F BLACK HOLES

F.1 *Schwarzschild black holes*

The Schwarzschild geometry refers to the geometry outside of a spherically symmetric static matter distribution as a generalisation to the Newtonian gravitational potential $\Phi = -GM/r$. Just like the latter follows from the solution of the vacuum ($\rho = 0$) Poisson-equation $\Delta\Phi = 0$ in the spherically symmetric case, K. Schwarzschild obtained his solution from the gravitational field equation in vacuum. $T_{\mu\nu} = 0$ implies directly $T = g^{\mu\nu}T_{\mu\nu} = 0$, such that the trace of the field equation becomes

$$g^{\mu\nu}R_{\mu\nu} - \frac{R}{2}g^{\mu\nu}g_{\mu\nu} = -R = -\frac{8\pi G}{c^4}g^{\mu\nu}T_{\mu\nu} = 0 \quad \rightarrow \quad R = 0 \tag{F.307}$$

restricting ourselves to scales $\ll 1/\sqrt{\Lambda}$, meaning that the cosmological constant can be neglected. The trace relation implies that the Ricci-scalar R vanishes for vacuum solutions as a general result. Then, what remains from the field equation is

$$R_{\mu\nu} = 0 \tag{F.308}$$

which is to be solved for a spherically symmetric, static case. It would be wrong to conclude from $R_{\mu\nu} = 0$ that there could not be any curvature: There can not be any Ricci-curvature $R_{\mu\nu}$ in a vacuum case, but the field equation does not restrict the Weyl-curvature $C_{\alpha\beta\mu\nu}$, in the same way as the classical Poisson-equation only restricts $\Delta\Phi = 0$ but not the traceless tidal field $\partial_i\partial_j\Phi - \Delta\Phi/3\,\delta_{ij}$. Although there is no parallel in Newtonian theory as there is no notion of general covariance, the Weyl-curvature must obey the differential Bianchi-identity, which acts as the dynamic equation of $C_{\alpha\beta\mu\nu}$.

Guided by isotropy and staticity as symmetries, a suitable ansatz for the metric could be

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \tag{F.309}$$

for an intuitive coordinate choice. Clearly, one would like to work with spherical coordinates r, θ, φ, augmented by a temporal ct-coordinate. But there is some fineprint attached to this: The ct-coordinate would be the conventional coordinate time at the location of an infinitely distant observer, where $A(r) \rightarrow 1$ asymptotically to recover Minkowskian space. In this asymptotically flat space, the coordinate time would be identical to the proper time of observers at rest relative to the black hole. The radial coordinate r has the same limit $B(r) \rightarrow \infty$ as $r \rightarrow \infty$ to make the spatial submanifold appear as a flat Euclidean space. Clearly, the dependence of A and B on the radial coordinate is there to encode curvature effects in the measurement of time intervals and radial distances, and these curvature effects do not depend on time, as a reflection of staticity.

The typical scaling $\propto 4\pi r^2$ of spheres of radius r is obtained by integration over the two angles at fixed r, which in turn is actually defining the radial coordinate! The area element is $dA = \sqrt{g_{\theta\theta}g_{\phi\phi}} = r^2\sin\theta d\theta d\phi$ such that

$$\int_{4\pi} dA = r^2\int_0^{\pi} d\theta\,\sin\theta\int_0^{2\pi} d\phi = 4\pi\, r^2 \tag{F.310}$$

i.e. the radial coordinate r is chosen in such a way that the scaling of surfaces of spheres with radius is defined just like in flat Euclidean space, despite the fact that there are curvature effects present. At least in this single coordinate direction, the effects of curvature have disappeared through a suitable coordinate choice.

Writing the metric in this coordinate choice in matrix form

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} \tag{F.311}$$

makes it apparent that it is diagonal, and the inverse can be found quickly using the determinant $\det(g_{\mu\nu}) = -A(r)B(r) \cdot r^4 \sin^2\theta$, such that

$$g^{\mu\nu} = \begin{pmatrix} A^{-1}(r) & 0 & 0 & 0 \\ 0 & -B^{-1}(r) & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -(r^2 \sin^2\theta)^{-1} \end{pmatrix} \tag{F.312}$$

Many of the Christoffel-symbols vanish due to the high degree of symmetry. If the metric does not change in a certain coordinate direction along x^α, $\partial_\alpha g_{\mu\nu}$ is zero and does not contribute to the Christoffel-symbol. It is a technical exercise to show that the nonzero $\Gamma^\alpha{}_{\mu\nu}$ are:

$$\Gamma^t{}_{tr} = \frac{A'}{2A} \qquad \Gamma^r{}_{\theta\theta} = -\frac{r}{B} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \tag{F.313}$$

$$\Gamma^r{}_{tt} = \frac{A'}{2B} \qquad \Gamma^r{}_{\phi\phi} = -\frac{r\sin^2\theta}{B} \qquad \Gamma^\phi{}_{r\phi} = \frac{1}{r} \tag{F.314}$$

$$\Gamma^r{}_{rr} = \frac{B'}{2B} \qquad \Gamma^\theta{}_{r\theta} = \frac{1}{r} \qquad \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \tag{F.315}$$

together with torsion-free condition $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}$, for switching the order of the two covariant indices. From those, we can compute the Riemann-tensor and determine the Ricci-tensor $R_{\beta\nu} = g^{\alpha\mu} R_{\alpha\beta\mu\nu}$ as its contraction: Again, the nonzero elements of $R_{\beta\nu}$ are:

$$R_{tt} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} = 0 \tag{F.316}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} \tag{F.317}$$

$$R_{\theta\theta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) \tag{F.318}$$

$$R_{\phi\phi} = R_{\theta\theta} \cdot \sin^2\theta \tag{F.319}$$

which comes out diagonal but not proportional to the metric, as a reflection of the presence of Weyl-curvature. Setting $R_{\mu\nu} = 0$ yields differential equations (the fourth involving $R_{\phi\phi} = 0$ is redundant because it is proportional to the third equation) whose solution will fix the two functions $A(r)$ and $B(r)$. Adding $B/A\times$ eqn. F.316 and adding it to eqn. F.317 yields

$$-\frac{A''}{2A}+\frac{A'}{4A}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{A'}{rA}+\frac{A''}{2A}-\frac{A'}{4A}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{B'}{rB}=0 \tag{F.320}$$

which immediately simplifies to

$$\frac{1}{r}\left(\frac{A'}{A}+\frac{B'}{B}\right)=0 \tag{F.321}$$

The term in brackets needs to vanish exactly for any choice of r, so one can determine

$$\frac{A'}{A}+\frac{B'}{B}=0\,\Big|\cdot AB, \quad BA'+AB'=\frac{d}{dr}(AB)=0 \tag{F.322}$$

such that the product AB needs to be constant, $AB=\alpha$. Substituting $B=\alpha/A$ and the derivative $B'=-\frac{\alpha}{A^2}A'$ into eqn. F.318,

$$R_{\theta\theta}=\frac{1}{B}-1+\frac{r}{2B}\left(\frac{A'}{A}-\frac{B'}{B}\right)=0 \tag{F.323}$$

from which one can isolate a differential equation for A,

$$A+rA'=\frac{d}{dr}(rA)=\alpha, \tag{F.324}$$

and therefore $rA=\alpha r+k$ with an integration constant k. In this way, we have obtained the two metric functions

$$A(r)=\alpha\left(1+\frac{k}{r}\right) \quad \text{and} \quad B(r)=\left(1+\frac{k}{r}\right)^{-1} \tag{F.325}$$

where the two constants α and k need to be identified by comparison with a Minkowski-metric that is weakly perturbed by a potential: In this way, we match up the two solutions in the weak field limit and make them consistent with each other.

$$ds^2=\left(1+\frac{2\Phi}{c^2}\right)c^2dt^2-\left(1-\frac{2\Phi}{c^2}\right)dr^2-r^2\left(d\theta^2+\sin^2\theta\, d\phi\right) \tag{F.326}$$

is the weakly perturbed line element with a potential Φ, in our case $\Phi=-GM/r$ generated by an isotropic matter distribution with mass M according to the Poisson-equation. Comparison with the Schwarzschild line element yields for the metric functions

$$\left(1+\frac{2\Phi}{c^2}\right)=A(r) \tag{F.327}$$

if $\alpha=c^2$ and $k=-2GM/c^2$.

