Z INDEX NOTATION

Z.1 *Vectors and linear forms*

Many quantities in physics have components, or internal degrees of freedom. This is particularly true in modern physics, with e.g. the realisation that the energy density $\rho c^2 = \mathrm{T}_t{}^t$ is part of the energy momentum-tensor $\mathrm{T}_\mu{}^\nu$ as a larger entity. The geometric picture is that there is a (vector)-space for all vectors $\boldsymbol{v} = v^i \boldsymbol{e}_i = \sum_i v^i \boldsymbol{e}_i$ which are decomposed into their components v^i with a basis $\boldsymbol{e}_i$, with the Einstein summation convention in place. Velocities $\boldsymbol{v}$, accelerations $\boldsymbol{a}$, the magnetic field **B** and the dielectric displacement **D** are examples of vectors. There is an associated (vector) space of linear forms $\boldsymbol{p} = p_i \boldsymbol{e}^i = \sum_i p_i \boldsymbol{e}^i$ which has identical geometric properties and is spanned by a basis $\boldsymbol{e}^i$. Examples of linear forms are, for instance, the canonical momentum $\boldsymbol{p}$, the gradient of a potential $\partial\Phi$, the electric field **E** or the magnetic induction **H**.

A very useful notation used throughout theoretical physics is the so-called abstract index notation, where one works entirely with the components of vectors and linear forms, with an implicitly assumed basis. By convention, one denotes vectors with a superscript, contravariant index v^i and linear forms with a subscript, covariant index p_j.

Canonically, one defines an orthogonality relation $\boldsymbol{e}^i \boldsymbol{e}_j = \delta^i_j$ between the basis vector of the vector space and the basis linear forms, such that the inner product between a vector $\boldsymbol{v}$ and a linear form $\boldsymbol{p}$ is given by

$$\boldsymbol{p} \cdot \boldsymbol{v} = p_i \boldsymbol{e}^i \, v^j \boldsymbol{e}_j = p_i v^j \, \boldsymbol{e}^i \boldsymbol{e}_j = p_i v^j \delta^i_j = p_i v^i. \tag{Z.637}$$

According to the Einstein sum convention (also called a contraction), an expression like $p_i v^i$ is to be interpreted as $\sum_i p_i v^i$, with an automatic implied summation over all index pairs which are appear as super- and subscripts.

A metric γ_{ij} is used for converting a vector v^j to its associated linear form $v_i = \gamma_{ij} v^j$, while the inverse metric γ^{ij} does the opposite: It translates a linear form p_j to its associated vector $p^i = \gamma^{ij} p_j$. Of course, making a linear form out of a vector and then translating it back to being a vector again can not change anything,

$$\gamma^{ij} \left(\gamma_{jk} v^k\right) = \underbrace{\gamma^{ij} \gamma_{jk}}_{=\delta^i_k} v^k = v^i \tag{Z.638}$$

and in this sense the metric and its inverse are related to each other:

$$\gamma^{ij} \gamma_{jk} = \delta^i_k. \tag{Z.639}$$

Instead of computing $p_i v^i$ directly as the contraction between a linear form p_i and the vector v^i, one can use the metric to generate the linear form p_i from a vector, $p_i = \gamma_{ij} p^j$ to arrive at

$$p_i v^i = \gamma_{ij} p^i v^i = \gamma^{ij} p_i v_j \tag{Z.640}$$

Alternatively, one can generate the vector $\upsilon^i = \gamma^{ij}\upsilon_j$ from the associated linear form υ_j using the inverse metric γ^{ij}. With this argument, one can say that the metric defines a scalar product between vectors, while the inverse metric defines a scalar product between linear forms. It is well worth it to differentiate carefully between the metric γ_{ij} and the Kronecker symbol δ^i_j, even in the case of Euclidean vector spaces. The Kronecker symbol renames indices of vectors or linear forms, but never changes them:

$$\upsilon^i = \delta^i_j \upsilon^j \quad \text{and} \quad p_i = \delta^j_i p_j \tag{Z.641}$$

Z.2 *Coordinates and differentials*

Coordinates are usually written as vectorial tuples x^i (in themselves, they are not vectors!), and this choice is purely conventional. The coordinates have the property that every entry of x^i can change independently from the others, so

