E GRAVITATIONAL FIELD EQUATION

E.1 *What should be realised in a gravitational field equation?*

The field equation for gravity should first of all be a tensorial relationship between curvature and the energy-momentum tensor the source of gravity, with a symmetric curvature tensor isolated from the full Riemann curvature. Tensorial relationships are necessary to have a consistent and well-defined transformation property of all terms in the field equation. The field equation should operate on a 4-dimensional background and allow for wave-like propagating solutions.

The field equation should obey covariant energy-momentum conservation. As a second order partial differential equation (because the Riemann-curvature is made from the second derivatives of the metric) it should be hyperbolic and allow modes to propagate on the light cone. In contrast to our first attempts at constructing a generalisation of the Poisson-equation within special relativity, there should be a natural explanation why $m = 0$ but why $\lambda \neq 0$. But nevertheless, the limit of the field equation for weakly perturbed, static spacetimes should fall back on the classical Poisson equation $\Delta\Phi = 4\pi G\rho + \lambda$ (I'm trying to make a point that the cosmological constant λ was always part of a classical theory).

It is a surprising result found by D. Lovelock that general relativity is unique as a relativistic theory of gravity for conserved energy and momentum in 4 dimensions with a second order hyperbolic and local field equation with a single dynamical field, the metric $g_{\mu\nu}$. It is an astonishing fact that the field equation of general relativity is as fundamental as the Maxwell equations with nothing more fundamental from which it could be derived. So all we can hope is to go through arguments why the equation is sensible and how physical concepts are realised. I should mention that there are ideas in relation to constructive gravity with the central idea that the theory for the material fields (like A^μ) already fixes the dynamics of the metric $g_{\mu\nu}$ up to the point that the gravitational field equation can be constructed from the Lagrange-density of the Maxwell-field.

E.2 *Construction of the field equation*

The first issue in the quest to link the Riemann curvature $R_{\alpha\beta\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$ is the different rank of the two tensors. The Ricci-curvature $R_{\beta\nu} = g^{\alpha\mu}R_{\alpha\beta\gamma\mu}$ would be (up to an overall sign) the only non-vanishing contraction of the Riemann-curvature and it would be symmetric as well, as can be shown with the algebraic Bianchi-identity,

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0 \tag{E.262}$$

for the cyclic permutation of β, $\mu\nu$ while keeping the first index α fixed. Applying a contraction with $g^{\alpha\mu}$ to the algebraic Bianchi-identity gets rid of the second term due to the antisymmetry of $R_{\alpha\mu\nu\beta}$ in $\alpha\mu$. Then,

$$g^{\alpha\mu}R_{\alpha\beta\mu\nu} + g^{\alpha\mu}R_{\alpha\nu\beta\mu} = g^{\alpha\mu}R_{\alpha\beta\mu\nu} - g^{\alpha\mu}R_{\alpha\nu\mu\beta} = R_{\beta\nu} - R_{\nu\beta} = 0 \tag{E.263}$$

again using antisymmetry, this time of $R_{\alpha\nu\beta\mu}$ in the second index pair, which shows the symmetry of the Ricci tensor, $R_{\beta\nu} = R_{\nu\beta}$. Then, the Ricci-scalar $R = g^{\beta\nu}R_{\beta\nu} = g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\mu\nu}$ as a contraction of the Ricci-tensor $R_{\beta\nu}$ is well-defined and not fixed to zero by any index exchange symmetry.

Would $R_{\mu\nu} \propto T_{\mu\nu}$ be viable field equation? Covariant energy-momentum conservation requires that $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$, but one can show that the divergence $g^{\alpha\mu}\nabla_\alpha R_{\mu\nu} \neq 0$, so that the field equation would be inconsistent. Instead, one needs a more elaborate curvature quantity: the Einstein-tensor $G_{\mu\nu}$. Starting from the differential Bianchi-identity

$$\nabla_\tau R_{\alpha\beta\mu\nu} + \nabla_\mu R_{\alpha\beta\nu\tau} + \nabla_\nu R_{\alpha\beta\tau\mu} = 0 \tag{E.264}$$

with cyclic permutation in $\tau, \mu\nu$ and α, β fixed, one can make the substitution $R_{\alpha\beta\tau\mu} = -R_{\alpha\beta\mu\tau}$ in the last term with the index antisymmetry in the second pair. Contraction with $g^{\alpha\mu}$ yields:

$$g^{\alpha\mu}\nabla_\tau R_{\alpha\beta\mu\nu} + g^{\alpha\mu}\nabla_\mu R_{\alpha\beta\nu\tau} - g^{\alpha\mu}\nabla_\nu R_{\alpha\beta\mu\tau} = 0 \tag{E.265}$$