With this identification, the Schwarzschild line element becomes

$$\mathrm{d}s^2 = \left(1 - \frac{2\mathrm{GM}}{\mathrm{c}^2 r}\right) \mathrm{c}^2\,\mathrm{d}t^2 - \frac{1}{1 - \frac{2\mathrm{GM}}{\mathrm{c}^2}} \mathrm{d}r^2 - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi\right) \tag{F.328}$$

where one can read off the Schwarzschild-radius

$$r_\mathrm{S} = \frac{2\mathrm{GM}}{c^2} \tag{F.329}$$

which assigns a length scale to the gravitational field generated outside a spherically symmetric and static matter distribution of mass M. $\mathrm{G}/c^2 \simeq 10^{-28}$m, so a good number to remember is a few hundred meters for the Schwarzschild radius of the Sun with $\mathrm{M}_\odot \simeq 10^{30}$kg. The Earth, with a considerably lower mass of $\mathrm{M}_\oplus \simeq 10^{24}$kg has a Schwarzschild radius smaller by a factor 10^6. But please be careful: The gravitational field outside of every spherically symmetric matter distribution is of Schwarzschild form, there is no requirement that the mass would be somehow concentrated to $r < r_\mathrm{S}$.

A subtle but very interesting point is that indirectly through the Newtonian solution, we have introduced a boundary condition: Spacetime becomes flat and Minkowskian at very large distances. This is necessary because by deactivating time-evolution of the gravitational field (because of the assumption of staticity) the field equation as a hyperbolic partial differential equation falls back onto an elliptical partial differential equation, which has only unique solutions if boundary conditions are specified. In our case, this would be a Dirichlet boundary condition. The situation is similar to the transition from $\Box\phi = 0$ as a hyperbolic PDE to $\Delta\phi = 0$ as an elliptical PDE if $\partial_{ct}\Phi = 0$.

I would like to emphasise that, as technically straightforward the comparison of the two line elements may seem, there is quite a lot happening from a conceptual point of view: With increasing radial distance one would expect the metric functions to approach one, but also the definition of the radial coordinate r approaches the Euclidean regime, so there is a smooth interpolation from the curved spacetime to a flat one. It is interesting that through this matching of the two models the mass gains a significance: Before, we only had two functions A and B, and the mass of the field-generating object was nowhere to be found. As long as one deals with spacetime as a curved manifold, the choice of coordinates is arbitrary and bears no physical significance. Curvature varies with changing r, but r is not an indication of distance, so one actually can not know how far away the black hole is, and neither how curvature, mass and distance are related. In the asymptotically Minkowskian spacetime, which is a vector space with normal coordinates, distance as coordinate difference has an absolute sense, so the decrease of field strength with distance is indicative of the mass.

Funnily enough, the same problem also arises in Newtonian gravity. For vacuum solutions, $\Delta\Phi = 0$ is to be solved, yielding $\Phi \propto 1/r$ in the spherically symmetric case, which is perfectly scale invariant. As the superposition principle holds here (for the Poisson-equation as a linear field equation), the scaling of Φ with M is natural, and G is there to fix the units. But nowhere there is a moment where the prefactors are determined by requiring Φ to have a specific value at a given distance, which would effective amount to a Dirichlet boundary condition.

At this point one should make clear that it'd be very wrong to say that "gravity becomes strong" at $r = r_\mathrm{S}$, or that curvature would start to dominate, or that classical gravity would need to get replaced by relativity. As a statement involving coordinates this can not be universally true. There are certain things that the infinitely distant observer can not compute at $r = r_S$, but this is a consequence of the coordinate choice, as spacetime is curved but perfectly regular, as none of the curvature invariants diverges. Certainly one would notice an increase in e.g. the Kretschmann-scalar $K = 48r_S^2/r^6$ moving towards smaller r from the Minkowski-regime where $K = 0$, but this is a relative statement between $r \to \infty$ and finite r.

As discussed in the chapter about the equivalence principle, any freely falling frame recovers a perfectly Minkowskian spacetime with a coordinate choice making sure that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\alpha{}_{\mu\nu} = 0$ locally. The amount of curvature the defines the size of this laboratory in which special relativity holds for instance by $\delta = (r_S^2/K)^{1/6}$. Even at r_S, motion of particles separated by less than δ is unaffected by curvature to first order. Of course, δ becomes less as r decreases.

F.2 *Birkhoff's theorem*

Up to this point we chose metric functions $A(r)$ and $B(r)$ of the Schwarzschild-metric to be functions of the radial coordinate only

$$ds^2 = A(t, r)\,dt^2 - B(t, r)\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{F.330}$$

Surprisingly, the result would have been identically the same if we had started with an allowed time-dependence $A(t, r)$ and $B(t, r)$, as spherical symmetry disallows time-dependences in vacuum. One might have noticed that up to now we used three equations from the diagonal vacuum field equation $R_{\mu\nu} = 0$ to fix just two functions A and B; please be reminded that $R_{\phi\phi} = 0$ is automatically already fulfilled by $R_{\theta\theta} = 0$. If a time-dependence of the two metric functions is introduced, the Christoffel-symbols and the Ricci-tensor become more complicated and contain time derivatives, but the vacuum field equation enforces then staticity with the additional third differential equation that is unused if one restricts A and B to be functions of r only: Somehow, the assumption of static gravitational fields is superfluous if one deals with a spherically symmetric vacuum solution.

This result is known as the Birkhoff-theorem: The fields outside spherically symmetric matter distributions need to be static and to be of the Schwarzschild-type. For instance, a radially pulsating spherically symmetric matter distribution would generate a perfectly static curved spacetime, and all that matters is the total mass M. A remainder of the Birkhoff-theorem is present in Newtonian gravity: There, the field of a spherically symmetric matter distribution was always computed as if the matter was concentrated at the central point, which arose as a peculiarity of the Poisson-equation. Many students ask at this point how a black hole can grow by accreting matter if there is no dynamical evolution of the gravitational field, which naively would be in contradiction with intuition, as a larger mass black hole should show a stronger gravitational field. Accretion and black hole growth is only possible if spherical symmetry is broken, though, even the case of accreting a spherically symmetric spherical shell of matter onto the black hole would not change the physical situation: Outside of the shell, the field stays static and corresponds to the combined masses of the shell and the black hole itself.

F.3 *Conformal scaling of the Schwarzschild solution*

The Schwarzschild-geometry is a spherically symmetric vacuum solution: As such, it possesses only Weyl-curvature and no Ricci-curvature. Weyl-curvature is invariant unter conformal transformations of the metric, $g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu}$ with a conformal factor $\Omega^2(x) > 0$, so the question naturally arises, what the physical significance of a conformally rescaled Schwarzschild solution actually might be. Conformal rescaling is present in classical gravity too, $\Delta\,\Phi = 0$ is invariant under $\Phi \to\ \Omega^2\,\Phi$, but we would rather call this mechanical similarity of a scale-free potential: Increasing the mass M can always be absorbed in Φ by going to larger distances r.

