B TOPOLOGICAL AND METRIC STRUCTURE OF SPACETIME

General relativity requires that the idea of a vector-space (with Lorentzian geometry) as a model for spacetime is given up. From the example of the perihelion precession of the planet Mercury we saw that the gravitational field around massive objects like the Sun is stronger compared to the prediction of a Newtonian theory: This is surprising, because the $1/r$-form of the potential is a direct consequence of the fact that surfaces of spheres scale $\propto r^2$, so typical for a Euclidean vector-space. The new model for spacetime that was pioneered by Albert Einstein and by David Hilbert was that of a manifold: A topological space with a metric and a differential structure, and ultimately, curvature as an expression of the gravitational field. The decisive property of curved manifolds is a locally defined, varying geometry, encapsulated by the metric, which becomes dependent on the coordinates.

The topological structure explains the connectivity of sets of spacetime points and introduces open sets, which are used to construct continuous mappings of the spacetime points onto their coordinates. Changes from one coordinate choice to another need to be invertible and differentiable (which is called a diffeomorphism). Adding a metric structure to the manifold allows the measurement of norms of vectors and the angle between them, and the construction of invariants. Finally, the construction of parallel transport and that of a covariant derivative allows statements about variations of vector- and tensor-fields defined on the manifold. We need to make sure that all these structures are compatible with each other.

B.1 *Metric structure of manifolds and coordinate transforms*

We have already encountered weak perturbations to the Minkowski-metric $\eta_{\mu\nu}$ mediated by the gravitational field in the limit of weak fields $|\Phi| \ll c^2$ (which is only valid in a particular coordinate choice!). A general metric tensor $g_{\mu\nu}$ defines an infinitesimal contribution $\mathrm{d}s^2$ to the line element,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{B.111}$$

between two points that have an infinitesimal coordinate difference $\mathrm{d}x^\mu$. With this definition, the metric tensor is symmetric as $\mathrm{d}s^2$ would not pick up any antisymmetric contribution in the contraction with $\mathrm{d}x^\mu \mathrm{d}x^\nu$.

The line element $\mathrm{d}s^2$ is scalar, under coordinate transformations we should obtain:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu} \cdot \frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}\mathrm{d}x'^\rho \mathrm{d}x'^\sigma \tag{B.112}$$

isolating the transformation rule for the metric to be

$$g_{\mu\nu} \cdot \frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} \equiv g'_{\rho\sigma}, \tag{B.113}$$

and is naturally inverse to that of vectors like $\mathrm{d}x^\mu$

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}x^\nu, \tag{B.114}$$

making sure that the coordinate transformation by the Jacobian and its inverse cancel each other,

$$\frac{\partial x^\mu}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\rho} = \frac{\partial x^\mu}{\partial x^\rho} = \delta^\mu_\rho \tag{B.115}$$

The interpretation of $\mathrm{d}s^2$ as the arc-length of a trajectory through spacetime is still that of proper time, $\mathrm{d}s^2 = \mathrm{c}^2\mathrm{d}\tau^2$, measured on a clock comoving with a massive particle. We will show that photons would follow null-lines, $\mathrm{d}s^2 = 0$, so that the definitions of light-cones and their associated causal structure is valid on metric manifolds in exactly the same way.

A metric defines a geometry by defining distances and angles: It is a mapping of a pair of vectors x, y onto a positive number obeying the three metric axioms:

1. $g(x,y) \geq 0$, if $g(x,y) = 0 \leftrightarrow x = y$ positive definiteness
   Because from a physical motivation, the classification of vectors into timelike, spacelike and lightlike is incredibly important, we will soften this axiom and allow negative values for $\mathrm{d}s^2 = g(\mathrm{d}x, \mathrm{d}x) = g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu$: This, ultimately, defines a pseudo-Riemannian geometry.

In relativity we're dealing with pseudo-metrics: $\mathrm{d}s^2$ is negative for space-like vectors, but those lie outside the light cone and do not correspond to causal processes.

2. $g(x,y) = g(y,x)$ symmetry
   This axiom is fulfilled by $g_{\mu\nu}$ being a symmetric, real valued tensor defining a quadratic form

3. $g(x,y) + g(y,z) \geq g(x,z)$ triangle inequality
   Again, there might be physical situations, where a "detour" is shorter than the direct path, and the classical example for this is the twin paradoxon: Lightlike vectors have smaller norms than timelike vectors.

You would not believe how much I'd like at this point to go off on a tangent about the necessity of a metric structure and the possibility of having geometries that are defined in different ways, for instance avoiding scalar products. Instead, I would just like to emphasise that the only metric geometry allowing for hyperbolic evolution of the field equations along an invariant light cone is the Lorentzian one.

It is important to clarify the relation between an arbitrary geometry $g_{\mu\nu}$ and the Lorentz-geometry $\eta_{\mu\nu}$: If one zooms in onto a single point of spacetime, it should have a locally Minkowskian shape and allow for the local choice of Cartesian coordinates (called normal coordinates in this context). Clearly, with a coordinate transform one can transform the metric

$$g'_{\rho\sigma}(x) = g_{\mu\nu}(x)\cdot\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} \tag{B.116}$$

at one point in such a way that it becomes diagonal with eigenvalues λ_μ, because it is symmetric. A rescaling of the coordinates $x^\mu \rightarrow x^\mu\sqrt{\lambda_\mu}$ would then make $g_{\mu\nu}$ identical to $\eta_{\mu\nu}$.

But it should not be possible to bring the entire manifold to a Lorentzian shape and to choose globally Cartesian coordinates: To show this, we need to overcome the idea that an arbitrary coordinate transform would be able to define just the right transform to ensure $g_{\mu\nu} = \eta_{\mu\nu}$ at every point.

Let's consider a general coordinate transform $x^\mu(x'^\nu)$ at a point P:

$$x^\mu(x') = x^\mu\big|_P + \frac{\partial x^\mu}{\partial x'^\nu}\Big|_P\left(x'^\nu - x'^\nu_P\right) \tag{B.117}$$

$$+ \frac{1}{2}\frac{\partial^2 x^\mu}{\partial x'^\nu \partial x^\rho}\Big|_P \cdot \left(x'^\nu \cdot x'^\nu_P\right)\left(x'^\rho - x'^\rho_P\right) \tag{B.118}$$

$$+ \frac{1}{3!}\frac{\partial^3 x^\mu}{\partial x'^\nu \partial x'^\rho \partial x'^\sigma}\Big|_P\left(x'^\nu - x'^\nu_P\right)\left(x'^\rho - x'^\rho_P\right)\left(x'^\sigma - x'^\sigma_P\right) + \cdots \tag{B.119}$$

and count the number of degrees of freedom that is provided at every order of the Taylor-expansion and see if they suffice to have $g = \eta$ and to make all derivatives of g appear at arbitrary order. If that would be the case, a coordinate transform could be found that diagonalises the metric at every point and makes it globally Minkowskian, across the entire manifold.

