D SOURCES OF THE GRAVITATIONAL FIELD

D.1 *Gravity and matter*

The source of gravity in the Poisson equation $\Delta\Phi = 4\pi G\rho$ as the field equation is the matter density ρ. As a scalar potential Φ is identical in all frames. To make the source consistent with the field, we need to assume in Newtonian gravity that the density ρ is identical in all frames, too, in contradiction with relativistic effects like mass increase and length contraction that would affect the matter density, and with the fact that from a moving frame of reference ρ would be perceived as a momentum density rather than a matter density. For Newtonian gravity this is all irrelevant as the Poisson equation states a relation between two absolute quantities. The continuity equation for the matter density

$$\partial_t \rho + \partial_i(\rho v^i) = 0 \tag{D.220}$$

is phenomenological and expresses the idea that matter is not arbitrarily created or annihilated, and the partial derivatives refer to spacetime as being Euclidean, but in the spirit of Galilean relativity, but weirdly with a static relation between ρ and Φ.

Electrodynamics building on Lorentzian relativity does things better: The source of the electromagnetic field $F^{\mu\nu}$ in Maxwell's equation $\partial_\mu F^{\mu\nu} = 4\pi/c\, \jmath^\nu$ is the 4-current density as a Lorentz-vector $\jmath^\mu$. Neither the charge density nor the current density are absolute but depend on the state of motion of the observer relative to the charge. As $\jmath^\mu$ is a timelike vector (because charges are tied to massive particles), it is always possible to boost into the rest-frame of a charge with a suitable Lorentz-transform, $\jmath^\mu \to \Lambda^\mu{}_\alpha \jmath^\alpha$. There should be a consistent transformation between all terms of a formula, so Maxwell's field equation

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} \jmath^\nu \tag{D.221}$$

implies, that the Faraday-tensor $F^{\mu\nu}$ should transform, too, $F^{\mu\nu} \to \Lambda^\mu{}_\alpha \Lambda^\mu{}_\beta F^{\alpha\beta}$ as well as the partial derivative $\partial_\mu = \Lambda_\mu{}^\alpha \partial_\alpha$, which inherits its transformation property from the coordinates. The relativistic charge density $\jmath^\mu$ is conserved,

$$\partial_\mu \jmath^\mu = \partial_{ct}(c\rho) + \partial_i \jmath^i = 0, \tag{D.222}$$

consistently in all Lorentz-frames, because $\partial_\mu \jmath^\mu$ is a scalar.

It is fun to notice that the Maxwell equation and the Lorentz-equation introduce a nice consistency between the fields and the charges: Multiplying the Lorentz-equation for the acceleration of a charge

$$\frac{du_\alpha}{d\tau} = \frac{q}{m} \eta^{\beta\gamma} F_{\alpha\beta} u_\gamma \tag{D.223}$$

with $\eta^{\alpha\delta} u_\delta$ yields the conservation of the normalisation of the velocity as a timelike vector

$$\eta^{\alpha\delta} u_\delta \frac{du_\alpha}{d\tau} = \frac{1}{2}\frac{d}{dt}\left(\eta^{\alpha\delta} u_\alpha u_\delta\right) = \frac{q}{m} \eta^{\alpha\delta} \eta^{\beta\gamma} F_{\alpha\beta} u_\gamma u_\delta = 0 \tag{D.224}$$

Simultaneously, acting on the Maxwell equation with the differentiation $\eta^{\gamma\nu}\,\partial_\gamma$

$$\eta^{\beta\mu}\,\partial_\beta\,F_{\mu\nu} = \frac{4\pi}{c} j_\nu \tag{D.225}$$

shows that the charge is conserved $\partial_\mu j^\mu = 0$

$$\eta^{\beta\mu}\,\eta^{\gamma\nu}\,\partial_\beta\partial_\gamma\,F_{\mu\nu} = \frac{4\pi}{c}\,\eta^{\gamma\nu}\,\partial_\gamma j_\nu = 0 \tag{D.226}$$

In both cases, the contraction of the antisymmetric tensor $F^{\mu\nu}$ with the symmetric tensors $\partial_\mu\partial_\nu$ and $\upsilon_\mu\upsilon_\nu$ implies the conservation.

