A MAXWELL-EQUATIONS

A.1 *Fields and test charges*

Fields are a very novel concept in physics as there can be systems without direct physical contact of their constituent parts which are nevertheless influenced *at a distance*, as Newton formulated it. As we do not have any direct sensory perception of fields, they are indirectly observed by the force they are capable of exerting on charged test particles. If a test charge q is exposed to the electric field E_i and the magnetic field B^i it experiences a change $\dot{p}_i$ in its momentum p_i:

$$\dot{p}_i = q\left(E_i + \frac{1}{c}\epsilon_{ijk}\upsilon^j B^k\right). \tag{A.1}$$

This Lorentz-force depends on the magnitude and direction of the velocity υ^i/c in units of a velocity scale c. It is possible to measure all components of E_i and B^i separately as one has the freedom of choosing the state of motion of the test charge. Clearly, as the velocity υ^i depends on the choice of frame, the measurement of E_i and B^i has to be frame-dependent as well. Therefore, with the concept of a test charge one links the dynamical and kinematical properties of fields to the mechanics of the test particles in a consistent way. Historically this was very important, as electrodynamics showed that Galilean mechanics for the motion of test particles is inconsistent with the fields, and needed to be replaced by Lorentzian, relativistic mechanics. It is important to realise that the two fields measured by a test charge are a linear form E_i for the electric component and a vector B^i for the magnetic component.

A.2 *Physical properties of the electric charge*

Electrodynamics is a continuum theory: One can imagine the electric charge density ρ to be a fluid so that arbitrarily small volumes contain arbitrarily small amounts of electric charge. There is no idea of charge carriers such as electrons or protons, and no concept of a quantisation of charge into multiples of an elementary charge. Charge is conserved, meaning that the fluid can move and change the charge density, but there is no spontaneous creation or annihilation of electric charge. This statement is necessarily an empirical property of charge-carrying matter.

If the local charge density increases, it must be necessarily due to converging current densities, as expressed in a continuity equation:

$$\partial_t\rho + \partial_i \jmath^i = 0 \tag{A.2}$$

implying that the charge q contained within a volume V only changes over time if there are electric currents I transporting the charge through the surface ∂V:

$$\frac{d}{dt}\int_V dV\,\rho = \frac{d}{dt}q = -\int_V dV\,\partial_i \jmath^i = -\int_{\partial V} dS_i\,\jmath^i = -I \tag{A.3}$$

as a consequence of the Gauß-theorem. The electric charge q appears as the volume integral over the charge density ρ, in the same way as the electric current I is the surface integral over the current density $\jmath^i$, projected along the surface normal dS_i. Specifically, if the currents $\jmath^i$ point outwards and are parallel to the surface normals

dS_i of ∂V, the enclosed charge q will decrease, which provides a good motivation for the minus-sign.

A.3 *Maxwell-equations*

The Maxwell-equations are the axiomatic foundation of classical electrodynamics and they define the relation between the distribution of electric charges and currents on one side and the electric and magnetic fields on the other, as well as the dynamical evolution of the fields themselves. They are a set of linear hyperbolic partial differential equations, formulated in terms of first derivatives ∂_i and ∂_{ct} of the fields with respect to the coordinates x^i and time t. Temporal derivatives always appear multiplied with a constant c, that will turn out to be the speed at which excitations in the electromagnetic field propagate.

Maxwell's equations involve the two physically measurable fields E_i and B^i as well as two auxiliary fields D^i and H_i. These auxiliary fields are sourced by the electric charge density ρ and the electric current density $\jmath^i$, and can only be related to E_i and B^i with an assumption on the physical properties of the medium in which the charges reside. At the time, Maxwell isolated his four equations from empirical observation of magnetic and electric phenomena, but they are much more than that: They open a path to a geometric description of Nature in terms of relativistic field theories.

A.3.1 Gauß-law for electric fields

The electric field D^i emanates from the electric charge density ρ, meaning that wherever there are electric charges, they act as sources of the electric field. The field lines diverge from a positive charge and converge on a negative charge. Mathematically speaking, the divergence $\partial_i D^i$ of the electric field is proportional to the charge density ρ with the prefactor 4π in the Gauß-system of units:

$$\partial_i D^i = 4\pi\rho \tag{A.4}$$

With the help of the Gauß-theorem the Maxwell-equation can be reformulated in integral form,

$$\int_V dV\, \partial_i D^i = \int_{\partial V} dS_i\, D^i = \psi = 4\pi \int_V dV\, \rho = 4\pi q \tag{A.5}$$

implying that there is a flux ψ of electric field lines through the surface ∂V of any volume V which contains a charge q.

An electric field D^i should be spherically symmetric around a point charge of magnitude q, certainly in the case of an isotropic medium. This means that the electric field lines should be perpendicular to the surface ∂V of a sphere of volume V containing the charge at the centre, and the electric field should be of equal strength everywhere on the surface. Then,

$$\int_{\partial V} dS_i\, D^i = 4\pi\, r^2\, D = 4\pi q \quad \rightarrow \quad D = \frac{q}{r^2} \tag{A.6}$$

with the familiar expression for the Coulomb-field $D \propto 1/r^2$ of a point charge.

In summary, electric field lines start necessarily on a positive charge and end at a negative charge, unless they form a closed loop.

A.3.2 Non-existence of magnetic charges

The magnetic field B^i behaves differently: There are no corresponding magnetic charges from which the magnetic field lines would emanate, so the divergence of the magnetic field is necessarily zero,

$$\partial_i B^i = 0. \tag{A.7}$$

In integral form the relation would read

$$\int_V dV\, \partial_i B^i = \int_{\partial V} dS_i\, B^i = \phi = 0 \tag{A.8}$$

showing clearly with the Gauß-theorem that the flux ϕ of magnetic field lines across the surface ∂V of a volume V is zero, as it can not contain any magnetic charges.