1. At lowest order, there are are more degrees of freedom provided by the coordinate transform to diagonalise the metric $g_{\mu\nu}$ and have unit diagonal entries: We can adjust the coordinate transform to make $g_{\mu\nu} = \eta_{\mu\nu}$ at the point P, because counting the degrees of freedom yields

$$\frac{\partial x^\mu}{\partial x'^\nu} \sim n^2 \tag{B.120}$$

because there are n choices for x and n independent choices for x'

$$g'_{\mu\nu} \sim \frac{n(n+1)}{2} \tag{B.121}$$

because the metric is a symmetric, real-valued $n \times n$ matrix. The counting shows that $n^2 > n(n+1)/2$ for every number of dimensions n, so there are enough degrees of freedom to adjust $g_{\mu\nu} = \eta_{\mu\nu}$ locally at P.

2. At second order, the number of degrees of freedom provided by the coordinate transform is exactly that needed to make the first derivatives of the metric vanish at P.

$$\frac{\partial^2 x^\mu}{\partial x'^\nu \partial x'^\rho} \sim \frac{n^2(n+1)}{2} \tag{B.122}$$

because the differentiations should not be counted twice for $\nu = \rho$, and

$$\frac{\partial g'_{\mu\nu}}{\partial x'^\rho} \sim \frac{n^2(n+1)}{2} \tag{B.123}$$

because there are n possible differentiations of a symmetric matrix. Surprisingly, the degrees of freedom provided by the coordinate transform suffice exactly to have the derivatives $\partial_\rho g_{\mu\nu}$ disappear locally at P.

3. At third order, the number of degrees of freedom provided by the coordinate transform falls short of the number needed to make the second derivatives of the metric at the point P disappear.

$$\frac{\partial^3 x^\mu}{\partial x'^\nu \partial x'^\rho \partial x'^\sigma} \sim n \cdot \frac{n(n+1)(n+2)}{6} \tag{B.124}$$

because all derivatives must be different, while

$$\frac{\partial g'_{\mu\nu}}{\partial x'^{\rho}\partial x'^{\sigma}} \sim \frac{n(n+1)}{2} \cdot \frac{n(n+1)}{2} \tag{B.125}$$

because both the metric and the double partial are symmetric. As $n^2(n+1)(n+2)/6 > n^2(n+1)^2/4$, the second derivatives of the metric can not be made to vanish at P in the general case.

Continuing this line of reasoning shows that the problem exacerbates: The numbers of degrees of freedom provided by the coordinate transforms always falls short of the degrees of freedom needed to make higher order derivatives of the metric vanish. From that we conclude that there can only be two cases: Either the manifold is already Lorentzian but with an unfortunate coordinate choice, in which case there is a global construction of normal coordinates, or the manifold has new properties expressed by the non-vanishing second derivatives of the metric: This is in fact the curvature, as a new intrinsic property of the manifold that exists in any coordinate choice. But even if that is the case, our argument shows that the spacetime structure is locally Lorentzian with a Minkowski-metric.

B.2 *Locally Minkowskian structure and the equivalence principle*

While this argument is elegant, we might ask if the coordinate choice that achieves a locally flat structure has a particular physical meaning: This is in fact the case, as an expression of the equivalence principle which stipulates that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\rho g_{\mu\nu} = 0$ in a freely falling frame of reference. In such a freely falling frame, one recovers (locally!) perfectly Lorentzian geometries and the laws of special relativity are valid, for instance Maxwell's equations as defined on a flat, Minkowskian spacetime. The "size" r of the freely falling laboratory in which special relativity applies at least approximatively is given by the requirement that curvature effects associated with the second derivatives of the metric can not be dominant:

$$\frac{1}{r^2} = \left|\frac{\partial^2 g}{\partial x^2}\right| \longrightarrow r = \left|\frac{\partial^2 g}{\partial x^2}\right|^{-\frac{1}{2}} \tag{B.126}$$

And we will see in a second that the Christoffel-symbols $\Gamma^{\alpha}{}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left[\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right]$ will be zero, due to their proportionality to ∂g, and that the covariant derivative $\nabla_\mu \upsilon^\alpha = \partial_\mu \upsilon^\alpha + \Gamma^{\alpha}{}_{\mu\beta}\,\upsilon^\beta$ reverts back to the partial derivative $\partial_\mu \upsilon^\alpha$.

B.3 *Vectors and fields on manifolds*

Let us start with the picture that a manifold as the continuum of spacetime points has been given coordinates by a suitable mapping, so every point P has coordinates, x^μ. Changing from one coordinate set x^μ to a new set x'^ν should be done in an invertible, differentiable way. The manifold itself is not a vector space, but we can define abstract fields on the manifold: If they have internal degrees of freedom, their components can be expressed in the local set of basis vectors spanning the tangent space (or cotangent space, if their degrees of freedom rather correspond to linear forms instead of vectors).

One of the easiest geometric objects we can define is a curve $C(\lambda) = x^\mu(\lambda)$ visiting the spacetime points x^μ as the (possibly affine) parameter λ evolves. If there is a scalar field $\phi(x^\mu)$ defined on the manifold, the rate at which the field amplitude changes along the curve $x^\mu\lambda$ would be given by

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\phi(x^\mu(\lambda)) = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\partial\phi}{\partial x^\mu} = u^\mu\frac{\partial\phi}{\partial x^\mu} \tag{B.127}$$

and we recognise in the last term the scalar multiplication of the field gradient $\partial\Phi/\partial x^\mu$ into the tangent vector $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$. In this sense, one can think of the tangent u^μ and of $\mathrm{d}x^\mu$ as vectors. In transforming from on set of coordinates to another set shows that the vector u^μ and the linear forms $\partial_\mu\phi$ transform consistently:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = u^\mu\frac{\partial\phi}{\partial x^\mu} = u^\nu\delta^\mu_\nu\cdot\frac{\partial\phi}{\partial x^\mu} = u^\nu\frac{\partial x'^\alpha}{\partial x^\nu}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial\phi}{\partial x^\mu} = u'^\alpha\frac{\partial\phi}{\partial x'^\alpha} \tag{B.128}$$

so that in fact the vector u^μ transforms with the Jacobian and the linear form $\partial_\mu\phi$ with the inverse Jacobian of the coordinate transform.

We can run all possible curves through the point x^μ and get a complete set of tangent vectors which would ultimately constitute, after proper orthonormalisation, a local basis set: the basis of the tangent space $T_P M$ at the point P with the coordinates x^μ: It is important to realise that the tangent space's basis set exists for a given choice of coordinates and that a different coordinate choice would induce a new basis set. In particular, a neighbouring point Q can have a different tangent space $T_Q M$. That implies that if we take the same abstract vector $\boldsymbol{\upsilon}$ and express it with the basis sets at $T_P M$ and $T_Q M$ in coordinates υ^μ, the tuples will in general differ, and that one needs a more elaborate concept of differentiating vectors than just partial derivatives: the covariant derivative.

Up to this point, the manifold has two structures: the topological structure which defines open sets and allows the definition of continuous coordinate mappings, and the metric structure which defines the geometry through a scalar product. The two structures are compatible with each other, as the definition of open sets with the metric is never in contradiction with the topology. The next step is the definition of a differentiable structure constructed with parallel transport.

