C DYNAMICS OF THE ELECTROMAGNETIC FIELD

C.1 *Potentials and wave equations*

Working with static fields was a tremendous simplification of the Maxwell-equations and yielded, at least under the assumption of Coulomb-gauge, Poisson-type relations between the potentials Φ and A_i and the sources ρ and $\jmath^i$, with easily computable fields E_i and B^i. In taking the detour over the potentials one enables the full toolkit around Green-functions including the treatment of boundary conditions. But the existence of potentials, clearly at this point unrelated to energies as in the electrostatic case, follows from the homogeneous Maxwell-equations in a much more general argument: The second Maxwell-equation $\partial_i B^i = 0$ suggests that there is a vector field A_i with $B^i = \epsilon^{ijk}\partial_j A_k$, as then $\partial_i B^i = \epsilon^{ijk}\partial_i\partial_j A_k = 0$ is automatically fulfilled. Consequently, the induction law $\epsilon^{ijk}\partial_j E_k + \partial_{ct} B^i = 0$ becomes $\epsilon^{ijk}\partial_j E_k + \partial_{ct}\epsilon^{ijk}\partial_j A_k = \epsilon^{ijk}\partial_j(E_k + \partial_{ct} A_k) = 0$, suggesting a potential Φ with $E_i + \partial_{ct} A_i = -\partial_i \Phi$ (the minus-sign is conventional).

Therefore, the homogeneous Maxwell-equations ensure the existence of potentials in the general case, which again are only determined up to a gauge transform: As before, we write $A_i \to A_i + \partial_i\chi$ (which leaves B^i invariant) and investigate the necessary changes to Φ: The electric field E_i is gauge-invariant only if

$$E_i = -\partial_i\Phi - \partial_{ct}A_i \to -\partial_i\Phi + \underbrace{\partial_{ct}\partial_i\chi}_{\text{for consistency}} - \partial_{ct}(A_i + \partial_i\chi) = E_i \tag{C.137}$$

i.e. if we include an additional term $\partial_{ct}\chi$, implying the transformation rule

$$\Phi \to \Phi - \partial_{ct}\chi \quad \text{alongside} \quad A_i \to A_i + \partial_i\chi \tag{C.138}$$

for consistency, keeping in mind that partial derivatives interchange, $\partial_{ct}\partial_i\chi = \partial_i\partial_{ct}\chi$.

While the homogeneous Maxwell-equations safeguard the existence of potentials, the inhomogeneous Maxwell-equations couple the fields to the charges, be it static or dynamic. But while the homogeneous Maxwell-equations make statements about the observable fields E_i and B^i and derive them from potentials Φ and A_i, the coupling to sources is clarified by the inhomogeneous Maxwell-equations in terms of the auxiliary fields D^i and H_i. Hence, constitutive relations are needed.

In fact, the first Maxwell-equation makes a statement about the divergence of D^i, which is given in terms of the potentials by $E_i = -\partial_i\Phi - \partial_{ct}A_i$, followed by $D^i = \epsilon^{ij}E_j = \epsilon\gamma^{ij}E_j$, where we assume an isotropic medium. Consequently,

$$\partial_i D^i = \epsilon\gamma^{ij}\partial_i E_j = -\epsilon\gamma^{ij}\partial_i\partial_j\Phi - \epsilon\gamma^{ij}\partial_{ct}\partial_i A_j = 4\pi\rho \tag{C.139}$$

where we recover the conventional Poisson-equation $\epsilon\Delta\Phi = -\epsilon\gamma^{ij}\partial_i\partial_j\Phi = -4\pi\rho$ in Coulomb-gauge, $\gamma^{ij}\partial_i A_j = 0$. The fourth Maxwell-equation links the magnetic field H_i to $\jmath^i$ and the time derivative of the electric fields $\partial_{ct}D^i$, implying with $B^i = \epsilon^{ijk}\partial_j A_k$ and the constitutive relation $H_i = \mu_{ij}B^j = \gamma_{ij}B^j/\mu$

$$\epsilon^{ijk}\partial_j H_k = +\partial_{ct}D^i + \frac{4\pi}{c}\jmath^i \quad \to \quad \frac{1}{\mu}\epsilon^{ijk}\partial_j\gamma_{kl}\epsilon^{lmn}\partial_m A_n = +\partial_{ct}\epsilon\gamma^{ij}E_j + \frac{4\pi}{c}\jmath^i. \tag{C.140}$$

The contracted Levi-Civita symbol can be expanded in terms of the Grassmann-idenity,

$$\frac{1}{\mu}\gamma_{kl}\epsilon^{ijk}\epsilon^{lmn}\partial_j\partial_m A_n = \frac{1}{\mu}\left(\gamma^{im}\gamma^{jn} - \gamma^{in}\gamma^{jm}\right)\partial_j\partial_m A_n = \frac{1}{\mu}\left(\gamma^{im}\partial_m\left[\gamma^{jn}\partial_j A_n\right] - \gamma^{in}\left[\gamma^{jm}\partial_j\partial_m A_n\right]\right) \quad \text{(C.141)}$$

where one recognises the Coulomb-gauge term in the first and the Laplace-operator in the second square bracket. Substitution of the expression $E_i = -\partial_i\Phi - \partial_{ct}A_i$ on the right side leads to

$$\partial_{ct}\epsilon\gamma^{ij}E_j + \frac{4\pi}{c}\jmath^i = -\partial_{ct}\epsilon\gamma^{ij}\partial_j\Phi - \epsilon\partial_{ct}^2\gamma^{ij}A_j + \frac{4\pi}{c}\jmath^i \quad \text{(C.142)}$$

By assuming a different gauge condition, namely Lorenz-gauge[1]

$$\epsilon\partial_{ct}\Phi + \frac{1}{\mu}\gamma^{ij}\partial_i A_j = 0 \quad \text{(C.143)}$$

the two field equations decouple into a perfectly symmetric shape. Starting with eqn. (C.139), one obtains by substitution of the Lorenz-gauge condition

$$-\gamma^{ij}\partial_i\partial_j\Phi + \epsilon\mu\partial_{ct}^2\Phi = \frac{4\pi}{\epsilon}\rho, \quad \text{(C.144)}$$

i.e. a perfectly viable wave equation for Φ, sourced by ρ/ϵ. The same procedure applied to eqns. (C.141) and (C.142) leads likewise to a wave equation,

$$-\gamma^{jm}\partial_j\partial_m\gamma^{in}A_n + \epsilon\mu\partial_{ct}^2\gamma^{ij}A_j = \frac{4\pi}{c}\mu\jmath^i \quad \text{(C.145)}$$

With the definition of the d'Alembert-operator

$$\Box = \epsilon\mu\partial_{ct}^2 - \Delta \quad \text{(C.146)}$$

as a generalisation to the Laplace-operator Δ for dynamic situations, the two equations can be written as

$$\Box\Phi = \frac{4\pi}{\epsilon}\rho \quad \text{and} \quad \Box A_i = \frac{4\pi}{c}\mu\gamma_{ij}\jmath^j \quad \text{(C.147)}$$

and become two decoupled linear partial hyperbolic differential equations, providing 4 relations between 4 sources and 4 potentials, all decoupled by virtue of the Lorenz-gauge condition.

[1] The Lorenz-gauge is named after Ludvig Lorenz while the Lorentz-transformation was proposed by Hendrik Antoon Lorentz, hence the different spelling.

Differential equations involving the d'Alembert-operator typically have wave-like solutions, propagating at the velocity c, in this case modified to $c/\sqrt{\epsilon\mu}$, where one recognises $n = \sqrt{\epsilon\mu}$ as the index of refraction:

$$\epsilon\mu\partial_{ct}^2 = \epsilon\mu\frac{\partial^2}{\partial(ct)^2} = \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2} = \left(\frac{c}{\sqrt{\epsilon\mu}}\right)^{-2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial(c't)^2} = \partial_{c't}^2 \quad \text{where} \quad c' = \frac{c}{\sqrt{\epsilon\mu}} \tag{C.148}$$

The gauge function χ for achieving Lorenz-gauge can be computed by considering the transformation of the expression $\epsilon\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j/\mu = 0$:

$$\epsilon\partial_{ct}\Phi + \frac{1}{\mu}\gamma^{ij}\partial_i A_j \rightarrow \epsilon\partial_{ct}\left(\Phi - \partial_{ct}\chi\right) + \frac{1}{\mu}\gamma^{ij}\partial_i\left(A_j + \partial_j\chi\right) =$$
$$\epsilon\partial_{ct}\Phi + \frac{1}{\mu}\gamma^{ij}\partial_i A_j - \epsilon\partial_{ct}^2\chi + \frac{1}{\mu}\Delta\chi = 0 \tag{C.149}$$

which is equivalent to

$$\Box\chi = \epsilon\mu\partial_{ct}^2\chi - \Delta\chi = \epsilon\mu\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j \tag{C.150}$$

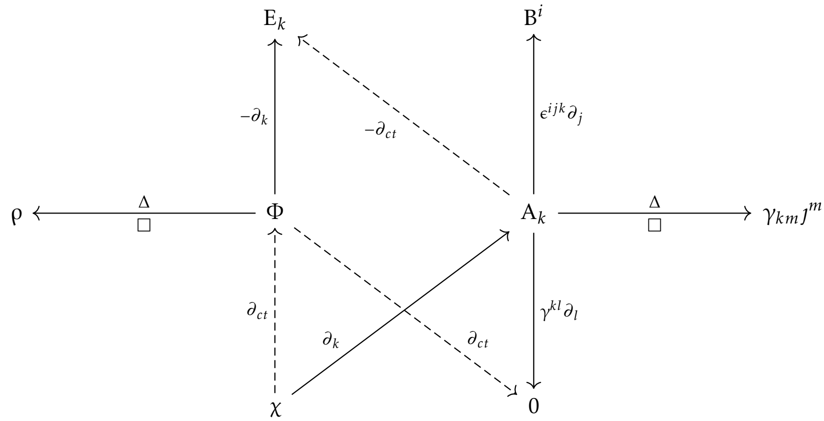
This is a wave-equation for χ, sourced by a possibly nonzero $\epsilon\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j/\mu$. As a hyperbolic partial linear differential equation, it has again a unique solution for χ, such that Lorenz-gauge can be imposed. Determining χ through $\Delta\chi = \gamma^{ij}\partial_i A_j/\mu$ for Coulomb-gauge in the static case and $\Box\chi = \epsilon\mu\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j$ for Lorenz-gauge in the dynamic case are completely analogous.