Applied to the Schwarzschild geometry, the metric transforms under conformal transformations as

$$g_{\mu\nu} \to \Omega^2(r)\,g_{\mu\nu} \quad \text{and consequently} \quad \mathrm{d}s^2 \to \Omega^2(r)\,\mathrm{d}s^2 \tag{F.331}$$

where only $\Omega^2(r)$ would respect the fundamental symmetry. Applied to the line element in Schwarzschild coordinates we get:

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right)\mathrm{c}\left(\Omega\,\mathrm{d}t\right)^2 - \frac{1}{1-\frac{r_\mathrm{S}}{r}}\left(\Omega\,\mathrm{d}r\right)^2 - \left(\Omega r\right)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}^2\phi\right) \tag{F.332}$$

with Schwarzschild radius $r_\mathrm{S} = \frac{2\mathrm{GM}}{\mathrm{c}^2}$. Absorbing the conformal factor in a redefinition of the coordinates yields then gives

$$\mathrm{d}s^2 = \left(1 - \frac{\Omega r_\mathrm{S}}{\mathrm{R}}\right)\mathrm{c}^2\mathrm{d}\tau^2 - \frac{1}{1-\frac{\Omega r_\mathrm{S}}{\mathrm{R}}}\mathrm{d}\mathrm{R}^2 - \mathrm{R}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{F.333}$$

if the conformal factor is *constant*, $\mathrm{R} = \Omega r \to \mathrm{dR} = \Omega\,\mathrm{d}r$, as well as $\tau = \Omega\,t \to \mathrm{d}\tau = \Omega\,\mathrm{d}t$. That is just the Schwarzschild line element for an upscaled mass,

$$r_\mathrm{S} \to \Omega \cdot r_\mathrm{S} = \frac{2\mathrm{G}}{\mathrm{c}^2}\left(\Omega\mathrm{M}\right), \tag{F.334}$$

i.e. the Schwarzschild solution is invariant under conformal transforms; $\mathrm{M} \to \Omega\mathrm{M}$ and $r_\mathrm{S} \to \Omega r_\mathrm{S}$ is absorbed by $r \to \Omega r$ and $t \to \Omega t$ as coordinate choices, so we have recovered a similarity transform and the class of Schwarzschild solutions for different masses is just related by a constant stretching of the spacetime by a factor of Ω, which is perhaps a bit surprising keeping in mind that r is not the Euclidean distance but a geometrically constructed radial coordinate. Of course, in classical gravity the same result would just be the scale-invariance of the potential, $\Phi = -\frac{\mathrm{GM}}{r}$.

Photon geodesics are invariant under conformal transforms:

$$\mathrm{d}s^2 = g_{\mu\nu}\,k^\mu k^\nu \to \Omega^2\,g_{\mu\nu}\,k^\mu k^\nu = 0 \tag{F.335}$$

as the conformal factor Ω^2 drops out, so we can explore the causal structure of Schwarzschild spacetimes in their generality even though there is no scale invariance as in the case of a classical Newtonian gravitational field $\Phi \propto -1/r$, because conformal invariance replaces that particular concept.

Constructing radial photon geodesics for the Schwarzschild geometry sets $\mathrm{d}\theta = 0,\ \mathrm{d}\phi = 0$ such that the photons only propagate along r as time passes: The geodesic equation does not predict any deviation as $\mathrm{d}^2\theta/\mathrm{d}\lambda^2 = 0$ as well as $\mathrm{d}^2\phi/\mathrm{d}\lambda^2 = 0$, perfectly in agreement with intution - there should not be any acceleration in the angular directions for a radially moving photon in a spherically symmetric field. The Schwarzschild line element

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right)\mathrm{c}^2\mathrm{d}t^2 - \frac{1}{1-\frac{r_\mathrm{S}}{r}}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) = 0 \tag{F.336}$$

with $big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big) = 0$. Therefore, the photon visits the coordinates r as measured by the passage of time t as measured by an infinitely distant observer as

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\mathrm{c} - \left(1 - \frac{r_\mathrm{S}}{r}\right) \tag{F.337}$$

with + for outgoing and − for infalling photons. This can be solved for the trajectory $t(r)$,

$$\pm\mathrm{c}\,\mathrm{d}t = \frac{\mathrm{d}r}{1-\frac{r_\mathrm{S}}{r}} \tag{F.338}$$

and integrated to give

$$\pm\mathrm{c}t = r + r_\mathrm{S}\cdot\ln(r - r_\mathrm{S}) + \mathrm{const} \quad \text{using} \quad \int\frac{\mathrm{d}r}{1-\frac{r_\mathrm{S}}{r}} = r + r_\mathrm{S}\ln(r-r_\mathrm{S}) + \mathrm{const} \tag{F.339}$$

At large distances $r \to \infty$, $r + r_\mathrm{S}\ln(r - r_\mathrm{S})$ approaches r and the light cone becomes Minkowskian $\pm ct = r$, but at $r = r_\mathrm{S}$, the light cone collapses as the effective speed of propagation of the photon approaches zero: It is unable to change the radial distance (into any direction!) as time (of the infinitely distant observer) passes:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\mathrm{c}\left(1 - \frac{r_\mathrm{S}}{r}\right) \to 0 \tag{F.340}$$

Therefore, photons would be unable to propagate away from a source at $r = r_\mathrm{S}$ and would certainly never reach an observer at $r > r_\mathrm{S}$: That's the reason why black holes are black. Please keep in mind that nowhere one would need concepts of energy loss or redshifting of photons; instead, it is much clearer to think of the null-condition $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu/\mathrm{d}\lambda\mathrm{d}x^\nu/\mathrm{d}\lambda = 0$ as being generally true, and the effective speed of propagation $\mathrm{d}r/\mathrm{d}t$ being dependent on the particular coordinate choice.

F.4 *Coordinate singularity at the Schwarzschild radius*

The Schwarzschild geometry has a diverging line element at $r = r_\mathrm{S}$, but that divergence only concerns the metric and has no physical implication: Firstly, it is not present in other coordinate choices and secondly, all curvature invariants stay finite at the Schwarzschild-radius, for instance $\mathrm{R} = 0$ for the Ricci-scalar and $\mathrm{K} = 48r_\mathrm{S}^2/r^6$ for the Kretschmann-scalar.

All apparent irregularities at r_S are only a consequence of the coordinate choice, and so is this curious switch between timelike and spacelike distances at r_S: The Schwarzschild line element in Schwarzschild coordinates assumes the form

$$\mathrm{d}s^2 = \left(1 - \frac{r_S}{r}\right) c^2 \mathrm{d}t^2 - \frac{1}{1 - \frac{r_S}{r}} \mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right) \tag{F.341}$$

where the first prefactor shows this behaviour,

$$\left(1 - \frac{r_S}{r}\right) \begin{cases} > 0 \text{ if } r > r_S \\ < 0 \text{ if } r < r_S \end{cases} \tag{F.342}$$

whereas the second prefactor shows exactly the opposite behaviour,

$$-\frac{1}{1 - \frac{r_S}{r}} \begin{cases} < 0 \text{ if } r > r_S \\ > 0 \text{ if } r < r_S \end{cases} \tag{F.343}$$

so that $c\mathrm{d}t > 0$ and $\mathrm{d}r = 0$ imply a positive $\mathrm{d}s^2$ at $r > r_S$ and a negative $\mathrm{d}s^2$ at $r < r_S$, while $\mathrm{d}r > 0$ with $c\mathrm{d}t = 0$ would cause $\mathrm{d}s^2$ to be negative at $r > r_S$ and positive at $r < r_S$, interchanging the classification of timelike and spacelike vectors.