B.4 *Parallel transport and the covariant derivative*

Parallel transport generates a perfect copy of an abstract vector at a different spacetime position. After defining coordinates and therefore entries of a vector tuple, the parallel transported copy $\upsilon^\mu_\parallel(x+\delta x)$ of the vector $\upsilon^\mu(x)$ at a new, infinitesimally distant position $x+\delta x$ is given by

$$\upsilon^\mu_\parallel(x+\delta x) = \upsilon^\mu(x) - \Gamma^\mu{}_{\alpha\beta}\,\upsilon^\alpha(x)\cdot\delta x^\beta + \cdots \tag{B.129}$$

at lowest order. It is conventional to use a minus-sign in front of the Christoffel-symbol $\Gamma^\mu{}_{\alpha\beta}$, which generates the transformation rule for the vector υ^μ, because we have a different set of tangent vectors at $x+\delta x$ compared to the point x, and therefore different expansions of the same vector into two different basis sets. In fact, the best way to visualise the Christoffel-symbol is to think of $\Gamma^\mu{}_{\alpha\beta}$ as a transformation matrix in the indices α and μ acting on the components υ^α for shifts in any possible direction δx^β.

There are cases where the connection is trivially zero, that is when index by index the components of the parallel-transported vector are identical to the original vector, which would be the case in a vector-space or a flat manifold with Cartesian coordinates. In the general case, the tangent spaces at x and $x + \delta x$ are not identical and have a different set of basis vectors, so the expansion of the abstract vector, although it is in fact identical at x and $x + \delta x$ under perfect parallel transport, needs to be different.

With this definition of parallel transport we can ask whether a vector field $\boldsymbol{\upsilon}$ has changed moving from x to $x + \delta x$, or equivalently, if it has a derivative. It is senseless just to compare the entries of the vectors as they exist in different tangent spaces, rather, we need to compare the vector field at $x + \delta x$ with a parallel transported version of υ taken from x to $x + \delta x$.

Taking the limit $\delta x^\beta \to 0$ to get the differential rate of change yields

$$\nabla_\beta \upsilon^\mu = \lim_{\delta x^\beta \to 0} \frac{\upsilon^\mu(x+\delta x) - \upsilon^\mu_{\parallel}(x+\delta x)}{\delta x^\beta} = \lim_{\delta x^\beta \to 0} \frac{\upsilon^\mu(x+\delta x) - \upsilon^\mu(x)}{\delta x^\beta} + \Gamma^\mu{}_{\alpha\beta} \cdot \upsilon^\alpha(x) \cdot \frac{\delta x^\beta}{\delta x^\beta} \tag{B.130}$$

such that the covariant derivative is given by

$$\nabla_\beta \upsilon^\mu = \partial_\beta \upsilon^\mu + \Gamma^\mu{}_{\alpha\beta}\, \upsilon^\alpha \tag{B.131}$$

if we substitute the partial derivative as the index-by-index comparison of the entries υ^μ at the two infinitesimally separated points. For scalar fields Φ there is no distinction between the covariant derivative and the conventional partial derivative, $\nabla_\mu = \partial_\mu \phi$ because there are no internal degrees of freedom whose entries would change if the set of basis vectors is different, hence the field can only have a derivative if it assumes a different value. Using Cartesian coordinates on a flat manifold allows the usage of the connection $\Gamma^\mu{}_{\alpha\beta} = 0$, because all tangent spaces are identical (or aligned) and vectors do not change their entries moving from one tangent space to another, therefore $\upsilon^\mu_{\parallel}(x + \delta x) = \upsilon^\mu(x)$ in parallel transport and consequently, $\nabla_\beta \upsilon^\mu = \partial_\beta \upsilon^\mu$.

Higher-order tensors require a Christoffel-symbol for every index

$$\nabla_\beta T^{\mu\nu} = \partial_\beta T^{\mu\nu} + \Gamma^\mu{}_{\alpha\beta}\, T^{\alpha\nu} + \Gamma^\nu{}_{\alpha\beta}\, T^{\mu\alpha} \tag{B.132}$$

because their basis set is the Cartesian product of the basis of $T_P M$, one factor for each index.

The covariant differentiation can be constructed for linear forms (or covariant vectors) in a way that is compatible with the differentiation of (contravariant) vectors: Because a product $\upsilon^\mu w_\mu = g_{\mu\nu} \upsilon^\mu w^\mu$ would be scalar, the covariant derivative reverts back into a partial one:

$$\nabla_\beta\left(\upsilon^\mu w_\mu\right) = \partial_\beta\left(\upsilon^\mu w_\mu\right) = \partial_\beta \upsilon^\mu \cdot w_\mu + \upsilon^\mu \cdot \partial_\beta w_\mu. \tag{B.133}$$

If we require the covariant differentiation to obey a Leibnitz-rule, the last term can be written as:

$$\nabla_\beta\left(\upsilon^\mu w_\mu\right) = \nabla_\beta \upsilon^\mu \cdot w^\mu + \upsilon^\mu \nabla_\beta w_\mu = \left(\partial_\beta \upsilon^\mu + \Gamma^\mu{}_{\alpha\beta}\, \upsilon^\alpha\right) w_\mu + \upsilon^\mu \nabla_\beta w_\mu \tag{B.134}$$

Then, the term $\partial_\beta \upsilon^\mu \cdot w_\mu$ drops out and renaming the indices $\mu \leftrightarrow \alpha$

$$\upsilon^\mu\left(\nabla_\beta w_\mu\right) = \upsilon^\mu \cdot \partial_\beta w_\mu - \Gamma^\mu_{\ \alpha\beta}\, \upsilon^\alpha w_\mu = \upsilon^\mu \partial_\beta w_\mu - \Gamma^\alpha_{\ \mu\beta}\, \upsilon^\mu w_\alpha = \upsilon^\mu\left(\partial_\beta w_\mu - \Gamma^\alpha_{\ \mu\beta}\, w_\alpha\right) \tag{B.135}$$

gives the final result

$$\nabla_\beta w_\mu = \partial_\beta w_\mu - \Gamma^\alpha_{\ \mu\beta}\, w_\alpha \tag{B.136}$$

for the covariant derivative of a linear form, with a minus-sign instead of a plus-sign.