It is important to realise that the gauge freedom only provides a mathematical convenience for computing the potentials from the sources, and it can be used to set terms in the potential equations to zero. Nowhere there is anything physical happening: Purely by the act of imagining a new gauge condition the physically observable fields can not change. In addition, it is just practicality that persuades us to use Coulomb-gauge for the static case and Lorenz-gauge for the dynamical case, and not a physical requirement. In fact, it is perfectly reasonable to use the Coulomb-gauge $\gamma^{ij}\partial_i A_j = 0$ for the dynamical equations. Then, eqns. (C.139) and (C.141) become

$$\Delta\Phi = -\frac{4\pi}{\epsilon}\rho \quad \text{and} \quad \Delta A_i - \partial_{ct}^2 A_i = -\frac{4\pi}{c}\mu\gamma_{ij}j^j + \epsilon\partial_i\partial_{ct}\Phi \tag{C.151}$$

and deserve some explanation: The Poisson-equation provides an instantaneously changing Φ at any distance from the dynamically changing source ρ, while there is a wave-equation linking A_i to $\gamma_{ij}j^i$. But A_i depends as well on $\partial_i\partial_{ct}\Phi$ as a dynamic, vectorial source, hence the two equations are not yet fully decoupled. Coulomb-gauge might still be attractive though, because of the particularly easy expression for Φ!

The relationship between source, potential and fields are summarised for the case of static fields in Coulomb-gauge and for the dynamical case in Lorenz-gauge, where additional terms are indicated by dashed arrows:



(C.152)

The fields E_i and B^i are obtained from the potentials Φ and A_i by differentiation, and applying a second differentiation gives the sources ρ and $\jmath^i$. The direct path from the potentials to the sources is given by application of the Δ. The dynamic case is slightly more complicated, as E_i obtains a contribution $-\partial_{ct}A_i$ and as $\epsilon^{ijk}\partial_j H_k$ not only depends on $\jmath^i$, but also on $\partial_{ct}D^i$. The gauge function χ transforms only A_i in the static case, but both A_i and Φ in the dynamical case.

While we already know the Green-function inverting Δ from electrostatic and magnetostatic potentials and have encountered a systematic way of its construction, we now have to turn to $\Box$ and find a suitable time-dependent Green-function: the Liénard-Wiechert potentials.

C.2 *Solving the wave equation for potentials*

Intuitively it is clear that a changed charge distribution does not immediately affect the fields at any distance, but that there needs to be some time passing until the field configuration has adjusted itself to changes in the source distribution. For this purpose, let's assume Lorenz-gauge $\epsilon\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j/\mu = 0$ such that the field equations become

$$\Box\Phi = 4\pi\rho \quad \text{and} \quad \Box A_i = \frac{4\pi}{c}\gamma_{ij}\jmath^j \tag{C.153}$$

These equations are decoupled hyperbolic partial differential equations, with the charge density ρ and the current density $\jmath^i$ as sources. Clearly, in vacuum ρ and $\jmath^i$ vanish, such that one falls back onto two homogeneous PDEs

$$\Box\Phi = 0 \quad \text{as well as} \quad \Box A_i = 0 \tag{C.154}$$

which can be solved with a plane wave ansatz

$$\Phi, A_i \propto \exp\left(\pm i(\omega t - k_i r^i)\right). \tag{C.155}$$

Substitution yields for both Φ and A_i the result that

$$\Box \exp\left(\pm \mathrm{i}(\omega t - k_k r^i)\right) = \left[\left(\pm\frac{\mathrm{i}\omega}{c}\right)^2 - \gamma^{ab}\left(\mp \mathrm{i}k_a\right)\left(\mp \mathrm{i}k_b\right)\right]\exp\left(\pm \mathrm{i}(\omega t - k_i r^i)\right) =$$

$$\left[-\left(\frac{\omega}{c}\right)^2 + \gamma^{ab}k_a k_b\right]\exp\left(\pm \mathrm{i}(\omega t - k_i r^i)\right) = 0 \quad \text{(C.156)}$$

i.e. the plane wave is a valid solution as long as the dispersion relation

$$\omega^2 = c^2\gamma^{ab}k_a k_b = (ck)^2 \quad \rightarrow \quad \omega = \pm ck \quad \text{(C.157)}$$

is fulfilled, which requires a strict proportionality between angular frequency ω and wave number k_a, with the speed of light c as the constant of proportionality. With this particular dispersion relation one can immediately show that the phase and group velocities are identical and have the value of c:

$$\upsilon_{\mathrm{ph}} = \frac{\omega}{k} = c = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \upsilon_{\mathrm{gr}} \quad \text{(C.158)}$$

which implies that wave packets in Φ and A_i propagate dispersion-free, i.e. without changing their shape. But perhaps more importantly, the wave equations suggest that excitations of the fields travel at a finite speed c in the potentials Φ and A_i (at least in Lorenz-gauge, the statement would not be true in Coulomb-gauge!).

C.3 *Wave equation for fields*

While propagation and the form of the propagation equations depends on the level of the potentials A_i and Φ on the assumed gauge, the fields E_i and B^i as physical observables can never depend on a certain gauge and always exhibit propagation at the speed of light c. In a vacuum situation with $\jmath^i = 0$ as well as $\rho = 0$ both fields are divergence-free $\partial_i D^i = \partial_i B^i = 0$ and the rotations are defined, up to a sign arising from duality invariance, by

$$\epsilon^{ijk}\partial_j E_k = -\partial_{ct} B^i \quad \text{and} \quad \epsilon^{ijk}\partial_j H_k = +\partial_{ct} D^i. \quad \text{(C.159)}$$

Taking a further rotation of any of the two equations, using rot rot $= \nabla$div $- \Delta$, setting the divergence-term to zero and substituting the time derivative of the other equation leads to

Including a source term for the wave equations for the fields themselves is a bit complicated, but we'll return to that issue later.

$$\left(\partial_{ct}^2 - \Delta\right)E_i = \Box E_i = 0 \quad \text{and, in parallel,} \quad \left(\partial_{ct}^2 - \Delta\right)H_i = \Box H_i = 0 \quad \text{(C.160)}$$

i.e. perfectly symmetric wave equations for the electric and magnetic fields, with excitations travelling at the speed of light c. The symmetry in the shape of the equations is perhaps not too surprising, as one can always replace the fields in a duality transform $E_i \rightarrow H_i$ and $H_i \rightarrow -E_i$ valid in vacuum. Solving the wave equations with a plane wave ansatz $\propto \exp\left(\pm \mathrm{i}(\omega t - k_i r^i)\right)$ is perfectly general: Due to the linearity of the PDEs, any field configuration can be written as a superposition of plane waves that solve the wave equation.

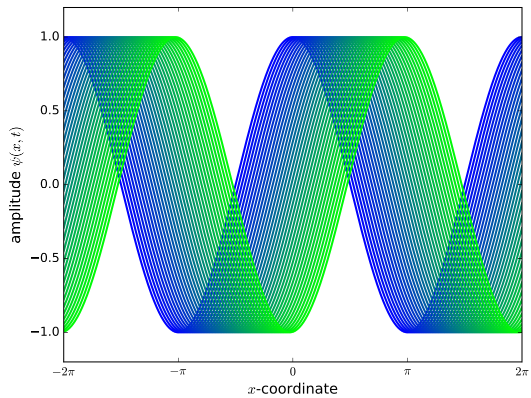


Figure 6: Harmonic wave as a function of x; the shading indicates evolution with time t.

Substituting the plane wave-ansatz into the divergences shows that

$$\gamma^{ij}k_i\mathrm{E}_j = 0 \quad \text{and} \quad \gamma^{ij}k_i\mathrm{H}_j = 0 \tag{C.161}$$

implying that the fields are transverse, i.e. the amplitudes are perpendicular to the direction of propagation, and substituting into the rotation equations suggests

$$\epsilon^{ijk}k_j\mathrm{E}_k = -\frac{\omega}{c}\mathrm{B}^i = -\frac{\omega}{\mu c}\gamma^{ij}\mathrm{H}_k \quad \text{as well as} \quad \epsilon^{ijk}k_j\mathrm{H}_k = +\frac{\omega}{c}\mathrm{D}^i = +\frac{\omega\epsilon}{c}\gamma^{ij}\mathrm{E}_j \tag{C.162}$$

such that the amplitudes of the fields themselves are perpendicular to each other. Please note that the statements of transversality and perpendicularity can not be independent: Pictorially, there is simply no other direction in which k could point: Multiplying the latter two equations with k_i already implies that $\gamma^{ij}k_i\mathrm{H}_j = \gamma^{ij}k_i\mathrm{E}_k = 0$. It is quite instructive to multiply with the linear forms H_i and E_i, leading to

$$\epsilon^{ijk}\mathrm{H}_ik_j\mathrm{E}_k = -\frac{\omega}{c}\mathrm{H}_i\mathrm{B}^i \quad \text{as well as} \quad \epsilon^{ijk}\mathrm{E}_ik_j\mathrm{H}_k = +\frac{\omega}{c}\mathrm{E}_i\mathrm{D}^i \tag{C.163}$$

showing that the volume of the rectangular cuboid spanned by the linear forms E_i, H_i and k_i is proportional to the energy densities, which are equal for a plane wave.