F.5 *Painlevé-Gullstrand-coordinates*

Solving the geodesic equation for a massive particle that is initially at rest at infinity, $\mathrm{d}r/\mathrm{d}\tau = 0$ and $\mathrm{d}t/\mathrm{d}\tau = 1$ by isolating the metric from the line element,

$$\mathrm{d}s^2 = \left(1 - \frac{r_S}{r}\right) c^2 \mathrm{d}t^2 - \frac{1}{1 - \frac{r_S}{r}} \mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right) \tag{F.344}$$

and computing the necessary Christoffel-symbols shows that massive particles do not cross the horizon as viewed by an observer at infinity, in fact their velocity defined as the rate at which they change the radial coordinate r in terms of t approaches zero:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -c\left(1 - \frac{r_S}{r}\right) \cdot \sqrt{\frac{r_S}{r}} \to 0 \quad \text{at} \quad r = r_S \tag{F.345}$$

At this point, Painlevé and Gullstrand came up with this idea: As one is completely free in choosing coordinates (as long as there is an invertible and differentiable way of changing between them), one can have a non-uniform time-coordinate $T(r) = t - a(r)$ with the differential $\mathrm{d}T = \mathrm{d}t - a'\,\mathrm{d}r$, prime denoting a differentiation with respect to the radial coordinate r.

Using this new coordinate in the Schwarzschild line element yields

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right) \mathrm{c}^2 \left(\mathrm{d}T + a'(r)\,\mathrm{d}r\right)^2 - \frac{1}{1 - \frac{r_\mathrm{S}}{r}}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{F.346}$$

with a non-diagonal term appearing in the metric,

$$\mathrm{d}s^2 = \left(1-\frac{r_\mathrm{S}}{r}\right)\mathrm{c}^2\mathrm{d}T^2+2a'\left(1-\frac{r_\mathrm{S}}{r}\right)\mathrm{c}\mathrm{d}T\,\mathrm{d}r+\left[\left(1-\frac{r_\mathrm{S}}{r}\right)a'^2-\frac{1}{1-\frac{r_\mathrm{S}}{r}}\right]\mathrm{d}r^2-r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right) \tag{F.347}$$

Up to here, the function $a(r)$ was unspecified, so we might set the term in front of the $\mathrm{d}r$-differential to unity,

$$\left[\left(1 - \frac{r_\mathrm{S}}{r}\right) a'^2 - \frac{1}{1 - \frac{r_\mathrm{S}}{r}}\right] \equiv -1 \tag{F.348}$$

provided that the differential equation

$$a' = -\frac{1}{1 - \frac{r_\mathrm{S}}{r}} \cdot \sqrt{\frac{r_\mathrm{S}}{r}} \tag{F.349}$$

has a solution. This is in fact the case as it is solved by

$$a(r) = r_\mathrm{S} \cdot \ln\left(\frac{y+1}{y-1} - 2y\right) \quad \text{with} \quad y = \sqrt{\frac{r}{r_\mathrm{S}}} \tag{F.350}$$

Then, the final form of the Schwarzschild line element in Painlevé-Gullstrand coordinates is given by:

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right) \mathrm{c}^2\,\mathrm{d}T^2 + 2a'\left(1 - \frac{r_\mathrm{S}}{r}\right) \mathrm{c}\,\mathrm{d}T\,\mathrm{d}r - \mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{F.351}$$

which is perfectly regular at $r = r_\mathrm{S}$ and allows tracking of particles through the Schwarzschild horizon: Solving a radial geodesic for a massive particle with $\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0$ and $\frac{\mathrm{d}T}{\mathrm{d}\tau} = 1$ as initial conditions shows that the particle approaches $\frac{\mathrm{d}r}{\mathrm{d}T} = -\mathrm{c}\sqrt{\frac{r_\mathrm{S}}{r}} = -\mathrm{c}$ at $r = r_\mathrm{S}$. It is weird to look at history and see that Painlevé and Gullstrand were criticised for their coordinate construction because they "assigned too much significance to the coordinates" when in fact they showed that coordinate choices adapted to a physical problem at hand were possible and sensible.

F.6 *Propagation of fields on a curved spacetime*

Light propagation on a curved spacetime differs technically in a very important point from Minkowski spacetimes with Cartesian coordinates. There, when the wave equation was formulated in terms of partial derivatives which required to compute differentiations of a wave-type ansatz $\phi = \exp(\pm \mathrm{i}k_\mu x^\mu)$, leading to $\partial x^\mu/\partial x^\alpha = \delta^\mu_\alpha$. The situation is very different on a manifold, where partial differentiations are replaced by covariant ones, and while $\partial_\alpha x^\mu$ remains well defined, $\nabla_\alpha x^\mu$ is a senseless operation: Covariant differentiations can only be applied to vectors and tensors (well, and scalars, $\nabla_\mu\phi = \partial_\mu\phi$), but the coordinates form only a tuple! Only transformation of infinitesimal coordinate differences dx^μ is well defined in terms of a Jacobian, as dx^μ has the properties of a vector, but the coordinates themselves do not have this property. In addition, even an expression like the scalar product $k_\mu x^\mu$ in $\exp(\pm ik_\mu x^\mu)$ is highly doubtful on a manifold, as it does not combine two vectors.

Please never try to apply covariant derivatives to coordinate tuples, $\nabla_\alpha x^\mu$ is not defined!

Formulating a wave equation for a free scalar field ϕ on a manifold starts inevitably at the action integral

$$S = \int d^4x\, \sqrt{-g} g^{\mu\nu}\, \nabla_\mu\phi\nabla_\nu\phi \tag{F.352}$$

and substitution into the Euler-Lagrange formula yields the covariant wave equation,

$$\nabla_\alpha \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi} = \frac{\partial\mathcal{L}}{\partial\phi} \quad \rightarrow \quad g^{\mu\nu}\,\nabla_\mu\nabla_\nu\phi = 0 \tag{F.353}$$

where the d'Alembert-operator is scalar, $\square = g^{\mu\nu}\nabla_\mu\nabla_\nu$. Defining the vector $\upsilon_\nu = \nabla_\nu\phi = \partial_\nu\phi$ which points into the direction of the field gradients of ϕ, gives

$$g^{\mu\nu}\,\nabla_\mu\left(\nabla_\nu\phi\right) = g^{\mu\nu}\,\nabla_\mu\upsilon_\nu = \nabla_\mu\, g^{\mu\nu}\,\upsilon_\nu = \nabla_\mu\upsilon^\mu = 0 \tag{F.354}$$

using metric compatibility, so that we can formulate the covariant divergence, with the suitable Christoffel-symbol, where two of the indices become equal.

$$\nabla_\mu\upsilon^\mu = \partial_\mu\upsilon^\mu + \Gamma^\mu_{\ \mu\alpha}\,\upsilon^\alpha \tag{F.355}$$

In particular, a Levi-Civita connection would have

$$\Gamma^\mu_{\ \mu\alpha} = \frac{g^{\mu\beta}}{2}\cdot\left[\partial_\mu\, g_{\beta\alpha} + \partial_\alpha\, g_{\mu\beta} - \partial_\beta\, g_{\mu\alpha}\right] = \frac{1}{2}\left[g^{\mu\beta}\,\partial_\mu\, g_{\beta\alpha} + g^{\mu\beta}\,\partial_\alpha\, g_{\mu\beta} - g^{\mu\beta}\,\partial_\beta\, g_{\mu\alpha}\right] \tag{F.356}$$

i.e. essentially

$$\Gamma^\mu_{\ \mu\alpha} = \frac{1}{2} g^{\mu\beta}\,\partial_\alpha\, g_{\mu\beta} \tag{F.357}$$

There is a curious relation between the covariant divergence and the covolume $g = \det(g_{\mu\nu})$. My third most favourite formula in theoretical physics says that

$$g = \det(g_{\mu\nu}) = \exp\ln\ \det(g_{\mu\nu}) = \exp\ \mathrm{tr}\ \ln(g_{\mu\nu}) \tag{F.358}$$

relating the logarithm of the determinant with the trace of the matrix-valued logarithm, which is easily checked in the principal axis frame. Then,

$$\partial_\alpha\, g = g\cdot\partial_\alpha \mathrm{tr}\ \ln(g_{\mu\nu}) = g\cdot\mathrm{tr}\,\partial_\alpha\ \ln(g_{\mu\nu})) = g\cdot\mathrm{tr}\left(g^{-1}\cdot\partial_\alpha\, g_{\mu\nu}\right) = g\cdot g^{\mu\nu}\cdot\partial_\alpha\, g_{\mu\nu} \tag{F.359}$$

using the linearity of the derivative as well as the inverse metric. With the derivative of the square root one then obtains

$$g^{\mu\nu}\,\partial_\alpha\, g_{\mu\nu} = \frac{1}{g}\partial_\alpha\, g, \quad \text{and therefore} \quad \frac{1}{2} g^{\mu\nu}\,\partial_\alpha\, g_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\alpha\,\sqrt{-g}. \tag{F.360}$$