Using metric compatibility $\nabla_\alpha g_{\mu\nu} = 0$ in the last term, followed by a contraction with $g^{\beta\tau}$ then introduces the Ricci-scalar R, because $g^{\alpha\mu}g^{\beta\tau}R_{\alpha\beta\mu\tau} = R$. The first term gives $g^{\alpha\mu}g^{\beta\tau}\nabla_\tau R_{\alpha\beta\mu\nu} = g^{\beta\tau}\nabla_\tau R_{\beta\nu}$, which is the divergence of the Ricci-tensor. The most complicated term is the middle one: Starting from the algebraic Bianchi-identity $R_{\alpha\beta\nu\tau} + R_{\alpha\nu\tau\beta} + R_{\alpha\tau_\beta\nu} = 0$ one can construct the argument that $R_{\beta\alpha\nu\tau} = R_{\alpha\nu\tau\beta} + R_{\alpha\tau\beta\nu}$ using the antisymmetry in the first index pair of the first term, followed by the contraction of $\nabla_\mu R_{\beta\alpha\nu\tau}$, over $\beta\tau$ and $\alpha\mu$, which comes out as $g^{\beta\tau}g^{\alpha\mu}[R_{\alpha\nu\tau\beta} + R_{\alpha\tau\beta\nu}]$, where the first term vanishes due to the (anti)symmetry of the indices and only $g^{\alpha\mu}\nabla_\mu R_{\alpha\nu}$ is left over, with an additional overall minus-sign. Realising that this term is, like the first one, the divergence of the Ricci-tensor albeit with different (internal) indices, the final result is:

$$2g^{\alpha\mu}\nabla_\mu R_{\alpha\nu} - \nabla_\nu R \quad \rightarrow \quad g^{\alpha\mu}\nabla_\mu\left[2R_{\alpha\nu} - R g_{\alpha\nu}\right] = 0 \tag{E.266}$$

indicating that this particular combination of the Ricci-tensor, the Ricci-scalar and the metric is divergence-free and could appear in the field equation. Commonly, one defines the Einstein-tensor $G_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} \tag{E.267}$$

for this purpose, which inherits its symmetry from $R_{\mu\nu}$ and $g_{\mu\nu}$. It is a memorable result that the trace of $G_{\mu\nu}$

$$g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}R_{\mu\nu} - \frac{R}{2}g^{\mu\nu}g_{\mu\nu} = R - \frac{R}{2}\delta^\mu_\mu = R - 4\frac{R}{2} = -R. \tag{E.268}$$

is just the negative Ricci-scalar R.

Realising that the metric is the second rank-2 tensor with vanishing divergence due to metric compatibility suggests as a possible gravitational field equation

💀 There is no "derivation" of the field equation, it is so fundamental that we don't know any more fundamental principle from which it could originate!

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{E.269}$$

with two gravitational constants G and Λ. It is a second-order nonlinear hyperbolic partial differential equation which respects the local covariant energy-momentum conservation and constitutes 10 independent relations in 4 dimensions, due to the

Table 1: Numbers of entries of $T_{\mu\nu}$, $R_{\mu\nu}$ and $R_{\alpha\beta\mu\nu}$ as a function of the dimensionality n of spacetime: While the number of entries of $T_{\mu\nu}$ is simply determined by symmetry $T_{\mu\nu} = T_{\nu\mu}$, the entries of $R_{\mu\nu}$ and $R_{\alpha\beta\mu\nu}$ must be derived from the index exchange symmetries, for instance, that there can not be any curvature in 1 dimension, because the non-commutativity of the covariant derivative in different directions never arises: there is only one direction. In 3 dimensions, there can not be any curvature beyond Ricci-curvature and the gravitational field would only exist at locations where the energy-momentum tensor is nonzero. Only in 4 dimensions or more there are components of curvature beyond Ricci curvature and the gravitational field can exist away from the source.

dimension	*1*	*2*	*3*	*4*
$T_{\mu\nu}$	*1*	*3*	*6*	*10*
$R_{\mu\nu}$	*0*	*1*	*6*	*10*
$R_{\alpha\beta\mu\nu}$	*0*	*1*	*6*	*20*

symmetry of $R_{\mu\nu}$, $g_{\mu\nu}$ and $T_{\mu\nu}$. Hyperbolicity of the field equation is a consequence of the sign-change in the signature $(+,-,-,-)$ of the metric $g_{\mu\nu}$, which falls back onto the Minkowskian-metric in freely-falling frames, $g_{\mu\nu} = \eta_{\mu\nu}$, and ultimately, hyperbolicity will allow for wave-type solutions: gravitational waves!