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j \tag{Z.642}$$

because x^i only changes in the direction of x^i at unit speed, whereas x^i remains constant if x^j is changed. This is encapsulated by the Kronecker symbol δ^i_j. But in this sense, derivatives with respect to the coordinates $\partial_i = \partial/\partial x^i$ are linear forms,

$$\frac{\partial x^i}{\partial x^j} = \frac{\partial}{\partial x^j} x^i = \partial_j x^i = \delta^i_j \tag{Z.643}$$

and the contraction

$$\partial_i x^i = \frac{\partial x^i}{\partial x^i} = \delta^i_i = n \tag{Z.644}$$

is sensibly defined and returns the dimensionality n. Then, the divergence $\partial_i \upsilon^i$ of a vector υ^i is defined in a straightforward way, and the divergence of a linear form would be $\gamma^{ij}\partial_i p_j = \partial_i \gamma^{ij} p_j = \partial_i p^i$ with the inverse metric.

This point can be illustrated better by considering a curve $x^i(\lambda)$ which runs through a scalar field Φ: The derivative of Φ along the curve as λ evolves, is

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\partial \Phi}{\partial x^i} = \dot{x}^i \partial_i \Phi \tag{Z.645}$$

by virtue of the chain rule. We interpret this expression as the projection, or scalar product between the gradient $\partial_i\Phi$ of the potential as a linear form with the velocity $\dot{x}^i = \upsilon^i = \mathrm{d}x^i(\lambda)/\mathrm{d}\lambda$ as a vector.

Let's try out a change of coordinates with an invertible and differentiable replacement $x^i(y^a)$: The chain rule suggests that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\partial \Phi}{\partial x^i} = \left(\frac{\mathrm{d}y^a}{\mathrm{d}\lambda}\frac{\partial x^i}{\partial y^a}\right)\left(\frac{\partial y^b}{\partial x^i}\frac{\partial \Phi}{\partial y^b}\right) = \frac{\mathrm{d}y^a}{\mathrm{d}\lambda}\underbrace{\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^i}}_{=\delta^b_a}\frac{\partial \Phi}{\partial y^b} = \frac{\mathrm{d}y^a}{\mathrm{d}\lambda}\frac{\partial \Phi}{\partial y^a} = \dot{y}^a \partial_a \Phi \tag{Z.646}$$

such that the rate $\mathrm{d}\Phi/\mathrm{d}\lambda$ is unchanged, no matter which coordinates have been used to compute the velocity and the gradient. This is achieved because the Jacobian $\partial x^i/\partial y^a$ used to transform the vectorial velocity and $\partial y^b/\partial x^i$ for the transformation of the potential gradient as a linear form are inverses to each other:

$$\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^i} = \frac{\partial y^b}{\partial y^a} = \delta^b_a \tag{Z.647}$$

by recognising that the expression originates from $\partial y^b/\partial y^a$ from an intermediate differentiation with respect to x^i as dictated by the chain rule. With the latter relation it becomes clear that even though the coordinates x^i are not (yet) a vector, the velocity $\upsilon^i = \mathrm{d}x^i/\mathrm{d}\lambda$ as the derivative is, and the gradient $\partial\Phi/\partial x^i$ is truly a linear form: Both have the correct transformation properties. Vectors such as the velocity transforms according to $\upsilon^i \rightarrow \mathrm{J}^i{}_a\upsilon^a = \partial x^i/\partial y^a\ \upsilon^a$, and linear forms inversely, $p_i \rightarrow \mathrm{J}^a{}_i p_a = \partial y^a/\partial x^i\ p_a$. Indeed, in differential geometry all quantities (scalars, vectors, linear forms, tensors of various rank and valence) are defined through their transformation behaviour.

The Kronecker symbol arises as the fundamental property of the coordinates y^a then makes sure that only equal indices are considered in multiplying $\dot{y}^a\delta^b_a\partial_b\Phi = \dot{y}^a\partial_a\Phi$. This neat cancellation would not automatically take place in scalar products between two vectors: Defining the Jacobian $\mathrm{J}^i{}_a = \partial x^i/\partial y^a$ suggests the transformation $\upsilon^i \rightarrow \mathrm{J}^i{}_a\upsilon^a$, and the scalar product $\gamma_{ij}\upsilon^i\upsilon^j$ can only be invariant if the metric transforms inversely (defining an orthogonal transform), $\gamma_{ij} \rightarrow \mathrm{J}_i{}^a\mathrm{J}_j{}^b\gamma_{ab}$ with the inverse Jacobian $\mathrm{J}_i{}^a$:

$$\mathrm{J}^i{}_a\mathrm{J}_i{}^b = \frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^i} = \delta^b_a \tag{Z.648}$$

such that scalar products are in fact invariant:

$$\mathrm{J}_i{}^a\mathrm{J}_j{}^b\gamma_{ab}\ \mathrm{J}^i{}_c\upsilon^c\ \mathrm{J}^j{}_d\upsilon^d = \mathrm{J}_i{}^a\mathrm{J}^i{}_c\ \mathrm{J}_j{}^b\mathrm{J}^j{}_d\ \gamma_{ab}\upsilon^c\upsilon^d = \delta^a_c\delta^b_d\gamma_{ab}\upsilon^c\upsilon^d = \gamma_{ab}\ \delta^a_c\upsilon^c\ \delta^b_d\upsilon^d = \gamma_{ab}\upsilon^a\upsilon^b. \tag{Z.649}$$

The same argument applies to the invariance of the scalar product $\gamma^{ij}p_ip_j$, only that the Jacobians now transforms the inverse metric γ^{ij} and the inverse Jacobians the linear forms p_i:

$$\mathrm{J}^i{}_a\mathrm{J}^j{}_b\gamma^{ab}\mathrm{J}_i{}^c p_c\mathrm{J}_j{}^d p_d = \mathrm{J}^i{}_a\mathrm{J}_i{}^c\ \mathrm{J}^j{}_b\mathrm{J}_j{}^d\ \gamma^{ab}p_cp_d = \delta^c_a\delta^d_b\gamma^{ab}p_cp_d = \gamma^{ab}\ \delta^c_a p_c\ \delta^d_b p_d = \gamma^{ab}p_ap_b \tag{Z.650}$$

The transformation properties of the metric and its inverse show that they are in fact tensors of rank 2.

Z.3 *Lagrange- and Hamilton-formalism in components*

If one chooses the coordinates to be summarised in a vectorial tuple x^i, the velocity $\dot{x}^i = \mathrm{d}x^i/\mathrm{d}t$ and the acceleration $\ddot{x}^i = \mathrm{d}^2x^i/\mathrm{d}t^2$ are vectors as well. The construction of a scalar quantity like the Lagrange function requires the metric γ_{ij} for the kinetic term,

$$\mathcal{L}(x^i, \dot{x}^i) = \frac{m}{2}\gamma_{ij}\dot{x}^i\dot{x}^j - \Phi(x^i) \tag{Z.651}$$

as well as the potential Φ. Variation with the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}^a} = \frac{\partial\mathcal{L}}{\partial x^a} \tag{Z.652}$$

leads to

$$\frac{\partial \mathrm{L}}{\partial\dot{x}^a} = \frac{m}{2}\gamma_{ij}\Big(\underbrace{\frac{\partial\dot{x}^i}{\partial\dot{x}^a}}_{=\delta^i_a}\dot{x}^j + \dot{x}^i\underbrace{\frac{\partial\dot{x}^j}{\partial\dot{x}^a}}_{=\delta^j_a}\Big) = \frac{m}{2}\left(\gamma_{aj}\dot{x}^j + \gamma_{ia}\dot{x}^i\right) = m\gamma_{aj}\dot{x}^j \tag{Z.653}$$

because the metric is symmetric, $\gamma_{ia} = \gamma_{ai}$, and any internal index in an expression can be renamed. Together with

$$\frac{\partial\mathcal{L}}{\partial x^a} = -\frac{\partial\Phi}{\partial x^a} \tag{Z.654}$$

one arrives at the Newtonian equation of motion

$$m\gamma_{aj}\ddot{x}^j = -\frac{\partial\Phi}{\partial x^a} \tag{Z.655}$$

which can be brought into a more familiar shape by multiplying both sides with the inverse metric γ^{ia}:

$$m\gamma^{ia}\gamma_{aj}\ddot{x}^j = m\delta^i_j\ddot{x}^j = m\ddot{x}^i = -\gamma^{ia}\frac{\partial\Phi}{\partial x^a} = -\gamma^{ia}\partial_a\Phi \quad\rightarrow\quad m\ddot{x}^i = -\gamma^{ia}\partial_a\Phi \tag{Z.656}$$

with $\gamma^{ia}\gamma_{aj} = \delta^i_j$ such that the inverse metric relates the gradient of the potential, itself a linear form, to the acceleration as a vector.