With this result one can write for the contracted Christoffel-symbol

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{\sqrt{-g}}\,\partial_\alpha \sqrt{-g} \tag{F.361}$$

and finally for the covariant divergence

$$\begin{aligned}\nabla_\mu\, \upsilon^\mu &= \partial_\mu\, \upsilon^\mu + \Gamma^{\mu}{}_{\mu\alpha}\,\upsilon^\alpha = \partial_\mu\,\upsilon^\mu + \frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g}\cdot\upsilon^\alpha \\ &\overset{\mu\leftrightarrow\alpha}{=} \partial_\mu \upsilon^\mu + \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\cdot\upsilon^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}\,\upsilon^\mu\Big)\end{aligned} \tag{F.362}$$

using the Leibnitz-rule. With the covariant divergence, the wave equation becomes

$$g^{\mu\nu}\,\nabla_\mu\nabla_\nu\,\phi = \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}\,\partial^\mu\,\phi\Big) = 0 \tag{F.363}$$

which is obviously not just $\partial_\mu\partial^\mu\phi = 0$; there is clearly an influence from the background onto wave propagation, for instance from the spacetime around a black hole.

We can solve the wave equation in the eikonal or geometric optics approximation, where the changes in the geometric properties of spacetime take place on spatial scales much larger than the wavelength, and temporal changes much larger than the frequency. Again, we need to navigate through the fact that the coordinate tuple x^μ is not a vector: Writing $\Phi = \mathrm{A}\exp(\mathrm{iS}/\epsilon)$ with an amplitude A, a phase $\mathrm{S}(x^\mu)$ as a function of the coordinates and a parameter ϵ which controls the rate of phase change (which must be high in comparison to the scales on which spacetime changes its geometry) one derives by heavy application of the Leibnitz-rule

$$\partial_\nu\,\Phi = \partial_\nu \mathrm{A}\exp\Big(\frac{\mathrm{iS}}{\epsilon}\Big) + \frac{\mathrm{i}}{\epsilon}\mathrm{A}\exp\Big(\frac{\mathrm{iS}}{\epsilon}\Big)\partial_\nu\,\mathrm{S} \tag{F.364}$$

as well as for the second derivative

$$\begin{aligned}\nabla_\mu\,\partial_\nu\,\Phi = \nabla_\mu\partial_\nu \mathrm{A}\exp\Big(\frac{\mathrm{iS}}{\epsilon}\Big) + \frac{\mathrm{i}}{\epsilon}\partial_\nu\,\mathrm{A}\exp\Big(\frac{\mathrm{iS}}{\epsilon}\Big)\nabla_\mu \mathrm{S} + \frac{1}{\epsilon}\nabla_\mu\,\mathrm{A}\exp\Big(\frac{\mathrm{i}}{\epsilon}\mathrm{S}\Big)\,\partial_\nu\mathrm{S} + \\ \frac{\mathrm{i}}{\epsilon}\mathrm{A}\exp\Big(\frac{\mathrm{iS}}{\epsilon}\Big)\nabla_\mu\partial_\nu \mathrm{S} + \Big(\frac{\mathrm{i}}{\epsilon}\Big)^2\,\mathrm{A}\exp\Big(\frac{\mathrm{iS}}{\epsilon}\Big)\nabla_\mu\,\mathrm{S}\partial_\nu\,\mathrm{S}\end{aligned} \tag{F.365}$$

Sorting the terms by powers in ϵ leads to $1/\epsilon^2$ as the dominating term for high ϵ (making sure the phase changes are fast) which exactly corresponds to geometric optics,

$$g^{\mu\nu}\,\nabla_\mu\nabla_\nu\,\Phi = \Big(\frac{\mathrm{i}}{\epsilon}\Big)^2\Phi g^{\mu\nu}\,\partial_\mu\,\mathrm{S}\,\partial_\nu\,\mathrm{S} = 0 \tag{F.366}$$

keeping in mind that the phase function is scalar. Defining the wave vector k^μ as the gradient in S one recovers the null-condition

$$g^{\mu\nu}\,k_\mu k_\nu = 0. \tag{F.367}$$

At order $1/\epsilon$ on obtains

$$g^{\mu\nu}\,\nabla_\mu\nabla_\nu\,\Phi = \left(\frac{\mathrm{i}}{\epsilon}\right)\left[g^{\mu\nu}\left(k_\mu\,\partial_\nu\,\mathrm{A} + \partial_\mu\,\mathrm{A}\,k_\nu\right)\exp\left(\frac{\mathrm{i}S}{\epsilon}\right) + g^{\mu\nu}\,\Phi\nabla_\mu k_\nu\right] = 0 \tag{F.368}$$

which suggest a relation how the amplitude of the wave is transported through spacetime,

$$2\,k_\mu\partial^\mu\,\mathrm{A} + \mathrm{A}\nabla_\mu\,k^\mu = 0 \tag{F.369}$$

Although we solved a wave-equation for a massless scalar field ϕ on a curved background, the essential results are applicable to the Maxwell-field A^μ as well.

F.7 *Causal structure of black holes*

The Schwarzschild-geometry is the unique solution for a spherically symmetric gravitational field in vacuum, but the particular choice of Schwarzschild coordinates, motivated by the passage of time for the infinitely distant observer for t and the Euclidean scaling of surfaces with radius r is unsuited to parameterise the metric at $r_\mathrm{S} = 2\mathrm{G}\mathrm{M}/c^2$: there exists a coordinate singularity. The issue is really only an unfortunate choice of coordinates as all curvature invariants stay finite for every finite r, and there is really only a divergence of the curvature invariants as convenient coordinate-independent quantifications of curvature at $r = 0$. Additionally, because the mass of the black hole in the Schwarzschild solution was injected into the derivation only at the stage of embedding the spacetime into an asymptotically flat Minkowski-spacetime with a weak perturbation caused by a Newtonian potential $\Phi = -\mathrm{G}\mathrm{M}/r$, the pecularity of the Schwarzschild radius only applies to the infinitely distant observer.

Instead of using Schwarzschild coordinates (ct, r, θ, φ) one can find much better coordinates by looking at the radial motion of null-geodesics, corresponding to ingoing or outgoing light rays. That is the foundational idea of Eddington-Finkelstein coordinates, and we need two sets of coordinate as required by differential geometry: non-flat manifolds need to be covered by at least two sets of coordinate maps. Null-geodesics are of course the expression of the causal structure of spacetime, as hyberbolic differential equations cause massless fields to propagate along the light cones and restrict massive fields to propagate strictly within the light cones.

