Omega_\Lambda} \rightarrow \mathrm{H}_0\sqrt{\Omega_\Lambda} \tag{G.489}$$

substantiating the idea that the cosmological fluids dominate in the order of decreasing equation of state w if the expansion is monotonic, $\dot{a} > 0$, i.e. if there is no recollapse of the Universe. Similarly, the second Friedmann-equation shows that the dynamics will be dominated by Λ because ρ is increasingly diluted, $\rho \rightarrow 0$:

$$\frac{\ddot{a}}{a} = -\frac{4\pi \mathrm{G}}{3}\rho + \frac{\Lambda c^2}{3} \tag{G.490}$$

in a way that $\ddot{a}$ becomes proportional to a as well as $\dot{a}$ as the expansion becomes exponential,

$$a(t) \sim \exp\left(\sqrt{\frac{\Lambda c^2}{3}}t\right) \tag{G.491}$$

leading to a deceleration of $q = -\ddot{a}a/\dot{a}^2 = -1$.

G.9 *Size and age of FLRW-universes*

It is a funny realisation that the age of the Universe as the elapsed time between $a = 0$ and $a = 1$ can be finite or infinite, depending on the cosmological model; in fact, whether the point $a = 0$ is reached in a finite past is determined by the matter and energy content of the FLRW-cosmology, and usually high densities of matter or radiation cause that time to be finite.

To be exact, the age of the Universe would be the elapsed coordinate time (and hence the proper time) of a comoving observer, who has the right to center the coordinate frame onto herself or himself. Then, $\mathrm{d}r = 0$ and the age of the Universe corresponds to the length of the observer's world line. $\mathrm{H} = \dot{a}/a$ implies $\mathrm{d}t = \mathrm{d}a/(a\mathrm{H})$ from the Hubble-Lemaître-function, and therefore

$$t = \int \mathrm{d}t = \int_0^1 \frac{\mathrm{d}a}{a\mathrm{H}(a)} = \frac{1}{\mathrm{H}_0}\int_0^1 \frac{\mathrm{d}a}{a\sqrt{\frac{\Omega_\gamma}{a^4} + \frac{\Omega_m}{a^3} + \Omega_\Lambda}} \tag{G.492}$$

where the inverse Hubble-Lemaître constant $1/\mathrm{H}_0 \simeq 10^{17}$s determines the scale of the age of the Universe. While fluids with an equation of state $w > -1/3$ tend to make the integral converge, very negative equation of state parameters $w < -1/3$ will cause infinite ts. A good example is a pure Λ-dominated Universe, where the Hubble-Lemaître-function is constant.

In this particular case,

$$t = \int \mathrm{d}t = \int_0^1 \frac{\mathrm{d}a}{a\mathrm{H}(a)} = \frac{1}{\mathrm{H}_0}\int_0^1 \frac{\mathrm{d}a}{a} = \frac{1}{\mathrm{H}_0}\int_0^1 \mathrm{d}\ln a \tag{G.493}$$

diverges logarithmically.

A related question is whether the Universe will exist an infinite time into the future. Coming back to the example with a Λ-dominated Universe as ours, the scale factor will increase exponentially in time, $a(t) = \exp(\sqrt{\Lambda}t)$, such that there is a finite a given at every time. The integral

$$t = \int \mathrm{d}t = \int_1^\infty \frac{\mathrm{d}a}{a\mathrm{H}(a)} = \frac{1}{\mathrm{H}_0}\int_1^\infty \frac{\mathrm{d}a}{a} = \frac{1}{\mathrm{H}_0}\int_1^\infty \mathrm{d}\ln a \tag{G.494}$$

is divergent, too. In contrast, high values of the equation of state parameter w will make the integral convergent. A weird example is an empty, hyperbolically curved universe with $\Omega_k = 1$ and $w = -1/3$. Then, $t = 1/\mathrm{H}_0 \int_0^1 \mathrm{d}a$ is exactly $1/\mathrm{H}_0$, so the age is finite and the Universe will continue to exist into the infinite future.