A.3.3 Faraday-law and induction

Electric field lines can be closed loops, too, and this is necessarily related to time-varying magnetic fields, as formulated by the Faraday-law

$$\epsilon^{ijk}\partial_j E_k = -\partial_{ct} B^i \tag{A.9}$$

with the most famous minus sign of physics: the Lenz-rule. It is a reflection of the hyperbolicity of the Maxwell-equations and despite many claims otherwise, it has little to do with energy conservation. Here, the speed of light c makes sure that the derivative ∂_i is dimensionally consistent to the derivative ∂_{ct}, because

$$\partial_{ct} = \frac{1}{c}\partial_t \tag{A.10}$$

has units of inverse length just as the spatial derivatives: Please keep in mind, that in the Gauß-system of units, all fields E_i, D^i, H_i and B^i have identical units. The corresponding integral form of the Faraday-law is derived by application of the Stokes-theorem,

$$\int_S dS_i\, \epsilon^{ijk}\partial_j E_k = U = \int_{\partial S} dr^i\, E_i = -\frac{d}{d(ct)}\int_S dS_i\, B^i = -\frac{d\phi}{d(ct)} \tag{A.11}$$

where one can identify the induced voltage U on the boundary ∂S of the surface S as being proportional to the rate of change of the magnetic flux ϕ with respect to ct. Integrating the rotation $\epsilon^{ijk}\partial_j E_k$ along an integration contour would yield the displacement work necessary to move a charge along this contour, which, normalised by the magnitude of the charge, is exactly the voltage.

A.3.4 Ampère-law

Magnetic fields are surely divergence-free, but can they be loops? Ampère's law answers this clearly in a positive way, as

$$\epsilon^{ijk}\partial_j \mathrm{H}_k = +\partial_{ct}\mathrm{D}^i + \frac{4\pi}{c}\jmath^i \tag{A.12}$$

implying that the rotation $\epsilon^{ijk}\partial_j \mathrm{H}_k$ is related to two phenomena: There can be a non-vanishing electric current density $\jmath^i$ with magnetic field lines looping around it, or the electric field is time-varying. Again, the Stokes-theorem allows to reformulate the Ampère-law in integral form,

$$\int_{\mathrm{S}} \mathrm{dS}_i\, \epsilon^{ijk}\partial_j \mathrm{H}_k = \int_{\partial \mathrm{S}} \mathrm{d}r^i\, \mathrm{H}_i = +\frac{\mathrm{d}}{\mathrm{d}(ct)}\int_{\mathrm{S}} \mathrm{dS}_i\, \mathrm{D}^i + \frac{4\pi}{c}\int_{\mathrm{S}} \mathrm{dS}_i\, \jmath^i = +\frac{\mathrm{d}\psi}{\mathrm{d}(ct)} + \frac{4\pi}{c}\mathrm{I} \tag{A.13}$$

such that magnetic field collected up on a closed loop $\partial\mathrm{S}$ becomes equal to the change of the electric flux ψ through the surface S and to the electric current I through that surface. In a static, cylindrically symmetric situation of a straight wire one would evaluate the integral as $\int \mathrm{d}r^i \mathrm{H}_i = 2\pi r\, \mathrm{H}$ on a circle with radius r, such that the magnetic field decreases $\mathrm{H} \propto 1/r$ with increasing distance from the wire.

A.4 *Linear media for electrodynamics*

Maxwell's equations allow to compute the electric and magnetic fields for a given distribution of the charge density ρ and the current density $\jmath^i$, and to localise the source distribution ρ and $\jmath^i$ for a given field configuration. In the general case, these relationships are defined for two auxiliary fields, an electric vector field D^i and a magnetic linear form H_i. These two excitations, D^i and H_i, are related to the sources ρ and $\jmath^i$ in purely geometric relations. For converting them into the measurable fields E_i and B^i, one needs to incorporate the properties of matter, in which the charges and currents as sources of the electric and the magnetic field are embedded.

	vector	*form*
vacuum	B^i	E_i
medium	D^i	H_i

Restricting the discussion to linear media one assumes a proportionality

$$\mathrm{D}^i = \epsilon^{ij}\mathrm{E}_j \quad \leftrightarrow \quad \mathrm{E}_i = \epsilon_{ij}\mathrm{D}^j \tag{A.14}$$

with a permissivity (or dielectric) tensor ϵ_{ij}, and in analogy a likewise linear relation

$$\mathrm{B}^i = \mu^{ij}\mathrm{H}_i \quad \leftrightarrow \quad \mathrm{H}_i = \mu_{ij}\mathrm{B}^j \tag{A.15}$$

with a permeability tensor μ_{ij}. These two relationships between the electric field pair E_i and D^i on one side and the magnetic field pair B^i and H_i on the other are referred to as constitutive relations. The permissivity tensor ϵ_{ij} and the permeability tensor μ_{ij} are both symmetric, positive definite tensors. As such, they act as a metric with the purpose of converting the vectors in linear forms and vice versa. This is made possible by the fact that ϵ^{ij} is inverse to ϵ_{ij}, with $\epsilon^{ij}\epsilon_{jk} = \delta^i_k$. Both tensors have a principal axis frame in which one observes that the fields become elementwise proportional to each other, scaled by the eigenvalues of ϵ_{ij} or μ_{ij}.

If all eigenvalues are equal, the medium is isotropic and the tensors become proportional to the Euclidean metric γ_{ij}, with an admittedly weird convention

$$\epsilon_{ij} = \frac{1}{\epsilon}\gamma_{ij} \quad \text{and} \quad \mu_{ij} = \frac{1}{\mu}\gamma_{ij}, \tag{A.16}$$

with the dielectric constant ϵ and the isotropic permeability μ. Vacuum is effectively described to be a medium with $\epsilon = 1 = \mu$. Because the Euclidean metric γ_{ij} mediates between the pairs E_i, D^i and H_i, B^i, their distinctiveness is lost in a vacuum situation without a medium. Positive definiteness makes sure that the observable electric and magnetic fields B^i and E_i are pointing in the same direction as the excitations H_i and D^i: The fields are attenuated in a medium but never reversed.