One issue needs considerable explanation: The Riemann-curvature as a complete characterisation of the spacetime curvature has 20 entries in 4 dimensions (reduced from $4^4 = 256$ to 20 by the index exchange symmetries), but the field equation only fixes half of the curvature, similarly to the Poisson equation $\Delta\Phi = 4\pi G\rho$, where only the trace $\Delta\Phi = \delta^{ij}\partial_i\partial_j\Phi$ of the tidal field tensor $\partial_i\partial_j\Phi$ is determined by the field equation. In electrodynamics, the field equation $\Box A^\mu = 4\pi/c\, j^\mu$ in Lorentz-gauge $\partial_\mu A^\mu = 0$ fixes 4 of the 10 derivatives $\partial_\alpha\partial_\beta A^\mu$, so this is really a common feature for all field theories. If this was not the case, we could have only Ricci-curvature, and it could only exist at places where the energy-momentum tensor is nonzero, $T_{\mu\nu} \neq 0$. Clearly, this would be a weird theory of gravity, as the field should be free to propagate away from the source into spacetime.

E.3 *Ricci- and Weyl-curvature*

In classical gravity, $\Delta\Phi = \delta^{ij}\partial_i\partial_j\Phi$ is invariant as the trace of the tidal field $\partial_i\partial_j\Phi$: It does not change under rotations of the coordinate system and links the potential to the source $4\pi G\rho$. Starting with Φ one obtains the gravitational acceleration $g_j = -\partial_j\Phi$, of which one can compute the divergence $\mathrm{div}\boldsymbol{g} = \delta^{ij}\partial_i g_j = -\delta^{ij}\partial_i\partial_j\Phi$ which tells you about a nonzero ρ at the point where Δ acts on Φ. Vice versa, however, does $\Delta\Phi = 0$ not imply that there is no gravitational field, it only implies that at that particular location there is no source, and clearly can gravity exist at locations outside the field generating matter, for instance on the surface of the Earth. This suggests that one would like to separate $\Delta\Phi$ from $\partial_i\partial_j\Phi$ and define the traceless shear

$$\tilde{\Phi}_{ij} = \partial_i\partial_j\Phi - \frac{\Delta\Phi}{3}\delta_{ij}. \tag{E.270}$$

$\tilde{\Phi}_{ij}$ are the components of the tidal shear that are sourced elsewhere and propagate to the point where the derivatives of Φ are computed.

E.4 *Curvature invariants*

There are two possible ways to quantify geometric properties of manifolds, or, in fact, tensorial or vectorial fields: Either, one is able to write down a relation between tensors of compatible rank and index structure, in which case all terms in an equation transform covariantly under coordinate transforms, or one can construct invariants by a full index contraction. Then, one obtains a scalar which is necessarily invariant under coordinate transforms and has to assume an identical value in all frames. That is the reason why scalars are so convenient: Their entries do not only for a given coordinate choice but are universally true. What one gives up, however, is a significant part of the information that gets lost in contraction. But sometimes, scalars have a physical interpretation and can isolate important information on a tensor.

In classical gravity, we can compute the tidal field $\partial_i\partial_j\Phi$ as the curvature analogue and build contractions of this quantity, for instance with the Euclidean metric: $\delta^{ij}\partial_i\partial_j\Phi = \Delta\Phi$ is rotationally invariant (reflecting the fundamental properties of Euclidean spaces, and proportional to $4\pi G\rho + \lambda$). Or, one constructs the quadratic quantity $\delta^{ai}\delta^{bj}\partial_a\partial_b\Phi\,\partial_i\partial_j\Phi$, which corresponds to the Frobenius-norm of Phi which is positive definite: We can conclude $\Phi = 0$ from a vanishing Frobenius-norm, but we can not do that from $\Delta\Phi = 0$, which only means that at that particular location no source of the field exists.