Up to this point, the connection has been arbitrary but we will now focus on Levi-Civita-connections: Those are metric-compatible and torsion-free, and can therefore be computed from the metric and its derivatives. A metric manifold with such a connection and the corresponding covariant derivative is referred to as a Riemannian geometry. It is important to achieve the compatibility between the metric and the differentiable structure of the manifold so that we can compute the connection coefficients from the metric itself. Scalar products $\upsilon^\mu w_\mu = g_{\mu\nu}\, \upsilon^\mu w^\nu$ between two vectors $\boldsymbol{\upsilon}$ and $\boldsymbol{w}$ that are parallel transported should be identical: The parallel transport of two abstract vectors only changes the tuples υ^μ and w^μ because the tangent spaces change and a different basis set is provided at every point. The scalar product is an abstract measure of the lengths and relative orientations of the two vectors and that statement should be invariant:

$$g(\boldsymbol{\upsilon}(x), \boldsymbol{w}(x)) = g(\boldsymbol{\upsilon}_\parallel(x + \delta x), \boldsymbol{w}_\parallel(x + \delta x)) \tag{B.137}$$

For that to be conserved, parallel transport by δx^β should not change anything, neither the length nor the relative orientation of the two vectors, $\upsilon_\parallel(x+\delta x) = \upsilon(x+\delta x)$, and $\delta x^\beta \nabla_\beta \upsilon^\mu$ is necessarily zero. Stating that the scalar product of parallel-transported vectors remains constant is equivalent to

$$\delta x^\beta\, \nabla_\beta\, g = \delta x^\beta\, \nabla_\beta\left(\upsilon^\mu w_\mu\right) = \delta x^\beta\, \nabla_\beta\left(g_{\mu\nu} \upsilon^\mu w^\nu\right) = 0 \tag{B.138}$$

As the covariant derivatives obeys a Leibnitz-rule, one can write

$$\delta x^\beta\, \nabla_\beta\left(g_{\mu\nu}\, \upsilon^\mu w^\nu\right) = \delta x^\beta\left(\nabla_\beta\, g_{\mu\nu} \cdot \upsilon^\mu w^\nu + g_{\mu\nu}\, \nabla_\beta\, \upsilon^\mu \cdot w^\nu + g_{\mu\nu}\, \upsilon^\mu\, \nabla_\beta w^\nu\right) \tag{B.139}$$

and therefore, as $\delta x^\beta \nabla_\beta \upsilon^\mu = 0$ and $\delta x^\beta \nabla_\beta w^\nu = 0$ as an expression of parallel transport,

$$\delta x^\beta\, \nabla_\beta\, g_{\mu\nu} \cdot \upsilon^\mu w^\nu = 0. \tag{B.140}$$

Because that statement must be valid for every index choice, we can isolate the metric compatibility condition

$$\nabla_\beta\, g_{\mu\nu} = 0, \tag{B.141}$$

stating that the covariant derivative of the metric must be zero. On the other hand, the metric is a covariant tensor, so its covariant derivative is explicitly given by

$$\nabla_\beta\, g_{\mu\nu} = \partial_\beta\, g_{\mu\nu} - \Gamma^\alpha_{\ \beta\mu}\, g_{\alpha\nu} - \Gamma^\alpha_{\ \beta\nu}\, g_{\mu\alpha} = 0. \tag{B.142}$$

As a second condition, we require symmetry of the Christoffel-symbol in the lower two indices,

$$\Gamma^{\alpha}_{\ \mu\nu} = \Gamma^{\alpha}_{\ \nu\mu}, \tag{B.143}$$

which is called the torsion-free condition of the connection. With that, we can write out eqn. B.142 with cyclically permuted indices (μ, ν, β):

$$\nabla_{\mu}\, g_{\nu\beta} = \partial_{\mu}\, g_{\nu\beta} - \Gamma^{\alpha}_{\ \mu\nu}\ g_{\alpha\beta} - \Gamma^{\alpha}_{\ \mu\beta}\ g_{\nu\alpha} = 0 \tag{B.144}$$

as well as

$$\nabla_{\nu} g_{\mu\beta} = \partial_{\nu}\, g_{\beta\mu} - \Gamma^{\alpha}_{\ \nu\mu}\ g_{\alpha\beta} - \Gamma^{\alpha}_{\ \nu\beta}\ g_{\mu\alpha} = 0 \tag{B.145}$$

and combine all three by computing B.144 +B.145 − B.142 = 0:

$$\partial_{\mu}\, g_{\beta\nu} - \Gamma^{\alpha}_{\ \mu\nu}\ g_{\alpha\beta} - \Gamma^{\alpha}_{\ \beta\mu}\ g_{\alpha\nu} + \partial_{\nu}\, g_{\mu\beta} - \Gamma^{\alpha}_{\ \mu\nu}\ g_{\alpha\beta} - \Gamma^{\alpha}_{\ \beta\nu}\ g_{\mu\nu} - \partial_{\beta}\, g_{\mu\nu} + \Gamma^{\alpha}_{\ \beta\mu}\ g_{\alpha\nu} + \Gamma^{\alpha}_{\ \beta\nu}\ g_{\mu\nu} = 0. \tag{B.146}$$

Finally, we solve for the Christoffel-symbol $\Gamma^{\alpha}_{\ \mu\nu}$:

$$\partial_{\mu}\, g_{\beta\nu} + \partial_{\nu}\, g_{\mu\beta} - \partial_{\beta}\, g_{\mu\nu} = 2\Gamma^{\alpha}_{\ \beta\nu}\, g_{\alpha\beta} \tag{B.147}$$

and isolate $\Gamma^{\gamma}_{\ \mu\nu}$ by multiplication with the inverse metric $g^{\beta\gamma}$,

$$\Gamma^{\alpha}_{\ \mu\nu}\ g_{\alpha\beta} g^{\beta\gamma} = \Gamma^{\gamma}_{\ \mu\nu} = \frac{g^{\beta\gamma}}{2}\left(\partial_{\mu}\, g_{\beta\nu} + \partial_{\nu}\, g_{\mu\beta} - \partial_{\beta}\, g_{\mu\nu}\right) \tag{B.148}$$

by using $g_{\alpha\beta} g^{\beta\gamma} = \delta^{\gamma}_{\alpha}$. Please keep in mind that

$$\upsilon^{\alpha} = \delta^{\alpha}_{\beta}\ \upsilon^{\beta} = g^{\alpha\beta}\ \upsilon_{\beta} = g^{\alpha\beta} g_{\beta\gamma}\ \upsilon^{\gamma} \ \rightarrow\ g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma} \tag{B.149}$$

as the defining equation for the inverse metric $g^{\mu\nu}$ for any metric $g_{\mu\nu}$. It is a standard exercise to show that the Christoffel-symbol $\Gamma^{\alpha}_{\ \mu\nu}$ is **not** a tensor, but that the covariant derivatives $\nabla_{\beta}\ \upsilon^{\mu}$ and $\nabla_{\beta}\ w_{\mu}$ are.