There are even bianisotropic media where dielectric effects are caused by magnetic fields and effects of permeability by electric fields:

$$D^i = \epsilon^{ij}E_j + \xi^{ij}H_j \quad \text{as well as} \quad B^i = \mu^{ij}H_j + \zeta^{ij}E_j, \tag{A.17}$$

with two additional tensors ξ^{ij} and ζ^{ij} in the constitutive relations. The connection between fields and the microscopic structure of matter can be extremely complicated, and only in simplified cases one will have a linear, instantaneous and isotropic response of the fields to the presence of matter. It is quite apparent that there is a time scale involved in the reaction of the fields E_i and B^i to the excitations H_i and D^i. Water, for instance, has a very high dielectric constant $\epsilon \simeq 80$ for static electric fields as a consequence of the polarity of the water molecules, but the dielectric constant for the rapidly changing electric field in visible light has decreased to a value of about $\epsilon \simeq 1.5$.

In contrast to the inhomogeneous equations that get modified in the presence of matter, the homogeneous relations $\partial_i B^i = 0$ and $\epsilon^{ijk}\partial_j E_k = -\partial_{ct}B^i$ are unchanged in matter. One can almost feel why this should be the case: The inhomogeneous equations predict, due to their shape, strong fields close to the sources, i.e. $D \propto 1/r^2$ and $H \propto 1/r$ which are diverging in the limit $r \to 0$ and which should evoke a strong response from the medium altering the fields. While this argument sounds very convincing, it neglects the fact that most materials follow linear relationships between the fields and the excitations, or equivalently, have constant dielectric and permeability tensors, and effectively do not distinguish between weak and strong fields: In fact, the situation is very puzzling to interpret as there are linear responses evoked by arbitrarily weak fields around sources, but not for induced fields!

For the purpose of this script we will always assume homogeneous media, such that the material properties do not change as a function of position and all derivatives $\partial_k\epsilon^{ij}$ and $\partial_k\mu^{ij}$ are zero. But this is a choice of convenience, as the Maxwell-equations would be prepared to deal with inhomogeneous media, at the expense of notational clarity: The Gauß-law $\partial_i D^i = 4\pi\rho$, for instance, becomes $\partial_i(\epsilon^{ij}E_j) = \partial_i\epsilon^{ij} \cdot E_j + \epsilon^{ij}\partial_i E_j = 4\pi\rho$, with a new term $\partial_i\epsilon^{ij}$ reflecting the change of the dielectric tensor with coordinate.

A.5 *Conservation of electric charge*

With the link between the fields and the distribution of charges (and currents) established by the Maxwell-equations it should be clear that the fields are not free and that they should reflect any dynamical laws to which the charges are subjected, for instance charge conservation: It will never be the case that the electrical field lines suddenly converge onto a single point in space and form a non-vanishing divergence, without there being an actual charge. Therefore, it should be possible to derive a charge conservation law from the field configuration! In fact, computing the divergence of the rotation $\epsilon^{ijk}\partial_j \mathrm{H}_k$ as defined by the Ampère-law,

$$\epsilon^{ijk}\partial_j \mathrm{H}_k = +\partial_{ct}\mathrm{D}^i + \frac{4\pi}{c} j^i \quad \rightarrow \quad \partial_i \epsilon^{ijk}\partial_j \mathrm{H}_k = \epsilon^{ijk}\partial_i\partial_j \mathrm{H}_k = \partial_i\partial_{ct}\mathrm{D}^i + \frac{4\pi}{c}\partial_i j^i = 0 \tag{A.18}$$

which is always zero: The contraction of ϵ^{ijk} which is antisymmetric in the index pair (ij) with the double derivative $\partial_i\partial_j$, which is symmetric in the index pair is necessarily zero. The exchangibility of the partial derivatives $\partial_i\partial_j = \partial_j\partial_i$ and hence the symmetry of the expression $\partial_i\partial_j$ is made sure by Schwarz's theorem.

This consideration leads to

$$\partial_{ct}\partial_i \mathrm{D}^i + \frac{4\pi}{c}\partial_i j^i = 0, \tag{A.19}$$

and by substituting the Gauß-law $\partial_i \mathrm{D}^i = 4\pi\rho$ to

$$4\pi\partial_{ct}\rho + \frac{4\pi}{c}\partial_i j^i = 0, \tag{A.20}$$

where ∂_{ct} interchanges with the divergence, as both are partial derivatives. In the last relation one recovers the conservation law in the shape of a continuity equation

$$\partial_t \rho + \partial_i j^i = 0 \tag{A.21}$$

In a very real sense, electrodynamics is the theory of electric and magnetic fields for conserved charges; if, by any mechanism, there would be spontaneous creation or decay of charges, or even teleportation of charges, the Maxwell-equations would need to be amended.

A.6 *Electromagnetic duality*

In vacuum, where the charge density ρ and the current density j^i are zero, the Maxwell-equations assume a very symmetric shape as all equations are purely homogeneous. The divergences read

$$\partial_i \mathrm{D}^i = \epsilon^{ij}\partial_i \mathrm{E}_j = \epsilon\gamma^{ij}\partial_i \mathrm{E}_j = 0 \quad \text{as well as} \quad \partial_i \mathrm{B}^i = \mu^{ij}\partial_i \mathrm{H}_j = \mu\gamma^{ij}\partial_i \mathrm{H}_j = 0, \tag{A.22}$$

and the rotations become:

$$\epsilon^{ijk}\partial_j \mathrm{E}_k = -\partial_{ct}\mathrm{B}^i = -\mu^{ij}\partial_{ct}\mathrm{H}_j = -\frac{1}{\mu}\gamma^{ij}\partial_{ct}\mathrm{H}_j \tag{A.23}$$

and

$$\epsilon^{ijk}\partial_j \mathrm{H}_k = +\partial_{ct}\mathrm{D}^i = +\epsilon^{ij}\partial_{ct}\mathrm{E}_j = \epsilon\gamma^{ij}\partial_{ct}\mathrm{E}_j, \tag{A.24}$$

where we introduced the permissivity and permeability tensors to map all fields to the two linear forms E_i and H_i. In isotropic media, Maxwell's equation exhibit invariance under the duality transform

$$\mathrm{E}_i \rightarrow +\mathrm{H}_i \quad \text{and} \quad \mu\mathrm{H}_i \rightarrow -\epsilon\mathrm{E}_i. \tag{A.25}$$

Clearly, there is no influence of the duality transform on the divergences, while the two equations involving rotations just interchange their roles. Duality is broken by the presence of ρ and $\jmath^i$. Without a medium, i.e for the case $\epsilon = 1 = \mu$, the duality transform takes on an even simpler form, $E_i \rightarrow \gamma_{ij}B^j$ and $B^i \rightarrow -\gamma^{ij}E_j$, and relates the vacuum fields directly to each other.