The central quantity for curvature in relativity is the Riemann-tensor $R_{\alpha\beta\mu\nu}$, with a range of possibilities to form a scalar. For instance, the Ricci-scalar $R = g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\mu\nu}$ would be a quantity analogous to $\Delta\Phi$, as it is proportional to the trace of the energy momentum tensor $T = g^{\mu\nu}T_{\mu\nu}$, minus 4Λ if the cosmological constant is included. That's clearly only the curvature that is generated locally by $T_{\mu\nu}$ (and by Λ), but not the complete curvature. In analogy to the Frobenius-norm one could think of Kretschmann-scalar $K = R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} = g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}R_{\alpha\beta\gamma\delta}R_{\mu\nu\rho\sigma}$.

The Weyl-tensor $C_{\alpha\beta\mu\nu}$ would correspond to the traceless tidal shear $\tilde{\Phi}_i$, because the locally generated part of the curvature has been eliminated. Then, clearly both $\delta^{ij}\tilde{\Phi}_{ij}$ and $g^{\alpha\mu}g^{\beta\nu}C_{\alpha\beta\mu\nu}$ vanish. But $\delta^{ai}\delta^{bj}\,\tilde{\Phi}_{ab}\tilde{\Phi}_{ij}$ is not required to be zero by $\Delta\Phi = 0$, and neither is the Weyl-scalar $C = C^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu} = g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}C_{\alpha\beta\gamma\delta}C_{\mu\nu\rho\sigma}$: It would serve as an invariant quantification of the curvature at a point of all gravitational fields that are sourced elsewhere. As such, the Weyl-curvature $C_{\alpha\beta\mu\nu}$ is a covariant generalisation of the traceless tidal tensor $\tilde{\Phi}_{ij}$.

E.5 *Weak and static gravity*

General relativity needs to be consistent with classical gravity in the limit of weak curvature and static gravitational fields consistent with a non-relativistic matter distribution at rest. The trace of the field equation is given by

$$g^{\mu\nu}R_{\mu\nu} - \frac{R}{2}g^{\mu\nu}g_{\mu\nu} = -R = -\frac{8\pi G}{c^4}g^{\mu\nu}T_{\mu\nu} = -\frac{8\pi G}{c^4}T \tag{E.271}$$

using $R = g^{\mu\nu}R_{\mu\nu}$, $T = g^{\mu\nu}T_{\mu\nu}$ and $g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = 4$, while the trace of the energy momentum tensor is given by

$$T = g^{\mu\nu}T_{\mu\nu} = g^{\mu\nu}\left[\left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu - p\cdot g_{\mu\nu}\right] = \left(\rho + \frac{p}{c^2}\right)g^{\mu\nu}\,u_\mu u_\nu - p g^{\mu\nu}g_{\mu\nu} = \rho c^2 - 3p \simeq \rho c^2 \tag{E.272}$$

if the matter is non-relativistic, $p \ll \rho c^2$, so that the Ricci-scalar just depends on the matter density,

$$R = -\frac{8\pi G}{c^2}\rho. \tag{E.273}$$

A weak perturbation an otherwise Minkowskian spacetime by a static gravitational potential Φ has the form

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij}\, dx^i dx^j \tag{E.274}$$

where the decomposition $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with the condition $|h_{\mu\nu}| \ll 1$ is only valid in that particular Cartesian coordinate choice. Then, the inverse metric can be approximated to be $g^{\mu\nu} \simeq \eta^{\mu\nu}$ with an error of the order h^2. The tt-component of the Ricci-tensor in general given by

$$R_{tt} = \partial_t \Gamma^{\mu}{}_{t\mu} - \partial_\mu \Gamma^{\mu}{}_{tt} + \Gamma^{\nu}{}_{t\mu}\Gamma^{\mu}{}_{\nu t} - \Gamma^{\nu}{}_{tt}\Gamma^{\mu}{}_{\mu\nu} \tag{E.275}$$

where the first term $\partial_t \Gamma^{\mu}{}_{t\mu} = 0$ for static fields, and the squared Christoffel-symbols $+\Gamma^{\nu}{}_{t\mu}\Gamma^{\mu}{}_{\nu t} - \Gamma^{\nu}{}_{tt}\Gamma^{\mu}{}_{\mu\nu}$ would contribute at order h^2, so we neglect them. The only contributing term is then

$$R_{tt} = -\partial_\mu \Gamma^{\mu}{}_{tt} = -\partial_i \Gamma^{i}{}_{tt} = -\partial_i\left(\frac{\delta^{ij}}{2}\left(-\partial_j h_{tt}\right)\right) = \frac{1}{2}\delta^{ij}\partial_i\partial_j h_{tt} = \frac{\Delta\Phi}{c^2} \tag{E.276}$$

because $h_{tt} = 2\Phi/c^2$. Collecting the results on the traces and the weak field, static limit then yields

$$R_{tt} = \frac{\Delta\Phi}{c^2} = \frac{4\pi G}{c^4}\rho c^2 \quad \rightarrow \quad \Delta\Phi = 4\pi G\rho, \tag{E.277}$$

which one recognises as the classic Poisson field equation.