The canonical momentum,

$$p_i = \frac{\partial\mathcal{L}}{\partial\dot{x}^i} \tag{Z.657}$$

is, by this reasoning, a linear form (and a function $p_j(\dot{x}^i)$ of the vectorial velocity $\dot{x}^i$, which can be inverted to yield $\dot{x}^i(p_j)$ for convex Lagrange-functions), so that the Legendre transform

$$\mathcal{H} = p_i\dot{x}^i(p_j) - \mathcal{L}(x^i, \dot{x}^i(p_j)) \tag{Z.658}$$

is sensibly defined and yields a scalar Hamilton function. The contraction of the vectorial velocity $\dot{x}^i$ with the linear form p_i appears naturally. And it provides an argument, why the canonical momentum $p_i = \partial\mathcal{L}/\partial\dot{x}^i$ is more than just the kinetic momentum $m\dot{x}^i$: On the contrary, with the definition of the canonical momentum p_i one obtains for a standard form of the Lagrange-function

$$p_i = m\gamma_{ij}\dot{x}^i \quad \text{and consequently,} \quad \dot{p}_i = -\partial_i\Phi \tag{Z.659}$$

from the Euler-Lagrange equation, showing how the metric is necessary, in one way or another, to mediate between velocity and acceleration as vectorial quantities on one side and momentum and potential gradient as linear forms on the other, even in the case of a standard kinetic term in the Lagrange-function. Hamilton's equations of motion

$$\dot{p}_i = -\frac{\partial\mathcal{H}}{\partial x^i} \quad \text{and} \quad \dot{x}^i = +\frac{\partial\mathcal{H}}{\partial p_i} \tag{Z.660}$$

remain consistent as the derivative with respect to a vector is a linear form, while the derivative with respect to a linear form returns again a vector: $\partial p_i/\partial p_j = \delta_i^j$ for p_i as a phase space coordinate. Please note how $\dot{p}_i$ as a linear form emerges from $-\partial\mathrm{H}/\partial x^i = -\partial\Phi/\partial x^i$ without a metric in contrast to equation (Z.655), in a consistent variant of Newton's second law: $\dot{p}_i = -\partial_i\Phi$.

Z.4 *Duals*

The cross product $\boldsymbol{x}\times\boldsymbol{y}$ between two vectors is defined in terms of their basis decomposition as

$$\boldsymbol{x}\times\boldsymbol{y} = x^j\boldsymbol{e}_j \times y^k\boldsymbol{e}_k = x^j y^k\;\boldsymbol{e}_j\times\boldsymbol{e}_k = x^j y^k\;\epsilon_{ijk}\boldsymbol{e}^i = \underbrace{\epsilon_{ijk}x^j y^k}_{=(\boldsymbol{x}\times\boldsymbol{y})_i}\boldsymbol{e}^i, \tag{Z.661}$$

with the Levi-Civita symbol as an expression of the right-handed orientation of the (orthogonal) basis system. Therefore, cross product $\epsilon_{ijk}x^j y^k$ is naturally a linear form, but is it possible to construct a naturally antisymmetric quantity out of the vectors x^j and y^k as a vectorial object? Clearly, the antisymmetric rank-2 tensor $x^j y^k - x^k y^j$ would be such a thing, and would be, up to a factor of two, equal to the cross product:

$$\epsilon_{ijk}\left(x^j y^k - x^k y^j\right) = \epsilon_{ijk}x^j y^k - \epsilon_{ikj}x^j y^k = \left(\epsilon_{ijk} - \epsilon_{ikj}\right)x^j y^k = 2\epsilon_{ijk}x^j y^k \tag{Z.662}$$

where in the first step the indices are interchanged $j \leftrightarrow k$, and then the property $\epsilon_{ijk} = -\epsilon_{ikj}$ is used. $(x^j y^k - x^k y^j)/2$ is called the dual, and the usability hinges heavily on the fact that the contraction of two antisymmetric objects is nonzero. The dual $x^j y^k - x^k y^j$ is a vectorial (antisymmetric) tensor that contains the same information as the linear form resulting from $\epsilon_{ijk}x^j y^k$. Duals can be defined for any antisymmetric tensor, for instance $\tilde{\mathrm{G}}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\mathrm{G}^{\mu\nu}$. They convert all Maxwell-equations into divergences, as

$$\epsilon^{ijk}\partial_j\mathrm{E}_k = \partial_j\left(\epsilon^{ijk}\mathrm{E}_k\right) = \partial_j\mathrm{E}^{ij} = -\partial_{ct}\mathrm{B}^i, \tag{Z.663}$$

as exemplified by the induction equation.