Maxwell's equations would straightforwardly be able to accommodate magnetic charges if they are in fact conserved, as can be seen from this argument. Purely driven by analogy and intuition, one can amend Maxwell's equations as

$$\partial_i D^i = 4\pi\rho \quad \text{and} \quad \partial_i B^i = 4\pi\tau \tag{A.26}$$

as well as

$$\epsilon^{ijk}\partial_j H_k = +\partial_{ct} D^i + \frac{4\pi}{c}\jmath^i \quad \text{and} \quad \epsilon^{ijk}\partial_j E_k = -\partial_{ct} B^i - \frac{4\pi}{c}\imath^i \tag{A.27}$$

by introducing a magnetic charge density τ and a magnetic current density $\imath^i$, making all Maxwell-equations inhomogeneous PDEs. Clearly, with vanishing τ and $\imath^i$ one would recover the true Maxwell-equations, one pair being homogeneous and the other pair being inhomogeneous. From $\epsilon^{ijk}\partial_i\partial_j E_k = 0$ one recovers a continuity equation of the magnetic charge

$$\partial_t \tau + \partial_i \imath^i = 0 \tag{A.28}$$

in complete analogy to the case for electric charges. This realisation is quite sensible: There are 2 scalar and 2 vectorial equations for 3 components for E_i and 3 components for B^i. They have to be determined by ρ as a scalar source and by $\jmath^i$ as a vectorial source, which might look odd, as there are more field components than source components ($6 > 4$), and again more equations than field components ($8 > 6$)! But the conservation of the source needs to be respected by the fields as well, reducing the effective number of equations by two: there is a conservation law for ρ and one for τ, which Nature has incidentally chosen to be zero (She has good reasons for doing so!), reducing the effective number of equations from 8 to 6. How exactly the 4 components of the source determine 6 components of the fields (clearly, they can't all be independent, otherwise the problem would be underdetermined) will be the topic of Sect. B on potential theory.

A summary of all quantities appearing in the Maxwell-equations is given in this diagram Fig. 1, for the general, hypothetical case of both magnetic and electric charges. For the actual Maxwell-theory with only electric charges, $\tau = 0 = \imath^i$.

A.7 *Maxwell-equations under discrete symmetries*

The Maxwell-equations show a curious and interesting behaviour under the three discrete symmetries: (*i*) charge conjugation $\mathcal{C}$, which replaces every positive charge $+q$ by a negative one $-q$, and vice versa, (*ii*) parity inversion $\mathcal{P}$, which mirrors the spatial coordinates $+x^i$ to $-x^i$, and (*iii*) time reversal $\mathcal{T}$, which replaces $+t$ by $-t$. Particularly relevant will be the classification of vectors (and linear forms) as being polar, $\mathcal{P}D^i = -D^i$ or axial, $\mathcal{P}B^i = +B^i$. Under the assumptions of a linear medium, the two pairs of fields will always be proportional to each other, $E_i = \epsilon_{ij}D^j$ and $H_i = \mu_{ij}B^j$ and must have pairwise identical behaviour under $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$.

Starting from the realisation that the position x^i behaves like a polar vector because its sign change under $\mathcal{P}$ leads to the implication that the differentiation ∂_i

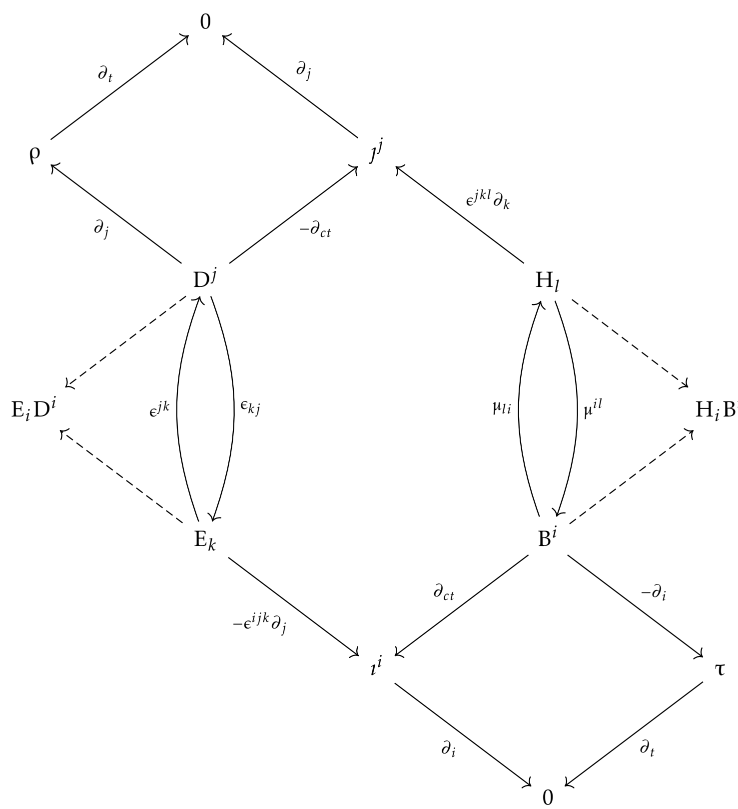


Figure 1: All quantities and their relationships within the Maxwell-equations.

behaves as $\mathcal{P}\partial_i = -\partial_i$. An identical argument applies to time reversal, leading to $\mathcal{T}\partial_{ct} = -\partial_{ct}$ for the time derivatives. The volume needed for computing the densities ρ and τ enters in an unoriented way, so it is unaffected by $\mathcal{P}$. The densities do change sign under $\mathcal{C}$, though. The currents $\jmath^i$ and $\imath^i$ change sign under $\mathcal{C}$ and reverse their direction of flow under both $\mathcal{P}$ and $\mathcal{T}$.