We would like these ideas to be realised for gravity as well: There should be a source of gravity with a proper covariant conservation law and a consistent transformation between the source and the field, all of course consistent with the Poisson equation in the limit of static sources and weak gravitational fields. With the knowledge of special relativity one notices a decisive difference between ρ as a charge density and ρ as a matter density: One can imagine that a cloud of charge gets Lorentz-contracted by a factor of γ as seen from an observer moving relative to the charge, implying that the charge density ρ is indeed the ct-component of a time-like Lorentz-vector. A cloud of matter seen from an observer moving relative to it would experience the same Lorentz-contraction, but there is relativistic mass increase in addition to it, introducing two instead of a single power of γ. This transformation property can not be reconciled with a single-indexed quantity like j^μ but requires a double indexed quantity: In fact, we will introduce the energy-momentum tensor $T^{\mu\nu}$ with $T^{tt} = \rho c^2$ in accordance with this idea.

D.2 *(Relativistic) fluids as sources of gravity*

Fluids are a continuum description of matter, i.e. a field where at every point the density and the velocity are defined: It is a valid picture to think of the fluid as being composed of small fluid elements across which the gradients of the fields do not vary strongly and linearisations apply. Fluid elements react to forces exerted by the surrounding fluid if their size is changed or if their shapes are distorted by gradients of the velocity field across the fluid element; in general there is a force $F^i = \sigma^{ij} dA_j$ acting on the surface element dA_j, parameterised by the shear tensor σ_{ij}, which is necessarily symmetric, $\sigma_{ij} = \sigma_{ji}$. While this relation is in general tensorial, the separation $\sigma_{ij} \rightarrow \sigma_{ij} + p\delta_{ij}$ would define a traceless anisotropic stress tensor σ_{ij} and the isotropic pressure p. Effects in the relation of anisotropic stress are parameterised by the shear viscosity and if in addition there are no viscous effects in relation to the change of volume of fluid element parameterised by the bulk viscosity, the fluid is ideal and only shows dynamic effects in relation with pressure p.

An ideal fluid is therefore characterised by density, pressure and velocity, and these quantities are assembled into the energy momentum tensor $T_{\mu\nu}$,

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - p\, g_{\mu\nu} \tag{D.227}$$

and we will convince ourselves retrospectively that this is the correct quantity, by showing the equivalence of covariant conservation of $T_{\mu\nu}$ by means of a continuity equation $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ and the equations of relativistic fluid mechanics.

The components of the energy momentum tensor,

$$T_{\mu\nu} = \begin{pmatrix} T_{tt} & T_{ti} \\ T_{jt} & T_{ij} \end{pmatrix} \tag{D.228}$$

contain the energy density T_{tt}, the energy flux in i–direction, T_{jt} being the component j of momentum density and T_{ij} the projection of the i–momentum in j-direction. In the local rest frame with Cartesian coordinates one would obtain $g_{\mu\nu} = \eta_{\mu\nu}$ as well as $u^\mu = (c, 0)^t$ such that

$$T_{\mu\nu} = \begin{pmatrix} \rho c^2 & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \tag{D.229}$$

The trace $g^{\mu\nu}T_{\mu\nu}$ has the value $\rho c^2 - 3p$, which likewise is true in any frame and any geometry: $g^{\mu\nu}T_{\mu\nu} = (\rho + p/c^2)g^{\mu\nu}u_\mu u_\nu - pg^{\mu\nu}g_{\mu\nu} = \rho c^2 - 3p$ because $g^{\mu\nu}u_\mu u_\nu = c^2$ and $g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = 4$. Many fluids are characterised by a fixed relation between pressure p and energy density ρc^2, which is referred to as the equation of state parameter $w = p/(\rho c^2)$. With the equation of state, the trace becomes $g^{\mu\nu}T_{\mu\nu} = (1 - 3w)\rho c^2$. A good way to remember this is the realisation that for photons the relationship $p = \rho c^2/3$ holds, implying that $g^{\mu\nu}T_{\mu\nu} = 0$ as $w = +1/3$, in accordance with a direct computation of the energy-momentum tensor from the Maxwell-Lagrange-density.

