Y COMPLEX CALCULUS

Y.1 *Aspects of complex differentiability*

Many of the integrals needed for the construction of a Green-function with the Fourier method are not solvable with elementary methods, i.e. integration by substitution, by parts or using partial fractions, for instance

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega \, \frac{1}{(ck)^2 - \omega^2} \exp\left(-\mathrm{i}\omega(t-t')\right) \tag{Y.584}$$

which shows two singularities at $\omega = \pm ck$. Methods from complex analysis, though, provide a pathway of doing that.

A function $g(z) = u(x,y) + \mathrm{i}v(x,y)$ maps a complex argument $z = x + \mathrm{i}y$ onto a complex value $g = u + \mathrm{i}y$. It is continuous in ζ if there is an $\epsilon > 0$ for every $\delta > 0$ such that $|g(z) - g(\zeta)| < \epsilon$ follows form $|z - \zeta| < \delta$. In other words, the limit

$$\lim_{\zeta \to z} |g(z) - g(\zeta)| = 0 \tag{Y.585}$$

does not depend on the way how ζ approaches z. The function $g(z)$ is complex differentiable in z, if the limit

$$\lim_{\zeta \to z} \frac{g(z) - g(\zeta)}{z - \zeta} = \frac{\mathrm{d}g}{\mathrm{d}z}(z) \tag{Y.586}$$

exists and is unique, or in other words: if the differential quotient is continuous.

Complex differentiability is a weird and very powerful concept. Historically, four different aspects have been discovered which turn out to be identical and merely different sides of the same idea: (i) complex differentiable, (ii) analytical, meaning that the Cauchy-Riemann differential equations hold, (iii) regular, defined as a vanishing loop integral over closed curves, and (iv) holomorphic, meaning that the function fulfils the residue theorem. An weidly enough, it blurs the boundaries between integration and differentiation, as exemplified by the Cauchy-theorem. Fundamentally, it is yet another example of the powerful concept of exact differentials.

Y.2 *Cauchy-Riemann differential equations*

In a complex differentiable function, the derivative does not depend on the direction how Δz, itself a complex number, approaches zero,

$$\frac{\mathrm{d}g}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z}. \tag{Y.587}$$

Therefore, the derivative in x-direction parallel to the real axis,

$$\lim_{\Delta x \to 0} \frac{g(z + \Delta x) - g(z)}{\Delta x} = \frac{\partial g}{\partial x} \tag{Y.588}$$

and the derivative in the y-direction parallel to the imaginary axis,

$$\lim_{\Delta y\to 0}\frac{g(z+\mathrm{i}\Delta y)-g(z)}{\mathrm{i}\Delta y}=\frac{1}{\mathrm{i}}\frac{\partial g}{\partial y} \tag{Y.589}$$

must be equal. Writing this relation in terms of the components of g yields

$$\frac{\partial g}{\partial x}=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}=\frac{1}{\mathrm{i}}\frac{\partial g}{\partial y}=\frac{1}{\mathrm{i}}\frac{\partial u}{\partial y}+\frac{\partial g}{\partial y} \tag{Y.590}$$

and with a subsequent separation of the real and imaginary parts on arrives at the Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial x}=+\frac{\partial v}{\partial y}\quad\text{and}\quad\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}. \tag{Y.591}$$

The notions of complex differentiability and the fulfilment of the Cauchy-Riemann differential equations is equivalent.

Y.3 *Complex line and loop integrals*

Given a curve Γ parameterised with λ running from point A with coordinates $z(a)$ to point B at $z(b)$, one can define a complex line integral by reducing it to an integral over the parameter by substitution,

$$\int_{\Gamma_{\mathrm{AB}}}\mathrm{d}z\,g(z)=\int_a^b\mathrm{d}\lambda\,\frac{\mathrm{d}z}{\mathrm{d}\lambda}g(z(\lambda)). \tag{Y.592}$$