B.5 *Geodesics as autoparallel curves*

A curve $x^{\mu}(\lambda)$ parameterised by λ can be autoparallel in the sense that the tangent $u^{\mu} = \mathrm{d}x^{\mu}/\mathrm{d}\lambda$ does not change, or equivalently, that the tangent vector u^{μ} is always a parallel transported version of itself along the curve. Then, writing $\dot{x}^{\mu} = u^{\mu} = \mathrm{d}x^{\mu}/\mathrm{d}\lambda$ for simplicity,

$$\dot{x}^{\beta}\nabla_{\beta}\dot{x}^{\alpha} = 0 \tag{B.150}$$

because $\dot{x}^{\mu}_{\parallel}(x + \delta x) = \dot{x}^{\mu}(x + \delta x)$. We can substitute the explicit form of the covariant derivative to get

$$\dot{x}^{\beta}\nabla_{\beta}\dot{x}^{\nu} = \dot{x}^{\beta}\left[\partial_{\beta}\dot{x}^{\alpha} + \Gamma^{\alpha}_{\ \beta\mu}\ \dot{x}^{\mu}\right] = \dot{x}^{\beta}\cdot\partial_{\beta}\dot{x}^{\alpha} + \Gamma^{\alpha}_{\ \beta\mu}\ \dot{x}^{\beta}\dot{x}^{\mu} = 0 \tag{B.151}$$

Rewriting the first term as a differentiation along λ yields

$$\dot{x}^\beta \partial_\beta \dot{x}^\alpha = \frac{dx^\beta}{d\lambda}\frac{\partial \dot{x}^\alpha}{\partial x^\beta} = \frac{d}{d\lambda}\left(\dot{x}^\alpha\right) = \ddot{x}^\alpha \tag{B.152}$$

which defines the standard form of the geodesic equation,

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu = 0. \tag{B.153}$$

A curve that obeys this equation of motion and follows an autoparallel line is called a geodesic. Geodesics generalise the concept of a straight line through Euclidean space to manifolds, where straight and autoparallel are equivalent. One would already suspect at this point that inertial motion, where no accelerations are felt, corresponds to motion along an autoparallel line. But at the same time, freely falling motion through a gravitational field would likewise be characterise by a feeling of perfect weightlessness and the absence of inertial forces: And one is correct in guessing that geodesics are in fact trajectories through spacetime followed by freely falling particles.

Because the rate at which particle pass by the coordinates does not need to be constant for inertial motion (imagine a particle drifting off-centre through Euclidean space with polar coordinates) we should not use the statement $\ddot{r} = \ddot{\varphi} = 0$ as a characterisation of inertial motion, possibly motivated by Newtonian thinking. Instead, autoparallelity condition would be the proper thing to do. And as the connection has been defined to be metric compatible, we immediately see that the modulus of the velocity, defined as the scalar product $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, is conserved.

It is possible to reverse-engineer Newton's equation of motion in a gravitational field with our knowledge of relativity and to rediscover the geodesic equation, adding perhaps some support for the idea on the connection between autoparallelity and geodesic motion: Newton's equation of motion reads

$$\ddot{x}^i + \partial^i\Phi = 0 \tag{B.154}$$

for a particle falling through the gravitational potential, where no accelerations can be felt. The dot denotes the derivative with respect to laboratory time, which for small velocities is equal to the proper time, $t = \tau$. Because we already suspect that the potential is measured in units of c^2 as suggested by the weak field-metric, one can write:

$$\ddot{x}^i + \partial^i \frac{\Phi}{c^2} \cdot c \cdot c = 0. \tag{B.155}$$

Perhaps the two cs are just the t-component of the 4-velocity in the slow motion limit,

$$\ddot{x}^i + \partial^i \frac{\Phi}{c^2}\dot{x}^t \dot{x}^t = 0 \tag{B.156}$$

with coordinates (a tuple!) and velocities (a vector!)

$$x^\mu = \begin{pmatrix} ct \\ x^i \end{pmatrix}, \quad \dot{x}^\mu = \begin{pmatrix} c \\ v^i \end{pmatrix}, \tag{B.157}$$

where the difference between coordinate time and proper time vanishes, and $\gamma = 1$. If we identify the Christoffel-symbol

$$\Gamma^{i}_{\ tt} = \partial^{i}\frac{\Phi}{c^2} \tag{B.158}$$

with a suitable derivative of the metric, one gets

$$\ddot{x}^{i} + \Gamma^{i}_{\ tt}\,\dot{x}^{t}\dot{x}^{t} = 0 \tag{B.159}$$

Finally, making everything covariant by replacing i with α and reinstating τ instead of t

$$\frac{\mathrm{d}^2x^{\alpha}}{\mathrm{d}\tau^2} + \Gamma^{\alpha}_{\ \mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} = 0 \tag{B.160}$$

one obtains the geodesic equation, with the affine parameter τ. This immediately poses the question if these statements are only true for a particular choice of the affine parameter. This is not the case, as geodesics are invariant under affine reparameterisations $\lambda \rightarrow \lambda'$!

We have seen that autoparallelity of the tangent vector is equivalent to the geodesic equation,

$$\dot{x}^{\beta}\nabla_{\beta}\dot{x}^{\mu} = 0 \quad \rightarrow \quad \ddot{x}^{\beta} + \Gamma^{\beta}_{\ \mu\nu}\,\dot{x}^{\mu}\dot{x}^{\nu} = 0 \tag{B.161}$$

where

$$\dot{x}^{\beta} = \frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}\lambda} \quad \text{and} \quad \ddot{x}^{\beta} = \frac{\mathrm{d}}{\mathrm{d}\lambda'}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}\lambda}\right) \tag{B.162}$$

yielding the following conversion

$$\frac{\mathrm{d}}{\mathrm{d}\lambda'}\left(\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}\right)\cdot\frac{\mathrm{d}x^{\beta}}{\mathrm{d}\lambda}\right) + \Gamma^{\beta}_{\ \mu\nu}\,\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\cdot\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda} = 0 \tag{B.163}$$

from the chain rule, and by applying the Leibnitz-rule,

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}\lambda'^2}\cdot\frac{\mathrm{d}x^{\beta}}{\mathrm{d}\lambda} + \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}\right)^2\frac{\mathrm{d}^2x^{\beta}}{\mathrm{d}\lambda^2} + \Gamma^{\beta}_{\ \mu\nu}\,\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\cdot\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}\cdot\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda} = 0 \tag{B.164}$$

such that

$$\frac{\mathrm{d}^2x^{\beta}}{\mathrm{d}\lambda^2} + \Gamma^{\beta}_{\ \mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\cdot\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda} = -\frac{\mathrm{d}^2\lambda}{\mathrm{d}\lambda'^2}\cdot\left(\frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda}\right)^2\cdot\frac{\mathrm{d}x^{\beta}}{\mathrm{d}\lambda} \tag{B.165}$$

If there is now a linear relationship between λ and λ', the derivative $\mathrm{d}^2\lambda/\mathrm{d}\lambda'^2$ vanishes, making sure that one recovers the geodesic equation in both parameters: In fact, there seems to be an entire class of affine parameters which are all equally suited to be used to define autoparallelity or the geodesic equation, all related by affine transformations $\lambda' = a\lambda + b$.

In classical mechanics with $\ddot{x}^{i} + \partial^{i}\Phi = 0$ as the equation of motion, this looks like nothing particular beyond mechanical similarity: $t \rightarrow at + b$ implies that $\ddot{x}$ acquires a factor a^{-2}, but Φ has units of velocity^2, so that it will have a factor of a^{-2}, too, which cancels. But we can make an interesting statement about the relativistic

Doppler-effect, which arises as a projection of a photon's wave vector k^μ onto the observer's world line with the tangent u^μ, $\omega = g_{\mu\nu}u^\mu k^\nu$. Clearly, reparameterisation of u^μ brings in a factor of a^{-1}, but the photon wave vector should not change, such that the frequency only changes by a single factor of a^{-1}: We can not work with the same affine parameter for photons and massive particles.