While the amplitudes E_i and H_i are always perpendicular to the direction of propagation, the analogous statement for vector potential A_i is only true under Coulomb-gauge, $\gamma^{ij}k_i\mathrm{A}_j = 0$: This is the reason why sometimes one refers to this gauge condition as transverse gauge. It is quite funny to go through all vector orientations for a duality transform. As plane electromagnetic waves are vacuum solutions, this transform must yield a physically sensible field configuration: Even the fact that k_i, E_i and H_i from a right-handed system in the sense that $\epsilon^{ijk}k_i\mathrm{E}_j\mathrm{H}_k$ is positive is conserved under duality transforms.

Fig. 6 shows how a wave of the type $\exp(\pm\mathrm{i}(\omega t - k_ir^i))$ propagates: Not only is it an oscillation in t at fixed r^i and an oscilation in r^i at fixed t, but the two are coupled: Defining the phase velocity $\upsilon_{\mathrm{ph}} = \omega/k$ makes the argument assume the form $\upsilon_{\mathrm{ph}} t - r$, and moving along with this velocity with the wave an observer would always perceive the phase function $\phi = \omega t - k_i r^i$ to be constant. The phase function ϕ has an interesting geometric shape as $k_i r^i - \omega t$ = const corresponds to the Hesse normal form of a plane, specifically in our case the plane of constant phase. As time progresses, this plane of constant phase moves along its surface normal k_i, which allows the identification of the wave "vector" k_i (actually a linear form) as the direction of propagation.

The wave "vector" should better be a linear form, as in quantum mechanics it is related to the momentum p_i, likewise a linear form, by $p_i = \hbar k_i$!

C.4 *Electromagnetic waves in matter and the telegraph equation*

Electromagnetic waves in matter experience two effects: Firstly, ϵ and μ can differ from one, such that one has to work with $\mathrm{D}^i = \epsilon^{ij}\mathrm{E}_j$ and $\mathrm{B}^i = \mu^{ij}\mathrm{H}_j$ in a potentially anisotropic way, and secondly, the electric field E_i might be able to move the charge carries in the medium, giving rise to a current density j^i, where the two are related by Ohm's law. It reads in its differential formulation

$$j^i = \sigma^{ij}\mathrm{E}_j \tag{C.164}$$

with the conductivity tensor σ^{ij}. As in the case of the dielectric constant ϵ and the permeability μ, the conductivity σ is scalar only in the case of isotropic media (perhaps one can imagine a somehow layered material as a counter example, in which the charges are movable at different rates in the different directions), and a linear relationship is essentially a first order approximation.

Faraday's induction law in an isotropic medium assumes the form

$$\epsilon^{ijk}\partial_j \mathrm{E}_k = -\partial_{ct}\mathrm{B}^i = -\mu\gamma^{ij}\partial_{ct}\mathrm{H}_j \tag{C.165}$$

and Ampère's law takes on the shape

$$\epsilon^{ijk}\partial_j \mathrm{H}_k = +\partial_{ct}\mathrm{D}^i + \frac{4\pi}{c} j^i = +\epsilon\gamma^{ij}\partial_{ct}\mathrm{E}_j + \frac{4\pi\sigma}{c}\gamma^{ij}\mathrm{E}_j \tag{C.166}$$

so that taking the time derivative of the first equation, and the rotation of the second equation, again by using the Grassmann-relation leads to a wave equation with a damping term, the so-called telegraph equation

$$\left(\epsilon\mu\partial_{ct}^2 - \Delta\right)\mathrm{H}_i = \Box\mathrm{H}_i = -\frac{4\pi\sigma\mu}{c}\partial_{ct}\mathrm{H}_i \tag{C.167}$$

The effective speed of propagation c' is given by

$$c' = \frac{c}{\sqrt{\epsilon\mu}} \simeq \frac{c}{\sqrt{\epsilon}} \tag{C.168}$$

as effectively all known transparent media have permeabilities close to one. The latter relation suggests that the refractive index n is given by $\sqrt{\epsilon}$, relating electrical to optical properties of a medium. The damping is determined by the conductivity σ: Non-conductive media do not show any attenuation of incident electromagnetic waves, but if the conductivity is nonzero, the motion of the charges in the medium dissipate the energy of the electromagnetic waves.

Dimensional analysis shows that the term

$$\frac{4\pi\sigma\mu}{c} = L_{\text{att}} \tag{C.169}$$

must have units of a length scale L_{att} on which the amplitude of the wave decreases by a factor $\exp(-1)$.

C.5 *Energy transport and the Poynting vector*

We have already seen that the electric and magnetic fields are real in the sense that they accelerate test charges and contain energy at the densities

$$w_{\text{el}} = \frac{E_i D^i}{8\pi} \quad \text{and analogously,} \quad w_{\text{mag}} = \frac{H_i B^i}{8\pi} \tag{C.170}$$

The corresponding energies, obtained by integration over space, would from a combined energy conservation law together with the mechanical energies. As the fields can dynamically evolve, the questions how energy is conserved by a dynamically evolving field configuration and how it is transported through space arises naturally.

A good starting point are the two inhomogeneous Maxwell-equations that contain time derivatives:

$$\epsilon^{ijk}\partial_j E_k = -\partial_{ct} B^i \quad \text{as well as} \quad \epsilon^{ijk}\partial_j H_k = +\partial_{ct} D^i + \frac{4\pi}{c} \jmath^i. \tag{C.171}$$

Multiplying the first equation with H_i and the second equation with E_i in the sense of a scalar product and subsequent subtraction yields

$$E_i\epsilon^{ijk}\partial_j H_k - H_i\epsilon^{ijk}\partial_j E_k = \frac{4\pi}{c} E_i \jmath^i + \underbrace{E_i\partial_{ct}D^i}_{=\frac{1}{2}\partial_{ct}(E_i D^i)} + \underbrace{H_i\partial_{ct}B^i}_{=\frac{1}{2}\partial_{ct}(H_i B^i)} \tag{C.172}$$

where the reshaping in the last two terms with the constitutive relation suggests substitution of the energy densities:

$$4\pi\partial_{ct} w_{\text{el}} = \frac{1}{2}\partial_{ct}(E_i D^i) = \frac{1}{2}\left(\partial_{ct}E_i \cdot D^i + E_i\partial_{ct}D^i\right) =$$
$$\frac{\epsilon^{ij}}{2}\left(\partial_{ct}E_i \cdot E_j + E_i\partial_{ct}E_j\right) = E_i\partial_{ct}\epsilon^{ij}E_j = E_i\partial_{ct}D^i, \tag{C.173}$$

relying on the symmetry of the dielectric tensor ϵ^{ij}, and likewise

$$4\pi\partial_{ct} w_{\text{mag}} = \frac{1}{2}\partial_{ct}(H_i B^i) = \frac{1}{2}\left(\partial_{ct}H_i \cdot B^i + H_i\partial_{ct}B^i\right) =$$
$$\frac{\mu^{ij}}{2}\left(\partial_{ct}H_i \cdot H_j + H_i\partial_{ct}H_j\right) = H_i\partial_{ct}\mu^{ij}H_j = H_i\partial_{ct}B^i, \tag{C.174}$$

for the magnetic fields with a symmetric permeability μ^{ij}. The left hand side of the equation can be written as

$$E_i\epsilon^{ijk}\partial_j H_k - H_i\epsilon^{ijk}\partial_j E_k = E_i\epsilon^{ijk}\partial_j H_k - H_k\epsilon^{kji}\partial_j E_i =$$
$$\epsilon^{ijk}(E_i\partial_j H_k + H_k\partial_j E_i) = \partial_j\epsilon^{ijk}E_iH_k = -\partial_i\epsilon^{ijk}E_jH_k \quad \text{(C.175)}$$

with renaming the indices $i \leftrightarrow k$ in the second term, before reordering $\epsilon^{kji} = \epsilon^{ikj} = -\epsilon^{ijk}$, with a cycling permutation in the first and an interchange in the second step. Defining the Poynting-vector P^i

Energy transport depends on the two linear forms E_i and H_i.

$$P^i = \frac{c}{4\pi}\epsilon^{ijk}E_jH_k \quad \text{(C.176)}$$

one arrives at the final result

$$\partial_i P^i = -E_i\jmath^i - \partial_t\left(w_{\mathrm{el}} + w_{\mathrm{mag}}\right) \quad \text{(C.177)}$$