Covering the same path in opposite direction yields the same numerical result, but with a negative sign

$$\int_{\Gamma_{\mathrm{BA}}}\mathrm{d}z\,g(z)=\int_b^a\mathrm{d}\lambda\,\frac{\mathrm{d}z}{\mathrm{d}\lambda}g(z(\lambda))=-\int_a^b\mathrm{d}\lambda\,\frac{\mathrm{d}z}{\mathrm{d}\lambda}g(z(\lambda)) \tag{Y.593}$$

If an integral does not depend on the particular path from A to B, one can assemble a trip from A to B on one path followed by a return trip from B to A on another path, with the two contributions cancelling each other, with the overall result being

$$\int_{\Gamma_{\mathrm{AB}}}\mathrm{d}z\,g(z)+\int_{\Gamma_{\mathrm{BA}}}\mathrm{d}z\,g(z)=\oint_{\Gamma}\mathrm{d}z\,g(z)=0 \tag{Y.594}$$

Just as before, traversing a closed loop in the opposite sense of rotation would yield an overall minus sign. Writing this relation component-wise

$$\oint_{\Gamma}\mathrm{d}z\,g(z)=\oint_{\Gamma}(\mathrm{d}x+\mathrm{i}\mathrm{d}y)\,(u+\mathrm{i}v)=\oint_{\Gamma}(u\mathrm{d}x-v\mathrm{d}y)+\mathrm{i}\oint_{\Gamma}(v\mathrm{d}x+u\mathrm{d}y)=0 \tag{Y.595}$$

Both terms can be reformulated as area integrals by virtue of Green's theorem, $\partial\mathcal{C} = \Gamma$,

$$\oint_\Gamma (u\mathrm{d}x - v\mathrm{d}y) = -\int_\mathcal{C} \mathrm{d}x\mathrm{d}y \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \quad \text{and} \quad \oint_\Gamma (v\mathrm{d}x + u\mathrm{d}y) = \int_\mathcal{C} \mathrm{d}x\mathrm{d}y \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{Y.596}$$

where one immediately recognises the Cauchy-Riemann equations in the integrands, making both results vanish. In summary,

$$\oint_\Gamma \mathrm{d}z\, g(z) = 0 \tag{Y.597}$$

for any complex differentiable function. ✈ Green's theorem, which allows the conversion of a loop integral to an area integral works for simply connected regions.

Y.4 *Residue theorem and holomorphic functions*

The Cauchy-theorem states that every value of a complex differentiable function inside a closed curve Γ is fixed by the values on that curve,

$$g(z) = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta)}{\zeta - z} \tag{Y.598}$$

Functions with that property are called holomorphic, which is synonymous to complex differentiable. In fact,

$$\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta)}{\zeta - z} = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta) \overbrace{-g(z) + g(z)}^{=0}}{\zeta - z} =$$

$$\frac{g(z)}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{1}{\zeta - z} + \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta) - g(z)}{\zeta - z} = g(z), \tag{Y.599}$$

after reordering the terms and using that $\oint \mathrm{d}\zeta\, g(z) \ldots = g(z) \oint \mathrm{d}\zeta \ldots$. The first term can be shown to be

$$\oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta} = \oint_\Gamma \mathrm{d}\ln\zeta = \int_0^{2\pi} \mathrm{d}\lambda\, \frac{\mathrm{d}\zeta}{\mathrm{d}\lambda} \frac{1}{\zeta} = \mathrm{i} \int_0^{2\pi} \mathrm{d}\lambda\, \exp(\mathrm{i}\lambda)\exp(-\mathrm{i}\lambda) = \mathrm{i} \int_0^{2\pi} \mathrm{d}\lambda = 2\pi\mathrm{i} \tag{Y.600}$$

after substitution $\zeta - z \rightarrow \zeta$, which can then be solved by choosing the unit circle $\zeta = \exp(\mathrm{i}\lambda)$ with $\mathrm{d}\zeta = \mathrm{i}\exp(\mathrm{i}\lambda)\mathrm{d}\lambda = \mathrm{i}\zeta\mathrm{d}\lambda$ as the integration contour. The second integral can be treated like this:

$$\left|\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta) - g(z)}{\zeta - z}\right| \leq \frac{1}{|2\pi\mathrm{i}|} \left|\oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta) - g(z)}{\zeta - z}\right| \leq \frac{\epsilon}{|2\pi\mathrm{i}|} \left|\oint_\Gamma \mathrm{d}\zeta\, \frac{1}{\zeta - z}\right| = \epsilon \tag{Y.601}$$

if the function g is continuous, which is quite obvious as it is already assumed to be complex differentiable: Then, the integration contour can be chosen to be small enough such that $|g(\zeta) - g(z)| < \epsilon$. In addition, the integral was already shown to be $2\pi i$. Overall, the second integral is bounded by ϵ, and does effectively does not contribute, as ϵ can be chosen to be arbitrarily small.

It is worth memorising the iconic result

$$\frac{1}{2\pi i}\oint_\Gamma \frac{d\zeta}{\zeta} = 1, \tag{Y.602}$$

but what about other powers in ζ? Clearly, for both positive and negative n, as long as $n \neq -1$,

$$\oint d\zeta\, \zeta^n = \int_0^{2\pi} d\lambda\, \frac{d\zeta}{d\lambda}\zeta^n = i\int_0^{2\pi} d\lambda\, \exp(i\lambda)\exp(in\lambda) = \\ i\int_0^{2\pi} d\lambda\, \exp(i(n+1)\lambda) = \frac{\exp(i(n+1)\lambda)}{n+1}\Big|_0^{2\pi} = 0 \tag{Y.603}$$

from elementary integration, again with the parameterised unit circle $\exp(i\lambda)$ as the integration contour. But alternatively, one could argue that the plane waves form an orthonormal system. Therefore, only for $n = -1$ one gets a nonzero result.

The Cauchy-theorem can be generalised to higher-order derivatives: Starting with a Taylor-expansion of $g(\zeta)$ around z,

$$g(\zeta) = g(z) + \frac{dg}{d\zeta}\Big|_z(\zeta - z) + \frac{d^2g}{dz^2}\Big|_z\frac{(\zeta - z)^2}{2} + \cdots \tag{Y.604}$$

Using the results from above, one can isolate $g(z)$ from the series by multiplying it with $1/(\zeta - z)$, followed by a loop integration comprising z:

$$\oint_\Gamma d\zeta\, \frac{g(\zeta)}{\zeta - z} = g(z)\underbrace{\oint_\Gamma d\zeta\, \frac{1}{\zeta - z}}_{=2\pi i} + \frac{dg}{dz}\Big|_z\underbrace{\oint_\Gamma d\zeta\, \frac{\zeta - z}{\zeta - z}}_{=0} + \frac{1}{2}\frac{d^2g}{dz^2}\Big|_z\underbrace{\oint_\Gamma d\zeta\, \frac{(\zeta - z)^2}{\zeta - z}}_{=0} + \cdots \tag{Y.605}$$

For accessing a higher order derivative, for instance dg/dz, one would need to multiply the series by $1/(\zeta - z)^2$ before integrating,

$$\oint_\Gamma d\zeta\, \frac{g(\zeta)}{(\zeta - z)^2} = g(z)\underbrace{\oint_\Gamma d\zeta\, \frac{1}{(\zeta - z)^2}}_{=0} + \frac{dg}{dz}\Big|_z\underbrace{\oint_\Gamma d\zeta\, \frac{\zeta - z}{(\zeta - z)^2}}_{=2\pi i} + \frac{1}{2}\frac{d^2g}{dz^2}\Big|_z\underbrace{\oint_\Gamma d\zeta\, \frac{(\zeta - z)^2}{(\zeta - z)^2}}_{=0} + \cdots \tag{Y.606}$$

This pattern generalises to the Cauchy-theorem for derivatives of $g(z)$,

$$\left.\frac{\mathrm{d}^n g}{\mathrm{d}z^n}\right|_z = \frac{n!}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta \, \frac{g(\zeta)}{(\zeta - z)^{n+1}}, \tag{Y.607}$$

with the interesting implication that derivatives of a complex differentiable function can be obtained through an integration process. If a function is complex differentiable once, it is complex differentiable arbitrarily often, in stark contrast to real differentiability.