The conservation law $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ for the energy momentum tensor is vectorial (in the index ν), in contrast to the corresponding law for the charge density $g^{\alpha\mu}\nabla_\alpha \jmath_\mu = 0$, which is a scalar expression. To make sense of it nonetheless, one can project the vector $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ onto the velocity u^μ and a plane perpendicular to it. Computing the gradient $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ yields

$$g^{\alpha\mu}\nabla_\alpha\Big[\Big(\rho+\frac{p}{c^2}\Big)u_\mu u_\nu - p\,g_{\mu\nu}\Big] = g^{\alpha\mu}\Big[\nabla_\alpha\Big(\rho+\frac{p}{c^2}\Big)\cdot u_\mu u_\nu + \Big(\rho+\frac{p}{c^2}\Big)\nabla_\alpha(u_\mu u_\nu) - \nabla_\alpha p\cdot g_{\mu\nu}\Big] = 0 \tag{D.230}$$

keeping in mind that metric compatibility states that $\nabla_\alpha g_{\mu\nu} = 0$ and that the product of velocities in the second term resolves to $g^{\alpha\mu}\nabla_\alpha(u_\mu u_\nu) = g^{\alpha\mu}\nabla_\alpha u_\mu\cdot u_\nu + g^{\alpha\mu}u_\mu\cdot\nabla_\alpha u_\nu$.

Computing $u^\nu g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ as the projection of the covariant conservation law onto u^ν yields, if applied to the form of eqn. D.230 :

$$= g^{\alpha\mu}\nabla_\alpha\Big(\rho + \frac{p}{c^2}\Big)\cdot u_\mu u^\nu u_\nu + g^{\alpha\mu}\Big(\rho + \frac{p}{c^2}\Big)\cdot\nabla_\alpha u_\mu\cdot u^\nu u_\nu - \\ g^{\alpha\mu}\Big(\rho + \frac{p}{c^2}\Big)\cdot u_\mu u^\nu\,\nabla_\alpha u_\nu\; - g^{\alpha\mu}\nabla_\alpha p\cdot g_{\mu\nu}\,u^\nu \tag{D.231}$$

where we can carry out a number of simplifications: $u_\nu u^\nu = c^2$ in the first and second term. Then, $\nabla_\alpha(u_\nu u^\nu) = 0 = u_\nu\nabla_\alpha u^\nu + \nabla_\alpha u_\nu\cdot u^\nu = 2\cdot u_\nu\nabla_\alpha u^\nu$ implies that the third term vanishes, and finally $g^{\alpha\mu}\nabla_\alpha p\cdot g_{\mu\nu}u^\nu = g^{\alpha\mu}g_{\mu\nu}\nabla_\alpha p\,u^\nu = \delta^\alpha_\nu\nabla_\alpha p\cdot u^\nu = \nabla_\alpha p\cdot u^\alpha$.

Therefore, one arrives at

$$u^\nu g^{\alpha\mu}\nabla_\alpha \mathrm{T}_{\mu\nu} = g^{\alpha\mu}\,\nabla_\alpha\left(\rho + \frac{p}{c^2}\right)\cdot c^2\,u_\mu + g^{\alpha\mu}\left(\rho + \frac{p}{c^2}\right)\nabla_\alpha u_\mu\cdot c^2 - \nabla_\alpha\,p\cdot u^\alpha = 0 \quad \text{(D.232)}$$

and lastly

$$u^\nu g^{\alpha\mu}\nabla_\alpha \mathrm{T}_{\mu\nu} = g^{\alpha\mu}\left[\nabla_\alpha\left(\rho\,c^2\cdot u_\mu\right) + p\nabla_\alpha u_\mu\right] = 0 \quad \text{(D.233)}$$

which is exactly the relativistic continuity equation. The non-relativistic limit is recovered by setting $\rho\,c^2 \gg p$ as well as $u^\mu = (c, \upsilon^i)^t$ with $\gamma = 1$, and using Cartesian coordinates implies $g^{\alpha\mu} = \eta^{\alpha\mu}$ as well as $\nabla_\alpha = \partial_\alpha$:

$$g^{\alpha\mu}\left[\nabla_\alpha(\rho c^2\cdot u_\mu) + p\,\nabla_\alpha u_\mu\right] = \eta^{\alpha\mu}\left[\partial_\alpha\,\rho c^2\cdot u_\mu + \rho c^2\partial_\alpha u_\mu + p\,\partial_\alpha u_\mu\right] \simeq c^2\cdot\eta^{\alpha\mu}\,\partial_\alpha(\rho\,u_\mu) = 0 \quad \text{(D.234)}$$

where the last term in the brackets reads

$$\partial_t\,\rho + \partial_i(\rho\,\upsilon^i) = 0 \quad \text{(D.235)}$$

in the preferred coordinate frame, which is exactly the continuity equation from classical continuum mechanics: But unlike classical mechanics, where continuity is an empirical finding, it results in relativity from the covariant conservation of $\mathrm{T}_{\mu\nu}$.