G.10 *Quintessence: dynamical fluids with varying w*

Up to this point, the equation of state $w = p/(\rho c^2)$ was a property of the fluid and expressed an intrinsic, unaltering property of the substance sourcing the gravitational field, for instance relativistic matter with $w = +1/3$ or nonrelativistic matter with $w = 0$. Interestingly, it was possible to map curvature as a property of spacetime onto a fluid with $w = -1/3$ or to think of the cosmological constant as a substance with $w = -1$. It is even possible to design an artificial fluid with a given Ω_X and an equation of state w_X that reproduces any expansion history $\mathrm{H}(t)$ that one might think of, if one has the freedom to choose a function $w_\mathrm{X}(t)$. Vice versa, it is an interesting question if one could take this one step further and not only generate any Hubble function $\mathrm{H}(t)$ with the freedom to choose $w_\mathrm{X}(t)$, but to set up a field that changes by interaction its gravitational properties such that it can vary its own equation of state: That is the foundational idea behind quintessence, the fifth substance after radiation, matter, curvature and the cosmological constant, substance meant here of course in a gravitational sense.

The quintessence construction starts with a scalar field ϕ which can only depend on t in accordance with the cosmological principle. ϕ can interact with itself in the sense of particle physics through the potential $\mathrm{V}(\phi)$, a suitable Lagrange-function would be

$$\mathcal{L}(\phi, \nabla_\alpha\phi, g_{\mu\nu}) = \frac{1}{2}\, g^{\alpha\beta}\nabla_\alpha\phi\,\nabla_\beta\phi - \mathrm{V}(\phi). \tag{G.495}$$

Apart from direct self-interaction the scalar field ϕ sources a gravitational field as it provides a nonzero energy momentum tensor, so as it evolves dynamically it does that in a varying geometry; in fact, it is best to think of the dynamical equations for ϕ and for $g_{\mu\nu}$ (or $a(t)$, which is the only degree of freedom in the metric if the FLRW-symmetries apply) as a coupled system with a joint solution.

Substitution into the corresponding Euler-Lagrange-equation for a scalar field ϕ on an arbitrary and possibly curved background yields a wave equation with a source term

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = -\frac{\mathrm{d}V}{\mathrm{d}\phi} \tag{G.496}$$

which is effectively a Klein-Gordon-equation with a driving term. It can be interpreted as the covariant divergence of the vector $\upsilon_\alpha = \nabla_\alpha\phi = \partial_\alpha\phi$,

$$g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta\,\phi = g^{\alpha\beta}\,\nabla_\alpha\,\upsilon_\beta = \nabla_\alpha\left(g^{\alpha\beta}\,\upsilon_\beta\right) = \nabla_\alpha\,\upsilon^\alpha = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\,\partial^\alpha\phi\right) \tag{G.497}$$

The covolume $\sqrt{-g}$ is quickly derived for the FLRW-metric to be $\sqrt{-g} = ca^3(t)$ and cosmological principle makes sure that there are only variations along the ct-direction, such that $\partial_\mu \to \partial_t$:

$$\frac{1}{\sqrt{-g}}\,\partial_\alpha\left(\sqrt{-g}\,\partial^\alpha\phi\right) = \frac{1}{a^3}\cdot\left(3a^2\,\dot{a}\,\dot{\phi} + a^3\ddot{\phi}\right) = \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} \tag{G.498}$$

such that the final Klein-Gordon-equation on a FLRW-background with scale factor $a(t)$ reads

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -\frac{\mathrm{d}V}{\mathrm{d}\phi} \tag{G.499}$$

where we recognise the Hubble-Lemaître-function $H(t) = \dot{a}/a$. The energy-momentum tensor $T_{\mu\nu}$ as the source of the gravitational field can be derived from the Lagrange-function and is covariantly conserved as $\mathcal{L}$ does not explicitly depend on the coordinates, $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$,