The Cauchy-theorem can be applied in the solution of real-valued integrals that can not be solved (easily) by means of elementary integration. A classic example of this is

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{1}{1+x^2} = \arctan x\Big|_{-\infty}^{+\infty} = \pi \tag{Y.608}$$

where a solution is only possibly by using the rule of the derivative of the inverse function and trigonometric identities. Instead, one can perform a complex continuation,

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{1}{1+x^2} \to \int_{-\infty}^{+\infty} \mathrm{d}z \, \frac{1}{1+z^2} \tag{Y.609}$$

where x is interpreted as a complex-valued variable z. The denominator has two poles at $z = \pm\mathrm{i}$, allowing a decomposition into partial fractions,

$$\frac{1}{1+z^2} = \frac{1}{(1+z)(1-z)} = \frac{1}{2\mathrm{i}}\left(\frac{1}{z-\mathrm{i}} - \frac{1}{z+\mathrm{i}}\right), \tag{Y.610}$$

and the integration along the real axis from $-\infty$ to $+\infty$ can be extended by an semi-circular arc, which does not contribute to the value of the integrand, as its arc length increases with radius, but the value of the integrand decreases proportional to the squared radius. This arc now makes the integration a complex loop integral, so that we can write

$$\oint_\Gamma \mathrm{d}z \, \frac{1}{z^2+1} = \frac{1}{2\mathrm{i}} \underbrace{\oint_{\frown} \frac{\mathrm{d}z}{z-\mathrm{i}}}_{=2\pi} - \frac{1}{2\mathrm{i}} \underbrace{\oint_{\frown} \Gamma \frac{\mathrm{d}z}{z+\mathrm{i}}}_{=0} = \pi \tag{Y.611}$$

because only the pole at $z = +\mathrm{i}$ is contained inside the integration contour.

One would have arrived at exactly the same result if the arc had been closed at the bottom instead of the top: Then, the sense in which the curve Γ is traversed, is inverted, yielding a negative sign:

$$\oint_\Gamma \mathrm{d}z \, \frac{1}{z^2+1} = \frac{1}{2\mathrm{i}} \underbrace{\oint_{\smile} \frac{\mathrm{d}z}{z-\mathrm{i}}}_{=0} - \frac{1}{2\mathrm{i}} \underbrace{\oint_{\smile} \frac{\mathrm{d}z}{z+\mathrm{i}}}_{=-2\pi} = \pi \tag{Y.612}$$

as now the other pole at $z = -\mathrm{i}$ is caught by the integration contour.

Y.5 *Laurent-series*

In the example above we have already embedded a function of a single, real-valued variable into the complex plane, and consider it to a (differentiable) mapping between complex numbers. This idea can be generalised in analytical continuations of a complex function $g(z)$, in cases where it is known in a region around z_0 to a second region around z bounded by Δ. There, the Cauchy-relation

$$g(z) = \frac{1}{2\pi\mathrm{i}}\oint_\Delta \mathrm{d}\zeta\,\frac{g(\zeta)}{\zeta - z} \quad\text{and}\quad \frac{\mathrm{d}^n g}{\mathrm{d}z^n}\Big|_z = \frac{n!}{2\pi\mathrm{i}}\oint_\Delta \mathrm{d}\zeta\,\frac{g(\zeta)}{(\zeta - z)^{n+1}} \tag{Y.613}$$

for any Γ circling the point z allows to access the values of g and its derivatives. The function and its derivatives at z_0 can be used to construct a power series that extends from a region around z_0 to z and defines the continuation of the function in this *terra incognita* bounded by Δ.

