C DIFFERENTIAL STRUCTURE OF SPACETIME AND CURVATURE

C.1 *Riemann curvature tensor*

The connection, which establishes parallel transport of vectors and tensors across a manifold, defines the covariant derivative of these quantities because a proper rate of change can be measured through the comparison of e.g. a vector with the parallel transported counterpart. The Levi-Civita connection is singled out among all possible connections as the (i) metric compatible $\nabla_\alpha g_{\mu\nu} = 0$ and (ii) torsion-free $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}$ one, in which case the connection coefficients can be computed from the metric $g_{\mu\nu}$ and its first derivatives $\partial_\alpha g_{\mu\nu}$ alone. The metric structure of a manifold $g_{\mu\nu}$, with an additional differential structure ∇_α, defines the Riemann-geometry.

All these ideas and concepts are independent from actual curvature and are rather an expression of the choice of coordinates as they only use the metric and its first derivatives, for which there is always a coordinate transform to make them vanish locally, and because only second derivatives would contain information about curvature, we should use them to quantify it. Additionally, we would like to have a covariant quantification of curvature in the form of a tensor: the Riemann curvature. Only if the Riemann-curvature is nonzero, $R_{\alpha\beta\mu\nu} = 0$ as a properly covariant expression, the manifold is flat. None of the statements $g_{\mu\nu} = \eta_{\mu\nu}$, $\nabla_\alpha = \partial_\alpha$, or $\Gamma^\alpha{}_{\mu\nu} = 0$ are able to make a statement about curvature.

C.1.1 Riemann curvature in parallel transport

The order of parallel transport of vectors and tensors matters in shifts along different directions. Starting with the expression for parallel transport by $\delta\bar{x}^\beta$,

$$\upsilon^\mu(x + \delta\overline{x}) = \upsilon^\mu(x) - \Gamma^\mu{}_{\alpha\beta} \cdot \upsilon^\alpha(x)\, \delta\overline{x}^\beta \tag{C.192}$$

we can define two paths: first a shift by $\delta\bar{x}$ followed by a shift by δx,

$$\upsilon^\mu(x + \delta\overline{x}) + \delta x) = \upsilon^\mu(x + \delta\overline{x}) - \Gamma^\mu{}_{\alpha\beta}(x + \delta\overline{x}) \cdot \upsilon^\alpha(x + \delta\overline{x}) \cdot \delta x^\beta \tag{C.193}$$

which evaluates to

$$= \upsilon^\mu(x) - \Gamma^\mu{}_{\alpha\beta}(x) \cdot \upsilon^\alpha(x)\delta\overline{x}^\beta - \left[\Gamma^\mu{}_{\alpha\beta} + \frac{\partial \Gamma^\mu{}_{\alpha\beta}}{\partial x^\gamma} \cdot \delta\overline{x}^\gamma\right] \cdot \left[\upsilon^\alpha(x) - \Gamma^\alpha{}_{\gamma\delta}(x)\upsilon^\gamma(x)\delta\overline{x}^\delta\right] \cdot \delta x^\beta \tag{C.194}$$

with $\Gamma^\mu{}_{\alpha\beta}(x+\delta\bar{x}) = \Gamma^\mu{}_{\alpha\beta}(x) + \frac{\partial\Gamma^\mu{}_{\alpha\beta}}{\partial x^\gamma}(x) \cdot \delta\bar{x}^\gamma$ being the Taylor-expansion of the Christoffel-symbol at $x + \delta\bar{x}$. Alternatively, the two shifts can be interchanged, for a parallel transport first by δx and then by $\delta\bar{x}$.

$$\upsilon^\mu((x + \delta x) + \delta\overline{x}) = \upsilon^\mu(x + \delta x) - \Gamma^\mu{}_{\alpha\beta}(x + \delta x) \cdot \upsilon^\alpha(x + \delta x) \cdot \delta\overline{x}^\beta \tag{C.195}$$

yielding

$$\upsilon^\mu(x) - \Gamma^\mu{}_{\alpha\beta}(x) \cdot \upsilon^\alpha(x)\delta x^\beta - \left[\Gamma^\mu{}_{\alpha\beta}(x) + \frac{\partial \Gamma^\mu{}_{\alpha\beta}}{\partial x^\gamma}\delta x^\gamma\right]\left[\upsilon^\alpha(x) - \Gamma^\alpha{}_{\gamma\delta}(x)\upsilon^\gamma(x)\delta x^\delta\right] \cdot \delta\overline{x}^\beta \tag{C.196}$$

with an equivalent Taylor-expansion. Then, the change $\delta\upsilon^\mu$ in parallel transport to the point $x + \delta x + \delta\bar{x}$ along two different paths is given by