The coordinates ct, r of photons in radial motion where $\mathrm{d}\phi = \mathrm{d}\theta = 0$ fulfil the relation

$$\mathrm{c}t = -r - r_\mathrm{S}\cdot\ln(r - r_\mathrm{S}) + \mathrm{const.} \tag{F.370}$$

obtained by direct integration of the Schwarzschild line element $\mathrm{d}s^2 = 0$ for massless particles. The integration constant is defined to be a new coordinate p

$$p = \mathrm{c}t + r + r_\mathrm{S}\cdot\ln(r - r_\mathrm{S}) \tag{F.371}$$

with the differential $\mathrm{d}p$

$$\mathrm{d}p = \mathrm{c}\,\mathrm{d}t + \frac{\mathrm{d}r}{1 - \frac{r_\mathrm{S}}{r}} \tag{F.372}$$

that will be used to replace ct in the Schwarzschild line element,

$$ds^2 = \left(1 - \frac{r_S}{r}\right) c^2\, dt^2 - \frac{1}{1 - \frac{r_S}{r}} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{F.373}$$

to yield

$$ds^2 = \left(1 - \frac{r_S}{r}\right)\left[dp^2 - 2 \cdot \frac{1}{1 - \frac{r_S}{r}} dp\, dr + \frac{1}{(1 - \frac{r_S}{r})^2} dr^2\right] - \frac{1}{1 - \frac{r_S}{r}} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{F.374}$$

finally arriving at

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dp^2 - 2\, dp\, dr - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{F.375}$$

with $ds^2 = 0$ for photons. Clearly, any divergent behaviour of ds^2 at $r = r_S$ is avoided. The null-condition for radially moving photons suggests

$$\left(1 - \frac{r_S}{r}\right)\left(\frac{dp}{dr}\right)^2 = 2\frac{dp}{dr} \tag{F.376}$$

with two distinct solutions: $dp/dr = 0$, i.e. $p =$ const, and

$$\frac{dp}{dr} = \frac{2}{1 - \frac{r_S}{r}} \rightarrow \frac{p}{2} = r + r_S \ln(r - r_S) + \text{const} \tag{F.377}$$

With p one can define a new time coordinate t':

$$t' \equiv p - r = ct + r_S \cdot \ln(r - r_S \tag{F.378}$$

such that the line element reads

$$ds^2 = \left(1 - \frac{r_S}{r}\right) c^2\, dt'^2 - 2\, \frac{r_S}{r}\, dt'\, dr - \left(1 + \frac{r_S}{r}\right) dr^2 - r^2\left(\ \right) \tag{F.379}$$

It is perfectly regular for the entire Schwarzschild geometry and has for null-lines $ds^2 = 0$ the two branches

$$\begin{cases} ct' = -r + \text{const.} \\ ct' = r + 2r_S \cdot \ln(r - r_S) \end{cases} \tag{F.380}$$

In analogy, one can define retarded Eddington-Finkelstein coordinates q instead of advanced ones for outward moving radial photons,

$$dq = c\, dt - \frac{dr}{1 - \frac{r_S}{r}} \quad \text{(instead of + sign)} \tag{F.381}$$

It is important to realise that the Eddington-Finkelstein coordinates approach Minkowski-light cones at large distances from the black hole, and that they always consist of a linear branch and a nonlinear one, making the light cone tilt towards $r = r_S$: After the coordinate change, this replaces the closing up of the light cones in Schwarzschild coordinates. It illustrates the geometric origin of the event horizon. For large distances, the outward travelling photon can reach even larger distances, but at the Schwarzschild radius the outward travelling photon has to stay at $r = r_S$, and at smaller radii, the "outward" travelling photon actually moves towards smaller radii.

F.8 *Kruskal-coordinates*

Kruskal-coordiantes combine retarded and advanced Eddington-Finkelstein coordinates to construct effectively Minkowskian light cones.

$$\begin{cases} \mathrm{d}p = \mathrm{c}\,\mathrm{d}t + \alpha \mathrm{d}r \;\rightarrow\; \mathrm{c}\,\mathrm{d}t = \mathrm{d}p - \alpha\,\mathrm{d}r \\ \mathrm{d}q = \mathrm{c}\,\mathrm{d}t - \alpha\,\mathrm{d}r \;\rightarrow\; \mathrm{c}\,\mathrm{d}t = \mathrm{d}q + \alpha \mathrm{d}r \end{cases} \tag{F.382}$$

with $\alpha = 1/(1 - \frac{r_S}{r})$. As we need $c^2 \mathrm{d}t^2$ in the Schwarzschild line element, we could try out to substitute each of the two relations, each one providing one power of $c\mathrm{d}t$:

$$\mathrm{d}s^2 = \frac{1}{\alpha}\mathrm{c}^2\,\mathrm{d}^2 - \alpha \mathrm{d}r^2 = \frac{1}{\alpha}\mathrm{d}p\,\mathrm{d}q + \mathrm{d}p\,\mathrm{d}r - \mathrm{d}q\,\mathrm{d}r - \alpha \mathrm{d}r^2 - \alpha \mathrm{d}r^2 = \frac{1}{\alpha}\mathrm{d}p\,\mathrm{d}q \tag{F.383}$$

after substitution of $\mathrm{d}p\,\mathrm{d}r - \mathrm{d}q\,\mathrm{d}r = (\mathrm{d}p - \mathrm{d}q)\,\mathrm{d}r = 2\alpha \mathrm{d}r^2$, which is obtained by differencing both equations in eqn. F.382. Therefore, the Schwarzschild line element reads

$$\mathrm{d}s^2 = \left(1 - \frac{r_S}{r}\right)\mathrm{d}p\,\mathrm{d}q \tag{F.384}$$

which is obviously a modified line element for light cone coordinates, and in Minkowskian space at $r \rightarrow \infty$ one would recover $\mathrm{d}s^2 = \mathrm{d}p\mathrm{d}q$. In fact, reintroducing new spatial and temporal coordinates $(c\bar{t}, \bar{r})$ through the conversion

$$\begin{cases} \mathrm{c}\mathrm{d}\bar{t} = \frac{1}{2}(\mathrm{d}p + \mathrm{d}q) \\ \mathrm{d}\bar{r} = \frac{1}{2}(\mathrm{d}p - \mathrm{d}q) \end{cases} \tag{F.385}$$

gives a line element that is even more reminiscent of Minkowski-space, again with a prefactor approaching unity as $r \rightarrow \infty$.

$$\mathrm{d}s^2 = \left(1 - \frac{r_S}{r}\right)\left[\mathrm{c}^2\mathrm{d}\bar{t}^2 - \mathrm{d}\bar{r}^2\right] \tag{F.386}$$

At this point it might appear very surprising that one finds light cone coordinates with a conformal factor $1 - r_S/r$ for a non-conformally flat spacetime with clearly present Weyl-curvature $\mathrm{C}_{\alpha\beta\mu\nu}$! This contradiction is cleared up by realising that the argument only concerns the 2-dimensional submanifold in (ct, r), and there is no problem arising as 2-dimensional manifolds are always conformally flat because they are unable to support Weyl-curvature: The Riemann-tensor can then be written in terms of the Ricci-curvature alone.

☠ *Kruskal-coordinates with effectively Minkowski-light cones only exist for radially moving photons in the (ct, r)-submanifold, which is 2-dimensional and therefore necessarily conformally flat!*

Apart from that, there is a technical issue: The conformal factor $\Omega^2(r)$ is $1 - \frac{r_S}{r}$ is zero at $r = r_S$, but conformal factors are supposed to be strictly positive. To reach conformal flatness we can introduce yet another coordinate transform $(p, q) \rightarrow (\mathrm{P}, \mathrm{Q})$,

$$\mathrm{d}s^2 = \left(1 - \frac{r_S}{r}\right)\mathrm{d}p\,\mathrm{d}q = \left(1 - \frac{r_S}{r}\right)\frac{\mathrm{d}p}{\mathrm{d}\mathrm{P}}\frac{\mathrm{d}q}{\mathrm{d}\mathrm{Q}}\,\mathrm{d}\mathrm{P}\,\mathrm{d}\mathrm{Q} \tag{F.387}$$

If the coordinate transformation can be constructed in a way that the conformal factor $1 - \frac{r_S}{r}$ is absorbed into the coordinates, the line element would simply be $ds^2 = dPdQ$, and one would have reached conformal flatness in the submanifold:

$$\left(1 - \frac{r_S}{r}\right)\frac{dp}{dP}\frac{dq}{dQ} \sim 1 \tag{F.388}$$

Kruskal's really bright idea was the choice

$$P = +\exp\left(+\frac{p}{2r_S}\right),\ Q = -\exp\left(-\frac{q}{2r_S}\right) \rightarrow \frac{dP}{dp} = \frac{P}{2r_S},\ \frac{dQ}{dq} = +\frac{Q}{2r_S} \tag{F.389}$$