The function's values inside Δ are fixed by the Cauchy-theorem, and one can assemble an integration path consisting of two concentric loops Γ_1 (with radius r_1) and Γ_2 with radius r_2, joined by two bridges A_1 and A_2. This integration path replaces Δ, as it would result from continuous deformation within the holomorphic region. Then, $g(z)$ can be computed as

$$g(z) = \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_2} \mathrm{d}\zeta\frac{g(\zeta)}{\zeta - z} - \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_1} \mathrm{d}\zeta\frac{g(\zeta)}{\zeta - z}, \tag{Y.614}$$

because the contributions along A_1 and A_2 cancel each other due to the opposite direction in which they are traversed. Please note that the second loop Γ_1 contributes with a minus sign as the integration path is followed in a clockwise direction, i.e. in the mathematically negative sense. From the two integrals, the secoond one vanishes because of the Cauchy-theorem because z is outside Γ_1, but the first integral gives a non-vanishing result, with z being contained in Γ_2.

In our construction, the values of ζ traversed in the integration along the large loop Γ_2 have a modulus of r_2. Then, one can argue that

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0}\sum_n \left(\frac{z - z_0}{\zeta - z_0}\right)^n \tag{Y.615}$$

where in the last step we replaced the $1/(1-q)$-term with its corresponding geometric series. There is no issue of convergence of

$$\sum_n q^n = \frac{1}{1-q} \quad\text{because}\quad q = \left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{r}{r_2} < 1 \tag{Y.616}$$

Conversely, if ζ is situated on the loop Γ_1 with radius r_1, an analogous argument applies, as

$$\frac{1}{\zeta - z} = \frac{1}{z - z_0}\frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = \frac{1}{z - z_0}\sum_n \left(\frac{\zeta - z_0}{z - z_0}\right)^n. \tag{Y.617}$$

In this case, convergence of the geometric series is ensured by

$$\sum_n p^n = \frac{1}{1-p} \quad \text{where} \quad p = \left|\frac{\zeta - z_0}{z - z_0}\right| = \frac{r_1}{r} < 1 \tag{Y.618}$$

Collecting these results leads to

$$g(z) = \frac{1}{2\pi i}\oint_{\Gamma_2} d\zeta\, g(\zeta) \sum_n \frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}} - \frac{1}{2\pi i}\oint_{\Gamma_1} d\zeta\, g(\zeta)\sum_n \frac{(\zeta - z_0)^n}{(z-z_0)^{n+1}}. \tag{Y.619}$$

It is an interesting realisation that the two fractions are inverses of each other, leading to a natural continuation of the series towards negative n. Reordering integration and summation yields:

$$g(z) = \sum_n \left(\frac{1}{2\pi i}\oint_{\Gamma_2} d\zeta\, \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}\right) \times (z-z_0)^n - \sum_n \left(\frac{1}{2\pi i}\oint_{\Gamma_1} d\zeta\, \frac{g(\zeta)}{(\zeta - z_0)^{-n}}\right) \times (z-z_0)^{-(n+1)}. \tag{Y.620}$$

In summary, this result can be rewritten

$$g(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n \quad \text{with} \quad a_n = \frac{1}{2\pi i}\oint_{\Gamma} d\zeta\, \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}, \tag{Y.621}$$

for any close curve running between Γ_1 and Γ_2, where the minus-sign is cancelled by choosing a joint sense of rotation for the integration loop. This result is known as the Laurent-series, a power-law expansion of holomorphic functions, with its remarkable negative powers.