The Gauß-theorem allows to recast this differential conservation law into integral form,

$$\int_V dV\,\partial_i P^i = \int_{\partial V} dS_i P^i = -\int_V dV\,E_i\jmath^i - \frac{d}{dt}\int_V dV\,\left(w_{\mathrm{el}} + w_{\mathrm{mag}}\right) \quad \text{(C.178)}$$

such that the change in energy content of the inside the volume V is given by two terms. The first term describes the energy flux integrated over the surface ∂V: If the Poynting-vector P^i has a nonzero divergence and points outwards, the energy content will decrease. The second term is attributed to the dissipation inside the volume: Introducing Ohm's law in differential form,

$$\jmath^i = \sigma^{ij}E_j \quad \text{(C.179)}$$

with the conductivity tensor σ^{ij}, the integral over the volume $V = A\ell$

$$\int_V dV\,E_i\jmath^i = \int_V dV\,\sigma^{ij}E_iE_j = \sigma\int_V dV\,\gamma^{ij}E_iE_j \simeq \sigma E^2 V = \underbrace{\frac{\sigma A}{\ell}}_{=1/R}(\underbrace{E\ell}_{=U})^2 = \frac{U^2}{R} \quad \text{(C.180)}$$

This is exactly the energy per time interval which is dissipated into heat inside the volume V. Alternatively, one could have replaced E_i instead of $\jmath^i$, leading to

$$\int_V dV\,E_i\jmath^i = \int_V dV\,\sigma_{ij}\jmath^i\jmath^j = \frac{1}{\sigma}\int_V dV\,\gamma_{ij}\jmath^i\jmath^j \simeq \frac{\jmath^2 V}{\sigma} = \underbrace{\frac{\ell}{A\sigma}}_{=R}(\underbrace{\jmath A}_{=I})^2 = RI^2 \quad \text{(C.181)}$$

Only if the conductivity σ vanishes, or the resistance R is infinite, the terms is inactive and the ideal energy conservation law is given by

$$\partial_i P^i = -\partial_t\left(w_{\mathrm{el}} + w_{\mathrm{mag}}\right) \quad\rightarrow\quad \int_{\partial V} dS_i\,P^i = -\frac{d}{dt}\int_V dV\,\left(w_{\mathrm{el}} + w_{\mathrm{mag}}\right) \quad \text{(C.182)}$$

C.6 *Momentum transport and the Poynting linear form*

Similarly to the Poynting-law for energy conservation of the electromagnetic field there is an associated momentum conservation law: Starting at the Lorentz-equation for a volume element that contains a charge density ρ and a current density $\jmath^i$ allows to express the rate of change of the volume's momentum

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \int_V \mathrm{d}V \left(\rho E_i + \frac{1}{c}\epsilon_{ijk}\jmath^j B^k\right) \tag{C.183}$$

which would fall back onto eqn. (A.1) by setting $\mathrm{d}q = \rho\mathrm{d}V$, $q = \int \mathrm{d}q = \int \mathrm{d}V\,\rho$.

As in the calculation for the energy density of the electromagnetic fields we can replace the charge density ρ and the current density $\jmath^i$ by using the two inhomogeneous Maxwell-equations $\partial_i D^i = 4\pi\rho$ and $\epsilon^{jmn}\partial_m H_n = \partial_{ct}D^j + 4\pi/c\,\jmath^j$ leading to the change of the momentum associated with the fields themselves

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \frac{1}{4\pi}\int_V \mathrm{d}V \left(E_i\,\partial_j D^j + \epsilon_{ijk}\epsilon^{jmn}\partial_m H_n \cdot B^k - \partial_{ct}D^j \cdot B^k\right) \tag{C.184}$$

Aiming at making the expression more symmetric, it is clearly possible to add the term $H_i\partial_j B^j$ as $\partial_i B^i = 0$, and replacing the last term $\partial_{ct}D^j \cdot B^k$ using the Leibnitz-rule according to

$$\partial_{ct}\left(D^j B^k\right) = \partial_{ct}D^j \cdot B^k + D^j\partial_{ct}B^k. \tag{C.185}$$

Then, the penultimate term requires the time-derivative of the magnetic field, which suggests to substitute the induction equation $\partial_{ct}B^k = -\epsilon^{kmn}\partial_m E_n$:

$$\epsilon_{ijk}\partial_{ct}D^j \cdot B^k = \partial_{ct}\left(\epsilon_{ijk}D^j B^k\right) - \epsilon_{ijk}D^j\partial_{ct}B^k = \partial_{ct}\left(\epsilon_{ijk}D^j B^k\right) + \epsilon_{ijk}D^j\epsilon^{kmn}\partial_k\partial_m E_n \tag{C.186}$$

⚠ *Momentum transport depends on the vectorial fields* D^i *and* B^i.

The formula suggests a Poynting linear form Y_i

$$Y_i = \frac{c}{4\pi}\epsilon_{ijk}D^j B^k \tag{C.187}$$

analogous to the vector $P^i = c/(4\pi)\,\epsilon^{ijk}E_j H_k$, but composed of the two vectorial fields D^i and B^i. The missing c suggests that it has units of a momentum density, and hence it describes the momentum content associated with the fields inside volume.

The expression for the momentum change presents itself in a wonderfully symmetric form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_i + \int_V \mathrm{d}V\,Y_i\right) = \frac{1}{4\pi}\int_V \mathrm{d}V \left(E_i\partial_j D^j + H_i\partial_j B^j - \epsilon_{ijk}\epsilon^{kmn}\partial_m E_n \cdot D^j + \epsilon_{ijk}\epsilon^{jmn}\partial_m H_n \cdot B^k\right) \tag{C.188}$$

If the right side of this equation could be written as a divergence, one would recover the archetypical form of a continuity equation, this time for the momentum of the field configuration inside the volume V. As the left side of the equation is vectorial and because taking a divergence reduces the rank of a tensor by one, we are looking for a tensor of rank two on the right side. Treating the electric fields first, shows that

$$\mathrm{E}_i\partial_j\mathrm{D}^j - \epsilon_{ijk}\epsilon^{kmn}\partial_m\mathrm{E}_n\cdot\mathrm{D}^j = \mathrm{E}_i\partial_j\mathrm{D}^j - \left(\delta_i^m\delta_j^n - \delta_j^m\delta_i^n\right)\partial_m\mathrm{E}_n\cdot\mathrm{D}^j =$$
$$\mathrm{E}_i\partial_j\mathrm{D}^j - \partial_i\mathrm{E}_j\cdot\mathrm{D}^j + \partial_j\mathrm{E}_i\cdot\mathrm{D}^j = \underbrace{\mathrm{E}_i\partial_j\mathrm{D}^j + \partial_j\mathrm{E}_i\cdot\mathrm{D}^j}_{=\partial_j(\mathrm{E}_i\mathrm{D}^j)} - \underbrace{\partial_i\mathrm{E}_j\cdot\mathrm{D}^j}_{=\delta_j^i\partial_j(\mathrm{E}_k\mathrm{D}^k)/2}\,, \quad \text{(C.189)}$$

after a reordering of terms. While the first combination is just an application of the Leibnitz-rule, the rewriting of the last term deserves a more thorough argument:

$$\partial_i\mathrm{E}_j\cdot\mathrm{D}^j = \delta_i^j\partial_j\mathrm{E}_k\cdot\mathrm{D}^k = \delta_i^j\epsilon_{km}\partial_j\mathrm{E}_k\cdot\mathrm{E}_m = \frac{1}{2}\delta_i^j\epsilon^{km}\partial_j\left(\mathrm{E}_k\mathrm{E}_m\right) =$$
$$\frac{1}{2}\delta_i^j\partial_j\left(\epsilon^{km}\mathrm{E}_k\mathrm{E}_m\right) = \frac{1}{2}\delta_i^j\partial_j\left(\mathrm{E}_k\mathrm{D}^k\right). \quad \text{(C.190)}$$

The terms involving magnetic fields are treated in complete analogy up to a difference in sign, caused by the different contraction. This is quickly remedied by interchanging the indices $\epsilon^{ijk} = -\epsilon^{ikj}$:

$$\mathrm{H}_i\partial_j\mathrm{B}^j + \epsilon_{ijk}\epsilon^{jmn}\partial_m\mathrm{H}_n\cdot\mathrm{B}^k = \mathrm{H}_i\partial_j\mathrm{B}^j - \epsilon_{ikj}\epsilon^{jmn}\partial_m\mathrm{H}_n\cdot\mathrm{B}^k \quad \text{(C.191)}$$

The subsequent steps are identical:

$$\mathrm{H}_i\partial_j\mathrm{B}^j - \epsilon_{ikj}\epsilon^{jmn}\partial_m\mathrm{H}_n\cdot\mathrm{B}^k = \mathrm{H}_i\partial_j\mathrm{B}^j - \left(\delta_i^m\delta_k^n - \delta_k^m\delta_i^n\right)\partial_m\mathrm{H}_n\cdot\mathrm{B}^k =$$
$$\mathrm{H}_i\partial_j\mathrm{B}^j - \partial_i\mathrm{H}_k\cdot\mathrm{B}^k + \partial_k\mathrm{H}_i\cdot\mathrm{B}^k = \underbrace{\mathrm{H}_i\partial_j\mathrm{B}^j + \partial_k\mathrm{H}_i\cdot\mathrm{B}^k}_{=\partial_j(\mathrm{H}_i\mathrm{B}^j)} - \underbrace{\partial_i\mathrm{H}_k\cdot\mathrm{B}^k}_{=\delta_j^i\partial_j(\mathrm{H}_k\mathrm{B}^k)/2}\,. \quad \text{(C.192)}$$

where an identical argument applies to the last term:

$$\partial_i\mathrm{H}_j\cdot\mathrm{B}^j = \delta_i^j\partial_j\mathrm{H}_k\cdot\mathrm{B}^k = \delta_i^j\mu_{km}\partial_j\mathrm{H}_k\cdot\mathrm{H}_m = \frac{1}{2}\delta_i^j\mu^{km}\partial_j\left(\mathrm{H}_k\mathrm{H}_m\right) =$$
$$\frac{1}{2}\delta_i^j\partial_j\left(\mu^{km}\mathrm{H}_k\mathrm{H}_m\right) = \frac{1}{2}\delta_i^j\partial_j\left(\mathrm{H}_k\mathrm{B}^k\right). \quad \text{(C.193)}$$