We can resubstitute the conservation law eqn. D.233 into the divergence D.230 and see how we can isolate a statement about the conservation of momentum density. Again writing out $g^{\alpha\mu}\nabla_\alpha \mathrm{T}_{\mu\nu} = 0$ for the energy momentum tensor of an ideal fluid and writing out the expression fully gives:

$$g^{\alpha\mu}\nabla_\alpha \mathrm{T}_{\mu\nu} = g^{\alpha\mu}\Big[\nabla_\alpha(\rho u_\mu)u_\nu + \frac{p}{c^2}\nabla_\alpha u_\mu\cdot u_\nu + \nabla_\alpha p\cdot\frac{u_\mu u_\nu}{c^2} + \frac{p}{c^2}u_\mu\nabla_\alpha u_\nu + \rho\cdot u_\mu\nabla_\alpha u_\nu - \nabla_\alpha p\,g_{\mu\nu}\Big] \quad \text{(D.236)}$$

where the sum of the first two terms correspond exactly to the continuity equation D.233 (up to a pre-factor of c^2), and are therefore zero. Consequently,

$$g^{\alpha\mu}\nabla_\alpha \mathrm{T}_{\mu\nu} = g^{\alpha\mu}\Big[\Big(\frac{u_\mu u_\nu}{c^2} - g_{\mu\nu}\Big)\cdot\nabla_\alpha\,p + \left(\rho + \frac{p}{c^2}\right)\cdot u_\mu\nabla_\alpha u_\nu\Big] = 0 \quad \text{(D.237)}$$

In this way, one arrives at the relativistic Euler equation as an expression of momentum conservation:

$$g^{\alpha\mu}\Big[\Big(\frac{u_\mu u_\nu}{c^2} - g_{\mu\nu}\Big)\nabla_\alpha\,p + \left(\rho + \frac{p}{c^2}\right)u_\mu\nabla_\alpha\,u_\nu\Big] = 0 \quad \text{(D.238)}$$

First, we see that only pressure gradients perpendicular to the velocity are ever relevant,

$$g^{\alpha\mu}\left(\frac{u_\mu u_\nu}{c^2} - g_{\mu\nu}\right)\nabla_\alpha p = \nabla^\perp p \quad \text{(D.239)}$$

because one applies a projection operator on the gradient in pressure, projecting out the component of ∇_μ p perpendicular to u^μ, and secondly, if the motion of a fluid element proceeds along a geodesic with autoparallelity $u^\mu\nabla_\mu u_\nu = 0$ given,

$$\left(\rho + \frac{p}{c^2}\right)g^{\alpha\mu}u_\mu\nabla_\alpha u_\nu = \left(\rho + \frac{p}{c^2}\right)\,u^\mu\,\nabla_\mu u_\nu \quad \text{(D.240)}$$

that those pressure gradients must be zero! Pressure gradients would push a fluid element away from the geodesic that characterises free fall.

The nonrelativistic limit can be constructed by approximating the autoparallelity condition,

$$u^\mu \nabla_\mu u^\nu \simeq u^\mu \partial_\mu u^\nu = \mathrm{c}\partial_{ct} u^j + u^i \partial_i u^j \tag{D.241}$$

which shows that the nonlinearity of the Euler-equation has a relativistic origin, and furthermore for a flat background where $\nabla_\mu = \partial_\mu$ that

$$\rho\Big(\partial_t u^j + u^i \partial_i u^j\Big) = -\partial^j p \tag{D.242}$$

or equivalently, that

$$\partial_t u^j + u^i \partial_i u^j = -\frac{\partial^j p}{\rho} \tag{D.243}$$

which is the classical Euler-equation for ideal fluid mechanics. Allowing for weak, static Newtonian gravity one work with the approximation that pressure is scalar (actually it is only a partial trace of the energy momentum tensor!), so $\nabla_\alpha p = \partial_\alpha p$ and we obtain for the covariant derivative

$$u^\mu \, \nabla_\mu u_\nu = u^\mu\Big(\partial_\mu u_\nu - \Gamma^\alpha{}_{\mu\nu} \, u_\alpha\Big) \tag{D.244}$$

while Newtonian gravity is a weak and static perturbation to the line element,

$$\mathrm{d}s^2 = \Big(1 + \frac{2\Phi}{\mathrm{c}^2}\Big)\mathrm{c}^2\mathrm{d}t^2 - \Big(1 - \frac{2\Phi}{\mathrm{c}^2}\Big)\delta_{ij} \, \mathrm{d}x^i \mathrm{d}x^j \tag{D.245}$$

from which we isolate the two metric functions

$$g_{tt} = \Big(1 + \frac{2\Phi}{\mathrm{c}^2}\Big) \quad \text{and} \quad g_{ii} = -\Big(1 - \frac{2\Phi}{\mathrm{c}^2}\Big) \tag{D.246}$$