Then, the line element becomes

$$ds^2 = -\left(1 - \frac{r_S}{r}\right)\cdot 4r_S^2 \cdot \frac{dP}{P}\frac{dQ}{Q} = \left(1 - \frac{r_S}{r}\right)4r_S^2 \cdot \exp\left(-\frac{p}{2r_S} + \frac{q}{2r_S}\right) dP\, dQ \tag{F.390}$$

with the consistency condition

$$\frac{1}{2}(p - q) = r + r_S \cdot \ln(r - r_S) \tag{F.391}$$

With this coordinate transform, the line element reads

$$ds^2 = 4\left(1 - \frac{r_S}{r}\right)r_S^2 \cdot \exp\left(-\frac{r}{r_S}\right)\cdot \exp(-\ln(r - r_S))\, dP\, dQ \tag{F.392}$$

Further simplification with $\exp(-\ln(r - r_S)) = \frac{1}{r - r_S} = \frac{1}{r}\cdot\frac{1}{1 - \frac{r_S}{r}}$ then yields the Kruskal line element

$$ds^2 = 4\exp\left(-\frac{r}{r_S}\right)\frac{r_S^3}{r}\cdot dP\, dQ \tag{F.393}$$

where one power of r_S has been added for consistency, as the line element has the unit of a squared length and the coordinates (P, Q) are dimensionless. The conformal factor is then

$$\Omega^2(r) = 4\exp\left(-\frac{r_S}{r}\right)\frac{r_S^3}{r} \tag{F.394}$$

which is strictly positive and nonsingular everywhere with the exception of $r = 0$.

F.9 *Reissner-Nordström black holes*

Clearly, the Schwarzschild black hole with its highly symmetric spacetime as a solution to the vacuum field equation $R_{\mu\nu}$ is a very attractive starting point to find solutions for the relativistic field equation, which is simplified dramatically due to the symmetries and the absence of a source. There is, perhaps a bit surprisingly, an analytic solution for the gravitational field outside of a spherically symmetric matter distribution which is electrically charged: the Reissner-Nordström black hole. In this case, spherical symmetry and staticity is maintained, but the electric field emanating from the charge distribution can propagate and its own energy content can contribute to spacetime curvature in addition to the central mass. As such, the solution is not a pure vacuum solution and possesses Ricci-curvature, sourced by the nonzero energy-momentum-tensor of the electric field, alongside the Weyl-curvature propagating away from the matter distribution.

With the same symmetry assumptions of isotropy and staticity, which suggests the line element to be of the Schwarzschild type,

$$\mathrm{d}s^2 = A(r)\, c^2\, \mathrm{d}t^2 - B(r)\, \mathrm{d}r^2 - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right) \tag{F.395}$$

with two (possibly different) metric functions $A(r)$ and $B(r)$, one does not only to solve the gravitational field equation but also the vacuum Maxwell-equation

$$g^{\alpha\mu}\, \nabla_\alpha F_{\mu\nu} = 0 \quad \leftrightarrow \quad R_{\mu\nu} = -\frac{8\pi G}{c^4}\, T_{\mu\nu}(F) \tag{F.396}$$

in a self-consistent way: The first equation provides the field $F_{\mu\nu}$ in vacuum for a given spacetime geometry and defines source $T_{\mu\nu}$, which in turn sources the Ricci-curvature $R_{\mu\nu}$ and fixes the geometry. At this point, please keep in mind that the Maxwell-field, due to the masslessness of the photon, has a vanishing trace T of the energy momentum tensor.

The scalars T and R are zero here, but that does not imply that $T_{\mu\nu}$ and $R_{\mu\nu}$ are zero!

Let's begin with the Maxwell-equation in vacuum, $g^{\alpha\mu}\, \nabla_\alpha F_{\mu\nu} = 0$ or equivalently, $\nabla_\mu F^{\mu\nu} = 0$. Writing out the covariant divergence

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^{\mu}{}_{\mu\beta}\, F^{\beta\nu} + \Gamma^{\nu}{}_{\mu\beta}\, F^{\mu\beta} \tag{F.397}$$

shows that one term drops out, as a contraction of the symmetric Christoffel-symbol $\Gamma^{\nu}{}_{\mu\beta}$ with the antisymmetric field tensor $F^{\mu\beta}$. Then, the index structure suggests that we can bring in the divergence formula for the index μ, yielding

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^{\mu}{}_{\mu\beta}\, F^{\beta\nu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g} F^{\mu\nu}) \tag{F.398}$$

The covolume is readily computed to be $-g = ABr^4 \cdot \sin^2\theta$, implying that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\sqrt{AB}\, r^2 F^{rt}\right) = 0 \tag{F.399}$$

as all other derivatives vanish as a consequence of the assumed symmetries, with $F^{rt} = \partial^r A^t - \partial^t A^r$ being the only nonzero field component. To make things specific, we make the ansatz $A_\mu = \left(\Phi, 0, 0, 0\right)$ with the electrostatic potential Φ, where none of the entries of A_μ can depend on time. The field tensor with contravariant indices is then given by

$$F^{rt} = g^{r\mu} g^{tr}\, F_{\mu\nu} = g^{rr} g^{tt}\, F_{rt} = -\frac{E}{AB} \tag{F.400}$$

directly from $\partial_t F_{rt} = 0$ and the radial electric field $\partial_r F_{rt} = E$, along with the metric coefficients g^{rr} and g^{tt} (no summation is implied in the second last term!), finally suggesting the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2 E}{\sqrt{AB}}\right) = 0, \quad \text{solved by } E(r) = \sqrt{AB} \cdot \frac{k}{r^2} \tag{F.401}$$

At infinity, one recovers Minkowskian geometry, sp $A \to 1$, $B \to 1$ and k should be equal to $\frac{Q}{4\pi c}$ in Gaussian units to yield the static Coulomb-potential:

$$E(r) = \sqrt{AB} \cdot \frac{Q}{4\pi c r^2} \tag{F.402}$$

That would be the solution for the electric field for the - apart from symmetries - yet unknown background spacetime. The energy momentum tensor $T_{\mu\nu}$ of the Coulomb-field is given by

$$T_{\mu\nu} = g^{\rho\sigma} F_{\mu\rho} \cdot F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{F.403}$$

from the construction of $T_{\mu\nu}$ from the Maxwell-Lagrange density $S = \int d^4x \sqrt{-g} \cdot g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu}$ it comes out as naturally traceless,

$$T = g^{\mu\nu} T_{\mu\nu} = g^{\rho\sigma} g^{\mu\nu} F_{\mu\rho} F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} = 0 \tag{F.404}$$

with $g_{\mu\nu} g^{\mu\nu} = \delta^{\mu}_{\mu} = 4$.