Collecting all terms finally gives the sought-after divergence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_i + \int_{\mathrm{V}}\mathrm{dV}\,\mathrm{Y}_i\right) = \int_{\mathrm{V}}\mathrm{dV}\,\partial_j\mathrm{T}_i^{\ j} = \int_{\partial\mathrm{V}}\mathrm{dS}_j\,\mathrm{T}_i^{\ j} \quad \text{(C.194)}$$

where the Gauß-theorem was applied in the last step, yielding a surface integral over the Maxwell stress tensor $\mathrm{T}_i^{\ j}$

$$\mathrm{T}_i^{\ j} = \frac{1}{4\pi}\left(\mathrm{E}_i\mathrm{D}^j + \mathrm{H}_i\mathrm{B}^j - \frac{1}{2}\delta_i^j\left(\mathrm{E}_k\mathrm{D}^k + \mathrm{H}_k\mathrm{B}^k\right)\right) \quad \text{(C.195)}$$

The Maxwell-tensor is symmetric, ${T_i}^j = {T_j}^i$ in the case of isotropic media, but in general not: Examining $E_i D^j$, for instance, shows with the substitution

$$E_i D^i = \epsilon^{ij} E_i E_j = \epsilon_{ij} D^i D^j \tag{C.196}$$

that it can only be symmetric if ϵ^{ij} and $E_i E_j$, despite being both symmetric on their own, have coinciding eigensystems. This would be the case for isotropic media, as γ^{ij} and $E_i E_j$ are simultaneously diagonalisable. A straightforwardly mathematical condition would be a vanishing commutator $\left[\epsilon^{ij}, E_i E_j\right] = 0$.

It is striking that in an anisotropic medium the direction of energy transport and momentum transport are not collinear, as

$$Y_i = \frac{c}{4\pi} \epsilon_{ijk} D^j B^k = \frac{c}{4\pi} \epsilon_{ijk} \epsilon^{jm} E_m \mu^{kn} H_n \tag{C.197}$$

Forming the scalar product between the Poynting vector P^i and its associated linear form Y_i gives

$$Y_i P^i = \frac{c^2}{(4\pi)^2} \epsilon_{ijk} \epsilon^{imn} D^j B^k E_m H_n = \frac{c^2}{(4\pi)^2} \left(\delta_j^m \delta_k^n - \delta_k^m \delta_j^n\right) D^j B^k E_m H_n =$$
$$\frac{c^2}{(4\pi)^2} \left(D^m E_m B^n H_n - D^j H_j B^k E_k\right) = \frac{c^2}{(4\pi)^2} \left(\epsilon^{im} E_i E_m \mu^{jn} H_j H_n - \epsilon^{im} E_i H_m \mu^{jn} H_j E_n\right), \tag{C.198}$$

with the squared norms of the two fields in the first and the scalar products in the second term: This suggests that the scalar product $Y_i P^i$ is positive definite, as a result of the Cauchy-Schwarz inequality. After rewriting the expression in terms of the two constitutive tensors instead of the fields one arrives at

$$Y_i P^i = \frac{c^2}{(4\pi)^2} \epsilon^{im} \mu^{jn} \left(E_i H_j E_m H_n - E_i H_j E_n H_m\right) = \frac{c^2}{(4\pi)^2} \left(\epsilon^{im} \mu^{jn} - \epsilon^{in} \mu^{jm}\right) E_i H_j E_m H_n \tag{C.199}$$

For a plane wave with perpendicular electric and magnetic fields one would obtain, under the assumption of an isotropic medium, a vanishing second term, yielding the largest possible result for $Y_i P^i$, which indicates a parallel momentum and energy transport.

The trace $\mathrm{tr}(T) = \delta_j^i {T_i}^j = {T_i}^i$ computes to the negative energy density of the fields, as

$${T_i}^i = \frac{1}{4\pi} \left(E_i D^i + H_i B^i - \frac{\delta_i^j \delta_j^i}{2} \left(E_k D^k + H_k B^k\right)\right) = -\frac{1}{8\pi} \left(E_i D^i + H_i B^i\right) = -(w_{\mathrm{el}} + w_{\mathrm{mag}}) \tag{C.200}$$

as $\delta_i^j \delta_j^i = \delta_i^i = 3$. To be honest, this result can only be understood later, when we derive the Maxwell-tensor for electrodynamics as a relativistic field theory.

Looking at the mechanical aspect of the continuity equation for the momentum density as the change of momentum needs to be equal to the force acting on the volume element, and because dp_i is given as

$$\mathrm{d}\dot{p}_i = \mathrm{T}_i{}^j \mathrm{dS}_j \quad \rightarrow \quad \mathrm{T}_i{}^j = \frac{\partial \dot{p}_i}{\partial \mathrm{S}_j}, \tag{C.201}$$

one would associate $\mathrm{T}_i{}^j$ with a force per unit area: Those elements are referred to as stresses, of which the isotropic component is called radiation pressure. Depending on the field configuration, the stresses into different coordinate directions do not need to be equal. Commonly, one would expect radiation pressure to be exerted in the direction of propagation of an electromagnetic wave, but not perpendicularly to it. On the other hand, an isotropic superposition of plane electromagnetic waves, as for instance in a blackbody, can be assigned a radiation pressure. The combined term on the left side of the equation is the mechanical momentum p_i of the matter inside the volume V and the volume-integrated Poynting linear form Y_i as the momentum content of the electromagnetic field.

We will see that the four entities energy density $w = w_{\mathrm{el}} + w_{\mathrm{mag}}$, Poynting vector P^i or energy flux density, Poynting linear form Y_i or momentum density and the Maxwell stress tensor $\mathrm{T}_i{}^j$ can be assembled into a larger object, the energy momentum-tensor $\mathrm{T}_\mu{}^\nu$:

$$\mathrm{T}_\mu{}^\nu = \left(\begin{array}{c|c} w & \mathrm{Y}_i \\ \hline \mathrm{P}^j & \mathrm{T}_i{}^j \end{array}\right), \tag{C.202}$$

which will, when a combined derivative $\partial_\nu = (\partial_{ct}, \partial_j)$ is applied to it, yield energy conservation in the first column, and the three components of momentum conservation in the second, third and fourth columns. All conservation laws would then follow jointly from the divergence $\partial_\nu \mathrm{T}_\mu{}^\nu = 0$, for media of zero conductivity, and the entire tensor is traceless, $\delta^\mu_\nu \mathrm{T}_\mu{}^\nu = \mathrm{T}_\mu{}^\mu = 0 = w + \delta^i_j \mathrm{T}_i{}^j = w + \mathrm{T}_i{}^i$.

In summary, there is a clear notion of energy and momentum conservation in the electromagnetic field. One can associate energy and momentum densities to any field configuration, and as the configuration evolves dynamically, energy and momentum is transported through space in a way that is described by continuity equations. Possible dissipation can be described by Ohm's law, and would convert field energy into heat. The Poynting-vector plays the role as energy flux and is constructed from the linear forms E_i and H_i, while the transport of momentum density is encapsulated in the related linear form Y_i, which depends on the two vectors D^i and B^i. It is straightforward to see and not unexpected that for a plane electromagnetic wave the energy transport proceed along the wave vector, as P^i is collinear with k_i, which in turn is poynting (pardon me!) into the direction $\epsilon^{ijk}\mathrm{E}_j\mathrm{H}_k$. In metric spaces or spacetimes it is always possible to write the Maxwell stress-tensor and the energy-momentum tensor with one type of index, covariant for instance,

$$\mathrm{T}_{ij} = \gamma_{jk}\mathrm{T}_i{}^k \quad \text{and} \quad \mathrm{T}_{\mu\nu} = \eta_{\nu\alpha}\mathrm{T}_\mu{}^\alpha, \tag{C.203}$$

such that the traces read $\gamma^{ij}\mathrm{T}_{ij}$ and $\eta^{\mu\nu}\mathrm{T}_{\mu\nu}$, and the divergences $\gamma^{ai}\partial_a \mathrm{T}_{ij} = 0$ and $\eta^{\alpha\mu}\partial_\alpha \mathrm{T}_{\mu\nu} = 0$.