A good starting point are the third and fourth Maxwell-equations,

$$\epsilon^{ijk}\partial_j \mathrm{E}_k = -\partial_{ct}\mathrm{B}^i - \frac{4\pi}{c}\imath^i, \quad \text{and} \quad \epsilon^{ijk}\partial_j \mathrm{H}_k = +\partial_{ct}\mathrm{D}^i + \frac{4\pi}{c}\jmath^i \tag{A.29}$$

with the (possible) extension to include a (conserved) magnetic charge density τ and its associated magnetic current density $\imath^i$. They suggest that D^i and $\jmath^i$ on one side and B^i and $\imath^i$ on the other must have identical properties under the discrete symmetry transformations $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$. But at the same time it is clear that there is a fundamental difference in the behaviour of the electric and magnetic fields with respect to $\mathcal{P}$, as the right hand sides acquire additional minus signs because of the derivative ∂_i: Parity transforms affect electric and magnetic fields in opposite ways. Because the electric fields result from the gradient of a potential, $\mathrm{E}_i = -\partial_i\Phi$, they must be behave as polar vectors, $\mathcal{P}\mathrm{D}^i = -\mathrm{D}^i$, and the magnetic fields as axial vectors, $\mathcal{P}\mathrm{B}^i = +\mathrm{B}^i$.

The two divergences

$$\partial_i \mathrm{D}^i = 4\pi\rho \quad \text{as well as} \quad \partial_i \mathrm{B}^i = 4\pi\tau \tag{A.30}$$

make sure that the fields change sign under $\mathcal{C}$ along with the changes of the charges ρ and τ under $\mathcal{C}$. Far more interesting is $\mathcal{P}$: Because $\partial_i\mathrm{D}^i$ is parity-even, ρ must be scalar, $\mathcal{P}\rho = \rho$, but conversely, $\mathcal{P}\partial_i\mathrm{B}^i = -\partial_i\mathrm{B}^i$ implies a pseudoscalar magnetic charge $\mathcal{P}\tau = -\tau$. This translates to a more subtle difference in the transformation property of the currents $\jmath^i$ and $\imath^i$: The latter needs to be parity positive, $\mathcal{P}\imath^i = +\imath^i$ and therefore axial, while $\mathcal{P}\jmath^i = -\jmath^i$, with a polar electric current density, effectively ensuring the consistency of the two rotational Maxwell-equations.

The two conservation equations $\partial_t\tau + \partial_i\imath^i = 0$ and $\partial_t\rho + \partial_i\jmath^i = 0$ are likewise consistent because $\mathcal{T}$ changes both the time-derivatives as well as the direction of the currents, and parity inversion $\mathcal{P}$ changes τ because of its pseudoscalar property, but only the sign of ∂_i as $\mathcal{P}\imath^i$ is invariant: The change in sign of the pseudoscalar charge is cancelled by the inverted direction of flow of the magnetic current. In summary, it became clear that the Maxwell-equations show a transformation behaviour under $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$.

A.8 *Electrostatic potential*

Maxwell's equations clarify the relation between the field configuration and the distribution of the charges as sources of the fields. As such, they enable us to compute the field configuration from the source; in the easiest case this would be an electrostatic field around an electric point charge q. Using the Gauß-law in integral form

$$\int_V \mathrm{dV}\, \partial_i \mathrm{D}^i = \int_V \mathrm{dV}\, \epsilon^{ij}\partial_i \mathrm{E}_j = \int_{\partial V} \mathrm{dS}_i\, \epsilon^{ij}\mathrm{E}_j = 4\pi\epsilon r^2 \mathrm{E} = 4\pi \int_V \mathrm{dV}\, \rho = 4\pi q \tag{A.31}$$

		$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	$\mathcal{CP}$	$\mathcal{CT}$	$\mathcal{PT}$	$\mathcal{CPT}$
spatial derivative	∂_i	+	−	+	−	+	−	−
time derivative	∂_{ct}	+	+	−	+	−	−	−
electric charge density	ρ	−	+	+	−	−	+	−
electric current density	$\jmath^i$	−	−	−	+	+	+	−
magnetic charge density	τ	−	−	+	+	−	−	+
magnetic current density	$\imath^i$	−	+	−	−	+	−	+
dielectric displacement	D^i	−	−	+	+	−	−	+
electric field	E_i	−	−	+	+	−	−	+
magnetic induction	H_i	−	+	−	−	+	−	+
magnetic field	B^i	−	+	−	−	+	−	+

Table 1: Summary of the behaviour of all fields and sources in extended electrodynamics with electric and magnetic sources.

imposing spherical symmetry and working with an isotropic medium with dielectric constant ϵ (which implies $\epsilon^{ij} = \epsilon\gamma^{ij}$) leads to a radial field

$$E = \frac{q}{\epsilon\, r^2} \tag{A.32}$$

Clearly, the $1/r^2$-behaviour is a consequence of the growth of the surface area of spheres with increasing radius r, because the electric flux ϕ through every spherical shell is conserved. Positioning the charge q_1 at the position $\boldsymbol{r}_1$ and observing the field $\mathbf{E}$ at the position $\boldsymbol{r}$ would yield

$$E_i(\boldsymbol{r}) = \frac{q_1}{|\boldsymbol{r}-\boldsymbol{r}_1|^2}\,\gamma_{ij}\frac{(\boldsymbol{r}-\boldsymbol{r}_1)^j}{|\boldsymbol{r}-\boldsymbol{r}_1|} = q_1\,\gamma_{ij}\frac{(\boldsymbol{r}-\boldsymbol{r}_1)^j}{|\boldsymbol{r}-\boldsymbol{r}_1|^3} \tag{A.33}$$

where $\boldsymbol{r}-\boldsymbol{r}_1/|\boldsymbol{r}-\boldsymbol{r}_1|$ is a unit vector pointing from the charge q_1 to the observation point, converted with γ_{ij} into unit linear form. For a test charge, positive by convention, this would then yield a repulsive force for positive q_1 and an attractive force for negative q_1. The electric field of a collection of N charges q_n, $n = 1\ldots N$ follows by superposition, as the Maxwell-equations are linear:

$$E_i(\boldsymbol{r}) = \sum_{n=1}^{N} q_n\,\gamma_{ij}\frac{(\boldsymbol{r}-\boldsymbol{r}_n)^j}{|\boldsymbol{r}-\boldsymbol{r}_n|^3} \tag{A.34}$$