In fact, the wave vector as the tangent to the photon geodesic is normalised to zero, $g_{\mu\nu}k^\mu k^\nu = 0$, while there is a particular choice of the affine parameter for massive particles such that the tangent is normalised to c^2. With the proper time τ and tangents $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ one always obtains the normalisation $g_{\mu\nu}u^\mu u^\nu = c^2$. And, in both cases, geodesic motion conserves this normalisation as a consequence of metric compatibility $\nabla_\alpha g_{\mu\nu}$ and the autoparallelity condition $u^\alpha\nabla_\alpha u^\mu = 0$:

$$u^\alpha\nabla_\alpha\left(g_{\mu\nu}\, u^\mu u^\nu\right) = u^\alpha\nabla_\alpha\, g_{\mu\nu}\cdot u^\mu u^\nu + g_{\mu\nu}\, u^\alpha\nabla_\alpha\, u^\mu\cdot u^\nu + g_{\mu\nu}\, u^\mu u^\alpha\nabla_\alpha u^\nu = 0 \qquad \text{(B.166)}$$

B.6 *Geodesic motion through a variational principle*

Relativity surprises with the idea that the variational principles of classical mechanics have a clear geometric meaning: Particles move along trajectories in spacetime with extremised arc lengths. The central result of the last chapter was that autoparallelity leads to the geodesic equation and that autoparallel lines are straight in a general sense: But is straight equivalent to shortest? Writing down the action as the integrated arc length gives

$$S = \int_A^B \mathrm{d}s = \int_A^B \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda = \int_A^B \mathrm{d}\lambda\, L\left(x^\mu, \dot{x}^\mu, g_{\mu\nu}\right) \qquad \text{(B.167)}$$

with L being the generalised Lagrange function. A variation of the trajectory $x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda)$ by $\delta x^\mu(\lambda)$ generates a variation δS of the arc length,

$$\delta S = \int_A^B \mathrm{d}\lambda\left[\frac{\partial L}{\partial x^\alpha}\delta x^\alpha + \frac{\partial L}{\partial \dot{x}^\alpha}\delta\dot{x}^\alpha\right] \quad \text{with } \delta\dot{x}^\alpha = \frac{\mathrm{d}}{\mathrm{d}\lambda}\delta x^\alpha \qquad \text{(B.168)}$$

which can be recast into

$$\delta S = \int_A^B \mathrm{d}\lambda\left[\frac{\partial L}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\alpha}\right]\delta x^\alpha \qquad \text{(B.169)}$$

through an integration by parts, where no variation is done at the end points A and B. Then, the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial L}{\partial x^\alpha} \qquad \text{(B.170)}$$

can be isolated, as it applies to the generalised Lagrange function

$$L\left(x^\mu, \dot{x}^\mu, g_{\mu\nu}\right) = \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \qquad \text{(B.171)}$$

which depends on the trajectory and its tangent, apart from the metric itself defining the geometry. The derivatives can be directly computed, keeping in mind that the metric itself is a function of the coordinates, for the derivative with respect to the coordinates,

$$\frac{\partial L}{\partial x^\alpha} = \frac{1}{2L}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\cdot\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{B.172}$$

and for the derivative with respect to the velocities,

$$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{1}{2L}\cdot g_{\mu\nu}\left(\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha}\cdot\dot{x}^\nu + \dot{x}^\mu\cdot\frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha}\right) = \frac{1}{2L}\left(g_{\alpha\nu}\,\dot{x}^\nu + g_{\mu\alpha}\,\dot{x}^\mu\right) = \frac{1}{L}g_{\alpha\mu}\,\dot{x}^\mu \tag{B.173}$$

Substitution into the Euler-Lagrange-equation yields

$$\frac{d}{d\lambda}\left(\frac{1}{L}\cdot g_{\alpha\mu}\,\dot{x}^\mu\right) = -\frac{\dot{L}}{L^2}g_{\alpha\mu}\,\dot{x}^\mu + \frac{1}{L}\dot{g}_{\alpha\mu}\,\dot{x}^\mu + \frac{1}{L}g_{\alpha\mu}\,\ddot{x}^\mu = \frac{1}{L}\left[-\frac{\dot{L}}{L}g_{\alpha\mu}\,\dot{x}^\mu + \frac{\partial g_{\alpha\mu}}{\partial x^\nu}\cdot\dot{x}^\mu\dot{x}^\nu + g_{\alpha\mu}\ddot{x}^\mu\right] \tag{B.174}$$

where the derivative of the metric is given by the chain rule, $\dot{g}_{\alpha\mu} = \partial_\nu\,\dot{g}_{\alpha\mu}\cdot\dot{x}^\nu$, so that one arrives at

$$\frac{d}{d\lambda}\left(\frac{1}{L}\cdot g_{\alpha\mu}\,\dot{x}^\mu\right) = \frac{1}{L}\left[-\frac{\dot{L}}{L^2}g_{\alpha\mu}\,\dot{x}^\mu + \frac{\partial g_{\alpha\mu}}{\partial x^\nu}\cdot\dot{x}^\mu\dot{x}^\nu + g_{\alpha\mu}\,\ddot{x}^\mu\right], \tag{B.175}$$

which leads to

$$-\frac{\dot{L}}{L}g_{\alpha\mu}\,\dot{x}^\mu + \frac{\partial}{\partial x^\nu}g_{\alpha\mu}\,\dot{x}^\mu\dot{x}^\nu + g_{\alpha\mu}\,\ddot{x}^\mu = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu\dot{x}^\nu \tag{B.176}$$

with a symmetrisation $\frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu}\right)$ of the second term one then obtains

$$\ddot{x}_\alpha + \frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\dot{x}^\mu\dot{x}^\nu = \frac{\dot{L}}{L}\cdot\dot{x}^\alpha \tag{B.177}$$

Multiplying this relation with the inverse metric $g^{\beta\alpha}$ shows the emergence of the Christoffel symbol,

$$\ddot{x}^\beta + \frac{g^{\beta\alpha}}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\dot{x}^\mu\dot{x}^\nu = \frac{\dot{L}}{L}\cdot\dot{x}^\beta \tag{B.178}$$

which one can replace in the equation,

$$\ddot{x}^\beta + \Gamma^\beta_{\ \mu\nu}\,\dot{x}^\mu\dot{x}^\nu = \frac{\dot{L}}{L}\cdot\dot{x}^\beta = \frac{\ddot{S}}{\dot{S}}\dot{x}^\beta. \tag{B.179}$$

The arc length $S = \int d\lambda\, L$ has the derivatives $L = \dot{S}$ and $\dot{L} = \ddot{S}$. If in particular an affine parameter is chosen, then $\ddot{S} = 0$, and one obtains the classic geodesic equation,

$$\ddot{x}^\beta + \Gamma^\beta_{\ \mu\nu}\,\dot{x}^\mu\dot{x}^\nu = 0. \tag{B.180}$$

Conceptually, the geodesic equation joins straight, autoparallel motion and with the principle of external proper time or minimal arc length to arbitrary geometry, as the proper time is a preferred affine parameter because it has a measurable physical meaning.