Working towards the nonrelativistic limit we would replace $g^{\alpha\beta} = \eta_{\alpha\beta}$ but keep the derivative $\partial_\mu g_{\alpha\beta}$ with the exception $\partial_{ct} g_{\mu\nu} = 0$ as Newtonian fields are necessarily static. The derivatives of the metric then reflect potential gradients, $\partial_i \, g_{\mu\nu} = \pm\frac{2}{\mathrm{c}^2} \, \partial_i \Phi \delta_{\mu\nu}$ which become the Christoffel-symbol $\Gamma^i{}_{tt} \sim +\partial^j \Phi$. So ultimately, we arrive at the Euler-equation of classical ideal fluid mechanics including a gravitational potential Φ,

$$\partial_t u^j + (u^i \partial_i) u^j = -\frac{\partial^j p}{\rho} - \partial^j \Phi, \tag{D.247}$$

from the covariant divergence $g^{\alpha\mu} \nabla_\alpha \mathrm{T}_{\mu\nu} = 0$. Alternative to resubstituting we can take the vector $g^{\alpha\mu} \nabla_\alpha \mathrm{T}_{\mu\nu} = 0$ and project it straight away onto a plane perpendicular to u^ν, by means of a projection operator $\mathrm{P}_\perp^{\nu\rho} = u^\nu u^\rho / c^2 - g^{\nu\rho}$, to arrive at the Euler-equation.

D.3 *Fields as sources of gravity*

Relativistic gravity should be compatible with relativistic fields as well as fluids, similarly to electrodynamics which is equally valid for a classical charge density as a source or a charge density that is computed from the probability determined by the wave functions of the particles according to the Born-postulate: This is made sure by the fact that fields can be assigned an energy-momentum tensor as an expression of local energy density, momentum density and stress, which obeys automatically relativistic conservation laws as soon as the Lagrange-density $\mathcal{L}$ of the fields does not explicitly depend on the coordinate, meaning that the working principle of the fields should be identical everywhere and at every time.

A scalar field ϕ on an arbitrary, possibly curved spacetime with metric $g_{\mu\nu}$ for instance would be described by the Lagrange-function

$$\mathcal{L} = \mathcal{L}(\phi, \nabla_\alpha\phi, g_{\mu\nu}) \tag{D.248}$$

if its dynamics is universal, so that $\mathcal{L}$ depends on the field ϕ and its derivative $\nabla_\alpha\phi$ (which would of course be $= \partial_\alpha\phi$ as ϕ is scalar, but let's use the covariant formalism), but not explicitly on the coordinates x^μ. The action integral would read

$$S = \int d^4x\, \sqrt{-\det g}\, \mathcal{L} \tag{D.249}$$

where the additional factor $\sqrt{-\det g}$ makes sure that the volume element is invariant under coordinate transforms (we come to this in the next chapter). The field equation follows from variation according to Hamilton's principle $\delta S = 0$. Specifically,

$$\delta S = \int d^4x\, \sqrt{-\det g}\left(\frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\,\delta\nabla_\alpha\phi\right) \tag{D.250}$$

Using the interchangeability $\delta\nabla_\alpha\phi = \nabla_\alpha\delta\phi$ and integration by parts while keeping the variation on the boundary fixed gives

$$\delta S = \int d^4x\, \sqrt{-\det g}\left(\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_\alpha\,\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\right) = 0 \tag{D.251}$$

from which we extract the Euler-Lagrange equation, now in a covariant formulation ready to work on a curved background,

$$\nabla_\alpha\,\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi} = \frac{\partial\mathcal{L}}{\partial\phi} \tag{D.252}$$