E.6 *Weyl-curvature*

There is a very good physical reason to decompose the Riemann tensor $R_{\alpha\beta\mu\nu}$ as full quantification of curvature into two parts: The Ricci-curvature $R_{\beta\nu} = g^{\alpha\mu}R_{\alpha\beta\mu\nu}$, which appears in field equation as $R_{\beta\nu} - R/2\, g_{\beta\nu}$ and which is proportional to the energy-momentum tensor $T_{\beta\nu}$, and the remaining curvature components, which form the Weyl-tensor $C_{\alpha\beta\mu\nu}$ describing the curvature that has been sourced by energy and momentum elsewhere and has propagated to the spacetime point under consideration.

As already discussed, the field equation should not fully fix the curvature and set it to be proportional to the source of the field, which is a typical structure in all field equations. Electrodynamics, for instance, equates only 4 components of the $24 = 6 \times 4$ possible derivatives $\partial^\beta F^{\mu\nu}$ of $F^{\mu\nu}$ to be equal to the source $\jmath^\nu$ according to $\eta_{\beta\mu}\partial^\beta F^{\mu\nu} = 4\pi/c\, \jmath^\nu$.

But is there a constraint on the remaining 20 components? Yes, in fact through the Bianchi-identity,

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\nu} = 0 \quad \text{or, equivalently} \quad \eta^{\beta\mu}\partial_\beta \tilde{F}_{\mu\nu} = 0 \tag{E.278}$$

with the dual tensor $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}/2$.

Similarly, the off-trace parts of the curvature form the Weyl-tensor which obeys an analogous differential Bianchi-identity. In fact, it obeys the same antisymmetry relations as the Riemann-tensor, i.e.

$$C_{\alpha\beta\mu\nu} = -C_{\beta\alpha\mu\nu} = -C_{\alpha\beta\nu\mu} \tag{E.279}$$

as well as an algebraic Bianchi-identity

$$C_{\alpha\beta\mu\nu} + C_{\alpha\mu\nu\beta} + C_{\alpha\nu\beta\mu} = 0 \tag{E.280}$$

and

$$g^{\alpha\mu}\, C_{\alpha\beta\mu\nu} = 0 \tag{E.281}$$

and finally a differential Bianchi-identity

$$\nabla_\tau C_{\alpha\beta\mu\nu} + \nabla_\mu C_{\alpha\beta\nu\tau} + \nabla_\nu C_{\alpha\beta\tau\mu} = 0. \tag{E.282}$$

Let's construct a systematic decomposition of the Riemann curvature $R_{\alpha\beta\mu\nu}$: From any symmetric tensor $X_{\alpha\beta}$ one can derive the quantity $\tilde{X}_{\alpha\beta\mu\nu}$

$$\tilde{X}_{\alpha\beta\mu\nu} = A_{\alpha\mu}\, g_{\beta\nu} + A_{\beta\nu}\, g_{\alpha\mu} - A_{\alpha\nu}\, g_{\beta\mu} - A_{\beta\mu}\, g_{\alpha\nu}. \tag{E.283}$$

This definition of $\tilde{X}_{\alpha\beta\mu\nu}$ makes sure that the quantity fulfils the properties

$$\tilde{X}_{\alpha\beta\mu\nu} = -\tilde{X}_{\alpha\beta\mu\nu} = -\tilde{X}_{\beta\alpha\mu\nu} \quad \text{and} \quad \tilde{X}_{\alpha\beta\mu\nu} + \tilde{X}_{\alpha\mu\nu\beta} + \tilde{X}_{\alpha\nu\beta\mu} = 0 \tag{E.284}$$

i.e. effectively the index exchange symmetries of the Riemann-tensor, suggesting the ansatz

$$R_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + a \cdot \tilde{R}_{\alpha\beta\mu\nu} + b\, R \cdot \tilde{g}_{\alpha\beta\mu\nu} \tag{E.285}$$

with the Ricci-scalar R, and $\tilde{g}_{\alpha\beta\mu\nu}$ and $\tilde{R}_{\alpha\beta\mu\nu}$ from the metric $g_{\mu\nu}$ and the Ricci-tensor $R_{\mu\nu}$, respectively. Then, the two factors a and b can be determined through contraction.