C.7 *Time-dependent Green-functions and retardation*

Clearly, the propagation speed of excitations in the electromagnetic field is finite, so any change in the source configuration is not perceived instantaneously at any point at nonzero distance: In fact, the changed field configuration only arrives after a time

$c\Delta t = \Delta r$ at distance Δr, which is referred to as retardation. The same is true for the potentials Φ and A_i if one assumes Lorenz-gauge $\epsilon\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j/\mu = 0$, because then the wave equations for the potentials

$$\Box\Phi = 4\pi\rho \quad \text{and} \quad \Box A_i = \frac{4\pi}{c}\gamma_{ij}\jmath^j \tag{C.204}$$

are identical to those of the fields E_i and B^i. This particular form of an inhomogeneous wave equation, where we always verified that the homogeneous differential equation is solved by a plane wave, is referred to as the Helmholtz differential equation

$$\Box\psi(\boldsymbol{r}, t) = \left(\partial_{ct}^2 - \Delta\right)\psi(\boldsymbol{r}, t) = 4\pi q(\boldsymbol{r}, t) \tag{C.205}$$

where ψ could be either of the potentials Φ and A_i, and q the corresponding source, i.e. ρ or $\gamma_{ij}\jmath^j$. The Helmholtz differential equation is a hyperbolic linear partial differential equation of second order with an inhomogeneity. As a linear differential equation, a suitable solution strategy would be a Green-function, that depends both on space and time coordinates:

$$\Box G(\boldsymbol{r} - \boldsymbol{r}', t - t') = 4\pi\delta_D(\boldsymbol{r} - \boldsymbol{r}')\delta_D(t - t') \tag{C.206}$$

As before, the Green-function is the formal solution for the potential at $\boldsymbol{r}$ and t to a point-like source existing at $\boldsymbol{r}'$ and t', such that

$$\Box\psi(\boldsymbol{r}, t) = 4\pi q(\boldsymbol{r}, t) \quad \rightarrow \quad \psi(\boldsymbol{r}, t) = \int dV' \int dt'\, G(\boldsymbol{r} - \boldsymbol{r}', t, -t')q(\boldsymbol{r}', t') \tag{C.207}$$

in a convolution relation, which is consistent because of

$$\Box\psi(\boldsymbol{r}, t) = \int dV' \int dt'\, \Box G(\boldsymbol{r} - \boldsymbol{r}', t, -t')q(\boldsymbol{r}', t') = \\ 4\pi \int dV' \int dt' \delta_D(\boldsymbol{r} - \boldsymbol{r}')\delta_D(t - t')q(\boldsymbol{r}', t') = 4\pi q(\boldsymbol{r}, t) \tag{C.208}$$

as a consequence of the shifting relation of the δ_D-function.

In Fourier-space, the Green-function is given by

$$G(\omega, \boldsymbol{k}) = \int dV \int dt\, G(\boldsymbol{r} - \boldsymbol{r}', t - t')\exp(-i\omega(t - t'))\exp(-ik_i(\boldsymbol{r} - \boldsymbol{r}')^i) \tag{C.209}$$

with the inversion

$$G(\boldsymbol{r} - \boldsymbol{r}', t - t') = \frac{1}{(2\pi)^4}\int d\omega \int d^3k\, G(\omega, \boldsymbol{k})\exp(i\omega(t - t'))\exp(ik_i(\boldsymbol{r} - \boldsymbol{r}')^i) \tag{C.210}$$

so that the application of the d'Alembert-operator gives

$$\Box G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \Box\frac{1}{(2\pi)^4}\int\mathrm{d}\omega\int\mathrm{d}^3k\; G(\omega,\boldsymbol{k})\exp(\mathrm{i}\omega(t-t'))\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i) =$$
$$-\frac{1}{(2\pi)^4}\int\mathrm{d}\omega\int\mathrm{d}^3k\left[\frac{\omega^2}{c^2}-k^2\right]G(\omega,\boldsymbol{k})\exp(\mathrm{i}\omega(t-t'))\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i), \quad \text{(C.211)}$$

as ∂_{ct} acts on $\exp(\mathrm{i}\omega(t-t'))$, and ∂_a on $\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i)$, yielding $\mathrm{i}\omega/c$ and $\mathrm{i}k_a$ each twice; and we abbreviate $k^2 = \gamma^{ab}k_ak_b$:

$$\begin{array}{ccc} \psi(\boldsymbol{r},t) & \xleftarrow{\;\mathcal{F}^{-1}\;} & \psi(\boldsymbol{k},\omega) \\ \Big\downarrow \Box & & \Big\downarrow \omega^2/c^2-\gamma^{ij}k_ik_j \\ q(\boldsymbol{r},t) & \xrightarrow{\;\mathcal{F}\;} & q(\boldsymbol{k},\omega) \end{array} \quad \text{(C.212)}$$

On the other hand, this expression needs to be equal, according to eqn. (C.206), to the Fourier-representation of the δ_D-distributions,

$$\delta_D(t-t')\delta_D(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{(2\pi)^4}\int\mathrm{d}\omega\int\mathrm{d}^3k\;\exp(\mathrm{i}\omega(t-t'))\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i) \quad \text{(C.213)}$$

with all frequencies appearing at equal amplitude. By comparing the latter two expressions, one can extract the Fourier-transformed Green-function $G(\omega,\boldsymbol{k})$ to be

$$G(\omega,\boldsymbol{k}) = 4\pi\frac{c^2}{\omega^2-(ck)^2}. \quad \text{(C.214)}$$

But transforming back to configuration space reveals a problem: Formally, one writes down

$$G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \frac{1}{4\pi^3}\int\mathrm{d}\omega\int\mathrm{d}^3k\;\frac{c^2}{\omega^2-(ck)^2}\exp(\mathrm{i}\omega(t-t'))\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i) \quad \text{(C.215)}$$

where one encounters two singularities of the integrand at $\omega = \pm ck$ when performing the $\mathrm{d}\omega$-integration, for every value of k, as indicated by Fig. 7. This issue is most elegantly solved by the methods of complex integration.

For carrying out the $\mathrm{d}\omega$-integration one can extend the function to complex arguments and close the integration path along the real axis by a loop: In this way, one deals with a closed loop integral over a holomorphic function, where the two poles can be shifted inside the integration contour by adding $+\mathrm{i}\epsilon$ to them, which does not change the final result. Then, the value of the integral is entirely fixed by the values of the two residuals associated with the two poles:

$$\int\mathrm{d}\omega\;\frac{c^2}{(ck)^2-\omega^2}\exp(\mathrm{i}\omega(t-t')) \to -c^2\oint\mathrm{d}\omega\;\frac{\exp(\mathrm{i}\omega(t-t'))}{\omega^2-(ck)^2}, \quad \text{(C.216)}$$

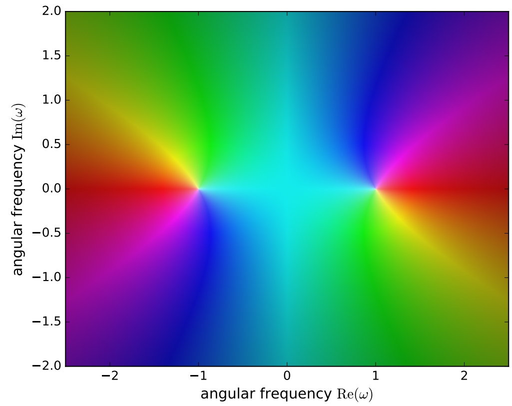


Figure 7: Function $1/(\omega^2 - (ck)^2)$ over the complex plane $\omega = \mathrm{Re}(\omega) + \mathrm{i}\,\mathrm{Im}(\omega)$, for $ck = 1$, with color indicating phase and hue indicating the absolute value. The two singularities at $\omega = \pm ck$ are clearly visible.

where the denominator factorises $(\omega^2 - (ck)^2) = (\omega + ck)(\omega - ck)$, by virtue of the binomial formula.

Let's investigate the residues at the two poles at $\omega_+ = \omega + ck$ and $\omega_-\omega - ck$ separately: Computing the residues requires the limits

$$\mathrm{Res}_+ = \lim_{\omega \to +ck} (\omega - ck) \frac{\exp(\mathrm{i}\omega(t - t'))}{(\omega + ck)(\omega - ck)} = -\frac{c}{2k} \exp(+\mathrm{i}ck(t - t')) \tag{C.217}$$

and

$$\mathrm{Res}_- = \lim_{\omega \to -ck} (\omega + ck) \frac{\exp(\mathrm{i}\omega(t - t'))}{(\omega + ck)(\omega - ck)} = +\frac{c}{2k} \exp(-\mathrm{i}ck(t - t')) \tag{C.218}$$

Cauchy's residue theorem now states that the value of the loop integral is equal to the sum of the residues, up to a factor of $2\pi\mathrm{i}$,

$$\oint \mathrm{d}\omega\, \frac{\exp(\mathrm{i}\omega(t - t'))}{\omega^2 - (ck)^2} = 2\pi\mathrm{i}\,(\mathrm{Res}_+ + \mathrm{Res}_-) =$$

$$\frac{2\pi}{\mathrm{i}} \left(\frac{c}{2k} \exp(+\mathrm{i}ck(t - t')) - \frac{c}{2k} \exp(-\mathrm{i}ck(t - t')) \right) = \frac{2\pi c}{k} \sin(ck(t - t')) \tag{C.219}$$

The remaining d^3k-integration reads:

$$G(\boldsymbol{r} - \boldsymbol{r}', t - t') = \frac{c}{2\pi^2} \int \mathrm{d}^3k\, \frac{\sin(ck(t - t'))}{k} \exp(\mathrm{i}k_i(\boldsymbol{r} - \boldsymbol{r}')^i) \tag{C.220}$$

and can be most sensibly carried out in spherical coordinates: $\mathrm{d}^3k = k^2\mathrm{d}k\mathrm{d}\mu\mathrm{d}\phi$, with azimuthal symmetry and μ being the cosine of the angle between $\boldsymbol{k}$ and $\boldsymbol{r} - \boldsymbol{r}'$.