Z.5 *Gauß- and Stokes-theorems*

The Gauß-theorem relates the volume integral over the divergence of a vector field to the integral of that particular vector field over the surface bounding the volume,

$$\int_V \mathrm{dV}\,\partial_i \mathrm{D}^i = \int_{\partial V} \mathrm{dS}_i\,\mathrm{D}^i \quad \text{and} \quad \int_V \mathrm{dV}\,\partial_i \mathrm{B}^i = \int_{\partial V} \mathrm{dS}_i\,\mathrm{B}^i, \tag{Z.664}$$

where in electrodynamics the relation gets applied to the two vector fields D^i and B^i. The surface element dS_i is a linear form, because it originates from the cross product of two vectors. Similarly, the Stokes-theorem relates the surface integral of the rotation of a field to the line integral along the boundary,

$$\int_S \mathrm{dS}_i\,\epsilon^{ijk}\partial_j \mathrm{E}_k = \int_{\partial S} \mathrm{d}r^i\,\mathrm{E}_i \quad \text{and} \quad \int_S \mathrm{dS}_i\,\epsilon^{ijk}\partial_j \mathrm{H}_k = \int_{\partial S} \mathrm{d}r^i\,\mathrm{H}_i, \tag{Z.665}$$

where in electrodynamics this becomes relevant for the two linear forms E_i and H_i. It is a bit remarkable that the assignment of vectors and linear forms to the fields in Maxwell's equations only needs as geometric objects the differential ∂_i and the associated surface element dS_i as linear forms, and never their possible vectorial counterparts. The Gauß-theorem gets only ever applied to the vectors D^i and B^i, whereas the application of the Stokes-theorem is restricted to the linear forms E_i and H_i. This, in fact, is a hint that electrodynamics would work even on non-metric spacetimes, because the metric (and its inverse) would be a mean to convert between the two types of fields.

Z.6 *Summary of co- and contravariant quantities in electrodynamics*

0. rank 0: scalars and pseudoscalars

Φ	electric potential
θ	axion field amplitude
ρ	electric charge density
dV	volume element

1. rank 1: vectors and linear forms

x^i	Euclidean coordinates	∂_i	coordinate differential
$\dot{x}^i$	velocity	p_i	momentum
$\ddot{x}^i$	acceleration	$\partial_i\Phi$	potential gradient
D^i	dielectric displacement	E_i	electric field
B^i	magnetic field	H_i	magnetic induction
		A_i	vector potential
P^i	Poynting vector	Y_i	Poynting linear form
$\jmath^i$	electric current density	dS_i	surface element

x^μ	Minkowski coordinates	∂_μ	coordinate differential
u^μ	4-velocity	p_μ	4-momentum
		A_μ	4-potential
$\jmath^\mu$	4-current density		

2. rank 2: co-, contravariant and mixed tensors

γ^{ij}	inverse Euclidean metric	γ_{ij}	Euclidean metric
ϵ^{ij}	permissivity tensor	ϵ_{ij}	inverse permissivity
μ^{ij}	permeability tensor	μ_{ij}	inverse permeability
σ^{ij}	conductivity		

$\eta^{\mu\nu}$	inverse Minkowski metric	$\eta_{\mu\nu}$	Minkowski metric
$G^{\mu\nu}$	excitation	$F_{\mu\nu}$	Faraday tensor
$\tilde{F}^{\mu\nu}$	Faraday dual	$\tilde{G}_{\mu\nu}$	excitation dual

δ_i^j, δ_ν^μ	Kronecker-symbol
${T_i}^j$	Maxwell stress tensor
${T_\mu}^\nu$	energy-momentum tensor
${\Lambda^i}_j$	endomorphism for vectors $v^i \to {\Lambda^i}_j v^j$
${\Lambda_i}^j$	endomorphisms for linear forms $p_i \to {\Lambda_i}^j p_j$