$$\delta\upsilon^\mu = \upsilon^\mu((x + \delta\bar{x}) + \delta x) - \upsilon^\mu((x + \delta x) + \delta\bar{x}) = R^\mu{}_{\alpha\beta\gamma} \cdot \upsilon^\alpha\, \delta x^\beta\, \delta\bar{x}^\gamma \tag{C.197}$$

where we can isolate the Riemann-curvature,

$$R^\mu{}_{\alpha\beta\gamma} = \frac{\partial}{\partial x^\beta}\Gamma^\mu{}_{\alpha\gamma} - \frac{\partial}{\partial x^\gamma}\Gamma^\mu{}_{\alpha\beta} + \Gamma^\mu{}_{\delta\beta}\,\Gamma^\delta{}_{\alpha\gamma} - \Gamma^\mu{}_{\delta\gamma}\,\Gamma^\delta{}_{\alpha\beta}, \tag{C.198}$$

after renaming $\gamma \leftrightarrow \beta$ in the second expression to have $\delta x^\beta\, \delta\bar{x}^\gamma$). Flat manifolds with vanishing Riemann curvature $R^\mu{}_{\alpha\beta\gamma} = 0$ would necessarily exhibit no change at all of the transported vector, i.e. $\delta\upsilon^\mu = 0$.

💀 *In a flat manifold the Riemann-tensor is zero in* **every** *coordinate choice.*

Of course, the contravariant index ν can be lowered with a contraction,

$$R_{\mu\alpha\beta\gamma} = g_{\mu\nu} R^\nu{}_{\alpha\beta\gamma\delta}. \tag{C.199}$$

And it is important to memorise the antisymmetry of the Riemann tensor in every index pair,

$$R_{\mu\alpha\beta\gamma} = -R_{\alpha\mu\beta\gamma} = -R_{\mu\alpha\gamma\beta} = +R_{\alpha\mu\gamma\beta} \tag{C.200}$$

as well as the algebraic Bianchi-identity,

$$R_{\mu\alpha\beta\gamma} + R_{\mu\beta\gamma\alpha} + R_{\mu\gamma\alpha\beta} = 0 \tag{C.201}$$

with cyclic index permutation of the last three indices while keeping the first index fixed.

C.1.2 Riemann-curvature from covariant derivatives

Covariant derivatives (into different direction) in contrast to partial derivatives, do not commute.

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\upsilon^\alpha = \left[\nabla_\mu, \nabla_\nu\right]\upsilon^\alpha = -R^\alpha{}_{\beta\mu\nu}\upsilon^\beta, \tag{C.202}$$

and the commutator defines, as before, the Riemann curvature $R^\alpha{}_{\beta\mu\nu}$. Concerning the index structure, it is best to remember that for every choice of μ and ν there is an transformation in α and β acting on the vector υ^β. As vectors are rotated in parallel transport with a Levi-Civita connection, α and β are an antisymmetric index pair because they effectively encode a rotation matrix. μ and ν are likewise an antisymmetric index pair, due to the commutator in the definition of the Riemann curvature, $[\nabla_\mu, \nabla_\nu] = -[\nabla_\nu, \nabla_\mu]$.

Acting on a vector υ^μ with covariant differentiation ∇_β yields

$$\nabla_\beta\upsilon^\mu = \partial_\beta\upsilon^\mu + \Gamma^\mu{}_{\beta\delta}\,\upsilon^\delta = t_\beta{}^\mu \tag{C.203}$$

with a tensor $t_\beta{}^\mu$ as a result. In further covariant differentiation ∇_γ one needs to watch out for co- and contravariant indices, with different signs in their respective Christoffel-symbols:

$$\nabla_\gamma\, t_\beta{}^\mu = \partial_\gamma\, t_\beta{}^\mu - \Gamma^\alpha{}_{\gamma\beta}\, t_\alpha{}^\mu + \Gamma^\mu{}_{\gamma\alpha}\, t_\beta{}^\alpha \tag{C.204}$$

Substituting eqn. C.203 into eqn. C.204 gives:

$$\nabla_\gamma(\nabla_\beta \upsilon^\mu) = \partial_\gamma\partial_\beta \upsilon^\mu - \partial_\gamma \Gamma^\mu{}_{\beta\delta} \cdot \upsilon^\delta - \Gamma^\mu{}_{\beta\delta}\, \partial_\gamma\, \upsilon^\delta - \Gamma^\alpha{}_{\gamma\beta} \cdot \left[\partial_\alpha \upsilon^\mu + \Gamma^\mu{}_{\alpha\delta}\, \upsilon^\delta\right] \tag{C.205}$$