Then,

$$G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \frac{c}{\pi}\int_0^\infty k\mathrm{d}k\ \sin(ck(t-t'))\int_{-1}^{+1}\mathrm{d}\mu\ \exp(\mathrm{i}\mu k\left|-\boldsymbol{r}-\boldsymbol{r}'\right|) \tag{C.221}$$

The dμ-integral has an elementary solution in term of the sine, so we arrive at

$$G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{2c}{\pi}\int_0^\infty \mathrm{d}k\ \sin(ck(t-t'))\sin(k\left|\boldsymbol{r}-\boldsymbol{r}'\right|) \tag{C.222}$$

The integral can be carried out by rewriting both sines as differences of complex exponentials, multiplying out the expression and integrate. For convenience, we abbreviate $\Delta t = t - t'$ and $\Delta r = |\boldsymbol{r}-\boldsymbol{r}'|$:

$$2\int_0^\infty \mathrm{d}k\ \sin(ck\Delta t)\sin(k\Delta r) = \int_{-\infty}^{+\infty}\mathrm{d}k\ \sin(ck\Delta r)\sin(k\Delta r) =$$

$$\frac{1}{(2\mathrm{i})^2}\int_{-\infty}^{+\infty}\mathrm{d}k\ \left(\exp\left(+\mathrm{i}ck\Delta t\right)-\exp\left(-\mathrm{i}ck\Delta t\right)\right)\times\left(\exp\left(+\mathrm{i}k\Delta r\right)-\exp\left(-\mathrm{i}k\Delta r\right)\right). \tag{C.223}$$

Rearranging the terms leads to

$$\ldots = \frac{1}{(2\mathrm{i})^2}\int_{-\infty}^{+\infty}\mathrm{d}k\ \exp(+\mathrm{i}k[c\Delta t+\Delta r]) + \exp(-\mathrm{i}k[c\Delta t+\Delta r]) -$$
$$\exp(+\mathrm{i}k[c\Delta t-\Delta r]) - \exp(-\mathrm{i}k[c\Delta t-\Delta r]), \tag{C.224}$$

where one recognises the sum and difference of the two frequencies $c\Delta t$ and Δr. The integrals are effectively the Fourier-representation of the δ_D-function,

$$\int_{-\infty}^{+\infty}\mathrm{d}k\ \exp(\mathrm{i}kx) = 2\pi\delta_D(x) \tag{C.225}$$

so that one arrives at

$$\ldots = \frac{4\pi}{(2\mathrm{i})^2}\delta_D(c\Delta t+\Delta r) - 4\pi\delta_D(c\Delta t-\Delta r) \tag{C.226}$$

as each term appears twice. By applying the scaling property of Dirac's δ_D-function

$$\delta_D(\alpha k) = \frac{1}{\alpha}\delta_D(k) \tag{C.227}$$

one arrives at

$$\ldots = -\frac{4\pi}{c}\delta_D\left(t - t' + \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}\right) + \frac{4\pi}{c}\delta_D\left(t - t' - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}\right) \tag{C.228}$$

where the factors c and π cancel with the corresponding factors in eqn. (C.222). Putting everything together yields as a final result for the Green-function

$$G_\pm(\boldsymbol{r} - \boldsymbol{r}', t - t') = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\Big[\underbrace{\delta_D(t - t' - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c})}_{\text{retarded}} - \underbrace{\delta_D(t - t' + \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c})}_{\text{advanced}}\Big] \tag{C.229}$$

with the conventional Green-function of Δ as a prefactor, modified by δ_D-functions. They take care of the fact that changes in the fields propagate at finite speed, such that the source configuration at distance $|\boldsymbol{r} - \boldsymbol{r}'|$ contributes to the potential at most at a time $|\boldsymbol{r} - \boldsymbol{r}'|/c$ *later* than t', which necessitates that one of the terms is discarded as being acausal: It would have the effect, that a source configuration at a time difference $|\boldsymbol{r} - \boldsymbol{r}'|/c$ in the *future* contributes to the fields. Finally, one arrives at the expression for the *retarded* Green-function $G_-(\boldsymbol{r} - \boldsymbol{r}', t - t')$,

$$G_-(\boldsymbol{r} - \boldsymbol{r}', t - t') = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\delta_D\left(t - t' - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}\right), \tag{C.230}$$

which serves for determining the potential $\psi(\boldsymbol{r}, t)$ from the source $q(\boldsymbol{r}', t')$,

$$\psi(\boldsymbol{r}, t) = \int \mathrm{d}V' \int \mathrm{d}t' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\delta_D\left(t - t' - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}\right) q(\boldsymbol{r}', t') = \int \mathrm{d}V' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} q\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}\right). \tag{C.231}$$

C.8 *Liénard-Wiechert potentials*

With the Green-functions for the d'Alembert-operator $\Box$,

$$G_\pm(\boldsymbol{r} - \boldsymbol{r}', t - t') = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\delta_D\left(t - t' \pm \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}\right) \tag{C.232}$$

it is possible to solve the wave equation

$$\Box\psi(\boldsymbol{r}, t) = 4\pi q(\boldsymbol{r}, t) \tag{C.233}$$

in a convolution relation,

$$\psi_\pm(\boldsymbol{r}, t) = \int \mathrm{d}t' \int \mathrm{d}V'\, G_\pm(\boldsymbol{r} - \boldsymbol{r}', t - t') q(\boldsymbol{r}', t') \tag{C.234}$$

where changes to the source configuration $q(\boldsymbol{r}', t')$ (to be interpreted as the charge distribution $\rho(\boldsymbol{r}, t)$ or the current density $\gamma_{ij}\jmath^i(\boldsymbol{r}, t)$) can only influence the fields (or potentials $\Phi(\boldsymbol{r}, t)$ and $A_i(\boldsymbol{r}, t)$, even though this statement depends on the gauge choice) after a time $|\boldsymbol{r} - \boldsymbol{r}'|/c$ has elapsed, and not instantaneously, due to the finite propagation speed c of excitations in the electromagnetic field.

Substituting the retarded Green-function G_- into the convolution relation for the potentials for obtaining them from the source distribution one arrives at

$$\psi(\boldsymbol{r},t) = \int \mathrm{d}V' \int \mathrm{d}t'\, G_-(\boldsymbol{r}-\boldsymbol{r}', t-t')q(\boldsymbol{r}',t') =$$
$$\int \mathrm{d}V' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\int \mathrm{d}t'\, \delta_D\left(t-t'-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)q(\boldsymbol{r}',t') = \int \mathrm{d}V' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}q\left(\boldsymbol{r}', t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \tag{C.235}$$

because the Dirac-δ_D fixes t' to the value $t-|\boldsymbol{r}-\boldsymbol{r}'|/c$. This expression applied to the potentials

$$\Phi(\boldsymbol{r},t) = \int \mathrm{d}V' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\rho\left(\boldsymbol{r}', t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \tag{C.236}$$

and

$$A_i(\boldsymbol{r},t) = \int \mathrm{d}V' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\gamma_{ij}\jmath^j\left(\boldsymbol{r}', t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \tag{C.237}$$

is referred to as the Liénard-Wiechert potentials, which provide a solution in the case a time-varying source distribution, taking retardation, i.e. the finite speed of propagation of the fields (or potentials in Lorenz gauge) into account. Clearly, in the limit $c \to \infty$ the fields and potentials would change instantaneously. Already now a causal structure becomes apparent, with a finite propagation speed at which the fields react to changes in the source. Taking the derivatives $B^i = \epsilon^{ijk}\partial_j A_k$ and $E_i = -\partial_i\Phi - \partial_{ct}A_i$ then leads to Jefimenko's equations, if one interchanges differentiation ∂_i and ∂_{ct} with the $\mathrm{d}V'$-integration for an expression for the fields for the case of time varying sources.

C.9 *Anatomy of partial differential equations*

Differential equations are the natural language in which laws of Nature are formulated: They set the rates of change of quantities into relation and depend crucially on initial and boundary conditions. Many different categories are relevant in the classification of differential equations:

- ordinary vs. partial:

  In ordinary differential equations, only derivatives with respect to a single variable or coordinate appear, whereas partial differential equations consist of derivatives with respect to two or more variables.

- homogeneous vs. inhomogeneous:

  If all terms depend on the field and its derivatives, the differential equation is homogeneous, but if a term appears that does not depend on the field or its derivatives, the equation is inhomogeneous.

- linear vs. nonlinear:

  If all terms in a differential equation are proportional to the field or its derivatives, the equation is linear, but if there are higher-order powers or nonlinear functions of the field, then the differential equation is nonlinear.

- derivative order:

  The highest derivative that appears in the differential equation sets the derivative order.

Given these definitions, the damped harmonic oscillator equation for the amplitude $x(t)$ with external driving $a(t)$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x(t) = a(t) \tag{C.238}$$

is an ordinary, inhomogeneous, linear differential equation of second order. The Schrödinger equation

$$\mathrm{i}\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + \Phi(r)\psi \tag{C.239}$$

on the other hand is a partial, homogeneous and linear differential equation, but its derivative order is likewise two.

It's well worth going through this categorisation as a checklist whenever you need to deal with ODEs/PDEs.

C.9.1 Hyperbolic, parabolic and elliptical differential equations

We have already encountered two partial differential equations of second order, the Laplace-equation

$$\Delta\Phi = \gamma^{ij}\partial_i\partial_j\Phi = 0 \tag{C.240}$$

as the field equation of electrostatics, and the wave equation

$$\Box\Phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\Phi = \left(\partial_{ct}^2 - \Delta\right)\Phi = 0 \tag{C.241}$$

of electrodynamics, here obtained in Lorenz-gauge. It suffices to consider the case of homogeneous partial differential equations because any inhomogeneity $\pm 4\pi\rho$ could be dealt with the Green-formalism. Comparing $\Box\Phi = 0$ as a wave equation with $\Delta\Phi = 0$ as a static field equation shows that the signs of the derivative operators $(+,-,-,-)$ and $(+,+,+)$ matter a lot, as one obtains oscillatory solutions for the wave equation, and (decreasing, at least in 3 dimensions or more) power-law solutions for the Poisson-equation. Please note that the choice of gauge does not have any influence at all on the derivative order (it is a statement involving only the first derivatives of the fields), but that it can change the character between hyperbolical and elliptical.