Y.6 *Residue theorem*

Looking at the Laurent series for $g(z)$,

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots + \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots, \tag{Y.622}$$

all terms belonging to positive indices $n \geq 0$ remain finite in the limit $z \to z_0$, while the terms for negative $n \neq 0$ are divergent. The function $g(z)$ would possess a pole of order $-n$ at z_0 if the Laurent series terminates at finite $-n$. Please note that the Laurent-series is constructed in a consistent way: Applying

$$\frac{1}{2\pi i}\oint_{\Gamma} dz\, \ldots \tag{Y.623}$$

to both sides yields for the terms with positive exponents $n \geq 0$,

$$\frac{1}{2\pi i}\oint_{\Gamma} dz\, (z-z_0)^n = 0, \tag{Y.624}$$

and similarly for the negative exponents with $n \geq 2$,

$$\frac{1}{2\pi i}\oint_\Gamma dz\,\frac{1}{(z-z_0)^n} = 0, \tag{Y.625}$$

whereas only the term for $n = -1$ yields a non-vanishing result, namely:

$$\frac{1}{2\pi i}\oint_\Gamma dz\,\frac{1}{z-z_0} = 1. \tag{Y.626}$$

The particular coefficient corresponding to $n = -1$ of the Laurent series,

$$a_{-1} = \frac{1}{2\pi i}\oint_\Gamma d\zeta\, g(\zeta), \tag{Y.627}$$

is called the residue of $g(z)$ at z_0, which needs to be located within Γ.

Y.7 *Conformal mappings*

Analytical (or complex differentiable, or regular, or holomorphic) functions automatically fulfil the Laplace-equation $\Delta g = 0$ in two dimensions and, as such, are viable solutions to the field equation ini vacuum. Starting with $g(z) = u(x,y) + iv(x,y)$ we write:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\underbrace{\frac{\partial u}{\partial v}}_{=\partial v/\partial y} = \frac{\partial}{\partial y}\underbrace{\frac{\partial v}{\partial x}}_{=-\partial u/\partial y} = -\frac{\partial^2 y}{\partial x^2} \quad\rightarrow\quad \Delta u = 0 \tag{Y.628}$$

taking advantage of the fact that partial differentiations interchange and substituting the Cauchy-Riemann equations twice. Conversely, one shows for the imaginary part

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\underbrace{\frac{\partial v}{\partial x}}_{=-\partial u/\partial y} = -\frac{\partial}{\partial y}\underbrace{\frac{\partial u}{\partial x}}_{=\partial v/\partial y} = -\frac{\partial^2 v}{\partial y^2} \quad\rightarrow\quad \Delta v = 0 \tag{Y.629}$$

from which we conclude that $\Delta u + i\Delta v = \Delta(u + iv) = \Delta g = 0$. In addition, as complex conjugation is a linear operation, it is valid that $\Delta g^* = 0$.

Clearly, the solution to the field equation $\Delta\Phi = 0$ in electrostatics in vacuum or to the field equations $\Delta A_i = 0$ for all three components A_i of the vector potential in Coulomb-gauge in magnetostatics, again in vacuum, could be represented by a holomorphic function. One needs to keep in mind, though, that g is a complex number with two components, whereas the potentials are real numbers. Hence the question arises, what the other component Ψ of $g = \Phi + i\Psi$ could represent!

If one were to identify Φ with the real value of g, it would need to represent the electric field $E_i = -\partial\Phi/\partial x^i$ as the gradient of Φ. It is possible to re-express the electric field as a complex number $E_x + iE_y = -\partial\Phi$ with the Wirtinger derivative instead of $E_i = -\partial_i\Phi$ in Cartesian coordinates. Additionally, there seems to be an auxiliary field Ψ, called the stream function, to be identified as $\Psi = v$.

The stream function is always perpendicular to lines of constant potential, which can be seen by this argument: The gradients ∇u and ∇v are clearly perpendicular,

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\underbrace{\frac{\partial v}{\partial x}}_{=-\partial u/\partial y} + \frac{\partial u}{\partial y}\underbrace{\frac{\partial v}{\partial y}}_{=+\partial u/\partial x} = 0 \qquad \text{(Y.630)}$$

by substituting the Cauchy-Riemann differential equations, and so would be the functions Φ and Ψ.