There is a number of interesting properties of gravity: Firstly, all objects experience the same acceleration irrespective of their mass; with acceleration being meant as the rate of the rate at which the coordinates pass by the object, not as a physical acceleration which is always absent in free fall. This is very much different for e.g. electrically charged particles experiencing electromagnetic fields. In this case, the arc length is computed with

$$S = \int_A^B \left(\mathrm{d}\tau + \frac{q}{m} g_{\mu\nu}\, A^\mu \mathrm{d}x^\nu\right) \tag{B.181}$$

with a vector potential A^μ. Clearly, the decisive quantity here is the specific charge q/m, and particles with different specific charge will follow different trajectories through the same field A^μ.

This specific charge for gravitational fields would correspond to the ratio between the gravitational mass as the coupling strength of massive particle to the gravitational field and the inertial mass. This ratio has been found to be unity at the level of 10^{-11}, giving a strong empirical indication of the universality of gravity. In fact, variation $\delta S = 0$ of (**) gives

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\ \mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \frac{q}{m} F^\alpha_{\ \mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{B.182}$$

so that any deviation from freely-falling motion must be proportional to the specific charge $\frac{q}{m}$, with $F^\alpha_{\ \mu} = g^{\alpha\beta}F_{\beta\mu}$.

Gravitational lensing is naturally explained by the geodesic equation even the photon has a vanishing mass, $m_\gamma = 0$. It is sufficient to use the geodesic equation for the wave vector k^μ of the photon as the force-free, gravitational left hand side of the geodesic equation allows for phenomena like gravitational lensing, effectively through

$$\frac{\mathrm{d}k^\alpha}{\mathrm{d}\lambda} + \Gamma^\alpha_{\ \mu\nu} k^\mu k^\nu = 0, \tag{B.183}$$

for the wave vector $k^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ for the affine parameter $\lambda \neq \tau$ parameterising the photon trajectory $x^\mu(\lambda)$.

Lastly, inertial motion through a vector space with Cartesian coordinates suggest a Euclidean straight line: $\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = 0 \rightarrow x^\alpha = a^\alpha\tau + b^\alpha$, because in Cartesian coordinates the metric is constant and the Christoffel-symbol vanishes.

Geodesic, autoparallel motion corresponds to freely falling particles, generalising the idea of inertial motion to curved manifolds, as a representation of gravitational fields. One should be careful, however, to associate $g_{\mu\nu} \neq \eta_{\mu\nu}$ or $\Gamma^\alpha_{\ \mu\nu} \neq 0$ to gravitational fields, as both statements can be true locally in a certain coordinate choice. Rather, one should think of geodesic motion as taking care of the coordinate choice by establishing autoparallelity of a straight line, irrespective of the presence of curvature or gravity. Both inertial motion and freely falling motion are, in addition, both characterised by a sensation of perfect weightlessness of an observer moving along with the particle, and are therefore, a priori indistinguishable.

B.7 *Equivalence and the relativistic origin of Newton's axioms*

The geodesic equation is a description of a straight line (in the autoparallel sense) through spacetime and should, as such, be a generalisation of the law of inertia and the Newtonian equation of motion. In fact, Newton's inertial law states that force-free motion proceeds at constant speed along a straight line, which is perfectly fulfilled by the geodesic equation: Straight actually means autoparallel, as the proper concept for more complicated coordinate choices, and the normalisation $g_{\mu\nu}u^\mu u^\nu = c^2$ of the velocity u^μ is conserved. Force-free in the Newtonian sense might pertain to both inertial motion through a flat spacetime or freely-falling motion through a curved spacetime: There is no fundamental difference between these two cases. To take things to extremes, one could say that Newton's first axiom is the definition of the word "straight": As soon as there are no accelerations measured, the trajectory is necessarily autoparallel.

We have already seen that the Newtonian equation of motion $\ddot{x}^i + \partial^i\Phi = 0$ with the gravitational potential Φ is hidden in geodesic equation for small velocities and weak fields, exactly the limit Newton could have been aware of. Writing $\ddot{x}^i + \partial^i\Phi = 0$ to allude at force-free motion is in the spirit of the geodesic equation $\ddot{x}^\alpha + \Gamma^\alpha{}_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu = 0$, and only non-gravitational forces would replace the zero on the right hand side, for instance an electromagnetic force,

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \frac{q}{m}g_{\mu\nu}\,\mathrm{F}^{\alpha\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{B.184}$$

for a particle with specific charge q/m experiencing electromagnetic fields $\mathrm{F}^{\alpha\nu}$. With this idea in mind, I personally don't like to speak about gravitational forces: Rather, I would call them gravitational accelerations which get modified by non-gravitational accelerations that are computed from the actual field with the specific charge q/m as the coupling constant, to yield an actual acceleration.

Ultimately, the third axiom actio = -reactio (with a minus-sign, as actio and reactio take place in opposing directions!) is the most interesting in view of relativity. It concerns non-geodesic motion with the appearance of inertial forces (reactio), which are opposed to the actual forces (actio) acting on a particle. To understand where this might come from we should first have a look at the way how classical inertial forces like the centrifugal force or the Coriolis-force are contained in the geodesic equation. In the slow-motion limit with $\tau = t$, $\gamma = 1$, fixed $u^t = c$ and $u^i = \upsilon^i$ we get

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t} = \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{tt}\,\mathrm{c}^2 + \Gamma^i_{mt}\,\upsilon^m\mathrm{c} + \Gamma^i_{tn}\,\mathrm{c}\upsilon^n + \Gamma^i_{mn}\,\upsilon^m\upsilon^n = \frac{q}{m}f^i, \tag{B.185}$$

with a non-gravitational acceleration $\frac{q}{m}f^i$. The Christoffel symbol is symmetric in the lower two indices $\Gamma^i{}_{mn} = \Gamma^i{}_{nm}$ because of the torsion-free condition and $\Gamma^i{}_{tt} = \partial^i\frac{\Phi}{\mathrm{c}^2}$ specifically would be the gradient of a classical gravitational potential. Then, fundamentally, there are two terms in the geodesic equation, $\Gamma^i{}_{mn}\,\upsilon^m\upsilon^n \sim (\boldsymbol{\Omega}\times\boldsymbol{\upsilon})\times\boldsymbol{\upsilon}$ corresponding to a centrifugal acceleration which is quadratic in the velocities, and the Coriolis acceleration $2\Gamma^i{}_{mt}\,\upsilon^m\mathrm{c} \sim 2\boldsymbol{\Omega}\times\boldsymbol{\upsilon}$ with the factor 2 appearing naturally out of the two identical terms linear in the velocity υ.