Transitioning to the continuum limit and replacing the discrete charges q_n at positions $\boldsymbol{r}_n$ with a continuous charge density $\rho(\boldsymbol{r})$ requires to replace summations by volume integrals

$$q = \sum_{n=1}^{N} q_n = \int_V dV'\,\rho(\boldsymbol{r}') \tag{A.35}$$

such that the total charge q in the system is respected. Similar relations should hold for any weighted integral and weighted sum, such that

$$E_i(\boldsymbol{r}) = \sum_{n=1}^{N} q_n\, \gamma_{ij} \frac{(\boldsymbol{r}-\boldsymbol{r}_n)^j}{|\boldsymbol{r}-\boldsymbol{r}_n|^3} = \int_V dV'\, \rho(\boldsymbol{r}')\, \gamma_{ij} \frac{(\boldsymbol{r}-\boldsymbol{r}')^j}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{A.36}$$

i.e. the electric field results by convolution of the charge density ρ with a vectorial integration kernel. An explicit calculation shows that

$$-\partial_i \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \gamma_{ij} \frac{(\boldsymbol{r}-\boldsymbol{r}')^j}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = +\partial_i' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{A.37}$$

with ∂_i acting on $\boldsymbol{r}$ and ∂_i' acting on $\boldsymbol{r}'$. Then,

$$E_i(\boldsymbol{r}) = \int_V dV'\, \rho(\boldsymbol{r}')\, \gamma_{ij} \frac{(\boldsymbol{r}-\boldsymbol{r}')^j}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = -\int_V dV'\, \rho(\boldsymbol{r}')\partial_i \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = $$
$$-\partial_i \int_V dV'\, \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\partial_i \Phi(\boldsymbol{r}) \tag{A.38}$$

with the electrostatic potential Φ,

$$\Phi(\boldsymbol{r}) = \int_V dV'\, \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{A.39}$$

which is clearly given as the superposition of the Coulomb-potentials of each element of the charge distribution ρ. Mathematically, this relation would be interpreted as a convolution of the distribution of sources $\rho(\boldsymbol{r})$ with a kernel $1/r$, which is in this context called the Green-function.

Here, the electric field E_i is a gradient field, $E_i = -\partial_i \Phi$, such that

$$\epsilon^{ijk}\partial_j E_k = -\epsilon^{ijk}\partial_j\partial_k\Phi = 0 \tag{A.40}$$

consistent with the induction law $\epsilon^{ijk}\partial_j E_k = -\partial_{ct}B^i = 0$ which yields a vanishing result in the static case. The gradient field E_i is rotationless because the contraction of the antisymmetric ϵ^{ijk} with the symmetric $\partial_j\partial_k$ is necessarily zero.

There is a clear interpretation of the electrostatic potential as the energy needed to displace a test charge in the electric field

$$W = -q\int_A^B dr^i\, E_i = q\int_A^B dr^i\, \partial_i\Phi = q\int_A^B d\Phi = q\,(\Phi(\boldsymbol{r}_B) - \Phi(\boldsymbol{r}_A)) \tag{A.41}$$

as the field is conservative. The integrand $dr^i\partial_i\Phi = d\Phi$ should be interpreted as the gradient $\partial_i\Phi$ projected onto dr^i. Combining the relation $E_i = -\partial_i\Phi$ between the electric field and the potential with the Gauß-law $\partial_i D^i = \epsilon^{ij}\partial_i E_j = 4\pi\rho$ yields the Poisson-equation

$$\partial_i D^i = -\epsilon^{ij}\partial_i\partial_j\Phi = -\Delta\Phi = 4\pi\rho \quad\rightarrow\quad \Delta\Phi = -4\pi\rho \tag{A.42}$$

introducing the Laplace-operator, which assumes the general form $\Delta = \epsilon^{ij}\partial_i\partial_j$, falls back on $\Delta = \epsilon\gamma^{ij}\partial_i\partial_j$ in an isotropic medium and ultimately on $\Delta = \gamma^{ij}\partial_i\partial_j$ in vacuum.

It might be a fun thought to use the positive definiteness of the metric ϵ^{ij} to carry out a Cholesky-decomposition to $\epsilon^{ij} = e_m{}^i\gamma^{mn}e_n{}^j$ such that the Laplace-operator becomes $\Delta = \epsilon^{ij}\partial_i\partial_j = e_m{}^i\gamma^{mn}e_n{}^j\,\partial_i\partial_j = \gamma^{mn}\,e_m{}^i\partial_i e_n{}^j\partial_j$. Then, a coordinate transform with $x^i \rightarrow e^m{}_i x^i$ leads to $\partial_i \rightarrow e_m{}^i\partial_i$, such that in the new coordinates the Laplace operator takes on the Euclidean form, $\gamma^{mn}\partial_m\partial_n = \Delta$. In essence, the effect of an anisotropic medium can be absorbed by a (linear) change in coordinates. In the particular case of an isotropic medium, this amounts to a mere rescaling or to the usage of a different unit of length or charge, as the two are degenerate in Coulomb's law: The situation of an electric field of a charge inside a medium can be mapped onto a different charge in vacuum

A bit more surprising might be the realisation that the equipotential surfaces of the electric field around a point charge in an anisotropic medium would be ellipsoids, but in the change of coordinate suggested by the Cholesky-decomposition of ϵ^{ij} would become perfect spheres!