There is a neat shortcut to this relation, by using the tools of ➹ Wirtinger-calculus: Motivated by the fact that the coordinates x and y are combined into a complex number $z = x + \mathrm{i}y$ (and its conjugate $z^* = x - \mathrm{i}y$), one can define the composite derivatives:

$$\partial \equiv \frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y} \quad \text{as well as} \quad \partial^* \equiv \frac{\partial}{\partial z^*} = \frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}. \qquad \text{(Y.631)}$$

Combination of the two derivatives leads directly to the Laplace operator, as both $\partial\partial^* g$ as well as $\partial^*\partial g$ are equal to Δg!

There is a neat application of conformal applications to potentials in vacuum in two dimensions. Commonly, potential problems are easy to solve in highly symmetric charge distributions, which makes the convolution with the Green-function relatively simple: In particular, a convolution of spherical symmetric charge distributions with spherically symmetric Green-functions give rise to the a spherically symmetric potential. To make this point more obvious, let's consider a circularly symmetric charge distribution in two dimensions. The potential is necessarily $\Phi \propto \ln r$ with the electric field $E_r = 1/r$ and $E_\varphi = 0$. A more complicated charge distribution would generate the potential $\Phi = \int \mathrm{d}^2r'\ \rho(\boldsymbol{r}) \ln(|\boldsymbol{r} - \boldsymbol{r}'|)$, with a potentially complicated d^2r'-integration.

The problem might be alleviated if a mapping of the old coordinates x, y to new coordinates u, v can be found which would not have an influence on the differential structure of the field equation.

This can in fact be achieved in two dimensions, where the coordinates can be combined into a complex number $z = x + \mathrm{i}y$, for vacuum solutions that obey the Laplace equation $\Delta\Phi = 0$. The Laplace-operator Δ transforms under coordinate change in a peculiar way and acquires just an overall strictly positive, position-dependent prefactor, which is called a conformal factor α^2. The vacuum field equation transforms as $\Delta\Phi \rightarrow \alpha\Delta\Phi = 0$ but clearly, the conformal factor α is irrelevant and drops out for vacuum solutions. Therefore, any vacuum solution in one set of coordinates is automatically a valid vacuum solution in the transformed coordinates. The necessary prerequisite is an analytical coordinate change.

To make things specific, let's consider the mapping

$$G(u, v) \rightarrow g(x, y) = G(u(x, y), v(x, y)) \qquad \text{(Y.632)}$$

and derive the Laplace-equation for g in the coordinates x, y in terms of the Laplace-equation for G in terms of u, v. For the first derivatives one obtains:

$$\frac{\partial g}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial G}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial G}{\partial v} \quad \text{as well as} \quad \frac{\partial g}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial G}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial G}{\partial v}. \qquad \text{(Y.633)}$$

Continuing with the second derivatives one arrives at

$$\frac{\partial^2 g}{\partial^2 x} = \left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 G}{\partial u^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 G}{\partial u \partial v} + \frac{\partial^2 u}{\partial x^2}\frac{\partial G}{\partial u} + \left(\frac{\partial v}{\partial x}\right)^2 \frac{\partial^2 G}{\partial v^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\frac{\partial^2 G}{\partial v \partial u} + \frac{\partial^2 v}{\partial x^2}\frac{\partial G}{\partial u} \tag{Y.634}$$

together with

$$\frac{\partial^2 g}{\partial^2 y} = \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 G}{\partial u^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2 G}{\partial u \partial v} + \frac{\partial^2 u}{\partial x^2}\frac{\partial G}{\partial u} + \left(\frac{\partial v}{\partial y}\right)^2 \frac{\partial^2 G}{\partial v^2} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\frac{\partial^2 G}{\partial v \partial u} + \frac{\partial^2 v}{\partial y^2}\frac{\partial G}{\partial u} \tag{Y.635}$$

These two expressions can be combined into

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \ldots = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]\left(\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2}\right) \tag{Y.636}$$

by making use of the interchangeability of the second partial derivatives and the Cauchy-Riemann differential equations. The prefactor in square brackets is the positive conformal factor. In a actual application the problem of performing the convolution of the Green-function with the charge distribution is then reduced to finding an analytical mapping between the simple and the complicated geometry.