This is in fact a surprising result: The velocity-dependent inertial accelerations appear as the non-relativistic limit of the geodesic equation, up to terms $\propto \upsilon^2$ because of the term $\Gamma^i{}_{\mu\nu}\,u^\mu u^\nu$. It seems to be the case that the velocity dependence of accelerations is natural, similar to the Lorentz force $\propto \boldsymbol{\upsilon}\times\mathbf{B}$. Here, υ^1 is the highest power that can be generated by $\frac{q}{m} g_{\mu\nu} F^{\alpha\nu} u^{\mu}$. The term $\Gamma^i_{\ tt} = \partial^i \frac{\Phi}{c^2}$ is an eternal source of confusion: The geodesic equation with such a term clearly refers to autoparallel motion along a straight line, but one tends to think of a curved trajectory, for instance when thinking about throwing a ball along a parabolic curve. But please keep in mind that there is a second definition of straightness corresponding to the Minkowski-space with the metric $\eta_{\mu\nu}$ that one might use instinctively instead of $g_{\mu\nu}$. Balls and planets follow autoparallel lines through spacetimes, and parabolas and elliptical orbits (besides, the parabola that is followed by a ball is only the second order Taylor-expansion of an elliptical orbit around the Earth's centre) are straight, otherwise Newton's first axiom could not be fulfilled.

If there is really the equivalence between inertial accelerations and gravitational accelerations, as made clear by Einstein's elevator argument, there should be a deep connection between the two. First of all, inertial motion in e.g. rotating or accelerating coordinate frames is the uninteresting case, because the geodesic equation makes the job of computing the rate of change of the passage of the coordinates perfectly and all we see are coordinate effects. It becomes more interesting if there is a non-gravitational force acting on a particle such that inertial forces appear as a consequence of, well, the change of the state of motion, but relative to what? At this point Mach's principle comes in and clarifies that inertial frames are defined in by the large-scale distribution of matter in the Universe. If there is a perfect equivalence between inertial and gravitational forces, we should be able to ask how inertial accelerations are sourced and what their gravitational origin is, after having thought of gravitational accelerations to be inertial: They vanish in freely falling frames and affect all objects in exactly the same way irrespective of their mass. Coming back to Newton's third axiom we should suspect that the inertial reactio is in fact gravitationally induced, because the state of motion changes relative to the masses in the Universe, and because there is an additional velocity dependent gravitational force acting on the particle.

A second striking example is the rotational flattening of the Sun, whose diameter at the equator is larger than the diameter taken at the poles, as a consequence of the centrifugal force acting on it due to its rotation. But how would you interpret the same observation from a frame co-rotating with the Sun? There, the entire universe rotates in the opposite direction and there is an additional component of the gravitational field which pulls on the Sun's equator.

There is an interesting remainder of the idea that accelerated frames and gravitational potentials are equivalent left in classical mechanics: A boost into a frame with constant acceleration a^i is defined by

$$x'^i = x^i + \frac{1}{2} a^i t^2 \quad \rightarrow \quad \dot{x}'^i = \dot{x}^i + a^i t, \tag{B.186}$$

such that the Lagrange-function L transforms accordingly,

$$L' = \frac{m}{2} \delta_{ij}\, \dot{x}'^i \dot{x}'^j = L + \frac{m}{2}\left(2\delta_{ij}\, \dot{x}^i a^j t + \delta_{ij}\, a^i a^j t^2\right) \tag{B.187}$$

The last term can be written as a total derivative, $t^2 = \mathrm{d}(t^3/3)/\mathrm{d}t$ and does not matter in the variation, as total derivatives of functions that only depend on time and coordinate (but not velocity) never have an influence on the variational principle.

The second term, however, can be rewritten using the fundamental theorem of calculus, as a differentiation of an integral,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}t\,\dot{x}^i \cdot t = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}t\,x^i = x^i \tag{B.188}$$

so that we can apply an integration by parts in the last step. Collecting these results yields

$$\mathrm{L}' = \mathrm{L} - m\,\delta_{ij}\,x^i a^j, \tag{B.189}$$

so that the Lagrange-function has acquired a new term that corresponds to a potential with a constant slope: This is classical equivalence between a linear potential and a frame accelerated at a constant rate.

B.8 *Geodesic deviation, curvature and gravity*

Geodesics as autoparallel lines through spacetime are the trajectories of freely falling particles. The geodesic equation computes the rates of change $\ddot{x}^\mu$ of the passage $\dot{x}^\mu$ of the coordinates past the particle, completely independent from the presence of curvature. Actually, neither the metric $g_{\mu\nu}$ nor the Christoffel-symbol $\Gamma^\alpha{}_{\mu\nu}$ do contain information about gravity, and neither does the covariant derivative ∇_μ: They are all constructed to deal with the arbitrariness of coordinate choices. In addition, we already know that the gravitational field does not exist at a single point, because both conditions $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\alpha{}_{\mu\nu} = 0$ can always be achieved **locally** by a coordinate transform.

💀 *$g_{\mu\nu} \neq \eta_{\mu\nu}$ does not imply that there is gravity, and neither does $\Gamma^\alpha{}_{\mu\nu} \neq 0$, and neither does $\partial_\mu \neq \nabla_\mu$!*

A possible idea would be to look at the relative motion of freely falling particles. Locally, every particle experiences perfect weightlessness, but that does not imply that the relative acceleration must be zero. Imagine two astronauts holding hands and falling through space(time) and following Keplerian orbits around the Earth. The astronaut on the lower orbit moves with a higher velocity according to Kepler's first law and would actually accelerate away from the astronaut in the higher orbit. Such an experiment could serve as an experiment to determine whether gravitational fields (or spacetime curvature) is present, because it is non-local and because it would be sensitive to the second derivatives $\partial^2 g$ of the metric, which partially resist coordinate transforms as they can not be made to vanish.

The quantity determining the relative acceleration between two freely falling particles is the Riemann-tensor,

$$\mathrm{R}^\alpha{}_{\mu\nu\beta} = \frac{\partial\Gamma^\alpha{}_{\mu\nu}}{\partial x^\beta} - \frac{\partial\Gamma^\alpha{}_{\mu\beta}}{\partial x^\nu} + \Gamma^\alpha{}_{\rho\beta}\,\Gamma^\rho{}_{\mu\nu} - \Gamma^\alpha{}_{\rho\nu}\,\Gamma^\rho{}_{\mu\beta}\,. \tag{B.190}$$

It is through $\partial\Gamma \sim \partial(g\partial g)$ composed of second derivatives of the metric which shows that it contains information about the manifold that can not be made to vanish by a coordinate transform. We will see in the next chapter that it contains all information about curvature of the manifold and the deviation from a Lorentzian geometry. In particular, the geodesic deviation equation

$$\frac{\mathrm{d}^2\upsilon^\alpha}{\mathrm{d}\lambda^2} = \mathrm{R}^\alpha{}_{\mu\nu\beta}\cdot u^\mu u^\nu \upsilon^\beta \tag{B.191}$$

defines the experiment one can test for the presence of gravitational fields. If there is no relative acceleration $d^2 v^\alpha / d\lambda^2 = 0$ for every index choice one must conclude that the Riemann curvature vanishes, $R^\alpha{}_{\mu\nu\beta} = 0$ and that the motion of the two test particles takes place in Minkowskian space, but possibly with a weird coordinate choice.