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\nabla^\mu\phi}\nabla_\nu\,\phi - g_{\mu\nu}\,\mathcal{L} \tag{G.500}$$

specifically for the particular Lagrange-function,

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\,\frac{1}{2}\,g^{\alpha\beta}\,\nabla_\alpha\phi\nabla^\beta\phi + V(\phi)g_{\mu\nu} \tag{G.501}$$

which we view in terms of the energy-momentum tensor of an ideal fluid,

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu - p g_{\mu\nu} \tag{G.502}$$

in order to be able to identify terms involving the field ϕ and its derivative $\dot{\phi}$ with the fluid-mechanical quantities ρ and p.

The actual entries of $T_{\mu\nu}$ can then be computed by enforcing the FLRW-symmetries, where only t-derivatives are present; we identify the tt-component with the energy density ρc^2 of an ideal fluid

$$T_{tt} = \left(\rho + \frac{p}{c^2}\right) u_t u_t - p \cdot g_{tt} = \rho c^2 \tag{G.503}$$

as $u_\mu = (c, 0)$ in the comoving frame, and the trace $g^{ij}T_{ij}$ with the pressure,

$$g^{ij}T_{ij} = \left(\rho + \frac{p}{c^2}\right) g^{ij} u_i u_j - p\, g^{ij} g_{ij} = +3a^2 p, \tag{G.504}$$

where the trace $g^{ij}g_{ij}$ only encompasses the diagonal elements of the metric and yields $-3a^2$, while the first term $g^{ij}u_i u_j$ does not contribute, as the spatial components of u_μ are zero: the fluid is at rest in the comoving frame. Comparing these two expressions with the energy-momentum tensor $T_{\mu\nu}$ of the field ϕ then yields for the tt-component

$$T_{tt} = \partial_t\phi \cdot \partial_t\phi - g_{tt} \cdot \frac{1}{2} g^{tt} \cdot \partial_t\phi\, \partial_t\phi + V(\phi) g_{tt} = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \rho c^2 \tag{G.505}$$

and correspondingly for the trace over the spatial components

$$g^{ij}T_{ij} = -g^{ij}g_{ij}\frac{1}{2}g^{tt}\partial_t\phi\partial_t\phi + g^{ij}g_{ij}V(\phi) = 3a^2\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) = 3a^2 p \tag{G.506}$$

Collecting the results yields for the equation of state w:

$$w = \frac{p}{\rho c^2} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \tag{G.507}$$

which is an amazingly interesting result: The coupled system of the Klein-Gordon-equation and the Friedmann-equation allows a simultaneous evolution of ϕ and a. ϕ and $\dot{\phi}$ determine the energy-momentum tensor $T_{\mu\nu}$ and define the two fluid properties ρc^2 and p that enter the Friedmann-equation as source properties. The Friedmann-equation in turn provides the solution for $a(t)$ for a given fluid, and $a(t)$ enters the Klein-Gordon-equation as $\dot{a}/a$. Phenomenologically, one obtains a time-varying equation of state w from the dynamics of the field ϕ: If the field is static, $\dot{\phi} = 0$ and the equation of state w is equal to -1. In this case, ϕ mimicks a cosmological constant. But it would be natural that ϕ is accelerated by the gradient in $V(\phi)$, as determined through the Klein-Gordon-equation, so $\dot{\phi}$ increases at the expense of $V(\phi)$, and w will move away from -1 towards less negative numbers. In summary, the coupled system of differential equations for $\phi(t)$ and $a(t)$ allow the construction of a Friedmann-universe with a dynamical fluid; the freedom to choose the equation of state function $w(t)$ is mapped onto the choice of the potential $V(\phi)$ and initial conditions for ϕ. Effectively, one obtains repulsive gravity in the limit $\dot{\phi} \ll V(\phi)$, making quintessence a possible explanation of dark energy.