The Cholesky-decomposition, valid for any symmetric, positive definite matrix, is given by $e^i{}_m\gamma^{mn}e^j{}_n$ for the inverse permissivity ϵ^{ij}, and by $e_i{}^m\gamma_{mn}e_j{}^n$ for the permissivity ϵ_{ij}

The Laplace-operator in the Poisson-equation can be used to localise charges: Evaluating $-\partial_i\Phi$ at any position it determines the electric field E_i, whose divergence $\epsilon^{ij}\partial_i\mathrm{E}_j = \partial_i\mathrm{D}^i$ must, therefore, reflect the amount of charge $4\pi\rho$ at that point, in accordance with the Gauß-law. This operation is rather straightforward in the discrete case of point charges:

$$\Phi(\boldsymbol{r}) = \sum_{i=1}^{n}\frac{q_i}{|\boldsymbol{r}-\boldsymbol{r}_i|} \quad\rightarrow\quad \Delta\Phi = \sum_{i=1}^{n} q_i\Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_i|} \propto -4\pi q_i \quad \text{if } \boldsymbol{r}=\boldsymbol{r}_i \tag{A.43}$$

and $\Delta\Phi = 0$ at any other position $\boldsymbol{r} \neq \boldsymbol{r}_i$. A direct calculation shows that this is in fact the case: Assuming spherical coordinates and positioning the charge at the origin implies indeed

$$\Delta\Phi = \Delta\frac{1}{r} = \frac{1}{r}\partial_r^2\left(r\frac{1}{r}\right) = 0 \tag{A.44}$$

for $r \neq 0$, but the expression can not be directly evaluated at the origin, as $1/r$ diverges. Instead, one can resort to averaging $\Delta\Phi$ over a small but finite integration volume V containing the charge and applying the Gauß-theorem:

$$\frac{1}{\mathrm{V}}\int_{\mathrm{V}}\mathrm{dV}\,\Delta\Phi = \frac{1}{\mathrm{V}}\int_{\mathrm{V}}\mathrm{dV}\,\epsilon^{ij}\partial_i\partial_j\Phi = \frac{1}{\mathrm{V}}\int_{\partial\mathrm{V}}\mathrm{dS}_i\,\underbrace{\epsilon^{ij}\partial_j\Phi}_{=\mathrm{D}^i\propto 1/r^2} = -\frac{1}{\mathrm{V}}\int_{\partial\mathrm{V}}\underbrace{r^2\mathrm{d}\Omega}_{=\mathrm{dS}}\,\frac{1}{r^2} = -\frac{4\pi}{\mathrm{V}}, \tag{A.45}$$

rewriting Δ as $\epsilon^{ij}\partial_i\partial_j$, making use of spherical symmetry and using $\nabla(1/r) = \partial_r(1/r) = -1/r^2$. This implies that the Laplace-operator Δ applied to $1/r$ yields either zero (if there is no charge at the location at which $\Delta\Phi$ is evaluated) or diverges (in the limit of $\mathrm{V}\rightarrow 0$ if one has caught the charge in the integration volume). These two results can be summarised using Dirac's δ_D-function

$$\Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -4\pi\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}'). \tag{A.46}$$

But electrodynamics is a continuum theory, and the computation has to work out for a charge density ρ as well: Integrating ρ over a volume V has to be the total charge q contained within that volume, such that the definitions

$$\rho(\boldsymbol{r}) = q\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}') \quad \rightarrow \quad \int_\mathrm{V} \mathrm{dV}\,\rho(\boldsymbol{r}) = q\int_\mathrm{V} \mathrm{dV}\,\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}') = q \tag{A.47}$$

become consistent due to the normalisation of the δ_D-function. Then,

$$\Phi(\boldsymbol{r}) = \int_\mathrm{V} \mathrm{dV}'\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{A.48}$$

is the solution for the potential, as it solves the Poisson-equation

$$\Delta\Phi(\boldsymbol{r}) = \int_\mathrm{V} \mathrm{dV}'\,\rho(\boldsymbol{r}')\Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \int_\mathrm{V} \mathrm{dV}'\,\rho(\boldsymbol{r}')(-4\pi)\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}') = -4\pi\rho(\boldsymbol{r}) \tag{A.49}$$

because of the shifting property of the δ_D-function

$$\int \mathrm{dV}'\,\rho(\boldsymbol{r}')\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}') = \rho(\boldsymbol{r}) \tag{A.50}$$

A collection of discrete point charges can be written as a charge density

$$\rho(\boldsymbol{r}) = \sum_{i=1}^{n} q_i\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}_i) \tag{A.51}$$

as a generalisation of equation (A.47), because

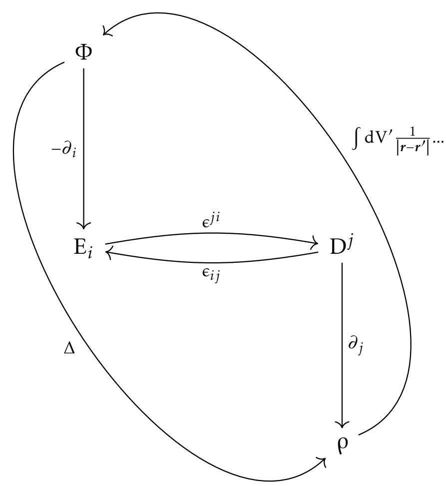
$$\Delta\Phi(\boldsymbol{r}) = \sum_{i=1}^{n} q_i\Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_i|} = \sum_{i=1}^{n} q_i(-4\pi)\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}_i) = -4\pi\rho(\boldsymbol{r}) \tag{A.52}$$

making the concept of discrete point charges and a continuous charge distribution compatible. Unitwise, the δ_D-function is an inverse volume, because it is normalised to unity, $\int \mathrm{dV}\,\delta_\mathrm{D}(\boldsymbol{r}) = 1$, such that $q_i\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}_i)$ becomes the charge density $\rho(\boldsymbol{r})$.