Next, it'd be great if an expression for the energy momentum tensor $T_{\mu\nu}$ would directly follow from the coordinate independent Lagrange-function $\mathcal{L}$, possibly along with a covariant conservation law in the form $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$. In fact, if $\mathcal{L}(\phi, \nabla_\alpha\phi)$ does not depend on position the variation $\delta\mathcal{L}$ is given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\,\delta\nabla_\alpha\phi = \nabla_\alpha\left[\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\,\delta\phi\right] + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \nabla_\alpha\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\cdot\delta\phi \tag{D.253}$$

where in the last step the Leibnitz-rule was used to introduce the derivative of the product $\partial\mathcal{L}/\partial\nabla_\alpha\phi\,\delta\phi$, which suggests that the Euler-Lagrange-equation should be substituted,

$$\delta\mathcal{L} = \nabla_\alpha\left[\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\delta\phi\right] + \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_\alpha\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\right)}_{=0}\delta\phi \quad\rightarrow\quad \delta\mathcal{L} = \nabla_\alpha\left[\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\delta\phi\right] \tag{D.254}$$

Next, we need to write the variation in $\mathcal{L}$ from an infinitesimal translation of the field $\delta\phi$ (because the Lagrange-density does not change itself as a function of coordinate, it can only change if the fields themselves are different!), i.e. to think of a way of actually generating the variation from an infinitesimal shift in the coordinates:

$$\phi\left(x^\mu + \delta x^\mu\right) = \phi\left(x^\mu\right) + \nabla_\nu\phi\left(x^\mu\right)\cdot\delta x^\nu + \ldots \tag{D.255}$$

again using covariant derivatives for generality. Then, the field variation $\delta\phi$ is given by

$$\delta\phi = \phi(x^\mu + \delta x^\mu) - \phi(x^\mu) = \nabla_\nu\phi\cdot\delta x^\nu = g^{\mu\nu}\,\nabla_\mu\phi\,\delta x_\nu \tag{D.256}$$

On the other hand, shifting the Lagrange function $\mathcal{L}$ by an amount δx_β is easily achieved by the displacement defined through the covariant derivative, $\delta x^\beta\nabla_\beta = g^{\alpha\beta}\delta x_\beta\nabla_\alpha$:

$$\delta\mathcal{L} = g^{\alpha\beta}\,\nabla_\alpha\mathcal{L}\cdot\delta x_\beta. \tag{D.257}$$

Combining both yields

$$\delta\mathcal{L} = g^{\alpha\beta}\,\nabla_\alpha\mathcal{L}\cdot\delta x_\beta = \nabla_\alpha\left[\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\cdot g^{\mu\nu}\,\nabla_\mu\phi\,\delta x_\nu\right] \tag{D.258}$$

As the same covariant derivative ∇_α acts on both terms, they can be combined to give

$$\nabla_\alpha\left[\mathcal{L}\cdot\delta x^\alpha - \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}g^{\mu\nu}\,\nabla_\mu\phi\,\delta x^\nu\right] = 0 \tag{D.259}$$

This equation would be perfect if it was independent of the shift δx, but it appears with different indices in the two terms. A possible remedy is a renaming $\delta x^\alpha = g^{\alpha\mu}\,g_{\mu\nu}\,\delta x^\nu = \delta^\alpha_\nu\,\delta x^\nu$, so that the formula becomes

$$\nabla_\alpha\left[g^{\alpha\mu}\,g_{\mu\nu}\,\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\nabla_\nu\phi\right]\delta x^\nu = g^{\alpha\mu}\,\nabla_\alpha\left[\mathcal{L}\,g_{\mu\nu} - \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\nabla_\nu\phi\right]\delta x^\nu = 0 \tag{D.260}$$

where we can identify the energy momentum tensor as computed for the field ϕ from its Lagrange-function $\mathcal{L}(\phi, \nabla_\alpha\phi, g_{\mu\nu})$,

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\nabla_\nu\phi - \mathcal{L}g_{\mu\nu} \tag{D.261}$$

including the conservation law $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ in a covariant formulation. The idea, that the energy-momentum tensor $T_{\mu\nu}$ mediates between the field and the gravitational field equation is very interesting: As soon as the dynamics of the fields are universal, $T_{\mu\nu}$ is defined , covariantly conserved, and computable from $\mathcal{L}$, irrespective of the actual substance. In this sense, general relativity is the gravitational theory of systems with conserved energy and momentum in the same way as Maxwell-electrodynamics is the electromagnetic theory for systems with conserved charges.