This decomposition can be used to show an extremely interesting algebraic property of the Weyl-curvature $C_{\alpha\beta\mu\nu}$ as the part of curvature that propagates: The tensor can only be nonzero in more than four dimensions, suggesting that gravity can only exist at locations where the energy momentum tensor is zero in less than four dimensions, entirely defeating the purpose of a field theory:

- $n = 1$

  no Riemann-curvature, $R_{\alpha\beta\mu\nu} = 0$, because of the exchange symmetry in e.g. the last two indices: There can't be any curvature in one dimensions, because the covariant derivatives always commute, as they apply only to a single direction.

- $n = 2$

  Riemann-curvature is always proportional to the Ricci-scalar and the metric, as two-dimensional manifolds are always maximally symmetric,

$$R_{\alpha\beta\mu\nu} = \frac{R}{2}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \tag{E.286}$$

- $n = 3$

  Riemann-curvature is proportional to the Ricci-tensor and the Ricci scalar, but the Weyl-tensor vanishes identically,

$$R_{\alpha\beta\mu\nu} = \left(g_{\beta\mu}\, R_{\alpha\nu} + g_{\alpha\nu}\, R_{\beta\mu} - g_{\beta\nu}\, R_{\alpha\mu} - g_{\alpha\mu}\, R_{\beta\nu}\right) + \frac{R}{2}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \tag{E.287}$$

  That implies that the full Riemann-curvature needs to vanish if $T_{\mu\nu}$ is linked to the Ricci-curvature as in the conventional field equation: There would not be vacuum solutions in 2 or 3 dimensions.

- $n = 4$

  Ricci- and Weyl-curvature can simultaneously exist

$$R_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + \frac{1}{2}\left(g_{\beta\mu}\, R_{\alpha\nu} + g_{\alpha\nu}\, R_{\beta\mu} - g_{\beta\nu}\, R_{\alpha\mu} - g_{\alpha\mu}\, R_{\beta\nu}\right) \tag{E.288}$$

  and $R_{\alpha\beta\mu\nu}$ can be nonzero even if $R_{\mu\nu}$ is zero as a consequence of $T_{\mu\nu} = 0$.

Spacetimes without Weyl-curvature, $C_{\alpha\beta\mu\nu} = 0$ (as for instance FLRW-spacetimes) are conformally flat and their metric can always be written as

$$g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu} \tag{E.289}$$

i.e. as originating with a (coordinate-dependent) conformal factor $\Omega^2(x) > 0$ from the flat Minkowski-metric: This implies that the light cone structure of these spacetimes is identical perfectly Minkowskian light cones: The conformal factor drops out in the condition $ds^2 = g_{\mu\nu}k^\mu k^\nu = \Omega(x)^2\eta_{\mu\nu}k^\mu k^\nu = 0$. That would be automatically the case in 2 and 3 dimensions.

A direct computation (which is very tedious) shows that Weyl-curvature is invariant under conformal transformations $g_{\mu\nu} \to \Omega(x)g_{\mu\nu}$ of the metric and that the Weyl-tensor maps onto itself: $C_{\alpha\beta\mu\nu} \to C_{\alpha\beta\mu\nu}$.

The differential Bianchi-identity is the dynamical equation for the Riemann curvature:

$$\nabla_\tau R_{\alpha\beta\mu\nu} + \nabla_\mu R_{\alpha\beta\nu\tau} + \nabla_\nu R_{\alpha\beta\tau\mu} = 0 \tag{E.290}$$

Contraction with $g^{\alpha\mu}$ then yields:

$$g^{\alpha\mu}\nabla_\tau R_{\alpha\beta\mu\nu} + g^{\alpha\mu}\nabla_\mu R_{\alpha\beta\nu\tau} + g^{\alpha\mu}\nabla_\nu R_{\alpha\beta\tau\mu} = 0 \tag{E.291}$$

Identifying the Ricci-scalar in the first (and after an index swap) in the last term yields:

$$g^{\alpha\mu}\nabla_\mu R_{\alpha\beta\nu\tau} = \nabla^\alpha R_{\alpha\beta\nu\tau} = \nabla_\nu R_{\beta\tau} - \nabla_\tau R_{\beta\nu}. \tag{E.292}$$