The energy momentum-tensor $T_{\mu\nu}$ now acts as the source of the gravitational field: The vanishing trace implies that the Ricci-scalar is zero, too (that is in fact identical to the Schwarzschild case as a vacuum solution), but the Ricci-scalar is otherwise liked to the energy momentum-tensor through the field equation:

$$R_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{F.405}$$

The expressions for the Ricci-tensor are identical to those in the Schwarzschild case, as the symmetries are identical:

$$R_{tt} = -4\pi G \cdot \frac{E^2}{B} \tag{F.406}$$

$$R_{rr} = +4\pi G \cdot \frac{E^2}{A} \tag{F.407}$$

$$R_{\theta\theta} = -4\pi G \cdot r^2 \frac{E^2}{AB} \tag{F.408}$$

only that the right side is nonzero due to the presence of the source $T_{\mu\nu}$. Specifically, the source components read:

$$T_{tt} == g^{\rho\sigma} F_{t\rho} F_{t\sigma} - \frac{1}{4} \cdot F_{\rho\sigma} F^{\rho\sigma} = -\frac{E^2}{2B} \tag{F.409}$$

$$T_{rr} = g^{\rho\sigma} F_{r\rho} F_{r\sigma} - \frac{1}{4} g_{rr} F_{\rho\sigma} F^{\rho\sigma} = \frac{E^2}{2A} \tag{F.410}$$

$$T_{\theta\theta} = g^{\rho\sigma} F_{\theta\rho} F_{\theta\sigma} - \frac{1}{4} g_{\theta\theta} F_{\rho\sigma} F^{\rho\sigma} = -r^2 \frac{E^2}{2AB} \tag{F.411}$$

using the form of the metric tensor and its inverse, the antisymmetry $F_{\mu\nu} = -F_{\nu\mu}$ of the field tensor, and the expression $F_{\rho\sigma} F^{\rho\sigma} = F_{rt} F^{rt} + F_{tr} F^{tr}$ for the trace. In both the Ricci-tensor and the energy momentum tensor, the information contained in the ϕ, ϕ-components is redundant with that in the θ, θ-component.

Proceeding with solving the field equation we obtain for the Ricci-tensor components specifically the same result as in the Schwarzschild case. Starting with B× eqn. F.407 + A× eqn. F.407 needs to vanish, from which one arrives at:

$$A'B + B'A = 0 \quad \rightarrow \quad \frac{d}{dr}(AB) = 0 \quad \rightarrow \quad AB = c^2. \tag{F.412}$$

Then, eqn. F.408 together with eqn. F.402 suggests that

$$A + rA' = c^2 \cdot \left(1 - \frac{G \cdot Q^2}{4\pi c^4} \cdot \frac{1}{r^2}\right) \tag{F.413}$$

which can be simplified using $A + rA' = \frac{d}{dr}(rA)$ to give

$$A(r) = c^2 \cdot \left[1 - \frac{2GM}{c^2 r} + \frac{GQ}{4\pi c^4 r^2}\right] \tag{F.414}$$

in analogy to the Schwarzschild case. Collecting all results and defining $q = \frac{GQ}{4\pi^2 c^4}$ yields the Reissner-Nordström line element,

$$ds^2 = \left(1 - \frac{r_S}{r} + \frac{q^2}{r^2}\right) c^2\, dt^2 - \frac{1}{1 - \frac{r_S}{r} + \frac{q^2}{r^2}} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{F.415}$$

as the spacetime geometry outside of a spherically symmetric matter and charge distribution. There are a couple of interesting observations to make: Firstly, the electric field $E(r)$ is really a Coulomb-field in Schwarzschild coordinates, as $AB = c^2$ one arrives at

$$E(r) = \frac{Q}{4\pi r^2} \tag{F.416}$$

which is perhaps not too surprising since the radial coordinate r in the Schwarzschild geometry is constructed to keep the scaling of surfaces $\propto 4\pi r^2$ fixed. Thinking of the Gauß-theorem applied to electrostatics one realises that the conservation of electric flux is made sure by diluting the field over larger and larger surfaces at increasing distance, such that the product remains constant. It is the particular construction of the Schwarzschild radial coordinate that this argument applies exactly despite curvature effects being present. Secondly, the new term proportional to q corresponds to the gravitational effect of the Coulomb field through its own energy content $\propto E^2$. Thirdly, all arguments on coordinate singularities apply likewise, as the Reissner-Nordström geometry has a finite curvature everywhere (except $r = 0$).

There is, however, a surprising result concerning the coordinate singularities: The line element implies a very interesting coordinate singularity structure:

$$ds^2 = \left(1 - \frac{r_S}{r} + \frac{q^2}{r^2}\right) c^2\, dt^2 - \frac{1}{1 - \frac{r_S}{r} + \frac{q^2}{r^2}} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{F.417}$$

shows that what matters are solutions to the quadratic equation $r^2 - rr_S + q^2 = 0$. If zeros exist, the metric function in front of the dr^2-differential can diverge.

The existence of solutions of a quadratic equation is decided by the value of the discriminant, $Delta = 1 - \frac{r_S}{r} + \frac{q^2}{r^2}$. If it is negative, there are no solutions and the spacetime is regular everywhere. If it is zero, then there is a single solution of the quadratic equation and a single singularity arises. The interesting case is a positive discriminant: Then, there are two zeros and consequently, coordinate singularities at two different radii. From a physical point of view, the solution of

$$r^2 - rr_S + q^2 = 0 \quad \rightarrow \quad r_\pm = \frac{r_S \pm \sqrt{r_S^2 - 4q^2}}{2} \tag{F.418}$$

is determined by the comparison of mass and charge,

$$\begin{cases} r_S^2 > 4q^2 \text{ 2 horizons} \\ r_S^2 = 4q^2 \text{ 1 horizon at } r_S \\ r_S^2 < 4q^2 \text{ no real valued solution} \rightarrow \text{interpretation unclear} \end{cases} \tag{F.419}$$

as the Schwarzschild radius increases with mass. Interestingly, a highly charged black hole has no horizons at all. There are analogies to Eddington-Finkelstein and Kruskal-coordinates that can deal with the double horizon structure, but their construction is very technical.

F.10 *Escape from a black hole*

Almost every student asks the question, after the causal structure of black holes is discussed, together with the impossible escape of photons form black hole if they are emitted inwards of $r_S = 2GM/c^2$, whether a sufficiently powerful spaceship can do that. Clearly, the spaceship is not in a state of freely falling motion but has non-gravitational accelerations acting on it. A short answer would be that light cones form the convex hull of all time-like geodesics, so the spaceship can at most travel inside the light cones, for which we have derived the causal structure, most clearly in e.g. Kruskal-coordinates.

Additionally, the causal structure is respected by electrodynamic forces: If they are added to the geodesic equation, they constitute a source term on the right hand side of the equation,

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha_{\ \mu\nu}\, u^\mu u^\nu = -\frac{q}{m} F^{\alpha\beta} u_\beta \tag{F.420}$$

with the velocity $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau$, using proper time τ as an affine parameter, which is perfectly admissible as an affine parameter. u^α is time-like, $g_{\mu\nu}u^\mu u^\nu = c^2$, and this normalisation is conserved. Using autoparallelity, the equation of motion is rewritten as

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\partial u^\alpha}{\partial x^\mu} + \Gamma^\alpha_{\ \mu\nu}\, u^\mu u^\nu = u^\mu\left(\frac{\partial u^\alpha}{\partial x^\mu} + \Gamma^\alpha_{\ \mu\nu}\right) = u^\mu \nabla_\mu u^\alpha = -\frac{q}{m} F^{\alpha\beta} u_\beta. \tag{F.421}$$

Multiplying both sides with u^α makes the electromagnetic term vanish, as a contraction of an antisymmetric tensor $F^{\alpha\beta}$ with a symmetric tensor $u_\alpha u_\beta$, while parallel transport conserves the normalisation of u^α because of metric compatibility; in a sense the two concepts are completely independent and do not interfere with each other. Explicitly,

$$u^\mu \nabla_\mu (u_\alpha u^\alpha) = u^\mu \nabla_\mu u_\alpha \cdot u^\alpha + u_\alpha u^\mu \nabla_\mu u^\alpha = u^\mu u^\alpha \left(\partial_\mu u_\alpha - \Gamma^\beta_{\mu\alpha}\, u_\beta\right) + u^\mu u_\alpha \left(\partial_\mu u^\alpha + \Gamma^\alpha_{\mu\beta}\, u^\beta\right) \tag{F.422}$$

and finally

$$u^\mu \nabla_\mu (u_\alpha u^\alpha) = u^\mu \partial_\mu \left(u_\alpha u^\alpha\right) = 0 \tag{F.423}$$

because of the normalisation $u_\alpha u^\alpha = c^2$.