The classification of differential equations borrows many ideas from curves, here in particular from the theory of conic sections. A quadratic form of two coordinates x and y would be given by

$$\begin{pmatrix} x \\ y \end{pmatrix}^t \underbrace{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}}_{=\mathrm{D}} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + bxy + cy^2 = \text{const.} \tag{C.242}$$

Depending on the structure of eigenvalues, which decide on the sign of the determinant of the (discriminant) matrix D, the quadratic form describes very different curves: If $b = 0$ (for simplicity) and $a = c = 1 > 0$ one obtains $x^2 + y^2 = \text{const}$, which can be rewritten in a parametric form by setting $x = \cos\varphi$ and $y = \sin\varphi$ such that the quadratic form describes a circle as a consequence of $\cos^2\varphi + \sin^2\varphi = 1$, and in the peculiar case of $a \neq c$ an ellipse. If $a = 1$ and $c = -1$, the quadratic form becomes $x^2 - y^2 = \text{const}$, i.e a hyperbola with the hyperbolic functions as parametric forms, using $\cosh^2\psi - \sinh^2\psi = 1$. More generally, the picture arises that $\det \mathrm{D} > 0$ for the elliptical conic section and conversely, $\det \mathrm{D} < 0$ for the hyperbolic conic section.

Applying this idea to the classification of partial differential equations, we start with a homogeneous second-order PDE for the field ϕ in two coordinates in full generality,

$$a(x,y)\frac{\partial^2}{\partial x^2}\phi(x,y) + b(x,y)\frac{\partial^2}{\partial x\partial y}\phi(x,y) + c(x,y)\frac{\partial^2}{\partial y^2}\phi(x,y) = \mathrm{A}(x,y)\phi(x,y) \quad \text{(C.243)}$$

and assemble the matrix D

$$\mathrm{D} = \begin{pmatrix} a(x,y) & \frac{1}{2}b(x,y) \\ \frac{1}{2}b(x,y) & c(x,y) \end{pmatrix} \quad \text{(C.244)}$$

The determinant of D then establishes, whether the PDE is elliptical, $\det \mathrm{D} > 0$, parabolic, $\det \mathrm{D} = 0$ or hyperbolic, $\det \mathrm{D} < 0$. A visual impression is provided by Fig. 8 which shows these curves, actually conic sections, for various choices of the parameters.

Sticking to 2 dimensions, a PDE like the Poisson-equation

$$\Delta\phi = \frac{\partial^2}{\partial x^2}\phi(x,y) + \frac{\partial^2}{\partial y^2}\phi(x,y) = 0 \quad \text{(C.245)}$$

would be elliptical, as the determinant of D would come out positive: $a = c = 1$ and $b = 0$: elliptical differential equations have only unique solutions after boundary conditions are specified. They can be of the Dirichlet-type, the Neumann-type or be of mixed type. Please note that vacuum boundary conditions, where the fields and their derivatives approach zero at infinity, are perfectly admissible. Typical solutions are decreasing (for Poisson-like problems, at least in 3 dimensions or higher) with increasing coordinates and parity invariant, as $(x,y) \to (-x,-y)$ does not change anything.

On the other hand, a wave-equation exhibits a sign change,

$$\Box\phi(t,x) = \frac{\partial^2}{\partial(ct)^2}\phi(t,x) - \frac{\partial^2}{\partial x^2}\phi(t,x) = 0 \quad \text{(C.246)}$$

with $a = 1$, $c = -1$ and $b = 0$ in these coordinates and would be hyperbolic as $\det \mathrm{D} < 0$. In this case, it is enough to specify initial conditions and the PDE evolves them in a well-defined and unique way into the future. Specification of boundary conditions as in the case of elliptical PDEs is unnecessary, and in contrast to elliptical PDEs, hyperbolic PDEs show typically wavelike-solutions.

There is clearly the notion of a light-cone due to retardation, which persists even when a change of coordinates is carried out: Switching to light-cone coordinates $\partial_u = \partial_{ct} + \partial_x$ and $\partial_v = \partial_{ct} - \partial_x$ brings the wave equation into the form

$$\Box\phi(u,v) = \frac{\partial^2}{\partial u\partial v}\phi(u,v) = 0 \quad \text{(C.247)}$$

⚠ Please go through all iconic PDEs in theoretical physics and classify them as elliptical, parabolic or hyperbolic partial differential equations!

this time with $a = c = 0$ and $b = 1$, but the determinant $\det \mathrm{D} < 0$ nonetheless. It is actually the case that the metric structure of spacetime, which we focus on in the next chapter, with the Minkowski-metric is uniquely suited for hyperbolic PDEs: It is even the fact. The Lorentzian spacetime is the only metric spacetime with naturally hyperbolic evolution!

C.9.2 Wave-equation and its reductions

Central to electrodynamic theory was the wave-equation

$$\Box\phi(\boldsymbol{r},t) = 4\pi q(\boldsymbol{r},t) \quad \text{with} \quad \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_{ct}^2 - \Delta \quad \text{and} \quad \Delta = \gamma^{ij}\partial_i\partial_j \tag{C.248}$$

as a linear, inhomogeneous, hyperbolical, partial differential equation of derivative order two. Separating out oscillations in time with an ansatz $\phi \propto \exp(\pm i\omega t)$ leads to the Helmholtz-equation

$$\Delta\phi + k^2\phi = -4\pi q(\boldsymbol{r},t) \tag{C.249}$$

with $k = \omega/c$. Under the stronger assumption of a static solution, where neither ϕ nor q depended on t, one arrives at the Poisson-equation,

$$\Delta\phi = -4\pi q(\boldsymbol{r}) \tag{C.250}$$

further reducing to the Laplace-equation

$$\Delta\phi = 0 \tag{C.251}$$

for the vacuum case with vanishing sources. In all cases is the incorporation of an inhomogeneity $q(\boldsymbol{r},t)$ straightforwardly possible by means of the Green-formalism.

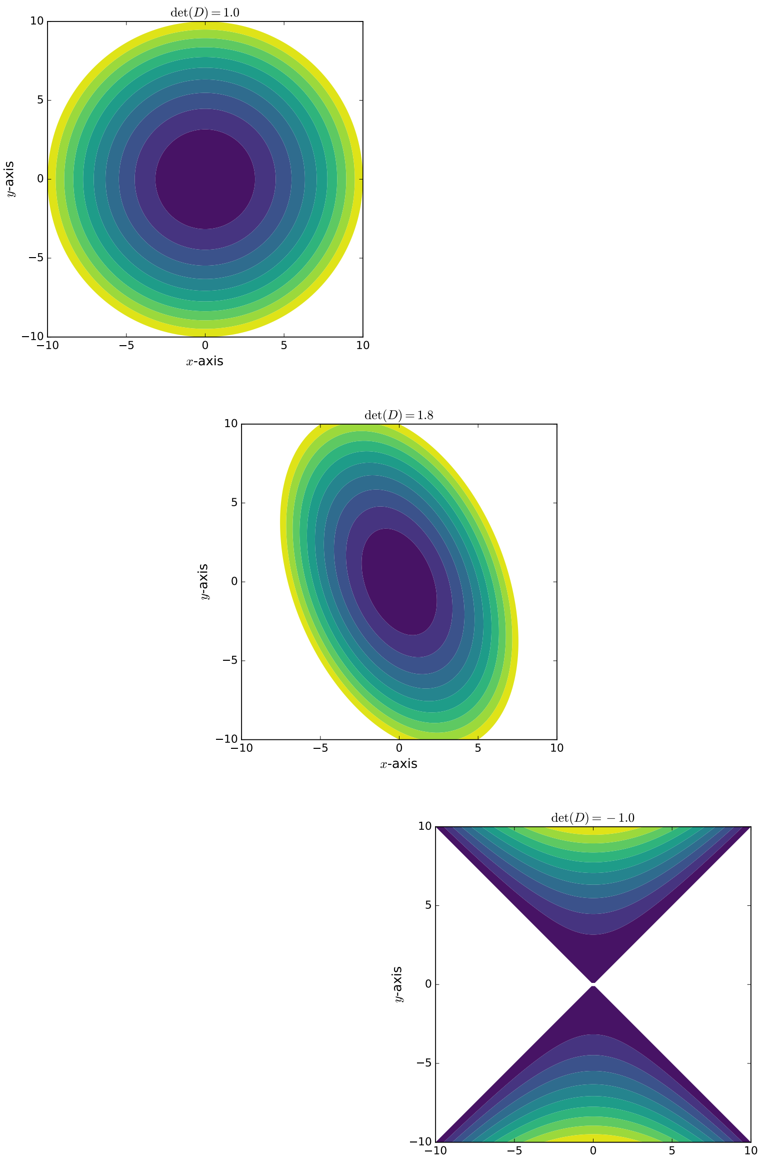


Figure 8: Conic sections: circles ($\det(\mathrm{D}) = 1$)*, ellipses* ($\det \mathrm{D} > 0$) *and hyperbolæ*($\det(\mathrm{D}) < 0$)*, from top to bottom.*