As the same differential Bianchi-identity applies to the Weyl-tensor $C_{\alpha\beta\mu\nu}$ as well, one obtains a very similar result

$$g^{\alpha\mu}\nabla_\mu R_{\alpha\beta\nu\tau} = \nabla^\alpha C_{\alpha\beta\nu\tau} = \nabla_\nu S_{\beta\tau} - \nabla_\tau S_{\beta\nu} = C_{\beta\nu\tau} \tag{E.293}$$

with the Schouten-tensor

$$S_{\beta\tau} = \frac{R_{\beta\tau}}{2} - \frac{R}{6} g_{\beta\tau} \tag{E.294}$$

and the Cotton-tensor

$$C_{\beta\nu\tau} = \nabla_\nu S_{\beta\tau} - \nabla_\tau S_{\beta\nu} \tag{E.295}$$

such that the differential Bianchi-identity assumes a shape that is in fact reminiscent of the field equation in Maxwell-electrodynamics! For vacuum both $R_{\beta\tau}$ and R vanish, such that the $S_{\beta\tau}$ is necessarily zero, implying that

$$g^{\alpha\mu}\nabla_\mu C_{\alpha\beta\nu\tau} = 0 \quad \text{invacuum.} \tag{E.296}$$

If there are is a field-generating energy momentum content $T_{\beta\tau} \neq 0$, one would obtain in a non-vacuum situation

$$g^{\alpha\mu}\nabla_\mu C_{\alpha\beta\nu\tau} = \nabla^\alpha C_{\alpha\beta\nu\tau} = C_{\beta\nu\tau} = \frac{4\pi G}{c^4} \cdot \left[\nabla_\nu T_{\beta\tau} - \nabla_\tau T_{\beta\nu} - \frac{1}{3}\left(\nabla_\tau T \cdot g_{\beta\nu} - \nabla_\nu T \cdot g_{\beta\tau}\right)\right] \tag{E.297}$$

similar to $g^{\alpha\mu}\nabla_\alpha F_{\mu\nu} = 4\pi/c\, \jmath_\nu$.

E.7 *Raychaudhuri-equation*

The Raychaudhuri-equation gives a very pictorial and intuitive impression of the effects of the two types of curvature (Ricci and Weyl). It's even possible to apply the concept to classical gravity, so let's do this first: A bundle of geodesics $x^i(t)$ with relative velocities υ^i

$$x'^i = x^i + \upsilon^i t \tag{E.298}$$

would exhibit relative motion

$$\frac{\partial x'^i}{\partial x^j} = \delta^i_j + \frac{\partial \upsilon^i}{\partial x^j} \cdot t \simeq \frac{\partial x'^i}{\partial x^j} = \exp\left(\frac{\partial \upsilon^i}{\partial x^j} \cdot t\right) \tag{E.299}$$

at order t, in the spirit of a Lie-generated transformation. The change of the volume elements from d^3x to d^3x' is given by

$$d^3x' = \det\left(\frac{\partial x'}{\partial x}\right) d^3x \tag{E.300}$$

with the Jacobian determinant of the coordinate change. Using my third most favourite formula,

$$\ln\, d^3x' = \ln\, \det\left(\frac{\partial x'}{\partial x}\right) + \ln\, d^3x \tag{E.301}$$

following from $\ln\, \det A = \ln \prod_i \lambda_i = \sum_i \ln\, \lambda_i = \mathrm{tr}\, \ln A$ for any non-singular matrix A one arrives at

$$\ln\, \det\left(\frac{\partial x'}{\partial x}\right) = \mathrm{tr}\, \ln\left(\frac{\partial x'}{\partial x}\right) = \mathrm{tr}\, \ln\, \exp\left(\frac{\partial \upsilon}{\partial x}\, t\right) = t\cdot \mathrm{tr}\left(\frac{\partial \upsilon}{\partial x}\right) \tag{E.302}$$

with the identification

$$\mathrm{tr}\, \frac{\partial \upsilon}{\partial x} = \delta^{ij}\, \partial_j \upsilon_i = -\delta^{ij}\, \partial_j \partial_i\, \Phi = -\Delta\Phi = -4\pi G\rho \tag{E.303}$$

such that the matter density ρ (appearing through the substitution of the Poisson equation $\Delta\Phi = 4\pi G\rho$) inside a cloud of freely falling test particles (made sure by the Newtonian equation of motion $\dot{\upsilon}^i + \partial_i\Phi = 0$) causes a negative change of the volume. Interestingly, the appearance of a cosmological constant λ would likewise contribute to the volume evolution, and we witness this actually in cosmology.