If one interchanges the order of differentiation and builds the antisymmetric combination $\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu$ one can isolate the Riemann tensor,

$$R^\mu{}_{\alpha\beta\gamma} = \frac{\partial}{\partial x^\beta}\Gamma^\mu{}_{\alpha\gamma} - \frac{\partial}{\partial x^\gamma}\Gamma^\mu{}_{\alpha\beta} + \Gamma^\mu{}_{\delta\beta}\,\Gamma^\delta{}_{\alpha\gamma} - \Gamma^\mu{}_{\delta\gamma}\,\Gamma^\delta{}_{\alpha\beta}. \tag{C.206}$$

as the partial derivatives of υ^μ drop out, according to $\partial_\gamma\partial_\beta \upsilon^\mu = \partial_\beta\partial_\gamma \upsilon^\mu$.

The two approaches are related to each other as parallel transport of a vector υ_α is performed using the covariant derivative as an operator, $\delta x^\beta\, \nabla_\beta$. One can think about extending this infinitesimal parallel transport to parallel transport operator for finite distances by exponentiation. Then, parallel transport with a shift operator

$$\exp(\delta x^\beta\, \nabla_\beta)\upsilon^\mu = \upsilon^\mu(x + \delta x) \tag{C.207}$$

produces a shifted vector, and shifts would follow the Baker-Hausdorff-Campbell formula,

$$\exp(\delta x^\beta\, \nabla_\beta)\exp(\delta x^\gamma\, \nabla_\gamma) \simeq \exp(\delta x^\beta\, \nabla_\beta + \delta \overline{x}^\gamma\, \nabla_\gamma)\exp\left(-\frac{1}{2}\cdot \delta x^\beta\, \delta \overline{x}^\gamma\, [\nabla_\beta, \nabla_\gamma]\right) \tag{C.208}$$

where translations into different directions would be sensitive to the presence of curvature in the case $[\nabla_\beta\,, \nabla_\gamma] \sim R^\mu{}_{\alpha\beta\gamma} \neq 0$.

Tensors that are derived from the Riemann-curvature by contraction with the metric include the Ricci-curvature $R_{\mu\beta}$

$$R_{\mu\beta} = g^{\alpha\nu}\, R_{\alpha\mu\nu\beta}, \tag{C.209}$$

where the contraction over the first and third index is the only sensible one, given the antisymmetry of the Riemann-tensor in the first and last index pair. Further contraction yields the Ricci-scalar R

$$R = g^{\mu\beta}\, R_{\mu\beta} = g^{\mu\beta}\, g^{\alpha\nu}\, R_{\alpha\mu\nu\beta} \tag{C.210}$$

which is a quantification of the (local) curvature, similarly to the Kretschmann-scalar K,

$$K = R^{\alpha\mu\nu\beta}\, R_{\alpha\mu\nu\beta} = g^{\alpha\gamma}\, g^{\mu\rho}\, g^{\nu\sigma}\, g^{\beta\delta}\, R_{\alpha\mu\nu\beta}\, R_{\gamma\rho\sigma\delta} \tag{C.211}$$

Both curvature scalars are independent from the coordinate choice and are a convenient quantification of curvature.

The Ricci-tensor and the Ricci-scalar define the Einstein-tensor $G_{\mu\nu}$,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \tag{C.212}$$

which is surprisingly the only rank-2 tensor with vanishing covariant divergence, $g^{\alpha\mu}\nabla_\alpha G_{\mu\nu} = 0$ (the other one being the metric itself, $g^{\alpha\mu}\nabla_\alpha g_{\mu\nu} = 0$, due to metric compatibility), as will become relevant in the next chapter.

C.1.3 What happens to vectors in parallel transport?

Levi-Civita connections are constructed to be metric-compatible which will imply that vectors, if transported around a closed loop, will conserve their norm. Then, the only way in which they can be affected in by curvature is a rotation: One can in fact make that determination because the transported vector is brought back into the original tangent space if the connection is torsion-free.