Clearly, both Φ and $\mathrm{E}_i = -\partial_i\Phi$ can exist at points where ρ vanishes, but at these positions, the divergence $\epsilon^{ij}\partial_i\mathrm{E}_j$ and consequently $\Delta\Phi$ are necessarily zero. From this point of view one could argue that the tensor $\partial_i\partial_j\Phi$ would naturally decompose into a traceless part and a trace,

$$\partial_i\partial_j\Phi = \left(\partial_i\partial_j\Phi - \frac{\gamma_{ij}}{3}\Delta\Phi\right) + \frac{\gamma_{ij}}{3}\Delta\Phi \tag{A.53}$$

where the trace $\Delta\Phi$ reflects the contribution to the field generated by the charge density at the same point where $\Delta\Phi$ is evaluated, whereas the traceless part is the contribution to the electrical field sourced elsewhere. The relationships between source ρ, potential Φ and the fields E_i and D^i is summarised concisely in this diagram:



(A.54)

A.9 *Potential energy of a static charge distribution*

The potential Φ is in the *electrostatic* case related to the energy needed to displace a charge in the electric field. If one assembles a charge distribution, one would need to invest energy for doing so, as all charges would need to be moved from infinity (where the potential vanishes) to their dedicated positions. Let's do this step by step: The first charge q_1 is located at $\boldsymbol{r}_1$ and generates a potential Φ_1 at position $\boldsymbol{r}$

$$\Phi_1(\boldsymbol{r}) = \frac{q_1}{|\boldsymbol{r} - \boldsymbol{r}_1|} \tag{A.55}$$

Then, moving q_2 from infinity to $\boldsymbol{r}_2$ in the electric field that is already generated by q_1 requires the energy $W_2 = q_2\Phi_1(\boldsymbol{r}_2)$, and continuing with a third charge q_3 to be taken from infinity to $\boldsymbol{r}_3$ requires $W_3 = q_3\left(\Phi_1(\boldsymbol{r}_3) + \Phi_2(\boldsymbol{r}_3)\right)$, which generalises to

$$W_n = q_n \sum_{m=1}^{n-1} \Phi_m(\boldsymbol{r}_n) \tag{A.56}$$

Adding up the amounts of work W_n needed for assembling the charge distribution suggests for the total energy W

$$W_{\mathrm{el}} = \sum_{n=1}^{N} W_n = \sum_{n=1}^{N} q_n \sum_{m=1}^{n-1} \Phi_m(\boldsymbol{r}_n) = \sum_{n=1}^{N} q_n \sum_{m=1}^{n-1} \frac{q_m}{|\boldsymbol{r}_n - \boldsymbol{r}_m|} = \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{q_m q_n}{|\boldsymbol{r}_n - \boldsymbol{r}_m|} \tag{A.57}$$

with a correction factor 1/2 due to the double counting, where we implicitly avoid the case $n = m$. In the continuum limit the relation becomes

$$W_{el} = \frac{1}{2}\int_V dV \int_V dV' \frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{2}\int_V dV\, \rho(\boldsymbol{r})\Phi(\boldsymbol{r}) = -\frac{1}{8\pi}\int_V dV\, \Phi(\boldsymbol{r})\Delta\Phi(\boldsymbol{r}) \quad (A.58)$$

inserting the definition of the potential in the first and the Poisson equation in the second step, replacing ρ by $\Delta\Phi$. By making use of the Leibnitz-rule one can rewrite

$$\Phi\Delta\Phi = \Phi\epsilon^{ij}\partial_i\partial_j\Phi = \epsilon^{ij}\partial_i(\Phi\partial_j\Phi) - \epsilon^{ij}\partial_i\Phi\cdot\partial_j\Phi \quad (A.59)$$

and arrive at the reformulation by virtue of the Gauß-theorem,

$$W_{el} = -\frac{1}{8\pi}\int_V dV\, \epsilon^{ij}\partial_i(\Phi\partial_j\Phi) + \frac{1}{8\pi}\int_V dV\, \epsilon^{ij}\partial_i\Phi\cdot\partial_j\Phi =$$

$$-\frac{1}{8\pi}\int_{\partial V} dS_i\, \epsilon^{ij}(\Phi\partial_j\Phi) + \frac{1}{8\pi}\int_V dV\, \epsilon^{ij}\partial_i\Phi\cdot\partial_j\Phi = \frac{1}{8\pi}\int_V dV\, \epsilon^{ij}E_iE_j \quad (A.60)$$

where the first term typically vanishes faster than the surface area ∂V increases, as $\Phi \propto 1/r$ and $\partial\Phi \propto 1/r^2$ at large distances from the charge distribution, dominating over the increase of $\partial V \propto r^2$. This result implies that the electric field can be assigned an energy density

$$w_{el} = \frac{\epsilon^{ij}E_iE_j}{8\pi} = \frac{E_iD^i}{8\pi} = \frac{\epsilon_{ij}D^iD^j}{8\pi} \quad \text{with} \quad W_{el} = \int_V dV\, w_{el}, \quad (A.61)$$

where the dielectric tensor now acts as a metric for computing the energy density from the fields, making it invariant under transformations. Positive definiteness of ϵ_{ij} (and consequently, of ϵ^{ij}) ensures that the energy density for electric fields comes out as positive. The energy density associated with magnetic fields is given in complete analogy by

$$w_{mag} = \frac{\mu^{ij}H_iH_j}{8\pi} = \frac{H_iB^i}{8\pi} = \frac{\mu_{ij}B^iB^j}{8\pi} \quad \text{with} \quad W_{mag} = \int_V dV\, w_{mag}. \quad (A.62)$$

A.10 *Boundary conditions for fields on surfaces*

Maxwell's equations allow a direct statement about the behaviour of the electric fields at boundaries in the static case: If a surface carries a charge surface density σ, an application of the Gauß-theorem to a small volume V situated on the surface yields

$$\int_V dV\,\partial_i D^i = \int_{\partial V} dS_i\, D^i = 4\pi\int_{\partial V} dS\,\sigma = 4\pi\sigma\Delta S = \Delta S(D_2^\perp - D_1^\perp) \quad\rightarrow\quad D_2^\perp = D_1^\perp + 4\pi\sigma \quad (A.63)$$

if the height of the integration volume is neglected; effectively one deals with a very flat box. Similarly, because $\epsilon^{ijk}\partial_jE_k = -\epsilon^{ijk}\partial_j\partial_k\Phi = 0$ for electrostatic fields,

$$\int_{S} \mathrm{d}S_i\, \epsilon^{ijk}\partial_j E_k = \int_{\partial S} \mathrm{d}r^i\; E_i = \Delta r\,(E_2^{\|} - E_1^{\|}) = 0 \quad \rightarrow \quad E_2^{\|} = E_1^{\|} \tag{A.64}$$

as an application of the Stokes-theorem to a small and flat area S perpendicular to the surface. The constitutive relations $D^i = \epsilon^{ij} E_j$ and its inverse then allow the computation of $E^{\perp}$ and $D^{\|}$.