The same intuition applies to a relativistic theory of gravity, as the Ricci-curvature is responsible to the volume change of a spacetime volume. The picture that emerges is that Ricci-curvature changes volumes while keeping their shape intact, and that Weyl-curvature changes shapes while conserving their volumes (at least to lowest order). In all theories this distinction is made by a decomposition into the trace and the traceless part of the curvature.

E.8 *Nonlinearity and locality*

The field equation of general relativity are nonlinear partial differential equations with the important consequence that the superposition principle does not apply, which was such a convenient tool in classical gravity for solving the Poisson equation $\Delta\Phi = 4\pi G\rho + \lambda$. There, it's always possible to separate the problems one faces when determining the potential Φ from ρ: the inversion of the differential operator, to account for boundary conditions (as the Poisson-equation is an elliptical partial differential equation) and the possibly complicated geometry of the source ρ. In the case of linear field theories one achieves that by means of a Green-function $G(\boldsymbol{r}, \boldsymbol{r}')$ as a solution to the field equation for a point charge $\delta_D(\boldsymbol{r} - \boldsymbol{r}')$, for simplicity on small scales where $\lambda = 0$ in a good approximation:

$$\Delta \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = 4\pi G \delta_D(\boldsymbol{r} - \boldsymbol{r}') \tag{E.304}$$

Using linearity, the equation can be multiplied with $\rho((r))$ and integrated over d^3r'. Effectively, this is exactly the expression of the superposition principle as one adds

Table 2: Compilation of the simplest solutions of general relativity together with their symmetries and peculiar physical properties. It should be emphasised that a coordinate choice has been taken which is particularly suited to the symmetry of the respective spacetimes.

	black holes	*grav. waves*	*FLRW-cosmologies*	*white dwarfs*
homogeneous	t	$r \pm ct$	r	t
isotropic	*yes*	*yes*	*yes*	*yes*
varies along	r	r,t	t	r
gravity	*strong*	*weak*	*strong*	*weak...strong*
scales	$r_S = \frac{2GM}{c^2}$	*linear physics*	$\rho_{crit} = \frac{3H_0^2}{8\pi G}$	*eqn. of state*
curvature	*Weyl*	*Weyl*	*Ricci*	*Weyl + Ricci*
sources	*vacuum solution*	*vacuum solution*	p, ρ *(ideal fluid)*	p, ρ *(ideal fluid)*

up the contributions to Φ at $\boldsymbol{r}$ from the source distribution $\rho(\boldsymbol{r}')$:

$$\int d^3r'\, \Delta\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = \Delta\int d^3r'\, \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = \Delta\Phi = 4\pi G\int d^3r'\, \rho(\boldsymbol{r}')\delta_D(\boldsymbol{r}-\boldsymbol{r}') = 4\pi G\rho(\boldsymbol{r}) \tag{E.305}$$

so that

$$\Phi(\boldsymbol{r}) = \int d^3r'\, \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{E.306}$$

is the required solution for the potential. Effectively, going from ρ to Φ relies on linearity, and going from Φ to ρ uses the locality of the equation as it determines the classical equivalent of Ricci-curvature. General relativity, however, is nonlinear, because pictorially the Christoffel-symbols contain terms of the type $g\partial g$, the Riemann curvature $(g\partial g)^2$ and $\partial(g\partial g)$, and finally the Ricci-curvature terms of the type $g(g\partial g)^2$ and $g\partial(g\partial g)$. Despite the nonlinearities, the field equation is still local, as it links the Ricci-part of the curvature to the energy-momentum tensor, as exemplified by the consideration of the change in volume of freely falling clouds of test particles in the Raychaudhuri-equation.

And I would like to mention, that the field equation of general relativity is a hyperbolic differential equation: Therefore, the solution is already unique if initial conditions are specified, while boundary conditions are not necessary. Hyperbolicity makes sure that excitations of the gravitational field are propagating along the light cones defined differentially by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0$. The nonlinearities of the field equation make it very difficult to find solutions for arbitrary $T_{\mu\nu}$, as one can not use the Green-method which would require linear superposition. But there are solutions for reasonable simple and symmetric cases, which are listed in Table. E.8 and which will be discussed in Sects. F, G and H.