We can compute explicitly that the norm of a vector $\boldsymbol{\upsilon}$ does not change, expressing parallel transport by δx^μ with the covariant derivative $\delta x^\mu \nabla_\mu$ as an operator acting on a geometric object like a vector or a scalar product. Bringing in the commutator of ∇_μ is a convenient way of interchanging the order of parallel transport from the starting point to the destination and to subtract the two results from each other: If the norm is conserved, the result should be zero.

$$g_{\alpha\beta}\upsilon^\alpha\upsilon^\beta \rightarrow \delta x^\mu \delta\bar{x}^\nu [\nabla_\mu, \nabla_\nu] \left(g_{\alpha\beta}\, \upsilon^\alpha\upsilon^\beta\right) = \delta x^\mu \delta\bar{x}^\nu \left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right)\left(g_{\alpha\beta}\, \upsilon^\alpha\upsilon^\beta\right) \quad \text{(C.213)}$$

Metric compatibility ensures that $\nabla g = 0$, so we obtain, dropping the common prefactor $\delta x^\mu \delta\bar{x}^\nu$,

$$= g_{\alpha\beta}\, \nabla_\mu\nabla_\nu \left(\upsilon^\alpha\upsilon^\beta\right) - g_{\alpha\beta}\, \nabla_\nu\nabla_\mu \left(\upsilon^\alpha\upsilon^\beta\right) \quad \text{(C.214)}$$

Expanding the expression with the Leibnitz-rule yields

$$= g_{\alpha\beta} \left(\nabla_\mu\nabla_\nu\, \upsilon^\alpha \cdot \upsilon^\beta + \nabla_\nu\, \upsilon^\alpha \cdot \nabla_\mu\, \upsilon^\beta + \nabla_\mu\, \upsilon^\alpha \cdot \nabla_\nu\, \upsilon^\beta + \upsilon^\alpha\, \nabla_\mu\nabla_\nu\, \upsilon^\beta\right) -$$
$$g_{\alpha\beta} \left(\nabla_\nu\nabla_\mu\, \upsilon^\alpha \cdot \upsilon^\beta + \nabla_\mu\, \upsilon^\alpha\, \nabla_\nu\, \upsilon^\beta + \nabla_\nu\, \upsilon^\alpha\, \nabla_\mu\, \upsilon^\beta + \upsilon^\alpha\, \nabla_\nu\nabla_\mu\, \upsilon^\beta\right) \quad \text{(C.215)}$$

and reordering the terms

$$= g_{\alpha\beta} \left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right) \upsilon^\alpha \cdot \upsilon^\beta + g_{\alpha\beta}\, \upsilon^\alpha \left(\nabla_\mu\nabla_n u - \nabla_\nu\nabla_\mu\right) \upsilon^\beta \quad \text{(C.216)}$$

Finally, identifying the Riemann curvature and renaming the indices in the second term gives:

$$= g_{\alpha\beta}\, R^\alpha{}_{\gamma\mu\nu}\, \upsilon^\gamma\upsilon^\beta + g_{\alpha\beta}\, \upsilon^\alpha\, R^\beta{}_{\gamma\mu\nu}\, \upsilon^\gamma = 2\, R_{\alpha\gamma\mu\nu}\, \upsilon^\alpha\upsilon^\gamma = 0, \quad \text{(C.217)}$$

which is zero as a consequence of the antisymmetry of the Riemann-tensor in the first index pair: The norm of υ^α is conserved.

In exactly the same way one can show that the scalar product $g_{\alpha\beta}\upsilon^\alpha w^\beta$ between two vectors υ^α and w^β is conserved. Indeed, repeating the entire calculation shows that

$$\left[\nabla_\mu\, , \nabla_\nu\right]\left(g_{\alpha\beta}\, \upsilon^\alpha w^\beta\right) = \left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right)\left(g_{\alpha\beta}\, \upsilon^\alpha w^\beta\right) = \ldots = R_{\alpha\gamma\mu\nu} \left(\upsilon^\alpha w^\gamma + \upsilon^\gamma w^\alpha\right) = 0, \quad \text{(C.218)}$$

keeping in mind that the tensor $\left(\upsilon^\alpha w^\gamma + \upsilon^\gamma w^\alpha\right)$ is perfectly symmetric.

With these results, we can revisit the defining equation of Riemann-curvature:

$$[\nabla_\mu, \nabla_\nu]v^\alpha = R^\alpha{}_{\beta\mu\nu}v^\beta \tag{C.219}$$

where the antisymmetry in the $\mu\nu$-index pair is obvious because of the commutator, $[\nabla_\mu, \nabla_\nu] = -[\nabla_\nu, \nabla_\mu]$. If a vector v^α is transported in a loop and compared to the original vector, it can not have changed its norm because of metric compatibility, and it exists (if the manifold is torsion-free) at the same point and can be decomposed in terms of the basis of the same tangent space. The only possible difference between the vectors is a rotation, and this is exactly the meaning of the Riemann-tensor (and which gives you a great way to memorise its antisymmetry in the index pair $\alpha\beta$): It is essentially a rotation matrix in $\alpha\beta$ for every $\mu\nu$-pair.